On the notion of guessing model

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Abstract

We introduce the notion of *guessing model*. This notion is a mean to attribute to accessible cardinals combinatorial properties which can be used in combination with inaccessibility to characterize various large cardinals ranging from supercompact to rank to rank embeddings. The majority of these large cardinals can be described by properties which are expressible in terms of elementary embedding $j : V_\gamma \to V_\lambda$. The key observation is that such embeddings are uniquely determined by the image structures $j[V_\gamma]$. These structures will be the prototypes *guessing models*. We shall show that by the same elementarity argument by which the structure $j[V_\gamma]$ attributes combinatorial properties to the ordinal $j(crit(j))$, a guessing model M will attribute analogue combinatorial properties to the cardinal $\kappa_M = j_M(crit(j_M))$, where j_M is the inverse of the transitive collapse of M. κ_M will always be a regular cardinal but can consistently be a successor cardinal. Applications of our analysis will be proofs of the failure of the square principle and of the singular cardinal hypothesis assuming the existence of guessing models. In particular the failure of square shows that existence of guessing models is a very strong assumption in terms of large cardinal strength.

1 Guessing models

Definition 1.1. Let W be a transitive model of ZFC. $R \in W$ is a *suitable initial segment* if[1]:

- R is a transitive set,
- R is a model of all axioms of ZFC except eventually the replacement schema and the powerset axiom,

[1]We adopt standard terminology as taken for example from [1]. The reader who may feel unfamiliar with it may have a quick look to section 1.1 below.

- R satisfies either the replacement axiom or the powerset axiom,
- $P(X)^W \subseteq R$ for all $X \in R$.

In order to simplify notation and without loss of generality the reader may assume all along the paper that we are working in some transitive model W for ZFC with class many strongly inaccessible cardinal and that $R = W_\theta$ for some inaccessible cardinal θ.

Tipically $R = H_\theta^W$ for some W-regular uncountable cardinal or $R = W_\alpha$ for some ordinal α are the two kind of suitable initial segments R we shall be interested. Most of our results and definitions apply to a wider family of transitive structures R than those captured by the above definition but, in our current state of knowledge, it is not worth the prize to specify all the times the exact assumptions on these structures that we need to carry out the argument.

Let X be any set, we define, whenever this makes sense:

$$\kappa_X = \min\{\alpha \in X : \alpha \text{ is an ordinal and } X \cap \alpha \neq \alpha\}.$$

Definition 1.2. Let R be a suitable initial segment and $M \prec R$. Given a cardinal $\delta < \kappa_M$, $X \in M$ and $d \in P(X) \cap R$ we say that:

- d is (δ, M)-*approximated* if $d \cap Z \in M$ for all $Z \in M \cap P_\delta R$.
- d is M-*guessed* if $d \cap M = e \cap M$ for some $e \in M \cap P(X)$.

$M \prec R$ is a δ-*guessing model for* X if every (δ, M)-approximated subset of X is M-guessed.

$M \prec R$ is a δ-*guessing model* if for all $X \in M$, M is a δ-guessing model for X.

$M \prec R$ is a *guessing model* if for some $\delta < \kappa_M$, $M \prec R$ is a δ-guessing model.

We shall show in section 3, exploiting ideas of Magidor [4], that many large cardinal axioms above supercompactness are equivalent to the existence of appropriate $\aleph_0$-guessing models. For uncountable δ, the notion of δ-guessing model is motivated by the core results of [6] and [7]. For example one of the main results of [7] can be rephrased as follows:

> It is relatively consistent with the existence of a supercompact cardinals that there is W model of ZFC in which for eventually all regular θ there is an $\aleph_1$-guessing model $M \prec H_\theta^W$ with κ_M successor of a regular cardinal.

On the other hand in [6] it is shown that the proper forcing axiom PFA implies that for every regular $\theta \geq \aleph_2$ there are $\aleph_1$-guessing models $M \prec H_\theta$ with $\kappa_M = \aleph_2$.

In the two papers we have also backward results, for example one of the main results of [6] can be stated as follows [2]:

> Assume $V \subseteq W$ are a pair of transitive models of ZFC which have the κ-covering and κ-approximation property for some κ inaccessible in V. Then the existence of an $\aleph_1$-guessing models $M \prec W_\theta$ with $\kappa_M = \kappa$ implies that κ_M is at least a $|\theta|^V$-strongly compact cardinal in V.

The first two results above show that δ-guessing models for uncountable δ are a mean to transfer many *very large cardinal* features of *inaccessible* cardinals to regular *accessible* cardinals and the latter result above combined with the characterization we give in 3 of very large cardinals shows that this is a two way correspondance: the existence of a δ-guessing model model M in some transitive class model W of ZFC will most often be a sufficient conditions to show that κ_M is a very large cardinal in some transitive inner model V of W.

