

PREANTIPODES FOR DUAL-QUASI BIALGEBRAS

ALESSANDRO ARDIZZONI AND ALICE PAVARIN

ABSTRACT. It is known that a dual quasi-bialgebra with antipode H , i.e. a dual quasi-Hopf algebra, fulfils a fundamental theorem for right dual quasi-Hopf H -bicomodules. The converse in general is not true. We prove that, for a dual quasi-bialgebra H , the structure theorem amounts to the existence of a suitable map $S : H \rightarrow H$ that we call a preantipode of H .

CONTENTS

1. Introduction	1
2. Preliminaries	2
2.1. The category of (bi)comodules for a dual quasi-bialgebra.	2
2.2. An adjunction between ${}^H\mathfrak{M}_H^H$ and ${}^H\mathfrak{M}$	3
3. The notion of preantipode	3
References	9

1. INTRODUCTION

Let H be a bialgebra. It is well known that the functor $(-) \otimes H : \mathfrak{M} \rightarrow \mathfrak{M}_H^H$ determines an equivalence between the category \mathfrak{M} of vector spaces and the category \mathfrak{M}_H^H of right Hopf modules if and only if H has an antipode i.e. it is a Hopf algebra. The if part of this statement is the so-called structure (or fundamental) theorem for Hopf modules, which is due, in the finite-dimensional case, to Larson and Sweedler, see [LS, Proposition 1, page 82].

In 1989 Drinfeld introduced the concept of quasi-bialgebra in connection with the Knizhnik-Zamolodchikov system of partial differential equations. The axioms defining a quasi-bialgebra are a translation of monoidality of its representation category with respect to the diagonal tensor product. In [Dr], the antipode for a quasi-bialgebra (whence the concept of quasi-Hopf algebra) is introduced in order to make the category of its flat right modules rigid. Also for quasi-Hopf algebras a fundamental theorem was given first by Hausser and Nill [HN] and then by Bulacu and Caenepeel [BC]. If we draw our attention to the category of co-representations of H , we get the concepts of dual quasi-bialgebra and of dual quasi-Hopf algebra. This notions have been introduced in [Maj2] in order to prove a Tannaka-Krein type Theorem for quasi-Hopf algebras.

A fundamental theorem for finite-dimensional dual quasi-Hopf algebras can be obtained by duality using the results in [HN]. For an arbitrary dual quasi-Hopf algebra, the fundamental theorem is proved in [Sch2] as follows. Schauenburg proves first that a dual quasi-bialgebra satisfies the fundamental theorem if and only if the category of its finite-dimensional co-representations is rigid. On the one hand any dual quasi-Hopf algebra fulfils this property. On the other hand the converse is not true in general. Thus dual quasi-Hopf algebras do not exhaust the class of dual quasi-bialgebras satisfying the fundamental theorem.

1991 *Mathematics Subject Classification.* Primary 16W30.

Key words and phrases. Dual quasi-bialgebras, structure theorem, preantipode, right dual quasi-Hopf bicomodules.

This paper was written while the first author was member of GNSAGA and both authors were partially supported by MIUR within the National Research Project PRIN 2007. Part of the paper is included in the master degree thesis of the second author which was developed under the supervision of the first author.

It is remarkable that the equivalence giving the fundamental theorem in the case of ordinary Hopf algebras must be substituted, in the “quasi” case, by the equivalence between the category of left H -comodules ${}^H\mathfrak{M}$ and the category of right dual quasi-Hopf H -bicomodules ${}^H\mathfrak{M}_H^H$ (essentially this is due to the fact that, unlike the classical case, H is not a right H -comodule algebra but is still an H -bicomodule algebra). The main result of this paper, Theorem 3.9, establishes that such an equivalence amounts to the existence of a suitable map $S : H \rightarrow H$ that we call a preantipode. By the foregoing, any dual quasi-bialgebra with antipode (i.e. a dual quasi-Hopf algebra) admits a preantipode (in Theorem 3.10 we give a direct prove of this fact) but the converse is not true in general (see Remark 3.12). Finally, in Example 3.14, we construct a preantipode for a group algebra endowed with a normalized 3-cocycle.

2. PRELIMINARIES

In this section we recall the definitions and results that will be needed in the paper.

NOTATION 2.1. *Throughout this paper \mathbb{k} will denote a field. All vector spaces will be defined over \mathbb{k} . The unadorned tensor product \otimes will denote the tensor product over \mathbb{k} if not stated differently.*

2.2. **Monoidal Categories.** Recall that (see [Ka, Chap. XI]) a *monoidal category* is a category \mathcal{M} endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \rightarrow X$, $r_X : X \otimes \mathbf{1} \rightarrow X$, for every X, Y, Z in \mathcal{M} . The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*, that is the following relation

$$(U \otimes a_{V,W,X}) \circ a_{U,V,W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W \otimes X} \circ a_{U \otimes V,W,X}$$

holds true, for every U, V, W, X in \mathcal{M} . The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom*, that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in \mathcal{M} .

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra A in \mathcal{M} one can define the categories ${}_A\mathcal{M}$, \mathcal{M}_A and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively.

