KAZHDAN-LUSZTIG BASIS FOR GENERIC SPECHT MODULES

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Abstract. In this paper, we let $\mathscr{H}$ be the Hecke algebra associated with a finite Coxeter group W and with one-parameter, over the ring of scalars $\mathcal{A} = \mathbb{Z}(q, q^{-1})$. With an elementary method, we introduce a cellular basis of $\mathscr{H}$ indexed by the sets $E_J(J \subseteq S)$ and obtain a general theory of "Specht modules". Our main purpose is to provide an algorithm for W-graphs for the "generic Specht module", which associates with the Kazhdan and Lusztig cell (or more generally, a union of cells of W) containing the longest element of a parabolic subgroup W_J for appropriate $J \subseteq S$. As an example of applications, we show a construction of W-graphs for the Hecke algebra of type A.

Preliminaries

Let W be a finite Coxeter group with S the set of simple reflections, and let $\mathscr{H}$ be the corresponding Hecke algebra. We use a variation of the definition given in [3], taking $\mathscr{H}$ to be an algebra over $\mathcal{A} = \mathbb{Z}[q^{-1}, q]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q. Then $\mathscr{H}$ is a algebra generated by $(T_s)_{s\in S}$ subject to

$$T_s{}^2 = 1 + (q - q^{-1})T_s$$
$$\underbrace{T_rT_sT_r\cdots}_{m_{rs}\text{ factors}} = \underbrace{T_sT_rT_s\cdots}_{m_{rs}\text{ factors}}$$

(for all $r, s \in S$).

Moreover, $\mathscr{H}$ has $\mathcal{A}$-basis $\{ T_w \mid w \in W \}$ where $T_w = T_{s_1}T_{s_2}\cdots T_{s_l}$ whenever $s_1s_2\cdots s_l$ is a reduced expression for w, and

$$(1) \qquad T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + (q - q^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $w \in W$ and $s \in S$. We also define $\mathcal{A}^+ = \mathbb{Z}[q]$, the ring of polynomials in q with integer coefficients, and let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A}$ such that $\overline{q} = q^{-1}$. This involution on $\mathcal{A}$ extends to an involution on $\mathscr{H}$ satisfying $\overline{T_s} = T_s^{-1} = T_s + (q^{-1} - q)$ for all $s \in S$. This gives $\overline{T_w} = T_{w^{-1}}^{-1}$ for all $w \in W$. The map $\mathscr{H} \to \mathscr{H}, h \longmapsto \overline{h}$ is a ring involution such that

$$\overline{\sum_{w\in W} a_wT_w} = \sum_{w\in W} \bar{a_w}{T_{w^{-1}}}^{-1}, a_w \in \mathcal{A}.$$

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0.1. **Kazhdan-Lusztig basis.** There are two types of Kazhdan-Lusztig bases of $\mathscr{H}$, denoted by $\{C_w|w\in W\}$ and $\{C'_w|w\in W\}$ in the original article by Kazhdan-Lusztig [3]. It will be technically more convenient to work with the C-basis. The reason can be seen, for example, in Lusztig [5, chap.18]. The basis element C_w is uniquely determined by the conditions that $\overline{C_w}=C_w$ and $C_w\equiv T_w \mod \mathscr{H}_{>0}$, where $\mathscr{H}_{>0}:=\sum_{w\in W} q\mathcal{A}^+T_w$, see [5]. Or more clearly

$$C_w = T_w + \sum_{y\in W, y<w} p_{y,w}T_y,$$

where $\leq$ denotes the *Bruhat-Chevalley order* on W and $p_{y,w}\in q\mathcal{A}^+$ for all $y<w$ in W. We write $y<w$ if $y\leq w$ and $y\neq w$.

The polynomials $p_{y,w}$ are related to the polynomials $P_{y,w}$ of [3] (the *Kazhdan-Lusztig polynomials*) by $p_{y,w}(q)=(-q)^{\ell(w)-\ell(y)}\overline{P_{y,w}(q^2)}$. That is, to get $p_{y,w}$ from $P_{y,w}$ replace q by q^2, apply the bar involution, and then multiply by $(-q)^{\ell(w)-\ell(y)}$.

0.2. **Multiplication rules for C-basis.** For $s\in S, w\in W$, we have

$$T_sC_w = \begin{cases} -q^{-1}C_w, \text{if } sw<w \\ qC_w + \sum\limits_{y<w, sy<y} \mu(y,w)C_y, \text{if } sw>w. \end{cases} \tag{2}$$

The quantity $\mu(y,w)$, which is the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$, is the coefficient of q in $(-1)^{\ell(w)-\ell(y)}p_{y,w}$. However, since Kazhdan and Lusztig show that $\mu(y,w)$ is nonzero only when $\ell(w)-\ell(y)$ is odd, therefore $\mu(y,w)\in\mathbb{Z}$ can also be described as the coefficient of q in $-p_{y,w}$, as above.

The following notion of W-graph was introduced by Kazhdan and Lusztig in [3].

Definition of W-graph. Since we have slightly modified the definition of Hecke algebra used in [3], we are forced to also slightly alter the definition of W-graph. We define a *W-graph datum* to be a triple (Γ, I, μ) consisting of a set Γ (the vertices of the graph), a function

$$I:\gamma\mapsto I_\gamma$$

from Γ to the set of all subsets of S, and a function

$$\mu:\Gamma\times\Gamma\to\mathbb{Z}$$

such that $\mu(\delta,\gamma)\neq 0$ if and only if $\{\delta,\gamma\}$ is an edge of the graph. These data are subject to the requirement that $\mathcal{A}\Gamma$, the free $\mathcal{A}$-module on Γ, has an $\mathscr{H}$-module structure satisfying

$$T_s\gamma = \begin{cases} -q^{-1}\gamma & \text{if } s\in I_\gamma \\ q\gamma + \sum_{\{\delta\in\Gamma | s\in I_\delta\}} \mu(\delta,\gamma)\delta & \text{if } s\notin I_\gamma, \end{cases} \tag{3}$$

for all $s\in S$ and $\gamma\in\Gamma$. If τ_s is the $\mathcal{A}$-endomorphism of $\mathcal{A}\Gamma$ such that $\tau_s(\gamma)$ is the right-hand side of Eq. (3) then this requirement is equivalent to the condition that for all s, $t\in S$ such that st has finite order, we require that

$$\underbrace{\tau_s\tau_t\tau_s\ldots}_{m \text{ factors}} = \underbrace{\tau_t\tau_s\tau_t\ldots}_{m \text{ factors}}$$

where m is the order of st. (Note that the definition of τ_s guarantees that $(\tau_s+q^{-1})(\tau_s-q)=0$ for all $s\in S$.)

For simplicity, if (Γ, I, μ) is a W-graph datum, we say that Γ is W-graph. We call I_γ the *descent set* of the vertex $\gamma \in \Gamma$, and we call $\mu(\delta,\gamma)$ and $\mu(\gamma,\delta)$ the *edge weights* associated with the edge $\{\delta,\gamma\}$. In almost all the cases we consider it turns out that $\mu(\gamma,\delta) = \mu(\delta,\gamma)$.

0.3. **Cells in W-graphs.** Following [3], given any W-graph Γ we define a preorder relation $\leq$ on Γ as follows: for $\gamma, \gamma' \in \Gamma$ we say that $\gamma \leq_\Gamma \gamma'$ if there exists a sequence of vertices $\gamma = \gamma_0, \gamma_1, \cdots \gamma_n = \gamma'$ such that for each i $(1 \leqslant i \leqslant n)$, we have both $\mu(\gamma_{i-1}, \gamma_i) \neq 0$ and $I_{\gamma_{i-1}} \nsubseteq I_{\gamma_i}$. We shall refer to $\leq_\Gamma$ as the *Kazhdan-Lusztig preorder* on Γ.

Let $\sim$ be the equivalence relation on Γ associated to the Kazhdan-Lusztig preorder; thus $\gamma \sim \gamma'$ means that $\gamma \leq_\Gamma \gamma'$ and $\gamma' \leq_\Gamma \gamma$. The corresponding equivalence classes are called the *cells* of Γ.