By *very large cardinals* we intend large cardinals axioms which are currently out of reach using fine structural inner models, i.e. cardinals whose strength is at least in the range of strong compactness. In view of the above considerations guessing models appears to be of central interest in all consistency problems related to this type of large cardinal axioms.

1.1 Notation

The notation used is mostly standard and in most cases is recovered from [1]. If W is a transitive model of ZFC, for a cardinal θ in W we let H_θ^W be the set of $z \in W$ whose transitive closure has size less than θ in W, for an ordinal α we let W_α be the set of $z \in W$ of rank less than α. Ord denotes the class of all ordinals. If a is a set of ordinals, $\operatorname{otp} a$ denotes the order type of a. For a regular cardinal δ, $\operatorname{cof} \delta$ denotes the class of all ordinals of cofinality δ, and $\operatorname{cof}(< \delta)$ denotes those of cofinality less than δ. Given a set X and an ordinal δ, $P_\delta X = \{z \in P(x) : |z| < \delta\}$, $[X]^\delta = \{z \in P(X) : \operatorname{otp}(z \cap \mathrm{Ord}) = \delta\}$.

[2] see section 1.1 for the relevant yet undefined notions

Clearly for W a transitive model of ZFC, $(P_\delta X)^W = \{z \in W : W \models |z| < \delta\}$ similarly we shall denote the relativization of various sets to the appropriate transitive model.

Given a structure $\mathfrak{R} = \langle R, \in, P_i : i \in I\rangle$ we shall say that $M \prec R$ if $M \subseteq R$ and $\langle M, \in, P_i \cap M : i \in I\rangle$ is an elementary substructure of $\mathfrak{R}$.

For forcings, we write $p < q$ to mean p is stronger than q. Names either carry a dot above them or are canonical names for elements of V, so that we can confuse sets in the ground model with their names. Given a filter G on $\mathbb{P}$, $\sigma_G(\dot{A}) = \{\sigma_G(\dot{x}) : \exists p \in G\; p \Vdash \dot{x} \in \dot{A}\}$ is the standard interpretation of $\mathbb{P}$-names given by G.

The phrases *for large enough* θ and *for sufficiently large* θ will be used for saying that there exists a θ' such that the sentence's proposition holds for all $\theta \geq \theta'$.

For $f : P_\omega X \to X$ we let $\mathrm{Cl}_f := \{x \in P(X) \mid f[P_\omega x] \subset x\}$. The club filter on $P(X)$ is the normal filter generated by the sets Cl_f.

$S \subseteq P(X)$ is stationary if it is positive with respect to the club filter.

If $X \subset X'$, $R \subset P(X)$, $U \subset P(X')$, then the projection of U to X is $U \restriction X := \{u \cap X \mid u \in U\} \subset P(X)$ and the lift of R to X' is $R^{X'} := \{x' \in P(X') \mid x' \cap X \in R\} \subset P(X')$.

We shall need for reference and motivation of our results the following definitions:

Definition 1.3. Let $V \subseteq W$ be a pair of transitive models of ZFC.

- (V, W) satisfies the μ-covering property if the class $P_\mu^V V$ is cofinal in $P_\mu^W V$, that is, for every $x \in W$ with $x \subset V$ and $|x| < \mu$ there is $z \in P_\mu^V V$ such that $x \subset z$.

- (V, W) satisfies the μ-approximation property if for all $x \in W$, $x \subset V$, it holds that if $x \cap z \in V$ for all $z \in P_\mu^V V$, then $x \in V$.

A forcing $\mathbb{P}$ is said to satisfy the μ-covering property or the μ-approximation property if for every V-generic $G \subset \mathbb{P}$ the pair $(V, V[G])$ satisfies the μ-covering property or the μ-approximation property respectively.

We shall adopt the following definitions of forcing axioms:

Definition 1.4. Given a class of forcing notions Γ we let:

- $\mathsf{MA}(\Gamma)$ hold if for any poset $\mathbb{P} \in \Gamma$ and eventually all regular θ, there are stationarily many structures $M \prec H(\theta)$ of size $\aleph_1$ which have an M-generic filter G for $\mathbb{P}$.

- $\mathsf{MA}(\Gamma)^{+2}$ hold if for any poset $\mathbb{P} \in \Gamma$ and eventually all regular θ, given $\mathbb{P}$-names $\dot{S}_0$ and $\dot{S}_1$ for stationary subsets of ω_1 there are stationarily many structures $M \prec H(\theta)$ of size $\aleph_1$ which have an M-generic filter G for $\mathbb{P}$ and are such that $\sigma_G(\dot{S}_i)$ is stationary.