DEFINITION 2.3. A dual quasi-bialgebra is a datum $(H, m, u, \Delta, \varepsilon, \omega)$ where

- (H, Δ, ε) is a coassociative coalgebra;
 - $m : H \otimes H \rightarrow H$ and $u : \mathbb{k} \rightarrow H$ are coalgebra maps called multiplication and unit respectively; we set $1_H := u(1_{\mathbb{k}})$;
 - $\omega : H \otimes H \otimes H \rightarrow \mathbb{k}$ is a unital 3-cocycle i.e. it is convolution invertible and satisfies
- (1) $\omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) = m_{\mathbb{k}}(\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) * m_{\mathbb{k}}(\omega \otimes \varepsilon)$ and
- (2) $v(h \otimes k \otimes l) = \varepsilon(h)\varepsilon(k)\varepsilon(l)$ whenever $1_H \in \{h, k, l\}$.
- m is quasi-associative and unitary i.e. it satisfies
- (3) $m(H \otimes m) * \omega = \omega * m(m \otimes H)$,
- (4) $m(1_H \otimes h) = h$, for all $h \in H$,
- (5) $m(h \otimes 1_H) = h$, for all $h \in H$.

ω is called *the reassociator* of the dual quasi-bialgebra.

2.1. **The category of (bi)comodules for a dual quasi-bialgebra.** Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. It is well known that the category \mathfrak{M}^H of right H -comodules becomes a monoidal category as follows. Given a right H -comodule V , we denote by $\rho = \rho_V^r : V \rightarrow V \otimes H$, $\rho(v) = v_0 \otimes v_1$, its right H -coaction. The tensor product of two right H -comodules V and W is a comodule via diagonal coaction i.e. $\rho(v \otimes w) = v_0 \otimes w_0 \otimes v_1 w_1$. The unit is \mathbb{k} , which is regarded as a right H -comodule via the trivial coaction i.e. $\rho(k) = k \otimes 1_H$. The associativity and unit constraints are defined, for all $U, V, W \in \mathfrak{M}^H$ and $u \in U, v \in V, w \in W, k \in \mathbb{k}$, by

$$\begin{aligned} a_{U,V,W}^H(u \otimes v \otimes w) &:= u_0 \otimes (v_0 \otimes w_0) \omega(u_1 \otimes v_1 \otimes w_1), \\ l_U(k \otimes u) &:= ku \quad \text{and} \quad r_U(u \otimes k) := uk. \end{aligned}$$

The monoidal category we have just described will be denoted by $(\mathfrak{M}^H, \otimes, \mathbb{k}, a^H, l, r)$.

Similarly, the monoidal categories $({}^H\mathfrak{M}, \otimes, \mathbb{k}, {}^H a, l, r)$ and $({}^H\mathfrak{M}^H, \otimes, \mathbb{k}, {}^H a^H, l, r)$ are introduced. We just point out that

$$\begin{aligned} {}^H a_{U,V,W}(u \otimes v \otimes w) &:= \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes (v_0 \otimes w_0), \\ {}^H a^H_{U,V,W}(u \otimes v \otimes w) &:= \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes (v_0 \otimes w_0)\omega(u_1 \otimes v_1 \otimes w_1). \end{aligned}$$

REMARK 2.4. We know that, if $(H, m, u, \Delta, \varepsilon, \omega)$ is a dual quasi bialgebra, we cannot construct the category \mathfrak{M}_H , because H is not an algebra. Moreover H is not an algebra in \mathfrak{M}^H or in ${}^H\mathfrak{M}$. On the other hand $((H, \rho_H^l, \rho_H^r), m, u)$ is an algebra in the monoidal category $({}^H\mathfrak{M}^H, \otimes, \mathbb{k}, {}^H a^H, l, r)$ with $\rho_H^l = \rho_H^r = \Delta$. Thus, the only way to construct the category ${}^H\mathfrak{M}_H^H$ is to consider the right H -modules in ${}^H\mathfrak{M}^H$. Hence, we can set

$${}^H\mathfrak{M}_H^H := ({}^H\mathfrak{M}^H)_H.$$

The category ${}^H\mathfrak{M}_H^H$ is the so called category of right dual quasi-Hopf H -bicomodules [BC, Remark 2.3].

REMARK 2.5. [AMS, Example 1.5(a)] Let (A, m, u) be an algebra in a given monoidal category $(\mathcal{M}, \otimes, 1, a, l, r)$. Then the assignments $M \mapsto (M \otimes A, (M \otimes m) \circ a_{A,A,A})$ and $f \mapsto f \otimes A$ define a functor $T : \mathcal{M} \rightarrow \mathcal{M}_A$. Moreover the forgetful functor $U : \mathcal{M}_A \rightarrow \mathcal{M}$ is a right adjoint of T .

2.2. An adjunction between ${}^H\mathfrak{M}_H^H$ and ${}^H\mathfrak{M}$. We are going to construct an adjunction between ${}^H\mathfrak{M}_H^H$ and ${}^H\mathfrak{M}$ that will be crucial afterwards.