In this paper, the preorder $\leq_\Gamma$ is generated by *Kazhdan-Lusztig left preorder* [3]: $x \leq_{\mathcal{L}} y$ if C_x occurs with nonzero coefficient in the expression of $T_s C_y$ in the C-basis, for some $s \in S$. Their equivalence classes are called *left cells*, see [3, 5, 11] where *right cells* and *two-sided cells* are also defined.

0.4. **Left cell module.** Let $\mathfrak{C}$ be a left cell or, more generally, a union of left cells of W. We define an $\mathscr{H}$-module by $[\mathfrak{C}]_{\mathcal{A}} := \mathfrak{I}_{\mathfrak{C}}/\hat{\mathfrak{I}}_{\mathfrak{C}}$ where

$$\mathfrak{I}_{\mathfrak{C}} := \langle C_w | w \leqslant_{\mathcal{L}} z \text{ for some } z \in \mathfrak{C}\rangle_{\mathcal{A}}$$

$$\hat{\mathfrak{I}}_{\mathfrak{C}} := \langle C_w | w \notin \mathfrak{C}, w \leqslant_{\mathcal{L}} z \text{ for some } z \in \mathfrak{C}\rangle_{\mathcal{A}}$$

are the $\mathcal{A}$-spanned modules.

This paper is organized as follows. In Sect. 1 we introduce the indexing sets $D_J, \overline{D_J}$ for the basis of $\mathscr{H}$-module $\mathscr{H} C_{w_J}$, and E_J for the so called *general Specht module*. In Sect. 2, we obtain a version of cellular basis for $\mathscr{H}$ in general and set up the concept of *general Specht module*. In Sect. 3 we show the construction of W-graph basis by introducing a new family of *E_J-Kazhdan-Lusztig polynomials* p_{xy}, and show an inductive procedure for computing $p'_{xy}s$. In Sect.4 we consider an example of type A and discuss the applications of our results, we show the transition between Murphy basis and W-graph basis.

1. The indexing sets

For each $J \subseteq S$, let $\hat{J} = S\backslash J$(the complement of J) and define $W_J = \langle J\rangle$, the corresponding parabolic subgroup of W and let $w_J \in W_J$ be the unique element of maximal length . Let $\mathscr{H}_J$ be the Hecke algebra associated with W_J. As is well known, $\mathscr{H}_J$ can be identified with a subalgebra of $\mathscr{H}$.

1.1. **Sets D_J, $\overline{D_J}$ and E_J.** Let $D_J = \{\, w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\,\}$, the set of minimal coset representatives of W/W_J. The following lemma is well known, it is also an easy consequence of [19, Prop. 5.9].

Lemma 1.1 (Deodhar [2, Lemma 3.2]). *Let $J \subseteq S$ and $s \in S$, and define*

$$\begin{aligned} D_{J,s}^- &= \{\, d \in D_J \mid \ell(sd) < \ell(d)\,\},\\ D_{J,s}^+ &= \{\, d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \in D_J\,\},\\ D_{J,s}^0 &= \{\, d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin D_J\,\},\end{aligned}$$

so that D_J is the disjoint union $D^-_{J,s} \cup D^+_{J,s} \cup D^0_{J,s}$. Then $sD^+_{J,s} = D^-_{J,s}$, and if $d \in D^0_{J,s}$ then $sd = dt$ for some $t \in J$.

Define

$$E_J = \{\, d \in W \mid \ell(ds) < \ell(d) \text{ for all } s \in J \text{ and } \ell(ds) > \ell(d) \text{ for all } s \notin J \,\} \tag{4}$$

that is, E_J is the set of maximal coset representatives of W/W_J and the minimal ones of $W/W_{\hat{J}}$. Clearly $\sharp E_J = \sharp E_{\hat{J}}$, where $E_{\hat{J}}$ was introduced and written as Y_J in [7].

Let $\leq_{\mathscr{L}}$ denote the *left weak Bruhat order* on W. That is, $x \leq_{\mathscr{L}} y$ if and only if $y = wx$ for some $w \in W$ such that $\ell(y) = \ell(w) + \ell(x)$. McDonough-Pallikaros [15] also say that x is a *prefix of* y if $x \leq_{\mathscr{L}} y$. Given $x, y \in W$ let $[x, y]_{\mathscr{L}} = \{z \in w \mid x \leq_{\mathscr{L}} z \leq_{\mathscr{L}} y\}$ be the left interval they determine.

Let

$$\overline{D_J} = D_J w_J,$$

then

$$\overline{D_J} = \{\, d \in W \mid \ell(ds) < \ell(d) \text{ for all } s \in J\}$$

is the set of longest coset representatives of W_J in W. Thus,

$$E_J = \overline{D_J} \cap D_{\hat{J}},$$

and directly from the definition,

$$\overline{D_J} = \bigcup_{J \subseteq K \subseteq S} E_K,$$

where the union is disjoint.

Proposition 1.2. *Let $J \subseteq S$ and $s \in S$, we define*

$$E^-_{J,s} = \{\, d \in E_J \mid \ell(sd) < \ell(d) \text{ and } sd \in E_J \,\},$$
$$E^+_{J,s} = \{\, d \in E_J \mid \ell(sd) > \ell(d) \text{ and } sd \in E_J \,\},$$
$$E^0_{J,s} = \{\, d \in E_J \mid \ sd \notin E_J \,\}$$

so that E_J is the disjoint union $E^-_{J,s} \cup E^+_{J,s} \cup E^0_{J,s}$, then $sE^+_{J,s} = E^-_{J,s}$; let

$$E^{0,-}_{J,s} = \{\, d \in E_J \mid \ell(sd) < \ell(d) \text{ and } sd \notin E_J \,\},$$
$$E^{0,+}_{J,s} = \{\, d \in E_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin E_J \,\},$$

then $E^0_{J,s} = E^{0,-}_{J,s} \bigcup E^{0,+}_{J,s}$ (disjoint union); if $d \in E^{0,-}_{J,s}$ then $sd = dt$ for some $t \in J$, if $d \in E^{0,+}_{J,s}$ then $sd = dt$ for some $t \in \hat{J}$.

Proof. For any $d \in E_J$, we write $d = d'w_J$, where $d' \in D_J$ and w_J the longest element of W_J. Given $s \in S$, we have either $sd < d$ or $sd > d$.

Case(a): if $sd < d$ then we have either $sd \in E_J$ or $sd \notin E_J$. If $sd \in E_J$ then $d \in E^-_{J,s}$.

We now consider the case $sd \notin E_J$. Since $d \in E_J$(that is, $d \in \overline{D_J}$ and $d \in D_{\hat{J}}$)and $sd < d$, according to Lemma 1.1 we have $sd \in D_{\hat{J}}$. Thus $sd \notin \overline{D_J}$, that is $sd' \notin D_J$, this is the case $d' \in D^0_{J,s}$ in the statement of Lemma 1.1 , so we have $sd' > d'$ and $sd' = d't$ for some $t \in J$, and

$$sd = s(d'w_J) = (sd')w_J = (d't)w_J = (d'w_J)t' = dt'$$

where $t' = w_J t w_J \in J$. This is the case $d \in E^{0,-}_{J,s}$.

Case(b): if $sd > d$ then again we have either $sd \in E_J$ or $sd \notin E_J$. If $sd \in E_J$ then $d \in E^{+}_{J,s}$. we consider the case $sd \notin E_J$.

Since $sd = s(d'w_J) = (sd')w_J$, where $d' \in D^{+}_{J,s}$ (according to the above discussion, the case $d' \in D^{0}_{J,s}$ can not happen , and clearly $d' \notin D^{-}_{J,s}$). So $sd \in \overline{D_J}$, and by the assumption $sd \notin E_J$, we have $sd \notin D_{\hat{J}}$.

Applying Lemma 1.1 to the set $D_{\hat{J}}$, we have $sd = dt$ for some $t \in \hat{J}$, which is the case $d \in E^{0,+}_{J,s}$. □

For $w \in W$ we set $\mathcal{L}(w) = \{s \in S; sw < w\}, \mathcal{R}(w) = \{s \in S; ws < w\}$ and refer them to be the *left and right descent set* of w.

