

# Weak convergence of empirical copula processes under nonrestrictive smoothness assumptions

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## Abstract

Weak convergence of the empirical copula process is shown to hold under the assumption that the first-order partial derivatives of the copula exist and are continuous on certain subsets of the unit hypercube. The assumption is nonrestrictive in the sense that it is needed anyway to ensure the candidate limiting process to exist and have continuous trajectories. In addition, resampling methods based on the multiplier central limit theorem which require consistent estimation of the first-order derivatives continue to be valid. The price to pay for the weaker assumption is the loss of an explicit rate for the remainder term. Under certain growth conditions on the second-order partial derivatives, an almost sure rate can still be established. The conditions are verified for instance in the case of the Gaussian copula with full-rank correlation matrix, many Archimedean copulas, and many extreme-value copulas.

*Key words.* Archimedean copula; Brownian bridge; empirical copula; empirical process; extreme-value copula; Gaussian copula; multiplier central limit theorem; tail dependence; weak convergence.

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# 1 Introduction

A flexible and versatile way to model dependence is via copulas. A fundamental tool for inference is the empirical copula, which basically is equal to the empirical distribution function of the sample of multivariate ranks, rescaled to the unit interval. The asymptotic behaviour of the empirical copula process was studied in, amongst others, Stute (1984), Gänssler and Stute (1987) Chapter 5, van der Vaart and Wellner (1996) p. 389, Tsukahara (2000, 2005), Fermanian et al. (2004), Ghoudi and Rémillard (2004), and van der Vaart and Wellner (2007). Weak convergence is shown typically for copulas that are continuously differentiable on the closed hypercube, and rates of convergence of certain remainder terms have been established for copulas that are twice continuously differentiable on the closed hypercube. Unfortunately, for many (even most) popular copula families, even the first-order partial derivatives of the copula fail to be continuous at some boundary points of the hypercube.

*Example 1.1* (Tail dependence). Let  $C$  be a bivariate copula with first-order partial derivatives  $\dot{C}_1$  and  $\dot{C}_2$  and positive lower tail dependence coefficient,  $\lambda = \lim_{u \downarrow 0} C(u, u)/u > 0$ . On the one hand,  $\dot{C}_1(u, 0) = 0$  for all  $u \in [0, 1]$  by the fact that  $C(u, 0) = 0$  for all  $u \in [0, 1]$ . On the other hand,  $\dot{C}_1(0, v) = \lim_{u \downarrow 0} C(u, v)/u \geq \lambda > 0$  for all  $v \in (0, 1]$ . It follows that  $\dot{C}_1$  cannot be continuous in the point  $(0, 0)$ . Similarly for  $\dot{C}_2$ . For copulas with a positive upper tail dependence coefficient, the first-order partial derivatives cannot be continuous at the point  $(1, 1)$ .

Likewise, for the Gaussian copula with nonzero correlation parameter  $\rho$ , the first-order partial derivatives fail to be continuous at the points  $(0, 0)$  and  $(1, 1)$  if  $\rho > 0$  and at the points  $(0, 1)$  and  $(1, 0)$  if  $\rho < 0$ ; see also Example 6.2 below. As a consequence, the cited results on the empirical copula process do not apply to such copulas. This problem has been largely ignored in the literature, and unjustified calls to the above results abound. A notable exception is the paper by Omelka et al. (2009). On page 3031 of that paper, it is claimed that weak convergence of the empirical copula process still holds if the first-order partial derivatives are continuous at  $[0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ .

It is the aim of this paper to remedy the situation by showing that the earlier cited results on the empirical copula process actually do hold under a much less restrictive assumption, including indeed many copula families that were hitherto excluded. The assumption is nonrestrictive in the sense that it is needed anyway to ensure the candidate limiting process to exist and have continuous trajectories. The results are stated and proved in general dimensions. When specialized to the bivariate case, the condition is substantially

weaker still than the above-mentioned condition in Omelka et al. (2009).

Let  $F$  be a  $d$ -variate cumulative distribution function (cdf) with continuous margins  $F_1, \dots, F_d$  and copula  $C$ , that is,  $F(x) = C(F_1(x_1), \dots, F_d(x_d))$  for  $x \in \mathbb{R}^d$ . Let  $X_1, \dots, X_n$  be independent random vectors with common distribution  $F$ , where  $X_i = (X_{i1}, \dots, X_{id})$ . The empirical copula was defined in Deheuvels (1979) as

$$C_n(u) = F_n(F_{n1}^{-1}(u_1), \dots, F_{nd}^{-1}(u_d)), \quad u \in [0, 1]^d, \quad (1.1)$$

where  $F_n$  and  $F_{nj}$  are the empirical joint and marginal cdfs of the sample and where  $F_{nj}^{-1}$  is the marginal quantile function of the  $j$ th coordinate sample; see Section 2 below for details. The empirical copula  $C_n$  is invariant under monotone increasing transformations on the data, so it depends on the data only through the ranks. Indeed, up to a difference of order  $1/n$ , the empirical copula can be seen as the empirical cdf of the sample of normalised ranks, as for instance in Rüschemdorf (1976). For convenience, the definition in equation (1.1) will be employed throughout the paper.

The empirical copula process is defined by

$$\mathbb{C}_n = \sqrt{n}(C_n - C),$$

to be seen as a random function on  $[0, 1]^d$ . We are essentially interested in the asymptotic distribution of  $\mathbb{C}_n$  in the space  $\ell^\infty([0, 1]^d)$  of bounded functions from  $[0, 1]^d$  into  $\mathbb{R}$  equipped with the topology of uniform convergence. Weak convergence is to be understood in the sense used in the monograph by van der Vaart and Wellner (1996).

Although the empirical copula is itself a rather crude estimator of  $C$ , it plays a crucial role in more sophisticated inference procedures on  $C$ , much in the same way as the empirical cdf  $F_n$  is a fundamental object for creating and understanding inference procedures on  $F$  or parameters thereof. For instance, the empirical copula is a basic building block when estimating copula densities (Chen and Huang, 2007; Omelka et al., 2009) or dependence measures and functions (Schmid et al., 2010; Genest and Segers, 2010), for testing for independence (Genest and Rémillard, 2004; Genest et al., 2007; Kojadinovic and Holmes, 2009), for testing for shape constraints (Denuit and Scaillet, 2004; Scaillet, 2005; Kojadinovic and Yan, 2010), for resampling (Rémillard and Scaillet, 2009; Bücher and Dette, 2010), and so forth.

After some preliminaries in Section 2, the principal result of the paper is given in Section 3, stating weak convergence of the empirical copula process under the condition that for every  $j \in \{1, \dots, d\}$ , the  $j$ th first-order partial derivative  $\dot{C}_j$  exists and is continuous on the set  $\{u \in [0, 1]^d : 0 < u_j < 1\}$ .

The condition is non-restrictive in the sense that it is necessary for the candidate limiting process to exist and have continuous trajectories. In Section 4, the resampling method based on the multiplier central limit theorem proposed in Rémillard and Scaillet (2009) is shown to be valid under the same condition. Section 5 provides a refinement of the main result: under certain bounds on the second-order partial derivatives that allow for explosive behaviour near the boundaries, the almost sure error bound on the remainder term in Stute (1984) can be recovered up to a  $\log \log n$  term. Section 6 concludes the paper with a number of examples of copulas that do or do not verify certain sets of conditions.