2.6. Consider the functor $L : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}^H$ defined on objects by $L(\bullet V) := \bullet V^\circ$ where the upper empty dot denotes the trivial right coaction while the upper full dot denotes the given left H -coaction of V . The functor L has a right adjoint $R : {}^H\mathfrak{M}^H \rightarrow {}^H\mathfrak{M}$ defined on objects by $R(\bullet M^\bullet) := \bullet M^{coH}$, where $M^{coH} := \{m \in M \mid m_0 \otimes m_1 = m \otimes 1_H\}$ is the space of right H -coinvariant elements in M .

By Remark 2.5, the forgetful functor $U : {}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}^H, U(\bullet M^\bullet) := \bullet M^\bullet$ has a right adjoint, namely the functor $T : {}^H\mathfrak{M}^H \rightarrow {}^H\mathfrak{M}_H^H, T(\bullet M^\bullet) := \bullet M^\bullet \otimes \bullet H^\bullet$. Here the upper dots indicate on which tensor factors we have a codiagonal coaction and the lower dot indicates where the action takes place. Explicitly, the structure of $T(\bullet M^\bullet)$ is given as follows:

$$\begin{aligned} \rho_{M \otimes H}^l(m \otimes h) &:= m_{-1}h_1 \otimes (m_0 \otimes h_2), \\ \rho_{M \otimes H}^r(m \otimes h) &:= (m_0 \otimes h_1) \otimes m_1h_2, \\ (m \otimes h)l &:= \omega^{-1}(m_{-1} \otimes h_1 \otimes l_1)m_0 \otimes h_2l_2\omega(m_1 \otimes h_3 \otimes l_3). \end{aligned}$$

Define the functors $F := TL : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}_H^H$ and $G := RU : {}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}$. Explicitly $G(\bullet M^\bullet) = \bullet M^{coH}$ and $F(\bullet V) := \bullet V^\circ \otimes \bullet H^\bullet$ so that, for every $v \in V, h, l \in H$,

$$\begin{aligned} \rho_{V \otimes H}^l(v \otimes h) &= v_{-1}h_1 \otimes (v_0 \otimes h_2), \\ \rho^r(v \otimes h) &= (v \otimes h_1) \otimes h_2, \\ (v \otimes h)l &= \omega^{-1}(v_{-1} \otimes h_1 \otimes l_1)v_0 \otimes h_2l_2. \end{aligned}$$

The following result is essentially the right-hand version of [Sch2, Lemma 2.1]

THEOREM 2.7. *The functor $F : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}_H^H$ is a left adjoint of the functor G . Moreover, the counit and the unit of the adjunction are given respectively by $\epsilon_M : FG(M) \rightarrow M, \epsilon_M(x \otimes h) := xh$ and by $\eta_N : N \rightarrow GF(N), \eta_N(n) := n \otimes 1_H$, for every $M \in {}^H\mathfrak{M}_H^H, N \in {}^H\mathfrak{M}$. Moreover η_N is an isomorphism for any $N \in {}^H\mathfrak{M}$. In particular the functor F is fully faithful.*

3. THE NOTION OF PREANTIPODE

The main result of this section is Theorem 3.9, where we show that, for a dual quasi-bialgebra H , the adjunction (F, G) is an equivalence of categories if and only if H admits what will be called a preantipode.

DEFINITION 3.1. [Maj1, page 66] A dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ endowed with a coalgebra antimorphism

$$s : H \rightarrow H$$

and two maps α, β in H^* , such that, for all $h \in H$:

$$(6) \quad h_1 \beta(h_2) s(h_3) = \beta(h) 1_H$$

$$(7) \quad s(h_1) \alpha(h_2) h_3 = \alpha(h) 1_H$$

$$(8) \quad \omega(h_1 \otimes \beta(h_2) s(h_3) \alpha(h_4) \otimes h_5) = \varepsilon(h) = \omega^{-1}(s(h_1) \otimes \alpha(h_2) h_3 \beta(h_4) \otimes s(h_5))$$

REMARK 3.2. Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a dual quasi-Hopf algebra. In [BC, Remark 2.3] it is studied the problem of finding an isomorphism between M and $M^{coH} \otimes H$, for each $M \in {}^H\mathfrak{M}_H^H$. The idea is to consider the surjection $P_M : M \rightarrow M^{coH}$ defined, for all $m \in M$, by $P_M(m) = m_0 \beta(m_1) s(m_2)$. Bulacu and Caenepeel observe that a natural candidate for the bijection could be the map $\gamma_M : M \rightarrow M^{coH} \otimes H$, defined, for each $M \in {}^H\mathfrak{M}_H^H$, by setting $\gamma_M(m) = P_M(m_0) \otimes m_1$. Unfortunately there is no proof of the fact that γ_M is bijective.

Next result characterizes when the adjunction (F, G) is an equivalence of categories in term of the existence of a suitable map τ .