Lemma 1.3. [3][5, Prop.8.6] *Let $w, w' \in W$, then*

(a) if $w \leq_{\mathcal{L}} w'$, then $\mathcal{R}(w') \subseteq \mathcal{R}(w)$. If $w \sim_{\mathcal{L}} w'$, then $\mathcal{R}(w') = \mathcal{R}(w)$.

(b) if $w \leq_{\mathcal{R}} w'$, then $\mathcal{L}(w') \subseteq \mathcal{L}(w)$. If $w \sim_{\mathcal{R}} w'$, then $\mathcal{L}(w') = \mathcal{L}(w)$.

The linear map $\varepsilon_J : \mathscr{H}_J \to \mathcal{A}$ defined by $\varepsilon_J(T_w) = \epsilon_w q^{-\ell(w)}$ for any $w \in W_J$ is an algebra homomorphism, called the sign representation. We denote by $\mathrm{Ind}^S_J(\varepsilon_J)$, the $\mathscr{H}$-module obtained by induction from ε_J.

We now introduce the element C_{w_J} in the Kazhdan-Lusztig C-basis of $\mathscr{H}$. By [5, Cor. 12.2], it has the expression

$$C_{w_J} = \epsilon_{w_J} q^{\ell(w_J)} \sum_{w \in W_J} \epsilon_w q^{-\ell(w)} T_w.$$

Lemma 1.4. [8, Lemma 2.8] *The followings hold*

(a) For any $w \in W_J$, we have $T_w C_{w_J} = \epsilon_w q^{-\ell(w)} C_{w_J}$.

(b) We have $C^2_{w_J} = \epsilon_{w_J} q^{-\ell(w_J)} P_J C_{w_J}$, where $P_J = \sum_{w \in W_J} q^{2\ell(w)}$.

(c) The set $\overline{D_J} = D_J w_J$ is a union of left cells in W, we have

$$\overline{D_J} = \{w \in W \mid w \leq_{\mathcal{L}} w_J\},$$

and $[\overline{D_J}]_{\mathcal{A}} \cong \mathrm{Ind}^S_J(\varepsilon_J) \cong \mathscr{H} C_{w_J}$ (isomorphisms as left $\mathscr{H}$-modules).

Proposition 1.5. *For $J \subseteq S$, then*

(1) E_J is the left cell, or union of left cells with right descent set J.

(2) The Bruhat order $\leq$ for the elements of E_J is exactly the weak order $\leq_{\mathscr{L}}$. If $x, y \in E_J$ and $x \leq y$, then $\left[x, y\right]_{\mathscr{L}} \subseteq E_J$.

Proof. (1) is directly from Lemma 1.3 and 1.4.

(2) is from Prop. 1.2. □

Remark For convenience, in the following sections we still use the usual notations of Bruhat order $\leq, <$ for the weak Bruhat orders $\leq_{\mathscr{L}}, <_{\mathscr{L}}$ for the elements of E_J, unless indicated.

1.2. **Some multiplication rules.** For $J \subseteq S$, let $M^J = \mathscr{H} C_{w_J}$ be a $\mathscr{H}$-module, then

Lemma 1.6. *(1) Let $J \subseteq S$, then M^J is a free $\mathcal{A}$-module with basis*

$$\{T_w C_{w_J} \mid w \in D_J\}, \text{ or alternatively } \{T_w C_{w_J} \mid w \in \overline{D_J}\}.$$

the multiplication of $\mathscr{H}$ with respect to this basis:

$$T_s(T_w C_{w_J}) = \begin{cases} T_{sw} C_{w_J} + (q - q^{-1}) T_w C_{w_J} & \text{if } w \in D^{-}_{J,s} \text{ or } w \in \overline{D}^{-}_{J,s} \\ T_{sw} C_{w_J} & \text{if } w \in D^{+}_{J,s} \text{ or } w \in \overline{D}^{+}_{J,s} \\ -q^{-1} T_w C_{w_J} & \text{if } w \in D^{0}_{J,s} \text{ or } w \in \overline{D}^{0}_{J,s} \end{cases}$$

for all $s \in S$.

(2) For $w \in E_J$, we have :

$$T_s(T_w C_{w_J}) = \begin{cases} T_{sw}C_{w_J} + (q-q^{-1})T_w C_{w_J} & \text{if } w \in E^{-}_{J,s} \\ T_{sw}C_{w_J} & \text{if } w \in E^{+}_{J,s} \\ -q^{-1}T_w C_{w_J} & \text{if } w \in E^{0,-}_{J,s} \\ qT_w C_{w_J} + T_w C_{tw_J} & \text{if } w \in E^{0,+}_{J,s}, t = w^{-1}sw \in \hat{J} \end{cases}$$

Proof. (1) M^J is spanned by the elements $T_w C_{w_J}$, where $w \in W$; however, if $w = dv$ for $d \in D_J$ and $v \in W_J$, then $T_w C_{w_J} = \varepsilon_v q^{-\ell(v)} T_d C_{w_J}$. It follows that M^J is a free $\mathcal{A}$-module with the basis shown and it remains to verify the multiplication formulae.

According to Eq. (1) we immediately get the first two rules. By the multiplication formula for the C-basis elements(Eq. (2)), we have:

$$T_s C_{w_J} = \begin{cases} -q^{-1}C_{w_J} & \text{if } s \in J \\ qC_{w_J} + C_{sw_J} & \text{if } s \in \hat{J} \end{cases}$$

if $w \in D^0_{J,s}$, let $t = w^{-1}sw$ and $t \in J$ then $sw = wt < w$, we have

$$\begin{aligned} T_s(T_w C_{w_J}) &= \big[T_{sw} + (q-q^{-1})T_w\big]C_{w_J} \\ &= \big[T_{wt} + (q-q^{-1})T_w\big]C_{w_J} \\ &= \big[T_{wt}(T_t {T_t}^{-1}) + (q-q^{-1})T_w\big]C_{w_J} \\ &= T_w {T_t}^{-1} C_{w_J} + (q-q^{-1})T_w C_{w_J} \\ &= T_w\big[T_t + (q^{-1}-q)\big]C_{w_J} + (q-q^{-1})T_w C_{w_J} \\ &= -q^{-1}T_w C_{w_J}. \end{aligned}$$

(2) If $w \in E^{0,+}_{J,s}$ and $t = w^{-1}sw \in \hat{J}$, again by the multiplication rules for C_{w_J}

$$T_s(T_w C_{w_J}) = T_w(T_t C_{w_J}) = T_w(qC_{w_J} + C_{tw})$$

□

2. A cellular basis and generic Specht modules

The concept of "cellular algebras" was introduced by Graham-Lehrer [14]. It provides a systematic framework for studying the representation theory of non-semisimple algebras which are deformations of semisimple ones. The original definition was modeled on properties of the Kazhdan-Lusztig basis [3] in Hecke algebras of type A. There is now a significant literature on the subject, and many classes of algebras have been shown to admit a "cellular" structure, including Ariki-Koiki algebras, q-Schur algebras, Temperly-Lieb algebras, and a variety of other algebras with geometric connections.

As we discussed above, $\mathscr{H}$ is the one-parameter Hecke algebra associated to finite Weyl group W. Furthermore, if $\mathscr{H}$ is defined over a ground ring in which "bad" primes for W are invertible, Geck [9] used deep properties of the Kazhdan-Lusztig basis and Lusztig's **a**-function, he showed that $\mathscr{H}$ has a natural cellular structure in the sense of Graham-Lehrer.

For the purpose of this paper, we show a new version of cellular basis of $\mathscr{H}$. Thus, we also obtain a general theory of "Specht modules" for Hecke algebras of finite type.

We introduce an $\mathcal{A}$-linear anti-involution: $* : \mathscr{H} \longrightarrow \mathscr{H}$ by $T_w^* = T_{w^{-1}}$ for $w \in W$. Clearly, $C_{w_J}^* = C_{w_J}$; for any $J \subseteq S$ and let $x, y \in D_J$ (or $x, y \in \overline{D_J}$), we define $m_{xy} = T_x C_{w_J} T_y^*$. Then $m_{xy}^* = m_{yx}$. For convenience, we use the indexing set $\overline{D_J}$ in the following context.