## 2 Preliminaries

Let  $X_i = (X_{i1}, \dots, X_{id})$ ,  $i \in \{1, \dots, d\}$ , be independent random vectors with common cdf  $F$  whose margins  $F_1, \dots, F_d$  are continuous and whose copula is denoted by  $C$ . Define  $U_{ij} = F_j(X_{ij})$  for  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, d\}$ . The random vectors  $U_i = (U_{i1}, \dots, U_{id})$  constitute an iid sample from  $C$ . Consider the following empirical distribution functions: for  $x \in \mathbb{R}^d$  and for  $u \in [0, 1]^d$ ,

$$\begin{aligned} F_n(x) &= \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, x]}(X_i), & F_{nj}(x_j) &= \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, x_j]}(X_{ij}), \\ G_n(u) &= \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[0, u]}(U_i), & G_{nj}(u_j) &= \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[0, u_j]}(U_{ij}). \end{aligned}$$

Here, order relations on vectors are to be interpreted componentwise, and  $\mathbb{1}_A(x)$  is equal to 1 or 0 according to whether  $x$  is an element of  $A$  or not. Let  $X_{1:n,j} < \dots < X_{n:n,j}$  and  $U_{1:n,j} < \dots < U_{n:n,j}$  be the vectors of ascending order statistics of the  $j$ th coordinate samples  $X_{1j}, \dots, X_{nj}$  and  $U_{1j}, \dots, U_{nj}$ , respectively. The marginal quantile functions associated to  $F_{nj}$  and  $G_{nj}$  are

$$\begin{aligned} F_{nj}^{-1}(u_j) &= \inf\{x \in \mathbb{R} : F_{nj}(x) \geq u_j\} \\ &= \begin{cases} X_{k:n,j} & \text{if } (k-1)/n < u_j \leq k/n, \\ -\infty & \text{if } u_j = 0; \end{cases} \\ G_{nj}^{-1}(u_j) &= \inf\{u \in [0, 1] : G_{nj}(u) \geq u_j\} \\ &= \begin{cases} U_{k:n,j} & \text{if } (k-1)/n < u_j \leq k/n, \\ 0 & \text{if } u_j = 0. \end{cases} \end{aligned}$$

Some thought shows that  $X_{ij} \leq F_{nj}^{-1}(u_j)$  if and only if  $U_{ij} \leq G_{nj}^{-1}(u_j)$ , for all  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, d\}$  and  $u_j \in [0, 1]$ . It follows that the empirical copula in equation (1.1) is given by

$$C_n(u) = G_n(G_{n1}^{-1}(u_1), \dots, G_{nd}^{-1}(u_d)).$$

In particular, without loss of generality we can work directly with the sample  $U_1, \dots, U_n$  from  $C$ .

The empirical processes associated to the empirical distribution functions  $G_n$  and  $G_{nj}$  are given by

$$\alpha_n(u) = \sqrt{n}(G_n(u) - C(u)), \quad \alpha_{nj}(u_j) = \sqrt{n}(G_{nj}(u_j) - u_j),$$

for  $u \in [0, 1]^d$  and  $u_j \in [0, 1]$ . Note that  $\alpha_{nj}(0) = \alpha_{nj}(1) = 0$  almost surely. We have

$$\alpha_n \rightsquigarrow \alpha \quad (n \rightarrow \infty)$$

in  $\ell^\infty([0, 1]^d)$ , the arrow ‘ $\rightsquigarrow$ ’ denoting weak convergence in the sense of van der Vaart and Wellner (1996). The limit process  $\alpha$  is a  $C$ -Brownian bridge, i.e. a tight Gaussian process, centered and with covariance function

$$\text{cov}(\alpha(u, v), \alpha(u', v')) = C(u \wedge u', v \wedge v') - C(u, v)C(u', v'),$$

for  $(u, v), (u', v') \in ([0, 1]^d)^2$ ; here  $x \wedge y = (\min(x_1, y_1), \dots, \min(x_d, y_d))$ . Tightness of the process  $\alpha$  and continuity of its mean and covariance functions implies that there exists a version of  $\alpha$  with continuous trajectories. Without loss of generality, we assume henceforth that  $\alpha$  is such a version.

For  $j \in \{1, \dots, d\}$ , let  $e_j$  be the  $j$ th coordinate vector in  $\mathbb{R}^d$ . For  $u \in [0, 1]^d$  such that  $0 < u_j < 1$ , let

$$\dot{C}_j(u) = \lim_{h \rightarrow 0} \frac{C(u + he_j) - C(u)}{h},$$

be the  $j$ th first-order partial derivative of  $C$ , provided it exists.

**CONDITION 2.1.** *For each  $j \in \{1, \dots, d\}$ , the  $j$ th first-order partial derivative  $\dot{C}_j$  exists and is continuous on the set  $\{u \in [0, 1]^d : 0 < u_j < 1\}$ .*

Henceforth, assume Condition 2.1 holds. To facilitate notation, we will extend the domain of  $\dot{C}_j$  to the whole of  $[0, 1]^d$  by setting

$$\dot{C}_j(u) = \begin{cases} \limsup_{h \downarrow 0} \frac{C(u + he_j) - C(u)}{h}, & \text{if } u \in [0, 1]^d, u_j = 0; \\ \limsup_{h \downarrow 0} \frac{C(u) - C(u - he_j)}{h}, & \text{if } u \in [0, 1]^d, u_j = 1. \end{cases} \quad (2.1)$$

In this way,  $\dot{C}_j$  is defined everywhere on  $[0, 1]^d$ , takes values in  $[0, 1]$ , and is continuous on the set  $\{u \in [0, 1]^d : 0 < u_j < 1\}$ , by virtue of Condition 2.1.

### 3 Weak convergence

If the first-order partial derivatives  $\dot{C}_j$  exist and are continuous throughout  $[0, 1]^d$ , then from Fermanian et al. (2004) and Tsukahara (2005) we know that  $\mathbb{C}_n \sqrt{n}(C_n - C) \rightsquigarrow \mathbb{C}$  in  $\ell^\infty([0, 1]^d)$ , where

$$\mathbb{C}(u) = \alpha(u) - \sum_{j=1}^d \dot{C}_j(u) \alpha_j(u_j), \quad u \in [0, 1]^d.$$

By continuity of  $\dot{C}_j$  throughout  $[0, 1]^d$ , the trajectories of  $\mathbb{C}$  are continuous.

Now consider the same process  $\mathbb{C}$  but under Condition 2.1 and with the domain of the partial derivatives extended to  $[0, 1]^d$  as in equation (2.1). Since the trajectories of  $\alpha$  are continuous and since  $\alpha_j(0) = \alpha_j(1) = 0$  for each  $j \in \{1, \dots, d\}$ , the trajectories of  $\mathbb{C}$  are continuous, even though  $\dot{C}_j$  may fail to be continuous in points  $u \in [0, 1]^d$  such that  $u_j \in \{0, 1\}$ . The process  $\mathbb{C}$  is the weak limit in  $\ell^\infty([0, 1]^d)$  of the sequence of processes

$$\tilde{\mathbb{C}}_n(u) = \alpha_n(u) - \sum_{j=1}^d \dot{C}_j(u) \alpha_{nj}(u_j), \quad u \in [0, 1]^d.$$

The reason is that the map

$$\ell^\infty([0, 1]^d) \rightarrow \ell^\infty([0, 1]^d) : f \mapsto f - \sum_{j=1}^d \dot{C}_j \pi_j(f),$$

where  $(\pi_j(f))(u) = f(1, \dots, 1, u_j, 1, \dots, 1)$ , is linear and bounded.