If Γ is the family of CCC posets, we shall denote $\mathsf{MA}(\Gamma)$ by MA. If Γ is the family of proper posets, we shall denote $\mathsf{MA}(\Gamma)$ by PFA and $\mathsf{MA}(\Gamma)^{+2}$ by PFA^{+2}. If Γ is the family of stationary set preserving posets $\mathsf{MA}(\Gamma)$ is Martin's maximum MM. We refer the reader to [1] for the definition of the relevant Γ's. We recall however that any *CCC* partial order is proper and any proper partial order is stationary set preserving.

2 Basic properties of guessing models

The following are basic properties of guessing models[3]:

Proposition 2.1. *Let R be a suitable initial segment and $M \prec R$.*

1. *κ_M is a regular cardinal.*
2. *M is a 0-guessing model iff it is an $\aleph_0$-guessing model.*
3. *If M is a δ-guessing model, then it is also a γ-guessing model for all cardinal $\gamma \geq \delta$.*
4. *If M is a δ-guessing model and $2^{<\delta} < \kappa_M$, M is a 0-guessing model.*
5. *If M is a 0-guessing model, κ_M and $M \cap \kappa_M$ are strongly inaccessible cardinals.*
6. *If M is a δ-guessing model and for some regular cardinal $\gamma \leq \delta$, $2^{<\gamma} < \kappa_M$, then $M \cap \mathrm{Ord}$ is closed under suprema of sequences of length at most γ, in particular a guessing model M is always closed under countable suprema.*

Proof. **(1):** We first show κ_M is a cardinal: assume not, then by elementarity there is a bijection $\phi \in M$ between κ_M and $\delta = |\kappa_M| < \kappa_M$. Since $\delta < \kappa_M \cap M$, $\delta \subseteq M$, since $\phi \in M$, $\phi[\delta] = \kappa_M \subseteq M$ contradicting the very definition of κ_M.

[3] Property **(6)** is a rephrasing in the terminology of guessing models of a result by Weiss (see [7]).

Next we show κ_M is regular: assume not and fix $E \in M$ cofinal in κ_M of order type $\delta < \kappa_M$. Then since $\delta \in M \cap \kappa_M$, we have that $\delta \subseteq M$ and thus $E \subseteq M$. Now either $\kappa_M \subseteq M$ which contradicts the very definition of κ_M or κ_M is not the least ordinal in M such that $M \cap \kappa_M$ is bounded below κ_M which again contradicts the very definition of κ_M. Note that **(1)** holds for any $M \prec R$ and not just for guessing models.

(2): Immediate.

(3): Immediate.

(4): Observe that if $Z \in M$ and $|Z| < \delta$, $R \models |P(Z)| \leq 2^{<\delta}$. So there is a bijection ϕ from some ordinal $\alpha < \kappa_M$ and $P(Z)$. Then $P(Z) = \phi[\alpha] \subseteq M$: this follows since $\alpha \subseteq M$ because $\alpha < \kappa_M$ and $\alpha = \mathrm{dom}(\phi) \in M$. Thus if $d \in R \cap P(X)$ for some $X \in R$ and $Z \in M$ is any set of size less than δ, $d \cap Z \in P(Z) \subseteq M$. Thus any $d \in P(X) \cap R$ is (δ, M)-approximated for all $X \in M$. Since M is δ-guessing, any $d \in P(X) \cap R$ is M-guessed for any $X \in M$, Thus M is 0-guessing.

(5): We first show that $\kappa \cap M$ is a regular cardinal in R. Assume not and pick $C \subseteq \kappa \cap M$ in R of order type $\mathrm{cf}(\kappa \cap M) < \kappa \cap M$. Since M is 0-guessing, $C = E \cap M$ for some $E \in M$. Now it is not hard to check that:

$$M \models E \text{ is an unbounded subset of } \kappa_M \text{ of order type less than } \kappa_M.$$

For this reason there is a unique order preserving bijection $\phi \in M$ from some ordinal ξ less than κ_M into E. By elementarity $\xi \in M$. Since $\xi < \kappa_M$, $\xi \subseteq M$. Thus $E = \phi[\xi] \subseteq M$. Thus $C = E$ which implies that $\sup(\kappa_M \cap M) = \kappa_M$, contradicting the very definition of κ_M.

Now assume $2^\delta \geq \kappa_M \cap M$ for some $\delta < \kappa_M$. By elementarity, since $\delta \in M$, we get that $2^\delta \geq \kappa_M$. Now let $\phi : 2^\delta \to P(\delta)$ be a bijection in M. Let $X = \phi(\kappa_M \cap M)$. Then $X \subseteq \delta \subseteq M$. Since M is 0-guessing, $X = Y \cap M$ for some $Y \in P(\delta) \cap M$, since $Y \subseteq \delta \subseteq M$, $X = Y$, thus $\kappa_M \cap M = \phi^{-1}(Y) \in M$ which contradicts the very definition of κ_M. This proves that $\kappa_M \cap M$ is strongly inaccessible. Now by elementarity M models that κ_M is strong limit. Thus κ_M is strong limit and regular in R i.e. strongly inaccessible.