Remark If $J = \emptyset$ then $D_J = W$, as an $\mathcal{A}$-modules, $M^{\emptyset} = \mathscr{H}$ so the elements

$$\{m_{xy} \mid x, y \in \overline{D_\emptyset}\}$$

certainly span $\mathscr{H}$.

In order to show that $\mathscr{H}$ is cellular, we have to show that m_{xy} with $x, y \in \overline{D_J}$, can be written as an $\mathcal{A}$-linear combination of $\{m_{uv} \mid u, v \in E_K, J \subseteq K\}$.

Lemma 2.1. *For any $x \in \overline{D_J}$, we have*

$$T_x C_{w_J} = \sum_{x' \in E_J} r_{x'} T_{x'} C_{w_J} + \sum_{u \in E_K, J \subsetneq K} r_u T_u C_{w_K}.$$

where $r_{x'}, r_u \in \mathcal{A}$.

Proof. As we have found $\overline{D_J} = \bigcup_{J \subseteq K \subseteq S} E_K$, where the union is disjoint. If $x \in E_J$ there is nothing to prove; suppose that $x \notin E_J$, then $x \in E_K$ where $K \supsetneq J$. By Prop. 1.2 we have $x = w w_K$ and $w_K = g w_J$ where $w \in W$(or more exactly $w \in D_K$) and $g \in D_J^K = D_J \cap W_K$, with $\ell(x) = \ell(w) + \ell(w_K)$ and $\ell(w_K) = \ell(g) + \ell(w_J)$.

Since $T_g C_{w_J}$ is the sum of $C_{g w_J} = C_{w_K}$ and a linear combination of terms $C_{h w_J}$ where $h \in D_J^K$ and $h < g$ (this is the special case of [10, Prop.2.3]). On the other hand, $C_{h w_J}$ is the sum of $T_h C_{w_J}$ and an $\mathcal{A}$-linear combination of terms $T_f C_{w_J}$, where $f < h, f \in D_J^K$. As a result, $T_g C_{w_J}$ is the sum of C_{w_K} and an $\mathcal{A}$-linear combination of these terms $T_f C_{w_J}$. Thus

$$\begin{aligned}
T_x C_{w_J} &= T_{w(g w_J)} C_{w_J} \\
&= \epsilon_{w_J} q^{-\ell(w_J)} T_w (T_g C_{w_J}) \\
&= \epsilon_{w_J} q^{-\ell(w_J)} T_w \big(C_{w_K} + \sum_{f < g, f \in D_J^K} r_f T_f C_{w_J} \big) \\
&= r_w T_w C_{w_K} + \sum_{z \in \overline{D_J}, z < wg} r_z T_z C_{w_J}
\end{aligned}$$

where $r_w, r_f, r_z \in \mathcal{A}$. By induction, each term $T_z C_{w_J}$ has also the required form. □

Lemma 2.2. *Let $J \subseteq S$ and suppose that $x, y \in \overline{D_J}$, then there exist $r_{x'y}, r_{uv} \in \mathcal{A}$ such that*

$$m_{xy} = \sum_{x' \in E_J} r_{x'y} m_{x'y} + \sum_{u \in E_K, v \in \overline{D_K}, J \subsetneq K} r_{uv} m_{uv}.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} m_{xy} &= T_x C_{w_J} T_y^* \\ &= [\sum_{x' \in E_J} r_{x'} T_{x'} C_{w_J} + \sum_{u \in E_K, J \subsetneq K} r_u T_u C_{w_K}] T_y^* \\ &= \sum_{x' \in E_J} r_{x'} T_{x'} C_{w_J} T_y^* + \sum_{u \in E_K, J \subsetneq K} r_u T_u C_{w_K} T_y^* \end{aligned}$$

and

$$C_{w_K} T_y^* = (T_y C_{w_K})^*$$

where $T_y C_{w_K} \in \mathscr{H} C_{w_K}$, this implies $T_y C_{w_K} \in \langle T_v C_{w_K} \mid v \in \overline{D_K} \rangle_{\mathcal{A}}$, as required. □

Let $\Omega^{lex} = \{J \mid J \subseteq S\}$ be a set ordered lexicographically.

Theorem 2.3. *The Hecke algebra $\mathscr{H}$ is free as an $\mathcal{A}$-module with basis*

$$\mathcal{M} = \{m_{uv} \mid u, v \in E_J \textit{ for some } J \subseteq S\}.$$

Proof. We first show that $\mathcal{M}$ spans $\mathscr{H}$ by showing that whenever $x, y \in \overline{D_J}$ then m_{xy} can be written as a $\mathcal{A}$-linear combination of terms m_{uv} in $\mathcal{M}$. When $J = S$ this is clear because $\mathscr{H} C_{w_J} \mathscr{H} = \mathcal{A} C_{w_J}$. If $J \neq S$, by Lemma 2.2 , we have

$$m_{xy} = \sum_{x' \in E_J} r_{x'y} m_{x'y} + \sum_{(u,v), J \subsetneq K} r_{uv} m_{uv},$$

where $r_{x'}, r_{uv} \in \mathcal{A}$, and the second sum is over the pairs (u, v) where $u \in E_K$, $v \in \overline{D_K}$. However, $m_{xy}^* = m_{yx}$ so by induction on the elements of Ω^{lex} again (start with $J = S$, clearly $C_{w_J}^* = C_{w_J}$), m_{xy} can be written as an $\mathcal{A}$-linear combination of elements of $\mathcal{M}$. Finally, let $J = \emptyset$, then $\mathscr{H} = \mathscr{H} C_{w_\emptyset} \mathscr{H}$.

Therefore $\mathcal{M}$ spans $\mathscr{H}$.

By Wedderburn's theorem $dim(\mathscr{H}) = |W| = \sum_{J \subseteq S} |\mathcal{M}(J)|^2$, where

$$\mathcal{M}(J) = \{m_{uv} \mid u, v \in E_J \text{ for a fixed } J, J \subseteq S\}.$$

Hence the set $\mathcal{M}$ has the correct cardinality. □

Define $\hat{\mathscr{H}}^J$ to be the $\mathcal{A}$-module with basis

$$\{m_{uv} \mid w, v \in E_K \text{ for some } K \text{ such that } J \subset K \subseteq S\}.$$

where we write $J \subset K$ when $J \subseteq K$ and $J \neq K$. Similarly, we define $\mathscr{H}^J$ to be the $\mathscr{H}$-module with basis m_{uv} where $u, v \in E_K$ with $J \subseteq K \subseteq S$.

Theorem 2.4. *(1) The $\mathcal{A}$-linear map determined by*

$$m_{uv} \longmapsto m_{vu}$$

for all $m_{uv} \in \mathcal{M}$, is an anti-isomorphism of $\mathscr{H}$.

(2) Suppose that $h \in \mathscr{H}$ and that $u \in E_J$, there exist $r_u \in \mathcal{A}$ such that for all $v \in E_J$

$$h m_{uv} \equiv \sum_{w \in E_J} r_w m_{wv} \quad mod\ \hat{\mathscr{H}}^J.$$

Consequently, $\{\mathcal{M}, \Omega^{lex}\}$ is a cellular basis of $\mathscr{H}$.

Proof. (1) The $*$-endomorphism and the $\mathcal{A}$-linear map determined by $m_{uv} \longmapsto m_{vu}$ coincide since $m_{uv}^* = m_{vu}$ for all m_{uv} in $\mathcal{M}$. This proves (1) since $*$ is an anti-isomorphism of $\mathscr{H}$

(2) We argue by induction on $J \in \Omega^{lex}$. By (1), if $J = S$ then $\mathscr{H} C_{w_J} \mathscr{H} = \mathcal{A} C_{w_J}$, there is nothing to prove. Suppose that $J \subsetneq S$. First we consider $v = w_J$. Since $\mathcal{M}$ is a basis of $\mathscr{H}$, for any $h \in \mathscr{H}$ we may write

$$hm_{u,w_J} = \sum_{x,y \in E_K, K \subseteq S} r_{xy} m_{xy}$$

for some $r_{xy} \in \mathcal{A}$. Now hm_{u,w_J} belongs to M^J, clearly, if $r_{xy} \neq 0$ then $J \subseteq K$; further, if $J = K$ then we must also have $v = w_J$. Hence,

$$hm_{u,w_J} = \sum_{x \in E_J} r_x m_{x,w_J} \mod \hat{\mathscr{H}}^J \tag{5}$$

where $r_x = r_{x,w_J} \in \mathcal{A}$. This completes the proof of (2) when $v = w_J$.