PROPOSITION 3.1. *If Condition 2.1 holds, then*

$$\sup_{u \in [0, 1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| \xrightarrow{p} 0 \quad (n \rightarrow \infty).$$

As a consequence, in  $\ell^\infty([0, 1]^d)$ ,

$$\mathbb{C}_n \rightsquigarrow \mathbb{C} \quad (n \rightarrow \infty).$$

*Proof.* It suffices to show the first statement of the proposition. For  $u \in [0, 1]^d$ , put

$$R_n(u) = |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)|, \quad u \in [0, 1]^d.$$

If  $u_j = 0$  for some  $j \in \{1, \dots, d\}$ , then obviously  $\mathbb{C}_n(u) = \tilde{\mathbb{C}}_n(u) = 0$ , so  $R_n(u) = 0$  as well. Write

$$v_n(u) = (G_{n1}^{-1}(u_1), \dots, G_{nd}^{-1}(u_d)), \quad u \in [0, 1]^d.$$

We have

$$\begin{aligned}
\mathbb{C}_n(u) &= \sqrt{n}(C_n(u) - C(u)) \\
&= \sqrt{n}\{G_n(v_n(u)) - C(v_n(u))\} + \sqrt{n}\{C(v_n(u)) - C(u)\} \\
&= \alpha_n(v_n(u)) + \sqrt{n}\{C(v_n(u)) - C(u)\}.
\end{aligned} \tag{3.1}$$

Since  $\alpha_n$  converges weakly in  $\ell^\infty([0, 1]^d)$  to a  $C$ -Brownian bridge  $\alpha$ , whose trajectories are continuous, and since  $\sup_{u_j \in [0, 1]} |G_{nj}^{-1}(u_j) - u_j| \rightarrow 0$  almost surely, we have

$$\sup_{u \in [0, 1]^d} |\alpha_n(v_n(u)) - \alpha_n(u)| \xrightarrow{p} 0, \quad n \rightarrow \infty.$$

Consider the auxiliary function  $f : [0, 1] \rightarrow [0, 1]$  defined by

$$f(\lambda) = C(u + \lambda\{v_n(u) - u\}), \quad \lambda \in [0, 1].$$

If  $u \in (0, 1]^d$ , then  $v_n(u) \in (0, 1)^d$ , and therefore  $u + \lambda\{v_n(u) - u\} \in (0, 1)^d$  for all  $\lambda \in (0, 1]$  as well. By Condition 2.1, the function  $f$  is continuous on  $[0, 1]$  and continuously differentiable on  $(0, 1)$ . By the mean value theorem, there exists  $\lambda_n(u) \in (0, 1)$  such that

$$\sqrt{n}\{C(v_n(u)) - C(u)\} = \sum_{j=1}^d \dot{C}_j(u + \lambda_n(u)\{v_n(u) - u\}) \sqrt{n}(G_{nj}^{-1}(u_j) - u_j). \tag{3.2}$$

If one or more of the components of  $u$  are zero, then the above display remains true as well, no matter how  $\lambda_n(u) \in (0, 1)$  is defined, because both sides of the equation are equal to zero.

It is known since Kiefer (1970) that

$$\sup_{u_j \in [0, 1]} |\sqrt{n}(G_{nj}^{-1}(u_j) - u_j) + \alpha_{nj}(u_j)| \xrightarrow{p} 0, \quad n \rightarrow \infty.$$

Since  $0 \leq \dot{C}_j \leq 1$ , we find

$$\sup_{u \in [0, 1]^d} \left| \sqrt{n}\{C(v_n(u)) - C(u)\} + \sum_{j=1}^d \dot{C}_j(u + \lambda_n(u)\{v_n(u) - u\}) \alpha_{nj}(u_j) \right| \xrightarrow{p} 0$$

as  $n \rightarrow \infty$ . It remains to be shown that

$$\sup_{u \in [0, 1]^d} D_{nj}(u) \xrightarrow{p} 0 \quad (n \rightarrow \infty)$$

for all  $j \in \{1, \dots, d\}$ , where

$$D_{nj}(u) = |\dot{C}_j(u + \lambda_n(u)\{v_n(u) - u\}) - \dot{C}_j(u)| |\alpha_{nj}(u_j)|. \quad (3.3)$$

Fix  $\varepsilon > 0$  and  $\delta \in (0, 1/2)$ . Split the supremum over  $u \in [0, 1]^d$  according to the cases  $u_j \in [\delta, 1 - \delta]$  on the one hand and  $u_j \in [0, \delta) \cup (1 - \delta, 1]$  on the other hand. We have

$$\begin{aligned} \Pr\left(\sup_{u \in [0, 1]^d} D_{nj}(u) > \varepsilon\right) &\leq \Pr\left(\sup_{u \in [0, 1]^d, u_j \in [\delta, 1 - \delta]} D_{nj}(u) > \varepsilon/2\right) \\ &\quad + \Pr\left(\sup_{u \in [0, 1]^d, u_j \notin [\delta, 1 - \delta]} D_{nj}(u) > \varepsilon/2\right). \end{aligned}$$

Since  $\sup_{u \in [0, 1]^d} |v_n(u) - u| \rightarrow 0$  almost surely, since  $\dot{C}_j$  is uniformly continuous on  $\{u \in [0, 1]^d : \delta/2 \leq u \leq 1 - \delta/2\}$ , and since the sequence  $(\alpha_{nj})_n$  is tight in  $\ell^\infty([0, 1])$ , the first probability on the right-hand side of the previous display converges to zero. The second probability on the right-hand side of the previous display is bounded by

$$\Pr\left(\sup_{u_j \in [0, \delta) \cup (1 - \delta, 1]} |\alpha_{nj}(u_j)| > \varepsilon/2\right).$$

The lim sup of this probability as  $n \rightarrow \infty$  is bounded by

$$\Pr\left(\sup_{u_j \in [0, \delta) \cup (1 - \delta, 1]} |\alpha_j(u_j)| > \varepsilon/2\right).$$

As  $\alpha_j$  is a standard Brownian bridge, this probability can be made smaller than an arbitrarily chosen  $\eta > 0$  by choosing  $\delta$  sufficiently small. We find

$$\limsup_{n \rightarrow \infty} \Pr\left(\sup_{u \in [0, 1]^d} D_{nj}(u) > \varepsilon\right) \leq \eta.$$

As  $\eta$  was arbitrary, the claim is proven.  $\square$

## 4 Resampling

For purposes of hypothesis testing or confidence interval construction, resampling procedures are often required. In Fermanian et al. (2004), a bootstrap procedure for the empirical copula process is proposed, whereas in Rémillard and Scaillet (2009), a method based on the multiplier central limit theorem is employed. Yet another method is proposed in Bücher and Dette



(2010). In the latter paper, the finite-sample propoerties of all these methods are compared in a simulation study, and the multiplier approach by Rémillard and Scaillet (2009) is found to be best overall. As the latter approach requires estimation of the first-order partial derivatives, one may wonder if it is still valid under Condition 2.1, allowing for discontinuities on the boundaries. In this section it is shown that this is indeed the case.