PROPOSITION 3.3. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. The following assertions are equivalent.*

(i) *The adjunction (F, G) is an equivalence.*

(ii) *For each $M \in {}^H\mathfrak{M}_H^H$, there exists a \mathbb{k} -linear map $\tau : M \rightarrow M^{coH}$ such that:*

$$(9) \quad \tau(mh) = \omega^{-1}[\tau(m_0)_{-1} \otimes m_1 \otimes h] \tau(m_0)_0, \text{ for all } h \in H, m \in M,$$

$$(10) \quad m_{-1} \otimes \tau(m_0) = \tau(m_0)_{-1} m_1 \otimes \tau(m_0)_0, \text{ for all } m \in M,$$

$$(11) \quad \tau(m_0) m_1 = m \quad \forall m \in M.$$

(iii) *For each $M \in {}^H\mathfrak{M}_H^H$, there exists a \mathbb{k} -linear map $\tau : M \rightarrow M^{coH}$ such that (11) holds and*

$$(12) \quad \tau(mh) = m \varepsilon(h), \text{ for all } h \in H, m \in M^{coH}.$$

Proof. (i) \Rightarrow (ii) If (F, G) is an equivalence, then $\epsilon_M : M^{coH} \otimes H \rightarrow M$ is an isomorphism for each $M \in {}^H\mathfrak{M}_H^H$. Set, for any $m \in M$:

$$\epsilon_M^{-1}(m) = m^0 \otimes m^1 \in M^{coH} \otimes H.$$

We know that ϵ_M^{-1} is right H -colinear, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M^{-1}} & M^{coH} \otimes H \\ \rho_M^r \downarrow & & \downarrow \rho_{M^{coH} \otimes H}^r \\ M \otimes H & \xrightarrow{\epsilon_M^{-1} \otimes H} & (M^{coH} \otimes H) \otimes H \end{array}$$

In terms of elements this means that, for all $m \in M$,

$$(13) \quad m^0 \otimes (m^1)_1 \otimes (m^1)_2 = (m_0)^0 \otimes (m_0)^1 \otimes m_1.$$

Let us define $\tau : M \rightarrow M^{coH}$, $\tau(m) := m^0 \varepsilon(m^1)$. By applying $M^{coH} \otimes \varepsilon \otimes H$ to both terms in (13), we get

$$\epsilon_M^{-1}(m) = \tau(m_0) \otimes m_1.$$

We will now prove (9), (10) and (11).

From the H -linearity of ϵ_M^{-1} , the following diagram commutes:

$$\begin{array}{ccc} M \otimes H & \xrightarrow{\epsilon_M^{-1} \otimes H} & (M^{coH} \otimes H) \otimes H \\ \mu_M \downarrow & & \downarrow \mu_{M^{coH} \otimes H} \\ M & \xrightarrow{\epsilon_M^{-1}} & M^{coH} \otimes H \end{array}$$

i.e.

$$\omega^{-1}(\tau(m_0)_{-1} \otimes m_1 \otimes h_1) \tau(m_0)_0 \otimes m_2 h_2 = \tau(m_0 h_1) \otimes m_1 h_2, \text{ for all } m \in M, h \in H.$$

Applying now $M^{coH} \otimes \varepsilon$ on both terms, we obtain exactly (9).

ϵ_M^{-1} is also left H -colinear, that is:

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M^{-1}} & M^{coH} \otimes H \\ \downarrow \iota_{\rho_M} & & \downarrow \iota_{\rho_{M^{coH} \otimes H}} \\ H \otimes M & \xrightarrow{H \otimes \epsilon_M^{-1}} & H \otimes (M^{coH} \otimes H) \end{array}$$

i.e.

$$\tau(m_0)_{-1} m_1 \otimes \tau(m_0)_0 \otimes m_2 = m_{-1} \otimes \tau(m_0) \otimes m_1, \text{ for all } m \in M.$$

To obtain (10) we have to apply $H \otimes M^{coH} \otimes \varepsilon$ to both terms.

Finally, recalling the fact that $\epsilon_M \epsilon_M^{-1} = \mathbb{1}_M$, we have the following equality, for all $m \in M$,

$$m = \epsilon_M \epsilon_M^{-1}(m) = \epsilon_M(\tau(m_0) \otimes m_1) = \tau(m_0) m_1.$$

(ii) \Rightarrow (iii) It is trivial.

(iii) \Rightarrow (i) The only thing that we have to prove is the invertibility of ϵ_M , for any $M \in {}^H\mathfrak{M}_H^H$.

Let us define $\psi : M \rightarrow M^{coH} \otimes H$ by setting $\psi(m) := \tau(m_0) \otimes m_1$. Then, for all $m \in M$,

$$\epsilon_M \psi_M(m) = \tau(m_0) m_1 \stackrel{(11)}{=} m$$

and for all $m \in M^{coH}, h \in H$,

$$\psi_M \epsilon_M(m \otimes h) = \psi_M(mh) = \tau(m_0 h_1) \otimes m_1 h_2 \stackrel{m \in M^{coH}}{=} \tau(mh_1) \otimes h_2 \stackrel{(12)}{=} m \otimes h.$$

So ψ_M is the inverse of ϵ_M , for all $M \in {}^H\mathfrak{M}_H^H$.

From Theorem 2.7, we have that η_M is always an isomorphism, so we have the equivalence. \square

REMARK 3.4. Let $\tau : M \rightarrow M^{coH}$ be a \mathbb{k} -linear map such that (11) holds. Following the proof of Proposition 3.3, it is clear that a map τ fulfils (12) if and only if it fulfils (9) and (10).