Now, if $K \supsetneq J$ and $u, y \in E_K$ then $m_{uy} T_v^* = (T_v m_{yu})^* \in \mathscr{H}^K \subseteq \hat{\mathscr{H}}^J$ by induction on $J \in \Omega^{lex}$. Therefore, we can multiply the Eq. (5) on the right by T_v^*, to complete the proof. □

So we can now introduce the following:

Definition 2.5. Let $S^J = \langle T_u C_{w_J} + \hat{\mathscr{H}}^J \mid u \in E_J \rangle_{\mathcal{A}}$, then S^J is an $\mathscr{H}$-submodule of $\mathscr{H}^J / \hat{\mathscr{H}}^J$. We call this the *generic Specht module* of $\mathscr{H}$ associated with J.

The bar involution for S^J. For all x, $y \in E_J$ we define elements $R_{x,y} \in \mathcal{A}$ by the formula

$$\overline{T_y C_{w_J}} = \sum_{x \in E_J} R_{x,y} T_x C_{w_J} \mod \hat{\mathscr{H}}^J, \tag{6}$$

We can easily derive the following formulae which provide an inductive procedure for calculating these elements in S^J.

Proposition 2.6. *Let x, $y \in E_J$. If $s \in S$ is such that $\ell(sy) < \ell(y)$ then*

$$R_{x,y} (mod\ \hat{\mathscr{H}}^J) = \begin{cases} R_{sx,sy} & \text{if } x \in E_{J,s}^- \\ R_{sx,sy} + (q^{-1} - q) R_{x,sy} & \text{if } x \in E_{J,s}^+ \\ -q R_{x,sy} & \text{if } x \in E_{J,s}^{0,-} \\ q^{-1} R_{x,sy} & \text{if } x \in E_{J,s}^{0,+} \end{cases}$$

We may use induction on $\ell(y)$ to establish that $R_{x,y} = 0$ unless $x \leqslant_{\mathscr{L}} y$ in the weak Bruhat partial order on E_J; this follows from the fact that if $sy \leqslant_{\mathscr{L}} y$ and $x \leqslant_{\mathscr{L}} sy$ then both $x \leqslant_{\mathscr{L}} y$ and $sx \leqslant_{\mathscr{L}} y$. It is also easily seen that $R_{x,x} = 1$.

3. W-graphs for generic Specht modules

Let $\mathfrak{C}_{w_J}$ be a left cell, or more generally, a union of left cells containing w_J, then the transition between the bases of the left cell module $[\mathfrak{C}_{w_J}]_{\mathcal{A}}$ and the generic Specht module S^J is described as the following:

Theorem 3.1. *The $\mathscr{H}$-module S^J has a unique basis $\{ C_w \mid w \in E_J \}$ such that $\overline{C_w} = C_w$ for all $w \in E_J$, and*

$$C_w = \sum_{y \in E_J} P_{y,w} T_y C_{w_J} \ \ mod\ \hat{\mathscr{H}}^J$$

for some elements $P_{y,w} \in \mathcal{A}^+$ with the following properties:

(i) $P_{y,w} = 0$ *if* $y \nleq w$;
(ii) $P_{w,w} = 1$;
(iii) $P_{y,w}$ *has zero constant term if* $y \neq w$.

Comparing with the original Kazhdan-Lusztig's polynomials in [3], we called $\{P_{y,w} \mid y, w \in E_J\}$ the family of *E_J-relative Kazhdan-Lusztig polynomials.* We shall show that the basis $\{C_w \mid w \in E_J\}$ give S^J the structure of a W-graph. That is, there is a W-graph Λ with vertex elements $\{C_w \mid w \in E_J\}$. Before showing the proof of Theorem 3.1, we describe the edge weights and descent sets for Λ.

Given y, $w \in E_J$ with $y \neq w$, we define an integer $\mu(y, w)$ as follows. If $y < w$ then $\mu(y, w)$ is the coefficient of q in $-P_{y,w}$.

We write $y \prec w$ if $y < w$ and $\mu(y, w) \neq 0$.

The **(left) descent set** associated with the vertex element $C_w (w \in E_J)$ of Λ is

$$\begin{aligned} I(w) &= \{\, s \in S \mid \ell(sw) < \ell(w)\} \\ &= \{\, s \in S \mid w \in E^-_{J,s}\} \cup \{s \mid w \in E^{0,-}_{J,s}\,\} \end{aligned}$$

In accordance with the notation introduced in Section 2, we define

$$\begin{aligned} \Lambda^-_s &= \{\, w \in E_J \mid s \in I(w)\,\} \\ &= \{\, w \mid w \in E^-_{J,s} \text{ or } w \in E^{0,-}_{J,s}\}, \end{aligned}$$

and similarly $\Lambda^+_s = \{\, w \in E_J \mid s \notin I(w)\,\}$. Our proof of Theorem 3.1 will also incorporate a proof of the following result, which will be an important component of the subsequent proof that Λ is a W-graph.

Theorem 3.2. *Let $v \in E_J$. Then for all $s \in S$ such that $\ell(sv) > \ell(v)$ and $sv \in E_J$ we have*

$$T_s C_v = qC_v + C_{sv} + \sum_{z \in E_J} \mu(z, v) C_z,$$

where the sum is over all $z \in \Lambda^-_s$ such that $z \prec v$.

The following is the proof of Theorem 4.1.

Proof. Uniqueness is proved similarly with that of [3, Theorem 1.1], we omit the details.

Existence. We give a recursive procedure for constructing elements $P_{x,w}$ satisfying the requirements of Theorem 3.1. We start with the definition

$$P_{w_J, w_J} = 1$$

so that $\overline{C_w} = C_w$ holds for $w = w_J$, as do Conditions (i), (ii) and (iii).

Now assume that $w \neq w_J$ and that for all $v \in E_J$ with $\ell(v) < \ell(w)$ the elements $P_{x,v}$ have been defined (for all $x \in E_J$) so that the requirements of Theorem 3.1 are satisfied. Thus the elements C_v are known when $\ell(v) < \ell(w)$. We may choose $s \in S$ such that $w = sv$ with $\ell(w) = \ell(v) + 1$; note that $v \in E_J$ by Lemma 1.6. In accordance with the formula in Theorem 3.2 we define

$$C_w = (T_s - q)C_v - \sum_{\substack{z \prec v \\ z \in \Lambda^-_s}} \mu(z, v) C_z. \tag{7}$$

Since $\overline{T_s - q} = T_s - q$, induction immediately gives $\overline{C_w} = C_w$. We define $P'_{y,w}$ and $P''_{y,w}$ by

$$(T_s - q)C_v = \sum_{y\in E_J} P'_{y,w} T_y C_{w_J} \tag{8}$$

$$\sum_{z\prec v}\mu(z,v)C_z = \sum_{y\in E_J} P''_{y,w} T_y C_{w_J} \tag{9}$$

and define $P_{y,w} = P'_{y,w} - P''_{y,w}$.