For a bandwidth  $h_n \in (0, 1/2)$ , the partial derivative  $\dot{C}_j(u)$  can be estimated as follows: for  $u \in [0, 1]^d$ ,

$$\hat{C}_{nj}(u) = \begin{cases} \frac{C_n(u + h_n e_j) - C_n(u - h_n e_j)}{2h_n} & \text{if } u_j \in [h_n, 1 - h_n], \\ \hat{C}_{nj}(u_1, \dots, u_{j-1}, h_n, u_{j+1}, \dots, u_n) & \text{if } u_j \in [0, h_n), \\ \hat{C}_{nj}(u_1, \dots, u_{j-1}, 1 - h_n, u_{j+1}, \dots, u_n) & \text{if } u_j \in (1 - h_n, 1] \end{cases}$$

From the fact that  $F_{nj}^{-1}(u_j) = X_{[nu_j]:n,j}$ , it follows that

$$0 \leq \hat{C}_{nj} \leq \sup_{h_n \leq u_j \leq 1 - h_n} \frac{[n(u_j + h_n)] - [n(u_j - h_n)]}{2nh_n} \leq 1 + \frac{1}{2nh_n}. \quad (4.1)$$

LEMMA 4.1. *Suppose Condition 2.1 holds. If  $\lim_{n \rightarrow \infty} h_n = 0$  and  $\inf_n h_n \sqrt{n} > 0$ , then for any  $\delta \in (0, 1/2)$ , we have*

$$\sup_{u \in [0, 1]^d: u_j \in [\delta, 1 - \delta]} |\hat{C}_{nj}(u) - \dot{C}_j(u)| \xrightarrow{p} 0 \quad (n \rightarrow \infty).$$

*Proof.* For  $n$  large enough such that  $0 < h_n \leq \delta$ ,

$$\begin{aligned} \hat{C}_{nj}(u) &= \frac{C(u + h_n e_j) - C(u - h_n e_j)}{2h_n} \\ &\quad + \frac{1}{2h_n \sqrt{n}} \left( \mathbf{C}_n(u + h_n e_j) - \mathbf{C}_n(u - h_n e_j) \right), \end{aligned}$$

for all  $u \in [0, 1]^d$  such that  $u_j \in [\delta, 1 - \delta]$ . By the mean value theorem, there exists  $\lambda_n(u) \in [-1, 1]$  such that

$$\frac{C(u + h_n e_j) - C(u - h_n e_j)}{2h_n} = \dot{C}_j(u + \lambda_n(u) h_n e_j).$$

Write  $K = \sup_n 1/(2h_n \sqrt{n}) < \infty$ . We obtain

$$\begin{aligned} &\sup_{u \in [0, 1]^d: u_j \in [\delta, 1 - \delta]} |\hat{C}_{nj}(u) - \dot{C}_j(u)| \\ &\leq \sup_{u \in [0, 1]^d: u_j \in [\delta, 1 - \delta]} |\dot{C}_j(u + \lambda_n(u) h_n e_j) - \dot{C}_j(u)| \\ &\quad + K \sup_{u \in [0, 1]^d: u_j \in [\delta, 1 - \delta]} |\mathbf{C}_n(u + h_n e_j) - \mathbf{C}_n(u - h_n e_j)|. \end{aligned}$$

The first term on the right-hand side converges to zero by uniform continuity of  $\dot{C}_j$  on  $\{u \in [0, 1]^d : u_j \in [\delta, 1 - \delta]\}$ . The second term converges to zero in probability as  $\mathbf{C}_n$  converges weakly in  $\ell^\infty([0, 1]^d)$  to a stochastic process with continuous trajectories.  $\square$

Let  $\xi_1, \xi_2, \dots$  be an iid sequence of random variables with zero mean and unit variance, independent of the random vectors  $X_1, X_2, \dots$ . Define

$$\hat{\alpha}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i (\mathbb{1}\{X_{i1} \leq F_{n1}^{-1}(u_1), \dots, X_{id} \leq F_{nd}^{-1}(u_d)\} - C_n(u)),$$

$$\hat{\mathbf{C}}_n(u) = \hat{\alpha}_n(u) - \sum_{j=1}^d \hat{C}_{nj}(u) \hat{\alpha}_{nj}(u_j),$$

where, of course,  $\hat{\alpha}_{nj}(u_j) = \hat{\alpha}_n(1, \dots, 1, u_j, 1, \dots, 1)$ , with  $u_j$  appearing at the  $j$ th coordinate.

PROPOSITION 4.2. *Assume Condition 2.1. Then in  $(\ell^\infty([0, 1]^d))^2$ , we have*

$$(\mathbf{C}_n, \hat{\mathbf{C}}_n) \rightsquigarrow (\mathbf{C}, \hat{\mathbf{C}}) \quad (n \rightarrow \infty),$$

where  $\hat{\mathbf{C}}$  is an independent copy of  $\mathbf{C}$ .

*Proof.* In  $(\ell^\infty([0, 1]^d))^2$ , we have by Lemma A.1 in Rémillard and Scaillet (2009),

$$(\alpha_n, \hat{\alpha}_n) \rightsquigarrow (\alpha, \hat{\alpha}) \quad (n \rightarrow \infty), \quad (4.2)$$

where  $\hat{\alpha}$  is an independent copy of  $\alpha$ . Now define

$$\hat{\tilde{\mathbf{C}}}_n(u) = \hat{\alpha}_n(u) - \sum_{j=1}^d \dot{C}_j(u) \hat{\alpha}_{nj}(u_j).$$

The difference with  $\hat{\mathbf{C}}_n$  is that here, the true partial derivatives of  $C$  are used. By Proposition 3.1 and equation (4.2), we have

$$(\mathbf{C}_n, \hat{\tilde{\mathbf{C}}}_n) \rightsquigarrow (\mathbf{C}, \hat{\mathbf{C}}) \quad (n \rightarrow \infty)$$

in  $(\ell^\infty([0, 1]^d))^2$ . Moreover,

$$|\hat{\mathbf{C}}_n(u) - \hat{\tilde{\mathbf{C}}}_n(u)| \leq \sum_{j=1}^d |\hat{C}_{nj}(u) - \dot{C}_j(u)| |\hat{\alpha}_{nj}(u_j)|.$$

It suffices to show that each of the  $d$  terms on the right-hand side converges to 0 in probability, uniformly in  $u \in [0, 1]^d$ . The argument is similar as the

one at the end of the proof of Proposition 3.1. Pick  $\delta \in (0, 1/2)$  and split the supremum according to the cases  $u_j \in [\delta, 1 - \delta]$  and  $u_j \in [0, \delta) \cup (1 - \delta, 1]$ . For the first case, use Lemma 4.1 together with tightness of  $\hat{\alpha}_{nj}$ . For the second case, use the bound in equation (4.1) and the fact that the limit process  $\hat{\alpha}_j$  is a standard Brownian bridge, having continuous trajectories and vanishing in 0 and 1.  $\square$

## 5 Almost sure rate

If the second-order partial derivatives of  $C$  exist and are continuous on  $[0, 1]^d$ , then the original result by Stute (1984), proved in detail in Tsukahara (2000), reinforces the first claim of Proposition 3.1 to

$$\begin{aligned} \sup_{u \in [0, 1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| \\ = O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad (n \rightarrow \infty) \text{ almost surely.} \end{aligned} \quad (5.1)$$

A reasonable question concerns what can be said on the rate of the left-hand side under weaker conditions on the second-order partial derivatives, allowing these to explode near certain parts of the boundary of  $[0, 1]^d$ , as is the case for many copula families.

