A VANISHING THEOREM IN TWISTED DE RHAM COHOMOLOGY

ANA CRISTINA FERREIRA

Abstract. We prove a vanishing theorem for the twisted de Rham cohomology of a compact manifold.

1. Introduction

In this article, we show how to use connections with skew torsion to identify the operator $(d+H)+(d+H)^*$, where H is a three-form, with a cubic Dirac operator. In the compact case, if H is closed, we prove a vanishing theorem for twisted de Rham cohomology by means of a Lichnerowicz formula. As an application, we prove that for a compact non-abelian Lie group the cohomology of the complex defined by $d+H$, where H is the three-form defined by the Lie bracket, vanishes.

2. The Dirac operator

Let (M,g) be a Riemannian manifold. Suppose that ∇ is a connection on the tangent bundle of M and let T be its (1,2) torsion tensor. If we contract T with the metric we get a (0,3) tensor which we will still call the torsion of ∇. If T is a three-form then we say that ∇ is a connection with skew-symmetric torsion. Given any three-form H on M then there exists a unique metric connection with skew torsion H defined explicitly by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2}H(X,Y,Z)$$

where ∇^g is the Levi-Civita connection.

Fix a three-form H and consider the one-parameter family of affine connections

$$\nabla^s := \nabla^g + 2sH$$

(Notice that if $s=\frac{1}{4}$ we recover the connection with torsion H.) If M is spin, these connections lift to the spin bundle $\not{S}$ of M as

$$\nabla^s_X(\varphi) := \nabla^g_X(\varphi) + s(i_X H)\varphi$$

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where X is a vector field, φ is a spinor field, and $i_X H$ is acting by Clifford multiplication.

We may define the Dirac operator $\not{D}$ on $\not{S}$ with respect to ∇ by means of the following composition

$$\Gamma(M, \not{S}) \longrightarrow \Gamma(M, T^*M \otimes \not{S}) \longrightarrow \Gamma(M, TM \otimes \not{S}) \longrightarrow \Gamma(M, \not{S})$$

where the first arrow is given by the connection, the second by the metric and the third by the Clifford action. Suppose now that we have a complex vector bundle $\mathcal{W}$, we can form the tensor product $\not{S} \otimes \mathcal{W}$, which is usually called a twisted spinor bundle or a spinor bundle with values in $\mathcal{W}$. If $\mathcal{W}$ is equipped with a Hermitian connection $\nabla^{\mathcal{W}}$, we can consider the tensor product connection $\nabla \otimes 1 + 1 \otimes \nabla^{\mathcal{W}}$, again denoted by ∇, on $\not{S} \otimes \mathcal{W}$. We can define a Dirac operator on this twisted spinor bundle associated with the connection ∇ by the same formula, where the action of the tangent bundle by Clifford multiplication is only on the left factor.

We will need to make use of a Lichnerowicz type formula for the square of the Dirac operator. Such a formula first appeared in the literature in [3]. See also [1].

Theorem 2.1. **[Bismut,** [3]] *The rough Laplacian $\Delta^s = \nabla^{s*}\nabla^s$ and the square of the Dirac operator $D^{s/3}$ are related by*

$$(D^{s/3})^2 = \Delta^s + F^{\mathcal{W}} + \frac{1}{4}\kappa + s dH - 2s^2 ||H||^2,$$

where κ is the Riemannian scalar curvature and F is the curvature of the twisting bundle acting as $\sum_{i<j} F^{\mathcal{W}}(e_i, e_j) e_i e_j$ on $\not{S} \otimes \mathcal{W}$.

Notice that this formula relates the square of the Dirac operator $D^{s/3}$ and the Laplacian Δ^s. The Dirac operator $D^{1/3}$ is usually referred to as the cubic Dirac operator.

3. Twisted cohomology

Consider the spinor bundle with values in itself, that is, $\not{S} \otimes \not{S}$. Recall that for this we do not need a global spin structure. We have, in even dimensions, the following chain of isomorphisms

$$\not{S} \otimes \not{S} \simeq \not{S}^* \otimes \not{S} \simeq \mathrm{End}(\not{S}) \simeq \mathrm{Cl} \simeq \Lambda$$

where Cl denotes the Clifford bundle and Λ the bundle of exterior forms.