3.5. Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. As observed in Remark 2.4, (H, m, u) is an algebra in the monoidal category $({}^H\mathfrak{M}_H^H, \otimes, \mathbb{k}, {}^H_a H, l, r)$, where both comodule structures are given by Δ . Consider the functor T of 2.6 and set

$$H \widehat{\otimes} H := T({}^\circ H^\bullet) = {}^\circ H^\bullet \otimes {}^\bullet H^\bullet,$$

where in ${}^\circ H^\bullet$ the empty dot denotes the trivial left H -comodule structure and the full dot denotes the right coaction given by Δ . Explicitly, for $h, k, l \in H$, the structure of $H \widehat{\otimes} H$ is given by

$$\begin{aligned} \rho_{H \widehat{\otimes} H}^r(h \otimes k) &= (h_1 \otimes k_1) \otimes h_2 k_2, \\ \rho_{H \widehat{\otimes} H}^l(h \otimes k) &= k_1 \otimes h \otimes k_2, \\ (h \otimes k)l &= h_1 \otimes k_1 l_1 \omega(h_2 \otimes k_2 \otimes l_2). \end{aligned}$$

Set

$$\widehat{\epsilon}_H := \epsilon_{H \widehat{\otimes} H}^{F,G} : (H \widehat{\otimes} H)^{coH} \otimes H \rightarrow H \widehat{\otimes} H$$

so that, for $x_i \otimes y_i \in (H \widehat{\otimes} H)^{coH}$ (summation understood) and $h \in H$,

$$\widehat{\epsilon}_H((x_i \otimes y_i) \otimes h) = (x_i \otimes y_i) \cdot h = x_{i_1} \otimes y_{i_1} h_1 \omega(x_{i_2} \otimes y_{i_2} \otimes h_2).$$

Suppose now that $\widehat{\epsilon}_H$ is an isomorphism. Then, from the right H -colinearity of $\widehat{\epsilon}_H$, we deduce the right H -colinearity of $\widehat{\epsilon}_H^{-1}$, i.e., if we set

$$h^{[1]} \otimes h^{[2]} \otimes h^{[3]} := \widehat{\epsilon}_H^{-1}(h \otimes 1_H),$$

we have:

$$(14) \quad h^{[1]} \otimes h^{[2]} \otimes (h^{[3]})_1 \otimes (h^{[3]})_2 = (h_1)^{[1]} \otimes (h_1)^{[2]} \otimes (h_1)^{[3]} \otimes h_2.$$

Define

$$\beta(h) := h^{[1]} \otimes h^{[2]} \varepsilon(h^{[3]}) \in (H \widehat{\otimes} H)^{coH}.$$

Then, by applying $H \otimes H \otimes \varepsilon \otimes H$ on each side of (14), we get:

$$(15) \quad \widehat{\varepsilon}_H^{-1}(h \otimes 1_H) = \beta(h_1) \otimes h_2.$$

Let consider a new left coaction for $H \widehat{\otimes} H$:

$$\rho^l : H \widehat{\otimes} H \rightarrow H \otimes (H \widehat{\otimes} H), \rho^l(h \otimes k) := h_1 \otimes h_2 \otimes k = (\Delta \otimes H)(h \otimes k).$$

Let us verify that $\widehat{\varepsilon}_H$ is H -left colinear respect on this new coaction i.e. that the following commutes:

$$\begin{array}{ccc} (H \widehat{\otimes} H)^{coH} \otimes H & \xrightarrow{\widehat{\varepsilon}_H} & H \widehat{\otimes} H \\ \Delta \otimes H \otimes H \downarrow & & \downarrow \Delta \otimes H \\ H \otimes ((H \widehat{\otimes} H)^{coH} \otimes H) & \xrightarrow{H \otimes \widehat{\varepsilon}_H} & H \otimes (H \widehat{\otimes} H) \end{array}$$

In fact

$$\begin{aligned} (\Delta \otimes H) \circ \widehat{\varepsilon}_H(x_i \otimes y_i \otimes h) &= (\Delta \otimes H)(x_{i_1} \otimes y_{i_1} h_1 \omega(x_{i_2} \otimes y_{i_2} \otimes h_2)) \\ &= x_{i_1} \otimes x_{i_2} \otimes y_{i_1} h_1 \omega(x_{i_3} \otimes y_{i_2} \otimes h_2) \\ &= (H \otimes \widehat{\varepsilon}_H) \circ (\Delta \otimes H \otimes H)(x_i \otimes y_i \otimes h). \end{aligned}$$

Then also $\widehat{\varepsilon}_H^{-1}$ is left H -colinear with respect to this structure, i.e:

$$(\Delta \otimes H \otimes H) \circ \widehat{\varepsilon}_H^{-1}(x \otimes 1_H) = (H \otimes \widehat{\varepsilon}_H^{-1}) \circ (\Delta \otimes H)(x \otimes 1_H).$$

If we set $h^1 \otimes h^2 := \beta(h)$, the last displayed equality means

$$(16) \quad ((x_1)^1)_1 \otimes ((x_1)^1)_2 \otimes (x_1)^2 \otimes x_2 = x_1 \otimes (x_2)^1 \otimes (x_2)^2 \otimes x_3.$$

Let us define, for any $x \in H$,

$$S(x) := \varepsilon(x^1)x^2.$$

By applying $H \otimes \varepsilon \otimes H \otimes \varepsilon$ on each side of (16) it results:

$$(17) \quad \beta(x) = x_1 \otimes S(x_2).$$

Now let us deduce the properties of S from the bijectivity of $\widehat{\varepsilon}_H$ and its colinearity.