CONDITION 5.1. *For every  $i, j \in \{1, \dots, d\}$ , the second-order partial derivative  $\ddot{C}_{ij}$  is defined and continuous on the set*

$$V_{ij} = \{u \in [0, 1]^d : 0 < u_i < 1 \text{ or } 0 < u_j < 1\}.$$

Moreover, there exists a constant  $K > 0$  such that

$$|\ddot{C}_{ij}(u)| \leq K \min\left(\frac{1}{u_i(1-u_i)}, \frac{1}{u_j(1-u_j)}\right), \quad u \in V_{ij}.$$

It is useful to spell out Condition 5.1 in the bivariate case: besides existing and being continuous on the appropriate sets, the second-order partial derivatives should satisfy

$$\begin{aligned} \ddot{C}_{11}(u_1, u_2) &\leq K \frac{1}{u_1(1-u_1)}, & (u_1, u_2) &\in (0, 1) \times [0, 1], \\ \ddot{C}_{22}(u_1, u_2) &\leq K \frac{1}{u_2(1-u_2)}, & (u_1, u_2) &\in [0, 1] \times (0, 1), \end{aligned}$$

and

$$\begin{aligned} \ddot{C}_{12}(u_1, u_2) &\leq K \min\left(\frac{1}{u_1(1-u_1)}, \frac{1}{u_2(1-u_2)}\right), \\ & & (u_1, u_2) &\in [0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \end{aligned}$$

for some positive constant  $K$ . As shown in Section 6, Condition 5.1 holds for the bivariate Gaussian copula with correlation parameter  $\rho \in (-1, 1)$  and for bivariate extreme-value copulas whose Pickands dependence function  $A$  is twice continuously differentiable.

Under Condition 5.1, the rate in equation (5.1) can almost be recovered.

**PROPOSITION 5.2.** *If Condition 5.1 holds, then for every  $\varepsilon > 0$ , as  $n \rightarrow \infty$ ,*

$$\sup_{u \in [0,1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| = o(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/2+\varepsilon}) \quad \text{almost surely.}$$

*Proof.* From the proof of Proposition 3.1, in particular from the decompositions in equations (3.1) and (3.2), we have

$$\begin{aligned} & |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| \\ & \leq |\alpha_n(v_n(u)) - \alpha_n(u)| + \sum_{j=1}^d |\sqrt{n}(G_{n_j}^{-1}(u_j) - u_j) + \alpha_{n_j}(u_j)| + \sum_{j=1}^d D_{n_j}(u) \end{aligned}$$

with  $D_{n_j}(u)$  as defined in equation (3.3). As in the proof of Theorem 4.1 in Tsukahara (2000), each of the two sequences

$$\sup_{u \in [0,1]^d} |\alpha_n(v_n(u)) - \alpha_n(u)|, \quad \sup_{u_j \in [0,1]} |\sqrt{n}(G_{n_j}^{-1}(u_j) - u_j) + \alpha_{n_j}(u_j)|,$$

is  $O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4})$  as  $n \rightarrow \infty$ , almost surely. For the second sequence, this follows from Kiefer (1970). For the first sequence, this follows by noting that, by the law of the iterated logarithm for empirical distribution functions,

$$\begin{aligned} \sup_{u_j \in [0,1]} |G_{n_j}^{-1}(u_j) - u_j| &= \sup_{u_j \in [0,1]} |u_j - G_{n_j}(u_j)| \\ &= O(n^{-1/2}(\log \log n)^{1/2}) \quad (n \rightarrow \infty) \text{ almost surely,} \end{aligned} \tag{5.2}$$

and by properties of the oscillation modulus of the multivariate empirical process  $\alpha_n$  in (Stute, 1984, Theorem 1.7); see the end of the proof of Theorem 4.1 in Tsukahara (2000). Therefore, we only need to consider the term  $\sup_{u \in [0,1]^d} D_{n_j}(u)$ .

Let  $\delta_n = n^{-1/2} \log \log n$ . We split the supremum of  $D_{n_j}(u)$  over  $u \in [0, 1]^d$  according to the case  $u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]$  and  $u_j \in [\delta_n, 1 - \delta_n]$ .

Since  $0 \leq \hat{C}_j \leq 1$ , the supremum over  $u \in [0, 1]^d$  such that  $u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]$  is bounded by

$$\sup_{u \in [0,1]^d: u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]} D_{n_j}(u) \leq \sup_{u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]} |\alpha_{n_j}(u_j)|.$$

By Theorem 0.2 in Stute (1982), this is of the order

$$\begin{aligned} & \sup_{u \in [0,1]^d: u_j \in [0, \delta_n] \cup (1-\delta_n, 1]} D_{nj}(u) \\ &= O(\delta_n^{1/2} (\log \delta_n^{-1})^{1/2}) = O(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/2}) \\ & \quad (n \rightarrow \infty) \text{ almost surely.} \end{aligned} \quad (5.3)$$

Next let  $u \in [0, 1]^d$  be such that  $\delta_n \leq u_j \leq 1 - \delta_n$ . By Lemma 5.3 below, writing  $K_1 = K\sqrt{d}$ ,

$$\begin{aligned} D_{nj}(u) &= |\dot{C}_j(u + \lambda_n(u)\{v_n(u) - u\}) - \dot{C}_j(u)| |\alpha_{nj}(u_j)| \\ &\leq K_1 \max\left(\frac{1}{u_j(1-u_j)}, \frac{1}{G_{nj}(u_j)(1-G_{nj}(u_j))}\right) |v_n(u) - u| |\alpha_{nj}(u_j)|. \end{aligned}$$

Let  $a_n = (\log n)^{1/2} (\log \log n)^{1/2+\varepsilon}$  for some  $\varepsilon > 0$ . Note that  $\sum_{n=1}^{\infty} n^{-1} a_n^{-2} < \infty$ . By Csáki (1975) or Mason (1981),

$$\Pr\left(\sup_{0 < s < 1} \frac{|\alpha_{nj}(s)|}{(s(1-s))^{1/2}} > a_n \text{ infinitely often}\right) = 0.$$

It follows that, for all sufficiently large  $n$ ,

$$|\alpha_{nj}(u_j)| \leq (u_j(1-u_j))^{1/2} a_n, \quad u_j \in [0, 1].$$

Let  $I$  denote the identity function on  $[0, 1]$  and let  $\|\cdot\|$  denote the supremum norm. For  $u_j \in [\delta_n, 1 - \delta_n]$ ,

$$\begin{aligned} G_{nj}^{-1}(u_j) &= u_j \left(1 + \frac{G_{nj}^{-1}(u_j) - u_j}{u_j}\right) \geq u_j \left(1 - \frac{\|G_{nj}^{-1} - I\|}{\delta_n}\right), \\ 1 - G_{nj}^{-1}(u_j) &\geq (1 - u_j) \left(1 - \frac{\|G_{nj}^{-1} - I\|}{\delta_n}\right). \end{aligned}$$