(6): Assume not for some M. Observe that for such an M, $P_\gamma(X) \subseteq M$ for all $X \in M$ of size γ since $2^{<\gamma} < \kappa_M$ and any bijection in M between X and γ lifts to a bijection in M between $P_\gamma(X)$ and $2^{<\gamma} \in M \cap \kappa_M$.

Now let $\xi \in M$ have cofinality larger than γ and be such that $\sup(M \cap \xi) \notin M$ has cofinality at most γ. This means that $M \cap [\sup(M \cap \xi), \xi)$ is empty. Then for any $d \in M$ of size γ, $d \cap \xi$ is bounded below $\sup(M \cap \xi)$ else $\sup(M \cap \xi) \leq \sup(d \cap \xi) < \xi$

and $\sup(d \cap \xi) \in M$. Fix in R, $d^* = \{\alpha_\xi : \xi \in \gamma\} \subseteq M \cap \delta$ increasing and cofinal sequence converging to δ. Then $d^* \cap d \in R$ has order type less than γ for all $d \in M$ which have size γ and thus belongs to $P_\gamma d \subseteq M$. Thus d^* is a (δ, M)-approximated subset of M. This means that $d^* = d^* \cap M = e \cap M$ for some $e \in M \cap P(\xi)$. Now $M \models e$ is an unbounded subset of ξ, thus $\mathrm{otp}(e) \geq \mathrm{cf}(\xi)$, in particular $\mathrm{otp}(e \cap M) \geq \mathrm{otp}(\mathrm{cf}(\xi) \cap M) > \mathrm{otp}(\gamma \cap M) = \gamma = \mathrm{otp}(d^*)$. Thus $e \cap M \neq d^*$ which is the desired contradiction. □

Notice the immediate by-product of our results:

Remark 2.2. Assume $M \prec W_\theta$ is a δ-guessing model which is not a 0-guessing model. Then $2^{<\delta} \geq \kappa_M$.

Proof. This follows by the third item above. □

Thus existence of guessing models has effects on the exponential function. We shall see in section 6 that the existence of an $\aleph_1$-*internally unbounded* (see def 4.1) $\aleph_1$-guessing model M is an assumption strong enough to imply the SCH for all cardinals in M.

3 Large cardinals and $\aleph_0$-guessing models.

In this section we show that most of the large cardinal axioms present in the literature can be formulated in terms of the existence of the appropriate $\aleph_0$-guessing model.

3.1 Supercompactness

Magidor [4] has characterized supercompactness as follows:

Theorem 3.1 (Magidor). *κ is supercompact in V iff for every $\lambda \geq \kappa$ there is a non trivial elementary embedding $j : V_\gamma \to V_\lambda$ with $j(crit(j)) = \kappa$.*

The core of his argument can be rephrased in our setting as the following:

Lemma 3.2. *$M \prec V_\lambda$ is an $\aleph_0$-guessing model if and only if the transitive collapse of M is some V_γ.*

Proof. We prove just one direction, the other one is proved by a similar argument. Recall that $M \prec V_\lambda$ is an $\aleph_0$-guessing model iff it is a 0-guessing model. Now assume $M \prec V_\lambda$ is a 0-guessing model. We proceed by induction on $\beta \in M \cap \lambda$ to show that $M \cap V_\beta$ collapses to some V_{γ_β} via $\pi_M \restriction V_\beta$. This is clear if β is a limit ordinal since $\pi_M \restriction V_\beta = \bigcup_{\alpha<\beta} \pi_M \restriction V_\alpha = \bigcup_{\alpha<\beta} V_{\gamma_\alpha} = V_{\gamma_\beta}$.

Now consider the successor stage, i.e. $\beta = \alpha + 1$. $V_{\gamma_\beta} = V_{\gamma_\alpha+1} = P(\pi_M \restriction V_\alpha)$. Thus for every $Y \in V_{\gamma\beta}$ $Y = \pi_M[X_Y]$ for some $X_Y \in P(M \cap V_\alpha)$. Now M is a 0-guessing model Thus, since every $X \in P(V_\alpha \cap M)$ is 0-approximated, we have that for every Y, X_Y is M-guessed i.e. $X_Y = M \cap E_Y$ for some $E_Y \in M$. Clearly such an $E_Y \in V_{\alpha+1}$. In conclusion:

$$V_{\gamma_\beta} = \{\pi_M[E_Y] : E_Y \in V_\beta \cap M\} = \pi_M \restriction V_\beta$$

The conclusion follows. □

Note that if $M \prec V_\lambda$ and $\pi_M[M] = V_\gamma$ then $j = \pi_M^{-1}$ is an elementary embedding of V_γ in V_λ.