If $y \in E_J$ then

$$(T_s - q)T_y = \begin{cases} T_{sy} - qT_y & \text{if } y \in E^+_{J,s} \\ T_{sy} - q^{-1}T_y & \text{if } y \in E^-_{J,s} \\ T_y(T_t - q) & \text{if } y \in E^{0,-}_{J,s} \\ T_{sy} - qT_y & \text{if } y \in E^{0,+}_{J,s} \end{cases}$$

where we have written $t = y^{-1}sy$ in the case $y \in E^0_{J,s}$. Thus we see that

$$\begin{aligned}
(T_s - q)C_v = &\sum_{y\in E^+_{J,s}} P_{y,v}(T_{sy} - qT_y)C_{w_J} + \sum_{y\in E^-_{J,s}} P_{y,v}(T_{sy} - q^{-1}T_y)C_{w_J} \\
&+ \sum_{y\in E^{0,-}_{J,s}} P_{y,v}T_y(T_t - q)C_{w_J} + \sum_{y\in E^{0,+}_{J,s}} P_{y,v}(T_{sy} - qT_y)C_{w_J} \\
= &\sum_{y\in E^-_{J,s}} (P_{sy,v} - q^{-1}P_{y,v})T_y C_{w_J} + \sum_{y\in E^+_{J,s}} (P_{sy,v} - qP_{y,v})T_y C_{w_J} \\
&+ \sum_{y\in E^{0,-}_{J,s}} P_{y,v}(-q^{-1} - q)T_y C_{w_J} \\
&+ \sum_{y\in E^{0,+}_{J,s}} P_{y,v}\Big[(qT_y C_{w_J} + T_y C_{tw_J}) - qT_y C_{w_J}\Big]
\end{aligned}$$

Now comparing Eq. (8) with the expression for $(T_s - q)C_v$ obtained above we obtain the following formulas for the cases $y \in E^+_{J,s}$ (case (a)), $y \in E^-_{J,s}$ (case (b)), $y \in E^{0,-}_{J,s}$ and (case (c)) and $y \in E^{0,+}_{J,s}$ (case (d)):

$$P'_{y,w} = \begin{cases} P_{sy,v} - qP_{y,v} & \text{(case (a))}, \\ P_{sy,v} - q^{-1}P_{y,v} & \text{(case (b))}, \\ (-q - q^{-1})P_{y,v} & \text{(case (c))}, \\ 0 & \text{(case (d))}. \end{cases} \tag{10}$$

Since $C_z = \sum_{y\in E_J} P_{y,z} T_y C_{w_J}$, we have

$$\sum_{z\prec v, z\in\Lambda^-_s} \mu(z,v)C_z = \sum_{y\in E_J} \sum_{z\prec v, z\in\Lambda^-_s} \mu(z,v)P_{y,z}T_y C_{w_J}$$

and by comparison with Eq. (9)

$$P''_{y,w} = \sum_{\substack{z\prec v \\ z\in\Lambda^-_s}} \mu(z,v)P_{y,z}. \tag{11}$$

We may check that with $P'_{y,w}$ and $P''_{y,w}$ given by Eq's (10) and (11), the elements $P_{y,w} = P'_{y,w} - P''_{y,w}$ lie in $\mathcal{A}^+$ and satisfy Conditions (i), (ii) and (iii) of Theorem 3.1. We omit the details here. □

For convenience, let $\tilde{T_w} = T_w C_{w_J}$. Observe that the formula for C_w in Theorem 3.1 may be written as

$$C_w = \tilde{T_w} + \sum_{y<w, y\in E_J} P_{y,w}\tilde{T_y},$$

and inverting this gives

$$\tilde{T_w} = C_w + \sum_{y<w, y\in E_J} Q_{y,w} C_y \tag{12}$$

where the elements $Q_{y,w}$ (defined whenever $y < w$) are given recursively by

$$Q_{y,w} = -P_{y,w} - \sum_{\{z|y<z<w\}} Q_{y,z}P_{z,w}.$$

In particular, $Q_{y,w}$ is in $\mathcal{A}^+$, has zero constant term, and has coefficient of q equal to $\mu(y, w)$.

We now state our main result.

Theorem 3.3. *The basis $\{C_w \mid w \in E_J\}$ gives the generic Specht module S^J the structure of a W-graph, as described above.*

Proof. The proof is similar with [21, Theorem 2.6], modified appropriately. We start by using induction on $\ell(w)$ to prove that for all $s \in S$

$$T_s C_w = \begin{cases} -q^{-1}C_w & \text{if } w \in \Lambda_s^-, \\ qC_w + \sum\limits_{z\in E_J, z\in\Lambda_s^-} \mu(z,w)C_z & \text{if } w \notin \Lambda_s^-. \end{cases} \tag{13}$$

or more exactly
(14)

$$T_s C_w(\bmod \hat{\mathscr{H}}^J) = \begin{cases} -q^{-1}C_w & \text{if } w \in E_{J,s}^- \text{ or } w \in E_{J,s}^{0,-}, \\ qC_w + C_{sw} + \sum\limits_{z\in E_{J,s}^-, z<w} \mu(z,w)C_z & \text{if } w \in E_{J,s}^+. \\ qC_w + \sum\limits_{z\in E_{J,s}^-, z<w} \mu(z,w)C_z & \text{if } w \in E_{J,s}^{0,+}. \end{cases}$$

If $w \in E_{J,s}^+$ then $w \notin \Lambda_s^-$, and Eq. (13) follows immediately from Theorem 3.2 (applied with v replaced by w), since the only $z \in \Lambda_s^-$ with $\mu(z, w) \neq 0$ and $\ell(z) \geqslant \ell(w)$ is $z = sw$.

For the case $w \in E_{J,s}^{0,+}$, the term C_{sw} can not appear in the sum of Eq. (13).

If $w \in E_{J,s}^-$, which implies that $w \in \Lambda_s^-$, then writing $v = sw$ and applying Theorem 3.2 gives

$$C_w = (T_s - q)C_v - \sum \mu(z, v)C_z,$$

where $z \prec v$ and $z \in \Lambda_s^-$ for all terms in the sum. The inductive hypothesis thus gives $T_sC_z = -q^{-1}C_z$, and since we also have $T_s(T_s - q) = -q^{-1}(T_s - q)$ it follows that $T_sC_w = -q^{-1}C_w$, as required.

Now suppose that $w \in E^0_{J,s}$, and as usual let us write $sw = wt$. Suppose first that $t = w^{-1}sw \in J$, so that $w \in \Lambda_s^-$. By Eq. (12),

$$C_w = \tilde{T_w} - \sum_{\{y \mid y<w, y\in E_J\}} Q_{y,w} C_y,$$

and since $T_s T_w C_{w_J} + q^{-1} T_w C_{w_J} = T_w(T_t C_{w_J} + q^{-1} C_{w_J}) = 0$ we find that

$$T_s C_w + q^{-1} C_w = -\sum_{\{y \mid y<w, y\in E_J\}} Q_{y,w}(T_s C_y + q^{-1} C_y). \tag{15}$$

By the inductive hypothesis,

$$T_s C_y + q^{-1} C_y = \begin{cases} 0 & \text{if } y \in \Lambda_s^- \\ (q+q^{-1})C_y + \sum_{z\in\Lambda_s^-} \mu(z,y) C_z & \text{if } y \notin \Lambda_s^-, \end{cases}$$

and so Eq. (15) gives

$$T_s C_w + q^{-1} C_w = -\sum_{\substack{y\notin\Lambda_s^- \\ y<w}} Q_{y,w}(q+q^{-1})C_y \; + \; X \tag{16}$$

for some X in the $\mathcal{A}$-submodule spanned by the elements C_z for $z \in \Lambda_s^-$. Now since $T_s = T_s^{-1} + (q - q^{-1})$ it follows that

$$\begin{aligned}(T_s + q^{-1})C_w &= \overline{(T_s + q^{-1})C_w} \\ &= -\sum_{\substack{y\notin\Lambda_s^- \\ y<w}} \overline{Q_{y,w}}(q^{-1}+q)C_y \; + \; \overline{X},\end{aligned}$$

and comparing with Eq. (16) shows that for all y with $y < w (y \in E_J)$ and $y \notin \Lambda_s^-$,

$$\overline{Q_{y,w}} = Q_{y,w}. \tag{17}$$

Since $Q_{y,w}$ is in $\mathcal{A}^+$ and has zero constant term, Eq. (17) forces $Q_{y,w}$ to be zero whenever $y < w$ and $y \notin \Lambda_s^-$. Therefore the right hand side of Eq. (15) is zero, since $T_s C_y + C_y = 0$ whenever $y \in \Lambda_s^-$. So

$$T_s C_w = -q^{-1} C_w,$$

as required. □

.