If we take the induced Levi-Civita connection ∇^g on both factors of $\not{S} \otimes \not{S}$ and consider the tensor product connection $\nabla^g \otimes 1 + 1 \otimes \nabla^g$ we obtain the induced Levi-Civita connection, again denoted by ∇^g, on Λ. If we consider the associated Dirac operator D^g on $\not{S} \otimes \not{S}$ we get a familiar operator on Λ. In fact,

$$D^g = d + d^*$$

where d is the exterior differential and d^* is its formal adjoint, [5].

The same fact can be claimed for an odd-dimensional manifold. Consider the inclusion $M \hookrightarrow \mathbb{R} \times M$, $\$^+$ and $\$^-$ the half spinor bundles of $\mathbb{R} \times M$. The Clifford action by e_0, where e_0 is a unit vector field of $\mathbb{R}$, gives an isomorphism between $\$^+$ and $\$^-$ and thus we can regard $\$^+ \simeq \$^-$ as the spinor bundle of M. Under this identification, the Dirac operator associated to the Levi-Civita connection becomes

$$\$^+ \xrightarrow{D^g} \$^- \xrightarrow{e_0} \$^+$$

where e_0 denotes multiplication by e_0. Consider also the Levi-Civita connection on $\$$ and the twisted Dirac operator

$$\$^+ \otimes \$ \xrightarrow{D^g} \$^- \otimes \$ \xrightarrow{e_0} \$^+ \otimes \$.$$

Notice that the exterior bundle of M is $\Lambda \simeq \mathrm{Cl} \simeq \$^+ \otimes \$$, and so the twisted Dirac operator above is, in terms of differential forms, the restriction of the Laplacian $d+d^*$ on $\mathbb{R}\times M$ to forms that are independent of the coordinate t of $\mathbb{R}$, and can therefore be seen as the Laplacian on M.

We may now ask ourselves what happens if we introduce connections with skew torsion in this setting.

Theorem 3.1. *Let H be a three-form, and suppose that the left and right spinor factors are, respectively, equipped with the connections $\nabla^g + \frac{1}{12}H$ and $\nabla^g - \frac{1}{4}H$. Consider the tensor product of these two connections on $\$ \otimes \$$. The corresponding Dirac operator on Λ is given by*

$$D = (d + H) + (d + H)^*$$

where H is acting by exterior multiplication and $(d+H)^*$ is the formal adjoint of $d+H$ with respect to the metric, namely, $d^* + (-1)^{n(p+1)} * H *$ on Λ^p.

Proof — Let us consider first an even dimensional manifold. Take a p-form θ and identify it with $\varphi = \sum_r \varphi_r^+ \otimes \varphi_r^- \in \Gamma(M, \$\otimes\$)$. Then the Clifford left and right actions of a vector field e are given, respectively, by

$$\begin{array}{rclcl} e\varphi &=& \sum_r e\varphi_r^+ \otimes \varphi_r^- &=& e \wedge \theta - e \lrcorner \theta \\ \varphi e &=& \sum_r \varphi_r^+ \otimes e\varphi_r^- &=& (-1)^p (e \wedge \theta + e \lrcorner \theta) \end{array}$$

Using the summation convention, we have

$$\begin{array}{rcl} D(\varphi) &=& e_i \nabla^g_{e_i} \varphi_r^+ \otimes \varphi_r^- + e_i \varphi_1 \otimes \nabla^g_{e_i} \varphi_2 + \\ && \frac{1}{12} e_i (e_i \lrcorner H) \varphi_r^+ \otimes \varphi_r^- - \frac{1}{4} e_i \varphi_r^+ \otimes (e_i \lrcorner H) \varphi_r^- \\ &=& e_i \nabla^g_{e_i}(\varphi) + \frac{1}{12} e_i (e_i \lrcorner H)\varphi + \frac{1}{4} e_i \varphi (e_i \lrcorner H). \end{array}$$

Since $D^g(\varphi) = e_i \nabla^g_{e_i}(\varphi)$ corresponds to $(d+d^*)\theta$, it remains to see that $\frac{1}{12} e_i (e_i \lrcorner H)\varphi + \frac{1}{4} e_i \varphi (e_i \lrcorner \varphi)$ can be identified with $(H + H^*)\theta$.