Thus Magidor's theorem can be reformulated as follows:

Theorem 3.3 (Magidor). *κ is supercompact iff for every $\lambda \geq \kappa$ there is an $\aleph_0$-guessing model $M \prec V_\lambda$ with $\kappa_M = \kappa$.*

3.2 Hugeness

Recall that a cardinal κ is huge in V if for some $\delta > \kappa$ there is a normal fine ultrafilter[4] on $[\delta]^\kappa$.

Lemma 3.4. *Assume that for some λ there is an $\aleph_0$-guessing model $M \prec V_\lambda$ such that* $\operatorname{otp}(M \cap \lambda) \geq \kappa_M$*, then $\kappa = \pi_M(\kappa_M)$ is a huge cardinal. If moreover* $\operatorname{otp}(M \cap \lambda) \geq \kappa_M + 2$ *then also κ_M is huge.*

Proof. Let $\delta \leq \lambda$ be such that $\operatorname{otp}(M \cap \delta) = \kappa_M$. Let $j = \pi_M^{-1}$. Then $j : V_\gamma \to V_\lambda$ is elementary, $j(\kappa) = \kappa_M$ and $j(\kappa_M) = \delta$ (moreover if $\operatorname{otp}(M \cap \lambda) \geq \kappa_M + 2$, $\gamma \geq \kappa_M + 2$). Thus $M \cap \delta \in j([\kappa_M]^\kappa)$. Now define in V the ultrafilter $\mathcal{U}$ on $[\kappa_M]^\kappa$ by $A \in \mathcal{U}$ iff $M \cap \delta \in j(A)$. $\mathcal{U} \in V_{\kappa_M+2}$ witnesses that κ is huge in V. Moreover if $\gamma \geq \kappa_M + 2$, $\mathcal{U} \in V_\gamma$ and thus $j(\mathcal{U}) \in V_\lambda$ witnesses that κ_M is huge. □

With some more care one can also put conditions on M to guarantee that it witnesses n-hugeness of κ_M.

[4] An ultra filter $\mathcal{U}$ on $[\delta]^\kappa$ is fine if for all $\alpha < \delta$, $\{X \in [\delta]^\kappa : \alpha \in X\} \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ is normal if for all $A \in \mathcal{U}$ and all choice functions f on A there is $B \in \mathcal{U}$ such that f is constant on B.

3.3 Rank initial segment embeddings and beyond

The following fact is an immediate outcome of Magidor's observations:

Fact 3.5. $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary iff $j[V_{\lambda+1}] = M \prec V_{\lambda+1}$ is an $\aleph_0$-guessing model.

Thus the existence of an $\aleph_0$-guessing model $M \prec V_{\lambda+1}$ such that $\text{otp}(M \cap \lambda) = \lambda$ is an equivalent formulation of the axiom I_1.

Picking $R = L_\gamma(V_\lambda + 1)$ for a large enough γ it is not hard to define in terms of an $\aleph_0$-guessing model $M \prec R$ the axiom stating the existence of an elementary embedding of $L_\gamma(V_\lambda + 1)$ into itself with critical point smaller than λ.

4 Internal closure of guessing models

In this and in the next section, we come back to an analysis of the properties of guessing models and we also address some consistency issues regarding their existence.

If $M \prec V_\lambda$ is an $\aleph_0$-guessing model, κ_M is inaccessible and $P_\gamma M \subseteq M$ for all $\gamma \in M \cap \kappa_M$. Such a degree of closure cannot be achieved for $\aleph_1$-guessing models, however we can prove that such models have a reasonable degree of closure in most cases. To this aim we need to recall the following definitions:

Definition 4.1. Let R be a suitable initial segment. For a model $M \prec R$ and a cardinal δ, we say that M:

- is δ-internally unbounded if $M \cap P_\delta M$ is cofinal in the partial order $(P_\delta M, \subseteq)$,
- is δ-internally club if $M \cap P_\delta M$ is a club subset of $P_\delta M$,
- is δ-internally stationary if $M \cap P_\delta M$ is a stationary subset of $P_\delta M$.

We let $\mathcal{IC}^\delta R$ be the set of $M \prec R$ which are δ-internally club, $\mathcal{IS}^\delta R$ be the set of $M \prec R$ which are δ-internally stationary and $\mathcal{IU}^\delta R$ be the set of $M \prec R$ which are δ-internally unbounded.