4. Applications to type A

Throughout this section, we apply our results to the Hecke algebra of type A. Let $W = \mathfrak{S}_n$ be the symmetric group acting on the left on $\{1, 2, \cdots, n\}$. Another reference is the exposition by Mathas [6]. For $i = 1, 2, \cdots, n-1$ let s_i be the basic transposition $(i, i+1)$ and let $S = \{s_1, s_2, \cdots, s_{n-1}\}$, the generating set of $\mathfrak{S}_n$.

4.1. **Notations.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ be a partition of n with the notation $\lambda \vdash n$. A standard λ-tableau is a tableau whose entries are exactly $1, 2, \cdots, n$ and which has both increasing rows and increasing columns, the set is denoted $\mathbb{T}(\lambda)$. Let t^λ (resp. t_λ) be the λ-tableau in which the numbers $1, 2, \cdots$ appear in order from left to right (resp. top to bottom) and down along successive rows (resp. columns), then $t^\lambda, t_\lambda \in \mathbb{T}(\lambda)$. For a Young tableau t, we put

$$I(t) = \{i \mid 1 \leq i \leq n-1, i+1 \text{ is in a lower position than } i \text{ in } t\}$$

and call it the *descent set* of t. Let

$$I_0(t) = \{i \in I(t) \mid i+1 \text{ is in the left side of } i \text{ in } t\},$$
$$I_1(t) = \{i \in I(t) \mid i+1 \text{ is directly below } i \text{ in } t\}.$$

Lemma 4.1. [17] *For a standard tableau t of shape $\lambda \vdash n$,*

$$(1) I(t) = I_0(t) \cup I_1(t);$$
$$(2) I(t) \cup I(t') = \{1, 2, \ldots, n-1\};$$
$$(3) I_0(t) = \emptyset \text{ if and only if } t = t_\lambda;$$
$$(4) I_0(t') = \emptyset \text{ if and only if } t = t^\lambda.$$

The Young subgroup $\mathfrak{G}_\lambda = \mathfrak{G}_{\lambda_1} \times \cdots \times \mathfrak{G}_{\lambda_r}$ of $\mathfrak{G}_n$ is the row stabilizer of t^λ. Let D_λ be the set of distinguished left coset representatives of $\mathfrak{G}_\lambda$ in $\mathfrak{G}_n$, by Dipper-James [1] and Mathas [6], we have the following explicit description:

$$D_\lambda = \{w \in \mathfrak{G}_n \mid wt^\lambda \text{ is row-standard}\}.$$

As in [1, 12, 6], if t is a row-standard λ-tableau, the unique element $d \in D_\lambda$ such that $t = dt^\lambda$ will be denoted by $d(t)$. Let $w_{J(\lambda)}$ be the longest element of the Young subgroup $\mathfrak{G}_\lambda$, an element w_λ is defined by $t_\lambda = w_\lambda t^\lambda$.

Given partitions $\mu = (\mu_1, \mu_2, ...)$ and $\lambda = (\lambda_1, \lambda_2, ...)$ of n, we say μ *dominates* λ, and write $\lambda \trianglelefteq \mu$, if

$$\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \lambda_1 + \lambda_2 + \lambda_3 \leq \mu_1 + \mu_2 + \mu_3, ...$$

we write $\lambda \trianglelefteq \mu$ if $\lambda \trianglelefteq \mu$ and $\mu \neq \lambda$. The partial order $\trianglelefteq$ on the set of partitions(or shapes) of n will be referred to as the *dominance order*.

For a fixed $\lambda \vdash n$, $s, t \in \mathbb{T}(\lambda)$. We write $s \trianglelefteq t$ if $\ell(d(s)) \leqslant \ell(d(t))$, and $s \triangleleft t$ if $s \trianglelefteq t$ and $s \neq t$. We note that the notation here is different with [6][pp.31].

4.2. **Cells.** The cells of $W = \mathfrak{G}_n$ may be described in terms of the Robinson-Schensted correspondence. The correspondence is a bijection of S_n to pairs of standard tableaux (P, Q) of the same shape corresponding to partitions of n, so that if $w \longmapsto (P(w), Q(w))$ then $Q(w) = P(w^{-1})$. In particular, the involutions are the elements $w \in W$ for which $Q(w) = P(w)$. If $\lambda \vdash n$, the pair of tableaux corresponding to $w_{J(\lambda)}$ has the form $(t_{\lambda'}, t_{\lambda'})$. Hence, the tableaux corresponding to $w_{J(\lambda)}$ have shape λ', where λ' denotes the partition conjugate to λ.

If R is a fixed standard tableau then the set $\{w \in W : Q(w) = R\}$ is a left cell of W and the set $\{w \in W : P(w) = R\}$ is a right cell of W. See [3] and also [4] for an alternative proof of this result.

Lemma 4.2. *Let $\lambda \vdash n$ and $t \in \mathbb{T}(\lambda)$. The element of $\mathfrak{G}_n$, which corresponds to the pair of tableaux $(t^{\lambda'}, t_{\lambda'})$ under the Robinson-Schensted correspondence, is $w_\lambda w_{J(\lambda)}$.*

The following is the corollaries of the discussion in Section 1, see also in [15, Lemma 3.3] and Du [16, Lemma 1.2].

Lemma 4.3. *The followings hold(i) $w_\lambda w_{J(\lambda)} \in D_\lambda$, (ii) $dw_{J(\lambda)} \in D_\lambda$ for each prefix d of w_λ, (iii) $dw_{J(\lambda)} \in D_\lambda$ is in the same left cell as $w_{J(\lambda)}$ for each prefix d of w_λ.*

As in Section 1, we write $E_{J(\lambda)} = \{e \mid e = dw_{J(\lambda)} \text{ and } d \text{ is a prefix of } w_\lambda\}$, for any $s_i = (i, i+1) \in S$ we define

$$\begin{aligned}
E^-_{J(\lambda),s_i} &= \{\, e \in E_{J(\lambda)} \mid \ell(s_i e) < \ell(e) \text{ and } s_i e \in E_{J(\lambda)} \,\},\\
E^+_{J(\lambda),s_i} &= \{\, e \in E_{J(\lambda)} \mid \ell(s_i e) > \ell(e) \text{ and } s_i e \in E_{J(\lambda)} \,\},\\
E^0_{J(\lambda),s_i} &= \{\, e \in E_{J(\lambda)} \mid \ s_i e \notin E_{J(\lambda)} \,\}
\end{aligned}$$

so that $E_{J(\lambda)}$ is the disjoint union $E^-_{J(\lambda),s_i} \cup E^+_{J(\lambda),s_i} \cup E^0_{J(\lambda),s_i}$, then

$$s_i E^+_{J(\lambda),s_i} = E^-_{J(\lambda),s_i};$$

let

$$\begin{aligned}
E^{0,-}_{J(\lambda),s_i} &= \{\, e \in E_{J(\lambda)} \mid \ell(s_i e) < \ell(e) \text{ and } s_i e \notin E_{J(\lambda)} \,\},\\
E^{0,+}_{J(\lambda),s_i} &= \{\, e \in E_{J(\lambda)} \mid \ell(s_i e) > \ell(e) \text{ and } s_i e \notin E_{J(\lambda)} \,\},
\end{aligned}$$

then $E^0_{J(\lambda),s_i} = E^{0,-}_{J(\lambda),s_i} \bigcup E^{0,+}_{J(\lambda),s_i}$(disjoint union); if $e \in E^{0,-}_{J(\lambda),s_i}$ then $s_i e = et$ for some $t \in J(\lambda)$, if $e \in E^{0,+}_{J(\lambda),s_i}$ then $s_i e = et$ for some $t \in J(\hat{\lambda})$, where $J(\hat{\lambda}) = S \setminus J(\lambda)$.

We have the following observation

$$\begin{aligned}
E^-_{J(\lambda),s_i} &= \{\, d(t)w_{J(\lambda)} \mid t \in \mathbb{T}(\lambda), i \in I_0(t')\},\\
E^+_{J(\lambda),s_i} &= \{\, d(t)w_{J(\lambda)} \mid t \in \mathbb{T}(\lambda), i \in I_0(t)\},\\
E^{0,-}_{J(\lambda),s_i} &= \{\, d(t)w_{J(\lambda)} \mid t \in \mathbb{T}(\lambda), i \in I_1(t')\},\\
E^{0,+}_{J(\lambda),s_i} &= \{\, d(t)w_{J(\lambda)} \mid t \in \mathbb{T}(\lambda), i \in I_1(t)\},
\end{aligned}$$

Let

$$C_{w_{J(\lambda)}} = \epsilon_{w_{J(\lambda)}} q^{\ell(w_{J(\lambda)})} \sum_{w \in \mathfrak{S}_\lambda} \epsilon_w q^{-\ell(w)} T_w.$$

then the following statement is a corollary of Lemma 2.1.