From (17) and $\beta(x) \in (H \widehat{\otimes} H)^{coH}$, we have:

$$x_1 \otimes S(x_3)_1 \otimes x_2 S(x_3)_2 = x_1 \otimes S(x_2) \otimes 1_H.$$

By applying $\varepsilon \otimes H \otimes H$ on both sides, we obtain that

$$(18) \quad S(x_2)_1 \otimes x_1 S(x_2)_2 = S(x) \otimes 1_H, \text{ for all } x \in H.$$

From

$$\begin{aligned} x \otimes 1_H &= \widehat{\varepsilon}_H \widehat{\varepsilon}_H^{-1}(x \otimes 1_H) \\ &= \widehat{\varepsilon}_H(x_1 \otimes S(x_2) \otimes x_3) \\ &= x_{1_1} \otimes S(x_2)_1 x_3 \omega(x_{1_2} \otimes S(x_2)_2 \otimes x_4) \\ &= x_1 \otimes S(x_3)_1 x_4 \omega(x_2 \otimes S(x_3)_2 \otimes x_5) \end{aligned}$$

we get

$$x \otimes 1_H = x_1 \otimes S(x_3)_1 x_4 \omega(x_2 \otimes S(x_3)_2 \otimes x_5).$$

By applying $\varepsilon \otimes \varepsilon$ on both sides, we get

$$(19) \quad \omega(x_1 \otimes S(x_2) \otimes x_3) = \varepsilon(x), \text{ for all } x \in H.$$

From the left H -colinearity with respect to the usual coaction of $\widehat{\varepsilon}_H^{-1}$ we have:

$$\rho^l \circ \widehat{\varepsilon}_H^{-1}(x \otimes 1_H) = (H \otimes \widehat{\varepsilon}_H^{-1})(1_H \otimes x \otimes 1_H)$$

i.e.

$$S(x_2)_1 x_3_1 \otimes x_1 \otimes S(x_2)_2 \otimes x_3_2 = 1_H \otimes x_1 \otimes S(x_2) \otimes x_3.$$

Applying $H \otimes \varepsilon \otimes H \otimes \varepsilon$ on both sides, we obtain:

$$(20) \quad S(x_1)_1 x_2 \otimes S(x_1)_2 = 1_H \otimes S(x), \text{ for all } x \in H.$$

DEFINITION 3.6. A *preantipode* for a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ is a k -linear map $S : H \rightarrow H$ such that (18), (19) and (20) hold.

REMARK 3.7. Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. Then the following equalities hold

$$(21) \quad h_1 \varepsilon S(h_2) = \varepsilon S(h) 1_H = \varepsilon S(h_1) h_2 \text{ for all } h \in H.$$

LEMMA 3.8. Let $(H, m, u, \Delta, \varepsilon, \omega, S)$ be a dual quasi-bialgebra with a preantipode. For any $M \in {}^H \mathfrak{M}_H^H$ and $m \in M$, set

$$(22) \quad \tau(m) := \omega[m_{-1} \otimes S(m_1)_1 \otimes m_2] m_0 S(m_1)_2.$$

Then (22) defines a map $\tau : M \rightarrow M^{coH}$ which fulfills (9), (10) and (11).

Proof. To prove that τ is well-defined we have to check if $\text{Im} \tau \subseteq M^{coH}$.

Let us compute, for all $m \in M$,

$$\begin{aligned} \rho^\tau(\tau(m)) &= \omega(m_{-1} \otimes S(m_2)_1 \otimes m_3) m_0 S(m_2)_2 \otimes m_1 S(m_2)_3 \\ &\stackrel{(18)}{=} \omega(m_{-1} \otimes S(m_1)_1 \otimes m_2) m_0 S(m_1)_2 \otimes 1_H \\ &= \tau(m) \otimes 1_H. \end{aligned}$$

Now let us prove (11). For all $m \in M$,

$$\begin{aligned} \tau(m_0) m_1 &= \omega(m_{-1} \otimes S(m_1)_1 \otimes m_2) (m_0 S(m_1)_2) m_3 \\ &\stackrel{(3)}{=} \left[\begin{array}{c} \omega(m_{-1} \otimes S(m_1)_1 \otimes m_2) \omega^{-1}(m_{0-1} \otimes S(m_1)_2 \otimes m_3) \\ m_{0_0} (S(m_1)_3 m_4) \omega(m_{0_1} \otimes S(m_1)_4 \otimes m_4) \end{array} \right] \\ &= m_0 (S(m_2)_1 m_3) \omega(m_1 \otimes S(m_2)_2 \otimes m_3) \\ &\stackrel{(20)}{=} m_{0_1 H} \omega(m_1 \otimes S(m_2) \otimes m_3) \\ &\stackrel{(19)}{=} m 1_H. \end{aligned}$$