Write $H = H_{abc} e_a \wedge e_b \wedge e_c$ and observe that

$$H_{abc} e_a \wedge e_b \wedge e_c \wedge \alpha + H_{abc} e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \alpha))$$

is the same as $(H+H^*)\theta$ since the formal adjoint of exterior multiplication is interior multiplication. It is simple to see that $e_i(e_i \lrcorner H)\varphi = 3H\varphi$ and that the action of H is given by

$$H_{abc}(e_a \wedge e_b \wedge e_c \wedge \theta + e_a \wedge e_b \wedge (e_c \lrcorner \theta) + e_a \wedge (e_b \lrcorner (e_c \lrcorner \theta) + \dots$$

and that $e_i \varphi (e_i \lrcorner H)\theta$ is such that when we add

$$\frac{1}{12} e_i(e_i \lrcorner H)\theta = \frac{1}{4} H\theta$$

and

$$\frac{1}{4} e_i \theta (e_i \lrcorner H)$$

the mixed terms cancel and it amounts to

$$\frac{1}{4} H_{abc}(e_a \wedge e_b \wedge e_c \wedge \alpha + e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \alpha))$$

plus

$$\frac{3}{4} H_{abc}(e_a \wedge e_b \wedge e_c \wedge \alpha + e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \alpha))$$

which is then $(H + H^*)\theta$. The proof in the odd-dimensional case is perfectly analogous.

□

Remark 3.2. Notice that these are lifts of the metric connections on the tangent bundle with torsion $\frac{1}{3}H$ and $-H$. It is interesting to observe that these weights $\frac{1}{3}$ and -1 also appear in Bismut's proof of the local index theorem for non-Kähler manifolds, [3].

Suppose now that H is a closed three-form. In [2], Atiyah and Segal defined the concept of twisted de Rham cohomology. On the de Rham complex of differential forms Ω we can define the operator $d+H$. Note that

$$(d + H)^2 = d^2 + dH + Hd + H^2 = 0$$

since H is closed and of odd degree. The operator $d + H$ does not preserve form degrees but preserves the $\mathbb{Z}_2$-grading. We then have a 2-step chain complex and the cohomology of this complex is then the twisted de Rham cohomology.

The twisted de Rham complex is an elliptic complex so, on a compact manifold, Hodge theory applies. If H^+ and H^- are the cohomology groups then

$$H^{\pm} \simeq \{\theta \in \Omega^{\pm} : \ (d + H)\theta = 0 \quad \text{and} \quad (d + H)^*\theta = 0\}$$

or, in other words, each cohomology class has a unique representative in the kernel of D^2 where

$$D = (d + H) + (d + H)^*.$$

4. A vanishing theorem

We can use the Lichnerowicz formula of theorem 2.1 and also theorem 3.1 to prove the following

Theorem 4.1. *Let M be a compact spin manifold and let H be a closed three-form. Consider the Dirac operator $D^{1/12}$ on $\not{S}\otimes\not{S}$ associated with the connection*

$$\nabla = \nabla^{1/12} \otimes 1 + 1 \otimes \nabla^{1/4},$$

let $F^{-1/4}$ be the curvature of $\nabla^{-1/4}$ on $\not{S}$ and κ the Riemannian scalar curvature of M. If

$$F^{-1/4} + \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$$

acts as a positive endomorphism then the twisted de Rham cohomology for $d + H$ vanishes.

Proof — We start by observing that we need only to prove that the kernel of the operator $D^{1/12}$ is zero. Consider ψ a smooth section of $\not{S} \otimes \not{S}$. Since $dH = 0$, the Lichnerowicz formulas gives

$$(D^{1/12})^2\psi = \Delta^{1/4}\psi + F^{-1/4}\psi + \frac{1}{4}\kappa\psi - \frac{1}{8}\|H\|^2\psi.$$

Now take the inner product of this with ψ. Since the Dirac operator is self-adjoint and the Laplacian Δ is given by $\nabla^*\nabla$, we get

$$\int_M \|D^{1/12}\psi\|^2 \,\mathrm{dV} = \int_M \|\nabla^{1/4}\psi\|^2 + (F^{-1/4}\psi, \psi) + \frac{1}{4}\kappa\|\psi\|^2 - \frac{1}{8}\|H\|^2\|\psi\|^2 \,\mathrm{dV}.$$

Using the hypothesis that

$$F^{-1/4} + \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$$

is a positive endomorphism we conclude that $D^{1/12}\psi = 0$ if and only if $\psi = 0$.