By the law of the iterated logarithm, see (5.2),

$$\|G_{nj}^{-1} - I\| = o(\delta_n) \quad (n \rightarrow \infty) \text{ almost surely.}$$

We find that with probability one, for all sufficiently large  $n$  and for all  $u \in [0, 1]^d$  such that  $u_j \in [\delta_n, 1 - \delta_n]$ ,

$$D_{nj}(u) \leq 2K_1 (u_j(1-u_j))^{-1/2} |v_n(u) - u| a_n.$$

We use again the law of the iterated logarithm in (5.2) to bound  $|v_n(u) - u|$ . As a consequence, with probability one,

$$\begin{aligned} & \sup_{u \in [0,1]^d: u_j \in [\delta_n, 1-\delta_n]} D_{nj}(u) \\ &= O\left(\delta_n^{-1/2} (\log \log n)^{1/2} n^{-1/2} a_n\right) = O\left(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/2+\varepsilon}\right) \\ & \hspace{15em} (n \rightarrow \infty) \text{ almost surely.} \end{aligned} \quad (5.4)$$

The bound in (5.3) is dominated by the one in (5.4). The latter one being true for any  $\varepsilon > 0$ , we can replace the big ‘ $O$ ’ by a small ‘ $o$ ’.  $\square$

LEMMA 5.3. *If Condition 5.1 holds, then*

$$|\dot{C}_j(v) - \dot{C}_j(u)| \leq K\sqrt{d} \max\left(\frac{1}{u_j(1-u_j)}, \frac{1}{v_j(1-v_j)}\right) |v-u|,$$

for every  $j \in \{1, \dots, d\}$  and for every  $u, v \in [0, 1]^d$  such that  $0 < u_j < 1$  and  $0 < v_j < 1$ .

*Proof.* Fix  $j \in \{1, \dots, d\}$  and  $u, v \in [0, 1]^d$  such that  $0 < u_j < 1$  and  $0 < v_j < 1$ . By the mean-value theorem applied to the function  $[0, 1] \ni t \mapsto f(t) = \dot{C}_j(u + t(v-u))$ , we obtain that for some  $t^* \in (0, 1)$ ,

$$\dot{C}_j(v) - \dot{C}_j(u) = f(1) - f(0) = f'(t^*) = \sum_{i=1}^d (v_i - u_i) \ddot{C}_{ij}(u + t^*(v-u)).$$

As a consequence,

$$|\dot{C}_j(u) - \dot{C}_j(v)| \leq |v-u| \sup_{0 < t < 1} |\nabla \dot{C}_j(u + t(v-u))|,$$

where  $\nabla \dot{C}_j = (\ddot{C}_{1j}, \dots, \ddot{C}_{dj})$ . By Condition 5.1, for all  $i \in \{1, \dots, d\}$ ,

$$\ddot{C}_{ij}(w) \leq K \frac{1}{w_j(1-w_j)}, \quad w \in [0, 1]^d, 0 < w_j < 1.$$

But then

$$|\nabla \dot{C}_j(w)| \leq K\sqrt{d} \frac{1}{w_j(1-w_j)}, \quad w \in [0, 1]^d, 0 < w_j < 1.$$

Finally, since the function  $s \mapsto s^{-1}(1-s)^{-1}$  is convex on  $(0, 1)$ ,

$$\begin{aligned} |\dot{C}_j(u) - \dot{C}_j(v)| &\leq |v-u| K\sqrt{d} \sup_{0 < t < 1} \frac{1}{(u_j + t(v_j - u_j))(1 - u_j - t(v_j - u_j))} \\ &\leq |v-u| K\sqrt{d} \max\left(\frac{1}{u_j(1-u_j)}, \frac{1}{v_j(1-v_j)}\right). \end{aligned} \quad \square$$

*Remark.* The rate as stated in equation (5.1) was used in Genest and Segers (2009) in the context of bivariate extreme-value copulas; see the proof of their Theorem 3.2, equation (B.3). However, for bivariate extreme-value copulas, the first-order partial derivatives are in general not continuous in the corner points  $(0, 0)$  and  $(1, 1)$ , and the second-order derivatives diverge at these points. Fortunately, Condition 5.1 is verified (see Example 6.3 below) and the rate in Proposition 5.2 is sharp enough to make the proof of their Theorem 3.2 go through without further modifications [in fact, a rate of  $o_p(1/\log n)$  would already have been sufficient].

## 6 Examples

*Example 6.1* (Archimedean copulas). Let  $C$  be a  $d$ -variate Archimedean copula, that is,

$$C(u) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)), \quad u \in [0, 1]^d,$$

where the generator  $\phi : [0, 1] \rightarrow [0, \infty]$  is convex, decreasing, finite on  $(0, 1]$ , and vanishes at 1, whereas  $\phi^{-1} : [0, \infty) \rightarrow [0, 1]$  is its generalized inverse,  $\phi^{-1}(x) = \inf\{u \in [0, 1] : \phi(u) \leq x\}$ ; in fact, if  $d \geq 3$ , more conditions on  $\phi$  are required for  $C$  to be a copula, see McNeil and Nešlehová (2009).

Suppose  $\phi$  is continuously differentiable on  $(0, 1]$  and  $\phi'(0+) = -\infty$ . Then the first-order partial derivatives of  $C$  are given by

$$\dot{C}_j(u) = \frac{\phi'(u_j)}{\phi'(C(u))}, \quad u \in [0, 1]^d, 0 < u_j < 1.$$

If  $u_i = 0$  for some  $i \neq j$ , then  $C(u) = 0$  and  $\phi'(C(u)) = -\infty$ , so indeed  $\dot{C}_j(u) = 0$ . We find that Condition 2.1 is verified, so Propositions 3.1 and 4.2 apply.

In contrast,  $\dot{C}_j$  may easily fail to be continuous at some boundary points. For instance, if  $\phi'(1) = 0$ , then  $\dot{C}_j$  cannot be extended continuously to  $(1, \dots, 1)$ . Or if  $\phi^{-1}$  is long-tailed, i.e. if  $\lim_{x \rightarrow \infty} \phi^{-1}(x + y)/\phi^{-1}(x) = 1$  for all  $y \in \mathbb{R}$ , then  $\lim_{u_1 \downarrow 0} C(u_1, u_{-1})/u_1 = 1$  for all  $u_{-1} \in (0, 1]^{d-1}$ , whereas  $\dot{C}_1(u) = 0$  as soon as  $u_j = 0$  for some  $j \in \{2, \dots, d\}$ . It follows that  $\dot{C}_1$  cannot be extended continuously to the set  $\{0\} \times ([0, 1]^{d-1} \setminus (0, 1]^{d-1})$ .

Even worse,  $\dot{C}_j$  may fail to exist altogether when  $u_j = 0$ . For instance, let  $d = 2$  and let  $\phi(u) = -\int_u^1 \phi'(s) ds$  where  $\phi'$  is the piecewise linear function with knots  $\phi'(2^{-k}) = -2^k$  for  $k \in \{0, 1, 2, \dots\}$ ; see Theorem 2 in Charpentier and Segers (2007). Then one can show that  $C(u_1, u_2)/u_1$  fails to converge as  $u_1 \downarrow 0$  for  $u_2 \in (0, 1) \setminus \{2^{-1}, 2^{-2}, \dots\}$ .