Lemma 4.4. [1]*Mathas2 Let $\lambda \vdash n$, then $\mathscr{H} C_{w_{J(\lambda)}}$ is a free $\mathcal{A}$-module with basis*

$$\{T_{d(t)} C_{w_{J(\lambda)}} | t \text{ a row standard } \lambda\text{-tableau}\}.$$

Moreover, if t is row standard and $s = s_i t$ for some $1 \le i \le n-1$, then

$$T_i T_{d(t)} C_{w_{J(\lambda)}} = \begin{cases} T_{d(s)} C_{w_{J(\lambda)}}, \text{ if } i \in I_0(t) \\ T_{d(s)} C_{w_{J(\lambda)}} + (q - q^{-1}) T_{d(t)} C_{w_{J(\lambda)}}, \text{ if } i \in I_0(t') \\ -q^{-1} T_{d(t)} C_{w_{J(\lambda)}}, \text{ if } i \in I_1(t') \end{cases}$$

where $T_i := T_{s_i}$.

4.3. **Murphy basis and W-graph basis.** The following is a corollary of the main Theorems in Section 2.

Theorem 4.5. [12, 13] *For any $\lambda \vdash n$ and $s, t \in \mathbb{T}(\lambda)$, we define elements of $\mathscr{H}$ by*

$$m_{st} = T_{d(s)} C_{w_{J(\lambda)}} T_{d(t)^{-1}}$$

then the following hold (a) The set $\{m_{st} | s, t \in \mathbb{T}(\lambda)$ for some $\lambda \vdash n\}$ is an $\mathcal{A}$-basis of $\mathscr{H}$; (b) For any $\lambda \vdash n$, let $\mathscr{H}^\lambda$ be the $\mathcal{A}$-submodule of $\mathscr{H}$ spanned by all elements m_{st} where $s, t \in \mathbb{T}(\mu)$ for some $\lambda \trianglelefteq \mu$, then $\mathscr{H}^\lambda$ is a two-sided ideals in $\mathscr{H}$.

Note that the element that we denote by T_w corresponds to the element $q^{\ell(w)} T_w$ in Murphy's notation. Thus the element denoted by $C_{w_{J(\lambda)}}$ in the above statement is exactly as in Murphy's work, except the associated coefficient $\epsilon_{w_{J(\lambda)}} q^{\ell(w_{J(\lambda)})}$. However, this does not affect the validity of (a) and (b) since q is invertible in $\mathcal{A}$. The statement in (a) can be found in Murphy [12, Th.3.9] or Murphy [13, Th. 4.17]. The statement(b) is proved in [13, Th. 4.18].

Murphy also obtains the following result concerning the Specht modules of $\mathscr{H}$. For any $\lambda \vdash n$, let $\hat{\mathscr{H}}^\lambda$ be the $\mathcal{A}$-submodule of $\mathscr{H}$ spanned by all m_{st} where $s, t \in \mathbb{T}(\mu)$ for some $\mu \vdash n$ such that $\lambda \triangleleft \mu$. Thus, we have

$$\hat{\mathscr{H}}^\lambda = \sum_\mu \mathscr{H}^\mu$$

where the sum runs over all $\mu \vdash n$ such that $\lambda \triangleleft \mu$. In particular, $\hat{\mathscr{H}}^\lambda$ is a two-sided ideal and we have $\mathscr{H}^\lambda = \mathscr{H} C_{w_{J(\lambda)}} \mathscr{H} + \hat{\mathscr{H}}^\lambda$

Definition 4.6. [6] For $\lambda \vdash n$, the Specht module S^λ is defined to be the left $\mathscr{H}$-module $(\hat{\mathscr{H}}^\lambda + C_{w_{J(\lambda)}})\mathscr{H}$.

Note that $\hat{\mathscr{H}}^\lambda + C_{w_{J(\lambda)}}$ is an element of the $\mathscr{H}$-module $\mathscr{H}/\hat{\mathscr{H}}^\lambda$ so that S^λ is a submodule of $\mathscr{H}/\hat{\mathscr{H}}^\lambda$. As we defined it, the Specht module S^λ is isomorphic to the dual of the Specht module which Dipper and James [1] indexed by λ'.

For a standard λ-tableau t let $m_t = m_{tt^\lambda} + \hat{\mathscr{H}}^\lambda = T_{d(t)} C_{w_{J(\lambda)}} + \hat{\mathscr{H}}^\lambda$, We have

Theorem 4.7. [8, 13] *The Specht module S^λ is free as an $\mathscr{H}$-module with basis $\{m_t | t \in \mathbb{T}(\lambda)\}$, and $\mathscr{H}^\lambda / \hat{\mathscr{H}}^\lambda$ is a direct sum of $|\mathbb{T}(\lambda)|$ copies of S^λ.*

While

Lemma 4.8. [6] *Suppose $t \in \mathbb{T}(\lambda)$ such that $i \in I_1(t)$, then for all $s \in \mathbb{T}(\lambda)$*

$$T_i m_{st} \equiv q m_{st} + \sum_{v \triangleleft s} r_v m_{vt} \qquad \textit{mod } \hat{\mathscr{H}}^\lambda$$

for some $r_v \in \mathcal{A}$.

Corollary 4.9. *Let $t \in \mathbb{T}(\lambda)$ and $s = s_i t$ for some $1 \le i \le n-1$, then*

$$T_i m_t = \begin{cases} m_s, \textit{ if } i \in I_0(t) \\ m_s + (q - q^{-1}) m_t, \textit{ if } i \in I_0(t') \\ -q^{-1} m_t, \textit{ if } i \in I_1(t') \\ q m_t + \sum_{v \triangleleft t} r_v m_v \qquad \textit{mod } \hat{\mathscr{H}}^\lambda, \textit{ if } i \in I_1(t). \end{cases}$$

where $r_v \in \mathcal{A}$.

We apply with Theorem 4.1 and 4.3 to establish the transition between Murphy's basis and W-graph basis of the Specht module. We also note that in the references, the authors related the Kazhdan-Lusztig cell module and the corresponding Specht module in the case of symmetry group, group algebra and Hecke algebra of type A. See Naruse [17], Garsia-MacLarnan [18] and MacDonough and Pallicaros [15] ect.

Theorem 4.10. *For a fixed $\lambda \vdash n$, we define the elements of the C-basis for S^λ*

$$C_{d(s)w_{J(\lambda)}} = m_s - q \sum_{d(t)<d(s)} p_{t,s} m_t,$$
$$= T_{d(s)} C_{w_{J(\lambda)}} - q \sum_{t \lhd s} p_{t,s} T_{d(t)} C_{w_{J(\lambda)}} \quad mod(\hat{\mathscr{H}}).$$

where $s, t \in \mathbb{T}(\lambda)$ and $p_{t,s} \in \mathbb{Z}(q)$ will be defined recursively by
(18)

$$T_i C_{d(t)w_{J(\lambda)}} = \begin{cases} -q^{-1} C_{d(t)w_{J(\lambda)}}, \text{if } i \in I(t') \\ q C_{d(t)w_{J(\lambda)}} + \sum\limits_{i \in I(u'), u \lhd t} \mu(u,t) C_{d(u)w_{J(\lambda)}}, \text{if } i \in I_1(t) \\ q C_{d(t)w_{J(\lambda)}} + C_{s_i d(t) w_{J(\lambda)}} + \sum\limits_{i \in I(u'), u \lhd t} \mu(u,t) C_{d(u)w_{J(\lambda)}}, \text{if } i \in I_0(t) \end{cases}$$

where $\mu(u,t)$ is the constant term of the polynomial $p_{u,t}$.

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