□

5. An example

Let G be a compact, non-abelian Lie group equipped with a bi-invariant metric. Consider the one-parameter family of connections $\nabla^t_X(Y) = t[X, Y]$. Given t, the torsion of ∇^t is $(2t - 1)[X, Y]$. Notice that since the metric is ad-invariant, it means that these are metric connections and also that their torsion is skew-symmetric. Note also that if $t = \frac{1}{2}$ we get the Levi-Civita connection, since the torsion vanishes. The curvature of ∇^t is given by

$$R^{\nabla^t}(X, Y)Z = t^2[X, [Y, Z]] - t^2[Y, [X, Z]] - t[[X, Y], Z] = (t^2 - t)[[X, Y], Z],$$

by means of the Jacobi identity. For $t = 0$ and $t = 1$, we get two flat connections. These correspond, respectively, to the left and right invariant trivialization of the tangent bundle, [4].

Let us write the above one-parameter family of connections as

$$\nabla^{2s}_X(Y) = \nabla^g_X(Y) + 2s[X, Y].$$

Notice that the Levi-Civita connection corresponds now to the parameters $s = 0$ while the two flat connections correspond to $s = \pm\frac{1}{4}$.

Consider the lift of these connections to the spinor bundle $\$$ of G. Take the connection $\nabla^{1/12} \otimes 1 + 1 \otimes \nabla^{1/4}$ on $\Gamma(M, \$ \otimes \$)$. We know from theorem 3.1 that the Dirac operator $D^{1/12}$ then corresponds to $(d + H) + (d + H)^*$ on ΛG, where H is given by $H(X, Y, Z) = ([X, Y], Z)$. Note that H, being a bi-invariant form, is closed.

We need the following auxiliary lemma, which can be proved by direct computation.

Lemma 5.1. *Let G be a non-abelian Lie group equipped with a bi-invariant metric, then the scalar curvature κ of G is given by*

$$\kappa = \frac{1}{4} \sum_{ij} \|[e_i, e_j]\|^2$$

where $\{e_i\}$ is an orthonormal basis of the Lie algebra of G.

Theorem 5.2. *Let G be a compact, non-abelian Lie group equipped with a bi-invariant metric and let $H(X, Y, Z) = ([X, Y], Z)$ be the associated bi-invariant three-form. Then the twisted de Rham cohomology of $d + H$ vanishes.*

Proof — Since $F^{-1/4} = 0$, by means of theorem 4.1 we only need to show that the constant $\rho = \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$ is positive. We have already computed κ in lemma 5.1, so if we take the same orthonormal basis we get that

$$\|H\|^2 = \frac{1}{6} \sum_{ijk} |([e_i, e_j], e_k)|^2,$$

and using the Cauchy-Schwarz inequality

$$\|H\|^2 \leqslant \frac{1}{6} \sum_{ijk} \|[e_i, e_j]\|^2 \|e_k\|^2 = \frac{1}{6} \sum_{ij} \|[e_i, e_j]\|^2$$

So $\rho > \left(\frac{1}{16} - \frac{1}{48}\right) \sum_{ij} \|[e_i, e_j]\|^2 > 0$.

□

Remark 5.3. To see this result for connected, compact, simple groups in a different way, note that it is well known that by averaging, each cohomology class of G can be represented by a bi-invariant form. The de Rham cohomology ring $H^*(G)$ is an exterior algebra (more precisely $H^*(G)$ is an exterior algebra on generators in degree $2d_i - 1$, where each d_i is the degree of generators of invariant polynomials on the Lie algebra of G). The Killing form gives $H^3(G) = \mathbb{R}$. Consider now the twisted de Rham operator $d + H$. Since H is bi-invariant, the twisted cohomology classes can also be represented by bi-invariant forms. Since bi-invariant forms are closed, $(d+H)\alpha = H \wedge \alpha$. So if $H \wedge \alpha = 0$, since H is a generator, then $H \wedge \alpha = 0$ implies that $\alpha = H \wedge \beta$ for some β. Therefore, the twisted cohomology vanishes.

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References

[1] I. Agricola and T. Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Mathematische Annalen, 328:711-748, 2004.
[2] M. Atiyah and G. Segal, *Twisted K-theory and cohomology*, Nankai Tracts in Mathematics, 11:5-43, 2006.
[3] J. M. Bismut, *A local index theorem for non-Kähler manifolds*, Mathematishe Annalen, 284:681-699, 1989
[4] S. Kobayashi and K. Nomizu, *Foundations of Differentiable Geometry*, Vols I & II, Interscience Publishers, New York, 1969.
[5] H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.

Departmento de Matemática, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal
E-mail address: anaferreira@math.uminho.pt