*Example 6.2* (Gaussian copula). Let  $C$  be the  $d$ -variate copula Gaussian with correlation matrix  $R \in \mathbb{R}^{d \times d}$ , that is,

$$C(u) = \Pr\left(\bigcap_{j=1}^d \{X_j \leq \Phi^{-1}(u_j)\}\right), \quad u \in [0, 1]^d,$$

where  $X = (X_1, \dots, X_d)$  follows a  $d$ -variate Gaussian distribution with zero means, unit variances, and correlation matrix  $R$ ; here  $\Phi$  is the standard normal cdf and  $\Phi^{-1}$  is its inverse. Write  $\rho_j = \text{corr}(X_1, X_j)$  and suppose that  $|\rho_j| < 1$  for  $j \in \{2, \dots, d\}$ . We can write

$$X_j = \rho_j X_1 + \sqrt{1 - \rho_j^2} Y_j, \quad j \in \{2, \dots, d\}$$

where  $Y = (Y_2, \dots, Y_d)$  is independent of  $X_1$  and follows a  $(d - 1)$ -variate Gaussian distribution with zero means, unit variances, and a certain correlation matrix. Letting  $\varphi$  stand for the standard normal density function, we have

$$C(u) = \int_{-\infty}^{\Phi^{-1}(u_1)} \Pr\left(\bigcap_{j=2}^d \{\rho_j x + \sqrt{1 - \rho_j^2} Y_j \leq \Phi^{-1}(u_j)\}\right) \varphi(x) dx.$$

It follows that for all  $u \in [0, 1]^d$  such that  $0 < u_1 < 1$ ,

$$\dot{C}_1(u) = \Pr\left(\bigcap_{j=2}^d \left\{Y_j \leq \frac{\Phi^{-1}(u_j) - \rho_j \Phi^{-1}(u_1)}{\sqrt{1 - \rho_j^2}}\right\}\right).$$

We conclude that if the correlation matrix  $R$  is of full rank, then Condition 2.1 is verified and Propositions 3.1 and 4.2 apply.

Still, if  $0 < \rho_j < 1$  for all  $j \in \{2, \dots, d\}$ , then on the one hand we have  $\lim_{u_1 \downarrow 0} \dot{C}_1(u_1, u_{-1}) = 1$  for all  $u_{-1} \in (0, 1]^{d-1}$ , whereas on the other hand we have  $\dot{C}_1(u) = 0$  as soon as  $u_j = 0$  for some  $j \in \{2, \dots, d\}$ ; hence  $\dot{C}_1$  cannot be extended continuously to the set  $\{0\} \times ([0, 1]^{d-1} \setminus (0, 1]^{d-1})$ .

Let us verify Condition 5.1 in the bivariate case when the correlation parameter  $\rho$  satisfies  $|\rho| < 1$ . The copula density is given by

$$\dot{C}_{12}(u, v) = \frac{1}{\sqrt{1 - \rho^2}} \varphi\left(\frac{\Phi^{-1}(v) - \rho \Phi^{-1}(u)}{\sqrt{1 - \rho^2}}\right) / \varphi(\Phi^{-1}(v)).$$

Although this is not obvious from the above expression,  $\dot{C}_{12}$  is symmetric in  $u$  and  $v$ , by symmetry of  $C$  itself. It can be shown that  $\Phi(x)(1 - \Phi(x)) \leq \varphi(x)$



for  $x \in [-\infty, \infty]$ , which implies  $u(1-u) \leq \varphi(\Phi^{-1}(u))$  for  $u \in [0, 1]$ . It follows that

$$\dot{C}_{12}(u, v) \leq \frac{\varphi(0)}{\sqrt{1-\rho^2}} \frac{1}{v(1-v)}, \quad (u, v) \in [0, 1] \times (0, 1).$$

By symmetry, it then must also be true that

$$\dot{C}_{12}(u, v) \leq \frac{\varphi(0)}{\sqrt{1-\rho^2}} \frac{1}{u(1-u)}, \quad (u, v) \in (0, 1) \times [0, 1].$$

Similarly, we have

$$\dot{C}_{11}(u, v) = \frac{-\rho}{\sqrt{1-\rho^2}} \varphi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) / \varphi(\Phi^{-1}(u)),$$

whence

$$|\dot{C}_{11}(u, v)| \leq \frac{|\rho|\varphi(0)}{\sqrt{1-\rho^2}} \frac{1}{u(1-u)}, \quad (u, v) \in (0, 1) \times [0, 1],$$

together with the symmetric bound for  $|\dot{C}_{22}(u, v)|$ .

*Example 6.3* (Extreme-value copulas). Let  $C$  be a  $d$ -variate extreme-value copula, that is,

$$C(u) = \exp(-\ell(-\log u_1, \dots, -\log u_d)), \quad u \in (0, 1]^d,$$

where the tail dependence function or tail copula  $\ell : [0, \infty)^d \rightarrow [0, \infty)$  verifies

$$\ell(x) = \int_{\Delta_{d-1}} \max_{j \in \{1, \dots, d\}} (w_j x_j) H(dw), \quad x \in [0, \infty)^d,$$

with  $H$  a nonnegative Borel measure on the unit simplex  $\Delta_{d-1} = \{w \in [0, 1]^d : w_1 + \dots + w_d = 1\}$  satisfying the  $d$  constraints  $\int w_j H(dw) = 1$  for all  $j \in \{1, \dots, d\}$ ; see for instance Leadbetter and Rootzén (1988) or Pickands (1989). It can be verified that  $\ell$  is convex, is homogeneous of order 1, and that  $\max(x_1, \dots, x_d) \leq \ell(x) \leq x_1 + \dots + x_d$  for all  $x \in [0, \infty)^d$ .

Suppose that  $\ell$  is continuously differentiable on  $[0, \infty)^d$  with first-order partial derivatives  $\dot{\ell}_1, \dots, \dot{\ell}_d$ . Then the first-order partial derivatives of  $C$  are given by

$$\dot{C}_j(u) = \frac{C(u)}{u_j} \dot{\ell}_j(-\log u_1, \dots, -\log u_d), \quad u \in (0, 1]^d.$$

The properties of  $\ell$  imply that  $0 \leq \dot{\ell}_j \leq 1$  for all  $j \in \{1, \dots, d\}$ . Therefore, if  $u_i \downarrow 0$  for some  $i \neq j$ , then  $\dot{C}_j(u) \rightarrow 0$ , as required. Hence Condition 2.1 is verified and Propositions 3.1 and 4.2 apply.