Recall that the pseudo-intersection number $\mathfrak{p}$ is the minimal size of a family $X \subseteq P(\omega)$ which is closed under finite intersections and for which there is no infinite $a \subseteq \omega$ such that $a \subseteq^* b$ (i.e. $a \setminus b$ is finite) for all $b \in X$. We will show the following:

Lemma 4.2. *Assume $M \prec R$ for a suitable initial segment R is an $\aleph_1$-guessing model such that $\mathfrak{p} > |M|$. Then M is in $\mathcal{IU}^{\aleph_1}R$.*

Proof. Assume not and pick $M \prec R$ guessing model witnessing it. Pick x countable subset of M which is not covered by any countable set in M. The family $\{x \setminus z : z \in M \cap P_{\omega_1}M\}$ has the finite intersection property and has size at most $|M| < \mathfrak{p}$. Thus there is $y \subseteq x$ such that $y \cap z$ is finite for all countable $z \in M$. Thus y is M-approximated. Let $d \in M$ be such that $d \cap M = y$. Then d is countable, else, since $d \in M$ and $\omega_1 \subseteq M$, $d \cap M$ is uncountable and thus different from y. This means that $d = d \cap M = y$. This is impossible since $d \cap y$ is finite by choice of y. □

Theorem 4.3. *Assume* MM. *Then for evey regular $\theta \geq \aleph_2$ the following sets are stationary:*

1. *the set of $\aleph_1$-guessing models $M \prec H_\theta$ of size $\aleph_1$ which are $\aleph_1$-internally club,*
2. *the set of $\aleph_1$-guessing models $M \prec H_\theta$ of size $\aleph_1$ which are $\aleph_1$-internally unbounded but not $\aleph_1$-internally stationary,*
3. *the set of $\aleph_1$-guessing models $M \prec H_\theta$ of size $\aleph_1$ which are $\aleph_1$-internally stationary but not $\aleph_1$-internally club.*

For item **(1)** PFA suffices and for item **(3)** PFA^{+2} suffices.

Proof. In [6] we showed the following:

> Assume $\mathbb{P}$ is a poset with the ω_1-approximation and ω_1-covering properties which collapses $P(X)$ to $\aleph_1$, then there is in $V^{\mathbb{P}}$ a CCC-poset $\dot{\mathbb{Q}}_{\mathbb{P}}$ such that for eventually all θ, any model $M \prec H_\theta$ in V of size $\aleph_1$ which has a $\mathbb{P} * \dot{\mathbb{Q}}_{\mathbb{P}}$-generic filter, is guessing all $(\aleph_1, M)$-approximated subsets of X.

Now if $X = H_\lambda$ and $\mathbb{P}$ is a poset with the ω_1-approximation and ω_1-covering properties which collapses $P(X)$ to $\aleph_1$ and $M \prec H_\theta$ in V of size $\aleph_1$ has a $\mathbb{P} * \dot{\mathbb{Q}}_{\mathbb{P}}$-generic filter, we get that $M \cap H_\lambda \prec H_\lambda$ is $\aleph_1$-guessing of size $\aleph_1$.

Kruger in [2] and [3] has shown that for every λ there are stationary set preserving posets $\mathbb{P}_i$ for $i < 3$ all with the ω_1-approximation and ω_1-covering properties and all collapsing $P(H_\lambda)$ to $\aleph_1$ and that each one has the following property:

- any model $M \prec H_\theta$ in V of size $\aleph_1$ which has a $\mathbb{P}_0 * \dot{\mathbb{Q}}_{\mathbb{P}_0}$-generic filter, is such that $M \cap H_\lambda$ is internally club,
- any model $M \prec H_\theta$ in V of size $\aleph_1$ which has a $\mathbb{P}_1 * \dot{\mathbb{Q}}_{\mathbb{P}_1}$-generic filter, is such that $M \cap H_\lambda$ is internally unbounded but not internally stationary,
- any model $M \prec H_\theta$ in V of size $\aleph_1$ which have a $\mathbb{P}_2 * \dot{\mathbb{Q}}_{\mathbb{P}_2}$-generic filter is such that $M \cap H_\lambda$ is internally stationary but not internally club.

Actually $\mathbb{P}_1$ and $\mathbb{P}_2$ are semiproper while $\mathbb{P}_0$ is proper. Combining our and Kruger's results we can get the desired conclusion of the theorem. □

5 Isomorphism types of guessing models

In this section we will show that for guessing models M which are internally club, the isomorphism type is uniquely determined by the ordinal $M \cap \kappa_M$ and the order-type of the set of cardinals in M. In the case of 0-guessing models this is Magidor's result that any 0-guessing model $M \prec V_\lambda$ is isomorphic to some V_γ, however when we want to extend this result to $\aleph_1$-guessing models we must put some extra condition to constrain the variety of possible isomorphism types.

Given a set M we let $Card_M$ be the set of cardinals in M and $\chi_M : Card_M \to \sup M \cap \ Ord$ be the characteristic function of M which maps $\alpha \mapsto \sup(M \cap \alpha)$.