Let us check (12). For $m \in M^{coH}$, $h \in H$,

$$\begin{aligned} \tau(mh) &= \omega(m_{-1} h_1 \otimes S(m_1 h_3)_1 \otimes m_2 h_4) (m_0 h_2) S(m_1 h_3)_2 \\ &\stackrel{(*)}{=} \omega(m_{-1} h_1 \otimes S(h_3)_1 \otimes h_4) (m_0 h_2) S(h_3)_2 \\ &\stackrel{(3)}{=} \omega(m_{-1} h_1 \otimes S(h_3)_1 \otimes h_4) \omega^{-1}(m_{0-1} \otimes h_{2_1} \otimes S(h_3)_2) m_{0_0} (h_{2_2} S(h_3)_3) \\ &= \omega(m_{-1_1} h_1 \otimes S(h_4)_1 \otimes h_5) \omega^{-1}(m_{-1_2} \otimes h_2 \otimes S(h_4)_2) m_0 (h_3 S(h_4)_3) \\ &\stackrel{(18)}{=} \omega(m_{-1_1} h_1 \otimes S(h_3)_1 \otimes h_4) \omega^{-1}(m_{-1_2} \otimes h_2 \otimes S(h_3)_2) m_0 \\ &\stackrel{(1)}{=} \omega^{-1}(m_{-1_1} \otimes h_1 \otimes S(h_4)_1 h_5) \omega(h_2 \otimes S(h_4)_2 \otimes h_6) \omega(m_{-1_2} \otimes h_3 S(h_4)_3 \otimes h_7) m_0 \\ &\stackrel{(18)}{=} \omega^{-1}(m_{-1} \otimes h_1 \otimes S(h_3)_1 h_4) \omega(h_2 \otimes S(h_3)_2 \otimes h_5) m_0 \\ &\stackrel{(20)}{=} \omega(h_2 \otimes S(h_3) \otimes h_4) m_0 \\ &\stackrel{(19)}{=} \varepsilon(h) m, \end{aligned}$$

where in (*) we used that $m \in M^{coH}$. Now, by Remark 3.4, we conclude. \square

We are now able to state the main theorem characterizing when (F, G) is an equivalence.

THEOREM 3.9. For a dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ the following are equivalent.

- (i) The adjunction (F, G) of Theorem 2.7 is an equivalence of categories.

- (ii) $\widehat{\epsilon}_H$ is a bijection.
- (iii) There exists a preantipode.

Proof. (i) \Rightarrow (ii) It is trivial because $\widehat{\epsilon}_H = \epsilon_{\frac{F,G}{H \otimes H}}$.

(ii) \Rightarrow (iii) See Remark 3.5.

(iii) \Rightarrow (i) Let us define τ as in (22). Apply Lemma 3.8 and Proposition 3.3. □

THEOREM 3.10. *Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ be a dual quasi-Hopf algebra. Then*

$$S := \beta * s * \alpha$$

is a preantipode. Here $$ denotes the convolution product.*

Proof. Fix $h \in H$. Let us check (18):

$$\begin{aligned} S(h_2)_1 \otimes h_1 S(h_2)_2 &= \beta(h_2) s(h_3)_1 \alpha(h_4) \otimes h_1 s(h_3)_2 \\ &\stackrel{(*)}{=} s(h_3)_2 \alpha(h_4) \otimes h_1 \beta(h_2) s(h_3)_1 \\ &\stackrel{(6)}{=} s(h_2) \alpha(h_3) \otimes \beta(h_1) 1_H \\ &= S(h) \otimes 1_H. \end{aligned}$$

where in $(*)$ we used that s is a coalgebra anti-homomorphism. Let us prove (20):

$$\begin{aligned} S(h_1)_1 h_2 \otimes S(h_1)_2 &= \beta(h_1) s(h_2)_1 \alpha(h_3) h_4 \otimes s(h_2)_2 \\ &\stackrel{(*)}{=} \beta(h_1) s(h_2)_2 \alpha(h_3) h_4 \otimes s(h_2)_1 \\ &\stackrel{(7)}{=} \beta(h_1) \alpha(h_3) \otimes s(h_2). \\ &= 1_H \otimes S(h). \end{aligned}$$

Finally we have to verify (19):

$$\omega(h_1 \otimes S(h_2) \otimes h_3) = \omega(h_1 \beta(h_2) \otimes s(h_3) \otimes \alpha(h_4) h_5) \stackrel{(8)}{=} \varepsilon(h).$$

□

COROLLARY 3.11. *(Cf. right-hand version of [Sch2, Corollary 2.7]) Let $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ a dual quasi-Hopf algebra. Then the adjunction (F, G) of Theorem 2.7 is an equivalence of categories.*

REMARK 3.12. Let us show that the converse of Theorem 3.10 does not hold true in general. By [Sch1, Example 4.5.1], there is a dual quasi-bialgebra H which is not a dual quasi-Hopf algebra and such that the category ${}^H\mathfrak{M}_f$ of finite-dimensional left H -comodules is left and right rigid. Then, by the right-hand version of [Sch2, Theorem 3.1], the adjunction (F, G) of Theorem 2.7 is an equivalence of categories. By Theorem 3.9, H has a preantipode.

Nevertheless, for a finite-dimensional dual quasi-bialgebra, the existence of an antipode amounts to the existence of a preantipode. This follows by duality in view of [Sch2, Theorem 3.1].