Let us consider the bivariate case in somewhat more detail. The function  $A : [0, 1] \rightarrow [1/2, 1] : t \mapsto A(t) = \ell(1 - t, t)$  is called the Pickands dependence function of  $C$ . It is convex and satisfies  $t \vee (1 - t) \leq A(t) \leq 1$  for all  $t \in [0, 1]$ . The bivariate extreme-value copula  $C$  with Pickands dependence function  $A$  is given by

$$C(u, v) = \exp\{\log(uv) A(t)\} = (uv)^{A(t)},$$

where

$$t = t(u, v) = \begin{cases} \frac{\log(v)}{\log(uv)} & \text{if } (u, v) \in (0, 1]^2 \setminus \{(1, 1)\}, \\ 0 & \text{if } 0 = u < v \leq 1, \\ 1 & \text{if } 0 = v < u \leq 1. \end{cases}$$

Assume that  $A$  is continuously differentiable on  $[0, 1]$  with derivative  $A'$ . Put

$$\begin{aligned} \mu(t) &= A(t) - t A'(t) &= 1 - \int_0^t (A'(t) - A'(s)) ds, \\ \nu(t) &= A(t) + (1 - t) A'(t) &= 1 - \int_t^1 (A'(s) - A'(t)) ds. \end{aligned}$$

The first-order partial derivative of  $C$  with respect to  $u$  is equal to

$$\dot{C}_1(u, v) = \begin{cases} \frac{C(u, v)}{u} \mu(t) = (uv)^{A(t)-(1-t)} \mu(t) & \text{if } (u, v) \in (0, 1)^2, \\ 1 & \text{if } v = 1, u \in [0, 1], \\ 0 & \text{if } v = 0, u \in [0, 1], \\ \mu(1) v & \text{if } u = 1, v \in [0, 1], \\ v^{\nu(0)} & \text{if } u = 0, v \in (0, 1]. \end{cases}$$

It follows that  $\dot{C}_1$  may be discontinuous at the points  $(0, 0)$  or  $(1, 1)$ :

- $\dot{C}_1$  is not continuous at  $(1, 1)$  as soon as  $\mu(1) < 1$ , that is,  $A'(1) > 0$ , which always the case except when  $A \equiv 1$  (independence copula).
- $\dot{C}_1$  is not continuous at  $(0, 0)$  as soon as  $\nu(0) = 0$ , that is,  $A'(0) = -1$ , which is the case whenever the spectral measure associated to  $A$  has no atoms, for instance the Gumbel–Hougaard copula.

The analysis of  $\dot{C}_2(u, v)$  is similar.

We now verify that Condition 5.1 holds provided  $A$  is twice continuously differentiable on  $[0, 1]$  with second derivative  $A''$ . Then  $\mu$  and  $\nu$  are continuously differentiable with derivatives

$$\mu'(t) = -t A''(t), \quad \nu'(t) = (1 - t) A''(t).$$

The second-order partial derivatives of  $C$  on  $(0, 1)^2$  are given by

$$\begin{aligned}\dot{C}_{12}(u, v) &= \frac{C(u, v)}{uv} \left( \mu(t) \nu(t) - \frac{t(1-t)A''(t)}{\log(uv)} \right), \\ \dot{C}_{11}(u, v) &= \frac{C(u, v)}{u^2} \left( -\mu(t)(1-\mu(t)) + \frac{t^2 A''(t)}{\log(uv)} \right), \\ \dot{C}_{22}(u, v) &= \frac{C(u, v)}{v^2} \left( -\nu(t)(1-\nu(t)) + \frac{(1-t)^2 A''(t)}{\log(uv)} \right).\end{aligned}$$

On the boundaries, the situation is as follows:

$$\dot{C}_{12}(u, v) = \begin{cases} \nu(0) v^{-(1-\nu(0))} & \text{if } u = 0, v \in (0, 1], \\ \nu(0) & \text{if } v = 1, u \in [0, 1), \\ \mu(1) u^{-(1-\mu(1))} & \text{if } v = 0, u \in (0, 1], \\ \mu(1) & \text{if } u = 1, v \in [0, 1). \end{cases}$$

Moreover, if  $A''(1/2) > 0$ , then  $\lim_{w \rightarrow 1} \dot{C}_{12}(w, w) = \infty$ . We see that  $\dot{C}_{12}$  exists and is continuous on  $[0, 1]^2 \setminus \{(0, 0), (1, 1)\}$ , whereas at  $(0, 0)$  and  $(1, 1)$ , it may explode. Furthermore,

$$\dot{C}_{11}(u, v) = \begin{cases} 0 & \text{if } v \in \{0, 1\}, u \in [0, 1], \\ v(-\mu(1)(1-\mu(1)) + \frac{A''(1)}{\log(v)}) & \text{if } u = 1, v \in [0, 1), \\ -\infty & \text{if } u = 0, v \in (0, 1). \end{cases}$$

Similarly for  $\dot{C}_{22}$ .

Put  $\|A''\|_\infty = \sup_{t \in [0, 1]} A''(t)$ . From  $-\log(x) \geq 1 - x$ , it follows that

$$\frac{1}{-\log(uv)} = \frac{1}{-\log(u) - \log(v)} \leq \frac{1}{(1-u) + (1-v)} \leq \frac{1}{1-u} \wedge \frac{1}{1-v}.$$

Since  $C(u, v) \leq \min(u, v)$  and since  $(a \wedge b)(c \wedge d) \leq (ac) \wedge (bd)$  for positive numbers  $a, b, c, d$ , we find

$$\begin{aligned}0 \leq \dot{C}_{12}(u, v) &\leq \frac{u \wedge v}{uv} \left[ 1 + \frac{\|A''\|_\infty}{4} \left( \frac{1}{1-u} \wedge \frac{1}{1-v} \right) \right] \\ &\leq (1 + \|A''\|_\infty/4) \left( \frac{1}{u(1-u)} \wedge \frac{1}{v(1-v)} \right)\end{aligned}$$

Similarly,

$$\begin{aligned}0 \leq -\dot{C}_{11}(u, v) &\leq \frac{u \wedge v}{u^2} \left[ \frac{1}{4} + \|A''\|_\infty \left( \frac{1}{1-u} \wedge \frac{1}{1-v} \right) \right] \\ &\leq (1/4 + \|A''\|_\infty) \frac{1}{u} \left( \frac{1}{1-u} \wedge \frac{1}{1-v} \right).\end{aligned}$$

*Example 6.4* (If everything fails. . .). Sometimes, even Condition 2.1 does not hold. Consider the bivariate case. The first-order partial derivatives can be seen as conditional cdfs:

$$\dot{C}_1(u_1, u_2) = \Pr(U_2 \leq u_2 \mid U_1 = u_1).$$

It follows that  $\dot{C}_1$  will fail to be continuous on  $(0, 1) \times [0, 1]$  if the law of  $U_2$  given  $U_1 = u_1$  possesses atoms or if this law depends in a non-continuous way on  $u_1 \in (0, 1)$ . The first phenomenon occurs for instance for the Fréchet lower and upper bounds,  $C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$  and  $C(u_1, u_2) = \min(u_1, u_2)$ ; for Archimedean copulas whose generator  $\phi$  is not continuously differentiable or whose generator satisfies  $\phi'(0+) > -\infty$ ; and for bivariate extreme-value copulas whose Pickands dependence function  $A$  is not continuously differentiable. The second phenomenon occurs for instance for the checkerboard copula with Lebesgue density  $c = 2 \mathbb{1}_{[0, 1/2]^2 \cup [1/2, 1]^2}$ . In these cases, the candidate limiting process  $\mathbb{C}$  has discontinuous trajectories and the empirical copula process does not converge weakly in  $\ell^\infty([0, 1]^2)$  endowed with the topology of uniform convergence.

One may then wonder if weak convergence of the empirical copula process might still hold in a weaker topology than the one of uniform convergence, for instance, a Skorohod-type topology on the space of càdlàg functions on  $[0, 1]^2$ . Such a result would still be useful to derive for instance the asymptotic distribution of certain functionals of the empirical copula process, e.g. suprema or integrals such as appearing in certain test statistics.

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