This theorem intends to generalize Magidor's lemma 3.2 on the isomorphism type of 0-guessing models.

Theorem 5.1. *Assume M_0 and $M_1 \prec H_\theta$ are $\aleph_1$-guessing models which are internally club and moreover that:*

- $\kappa_{M_0} = \kappa_{M_1} = \kappa$,
- $M_0 \cap \kappa = M_1 \cap \kappa$,
- $2^{\aleph_0} \leq \kappa$,
- $\mathrm{otp}(Card_{M_0}) = \mathrm{otp}(Card_{M_1})$.

Then M_0 and M_1 are isomorphic.

Proof. The proof goes by induction on $\mathrm{otp}(Card_{M_0}) \setminus \kappa_M = \mathrm{otp}(Card_{M_1}) \setminus \kappa_M = \xi$.

Let $\{\alpha_i^\eta : \eta < \xi\} = Card_{M_i} \setminus \kappa_{M_i}$. We show that for any ordinal $\eta < \xi$, $(M_0 \cap \alpha_0^\eta, P(\alpha_0^\eta) \cap M_0, \in)$ is isomorphic to $(M_1 \cap \alpha_1^\eta, P(\alpha_1^\eta) \cap M_1, \in)$. This suffices, since it is well known that two submodels M_0, M_1 of H_θ such that $\mathrm{otp}(M_0) \cap Ord = \mathrm{otp}(M_1 \cap Ord)$ and which are isomorphic on sets of ordinals are fully isomorphic.

Base case: $\alpha_0 = \kappa = \kappa_M$

Clearly the identity map defines an isomorphism of the ordinal $M_i \cap \kappa$ with itself. Since $2^{\aleph_0} \leq \kappa$, there is a bijection ϕ in M_i between $P_{\omega_1}\kappa$ and κ. Using this bijection ϕ we get that also $M_0 \cap P_{\omega_1}\kappa = M_1 \cap P_{\omega_1}\kappa$. We extend the identity map on $M_0 \cap P_{\omega_1}\kappa$ to an isomorphism of $(M_0 \cap \kappa, P(\kappa) \cap M_0, \in)$ with $(M_1 \cap \kappa, P(\kappa) \cap M_1, \in)$ using the guessing property of each M_i as follows:

$d \in M_0 \cap P(\kappa)$ iff $d \cap M_0$ is $M_0 \cap \kappa$-approximated iff $d \cap M_1$ is $M_1 \cap \kappa$-approximated iff $d \cap M_1 = e(d) \cap M_1$ for some $e(d) \in M_1 \cap P(\kappa)$.

The mapping π_0 which is the identity on $M_0 \cap \kappa$ and sends $d \mapsto e(d)$ is an isomorphism of $(M_0 \cap \kappa, P(\kappa) \cap M_0, \in)$ with $(M_1 \cap \kappa, P(\kappa) \cap M_1, \in)$.

The idea is to extend step by step to all $\alpha_i^\eta \in Card_{M_i}$ this isomorphism first showing that $M_0 \cap P_{\omega_1}\alpha_0^\eta$ is isomorphic to $M_1 \cap P_{\omega_1}\alpha_1^\xi$ and then extend the isomorphism to the full structures $(M_i \cap \alpha, P(\alpha_i^\xi) \cap M_i, \in)$ using the key property of guessing models. We will need the assumption that the models are internally club to handle the limit stages of countable cofinality.

Now assume the induction has been carried up to some ordinal $\eta < \xi$ by defining a sequence of coherent and unique isomorphisms π_β of $(M_0 \cap \alpha_0^\beta, P(\alpha_0^\beta) \cap M_0, \in)$ with $(M_1 \cap \alpha_1^\beta, P(\alpha_1^\beta) \cap M_1, \in)$ for all $\beta < \eta$.

To define π_η we proceed by cases according to whether:

1. α_0^η is a limit cardinal of uncountable cofinality,
2. α_0^η is a successor cardinal,
3. α_0^η is a limit cardinal of countable cofinality.

α_0^η is a limit cardinal of uncountable cofinality

We start with the first case. First of all since $M_i \cap Ord$ is closed under countable suprema, we get that $\sup(M_i \cap \alpha_i^\eta)$ are ordinals of uncountable cofinality. This means that $\bigcup_{\beta<\eta} \pi_\beta \restriction M_0 \cap \beta$ defines an isomorphism of $M_0 \cap \alpha_0^\eta$ with $M_1 \cap \alpha_1^\eta$.