REMARK 3.13. Consider a dual quasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ and the associated preantipode S . Then the map τ defined by (22) becomes:

$$\begin{aligned} \tau(m) &= \omega(m_{-1} \otimes S(m_1)_1 \otimes m_2) m_0 S(m_1)_2 \\ &= \omega(m_{-1} \otimes \beta(m_1) s(m_2)_1 \otimes m_2) m_0 s(m_2)_2 \alpha(m_3) \\ &= \omega(m_{-1} \otimes \beta(m_1) s(m_3) \otimes m_5) m_0 s(m_2) \alpha(m_4) \\ &= \omega(m_{-1} \otimes s(m_3) \otimes m_5) m_0 \beta(m_1) s(m_2) \alpha(m_4) \\ &= \omega(m_{-1} \otimes s(m_1) \otimes m_3) P_M(m_0) \alpha(m_2), \end{aligned}$$

where P_M denotes the map recalled in Remark 3.2.

EXAMPLE 3.14. Let G be a group. Let $\theta : G \times G \times G \rightarrow \mathbb{k}^* := \mathbb{k} \setminus \{0\}$ be a normalized 3-cocycle on the group G in the sense of [Maj1, Example 2.3.2, page 54] i.e. a map such that, for all $g, h, k, l \in H$

$$\begin{aligned}\theta(g, 1_G, h) &= 1 \\ \theta(h, k, l)\theta(g, hk, l)\theta(g, h, k) &= \theta(g, h, kl)\theta(gh, k, l).\end{aligned}$$

Then θ can be extended by linearity to a reassociator $\omega : \mathbb{k}G \otimes \mathbb{k}G \otimes \mathbb{k}G \rightarrow \mathbb{k}$ making $\mathbb{k}G$ a dual quasi-bialgebra with usual underlying algebra and coalgebra structures. Moreover, by [AM, page 193], $\mathbb{k}G$ is indeed a dual quasi-Hopf algebra where $\alpha, \beta : \mathbb{k}G \rightarrow \mathbb{k}$ are defined on generators by $\alpha(g) := 1_{\mathbb{k}}$ and $\beta(g) := [\omega(g, g^{-1}, g)]^{-1}$ and the antipode $s : \mathbb{k}G \rightarrow \mathbb{k}G$ is given by $s(g) := g^{-1}$, for all $g \in G$. By Theorem 3.10, we have a preantipode on $\mathbb{k}G$, which is defined by $S := \beta * s * \alpha$ so that

$$S(g) := [\omega(g, g^{-1}, g)]^{-1}g^{-1}, \text{ for all } g \in G.$$

Note that, unlike the antipode, this preantipode is not an coalgebra anti-homomorphism as

$$S(g_2) \otimes S(g_1) = [\omega(g, g^{-1}, g)]^{-1}\Delta S(g), \text{ for all } g \in G.$$

REFERENCES

- [AM] H. Albuquerque, S. Majid, *Quasialgebra structure of the octonions*. J. Algebra **220** (1999), no. 1, 188–224.
- [AMS] A. Ardizzoni, C. Menini and D. Stefan, *Hochschild Cohomology And ‘Smoothness’ In Monoidal Categories*, J. Pure Appl. Algebra, Vol. **208** (2007), 297–330.
- [BC] D. Bulacu, S. Caenepeel, *Integrals for (dual) quasi-Hopf algebras. Applications*. J. Algebra **266** (2003), no. 2, 552–583.
- [Dr] V. G. Drinfeld, *Quasi-Hopf algebras*. (Russian) Algebra i Analiz **1** (1989), no. 6, 114–148; translation in Leningrad Math. J. **1** (1990), no. 6, 1419–1457.
- [HN] F. Haussler and F. Nill, *Integral theory for quasi Hopf algebras*, preprint (arXiv:math/9904164v2)
- [Ka] C. Kassel, *Quantum groups*, Graduate Text in Mathematics **155**, Springer, 1995.
- [LS] R. G. Larson, M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*. Amer. J. Math. **91** 1969 75–94.
- [Maj1] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995
- [Maj2] S. Majid, *Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results*. Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 219–232, Contemp. Math., **134**, Amer. Math. Soc., Providence, RI, 1992.
- [Sch1] P. Schauenburg, *Hopf algebra extensions and monoidal categories*. New directions in Hopf algebras, 321–381, Math. Sci. Res. Inst. Publ., **43**, Cambridge Univ. Press, Cambridge, 2002.
- [Sch2] P. Schauenburg, *Two characterizations of finite quasi-Hopf algebras*. J. Algebra **273** (2004), no. 2, 538–550.

UNIVERSITY OF FERRARA, DEPARTMENT OF MATHEMATICS, VIA MACHIAVELLI 35, FERRARA, I-44121, ITALY
E-mail address: alessandro.ardizzoni@unife.it
URL: <http://www.unife.it/utenti/alessandro.ardizzoni>

UNIVERSITY OF PADOVA, DEPARTMENT OF PURE AND APPLIED MATHEMATICS, VIA TRIESTE 63, PADOVA, I-35121, ITALY
E-mail address: apavarin@math.unipd.it
URL: http://www.math.unipd.it/?11=person&id_tipopersona=4&id_persona=487