Since α_i^η have uncountable cofinality $M_i \cap P_{\omega_1}\alpha_i^\eta = \bigcup_{\beta<\eta}(M_i \cap P_{\omega_1}\alpha_i^\beta)$. Thus we get that $\bigcup_{\beta<\eta} \pi_\beta \restriction M_0 \cap (\alpha_0^\beta \cup P_{\omega_1}\alpha_0^\beta)$ defines an isomorphism π^* of $(M_0 \cap P_{\omega_1}\alpha_0^\eta, \alpha_0^\eta \cap M_0)$ with $(M_1 \cap P_{\omega_1}\alpha_1^\eta, \alpha_1^\eta \cap M_1)$.

Now we can apply the same trick as before to extend the isomorphism π^* to $P(\alpha_0^\eta) \cap M_0$:

Pick d in this set. Then d is M_0 approximated, thus $d_1 = \pi^*[d \cap M_0]$ is M_1-approximated, thus $d_1 = e(d)$ for some unique $d \in M_1 \cap P(\alpha_1^\eta)$. Let π_η extend π^* by sending d to $e(d)$. Then π_η is the desired isomorphism of $(P(\alpha)\cap M_0, \alpha \cap M_0, \in)$ with $(P(\alpha) \cap M_1, \alpha \cap M_1, \in)$ which extends all the π_β.

α_i^η is the successor of α_i^β

We are given π_β isomorphism of $(P(\alpha_0^\beta) \cap M_0, \alpha_0^\beta \cap M_0, \in)$ with $(P(\alpha_1^\beta) \cap M_1, \alpha_1^\beta \cap M_1, \in)$. Any ordinal δ in $\alpha_i^{\beta+1}$ is coded by a binary relation on α_i^β whose transitive collapse is δ. Now let $\phi_i \in M_i$ be functions from $\alpha_i^{\beta+1}$ to $P(\alpha_i^\beta)$ such that for each $\gamma < \alpha_i^{\beta+1}$, $\phi_i(\gamma)$ codes γ.

Then we can extend π_β to π^* on $M_0 \cap \alpha_0^{\beta+1}$ as follows, $\pi^*(\gamma) = \delta$ iff $\phi_1(\delta) = \pi_\beta(\phi_0(\gamma))$. Notice that this also induces an isomorphism of $(M_0 \cap P_{\omega_1}\alpha_0^\eta, \alpha_0^\eta \cap M_0, \in)$ onto $(M_1 \cap P_{\omega_1}\alpha_1^\eta, \alpha_1^\eta \cap M_1, \in)$ which sends $a \in M_0 \cap P_{\omega_1}\alpha_0^\eta$ in $\pi^*[a]$.

Now we proceed as before: Pick d in $P(\alpha_0^\eta) \cap M_0$. Then d is M_0-approximated, thus $d_1 = \pi^*[d \cap M_0]$ is M_1-approximated, thus $d_1 = e(d)$ for some unique $d \in M_1 \cap P(\alpha_1^\eta)$. Let π_η extend π^* by sending d to $e(d)$. Then π_η is the desired isomorphism of $(P(\alpha_0^\eta) \cap M_0, \alpha_0^\eta \cap M_0, \in)$ with $(P(\alpha_1^\eta) \cap M_1, \alpha_1^\eta \cap M_1, \in)$ which extends π_β.

α_i^η is a limit cardinal of countable cofinality

Fix $(\beta_i : i < \omega) \in M_0 \cap M_1$ increasing sequence converging to η such that α_i^η are regular cardinals.

We get that $\bigcup_{i<\omega} \pi_{\beta_i}$ defines an isomorphism π^* of $(X_0 \cap M_0, \alpha_i^\eta \cap M_0, \in)$ with $(X_1 \cap M_1, \alpha_1^\eta \cap M_1, \in)$, where $X_i = \bigcup_{j<\omega} P_{\omega_1}\alpha_i^{\beta_j}$ is the family of countable and bounded subsets of α_i^η.

Now observe that, since both M_i are internally club, $M_i \cap P_{\omega_1}\alpha_i^\eta$ are club subsets C_i^* of $P_{\omega_1}(M_i \cap \alpha_i^\eta)$. By going to the order type ξ of $M_i \cap \alpha_i^\eta$ we get that both C_i^* collapse to club subsets C_i' of $P_{\omega_1}\xi$. Let $C = C_0' \cap C_1'$ and C_i be the clubs in $P_{\omega_1}M_i \cap \alpha_i^\eta$ which collapse to C.

Then every element in C_i belongs to M_i and π^* can be extended to an isomorphism of the structures $((X_i \cap M_i) \cup C_i, M_i \cap \alpha_i^\eta, \in)$. We want to extend π^* further to an isomorphism of the structures $(M_i \cap P_{\omega_1}\alpha_i^\eta, M_i \cap \alpha_i^\eta, \in)$. So pick $d \in C_i$ and consider the tree $T_d = \{e \cap d : e \in X\}$ ordered by $e <_d f$ iff $f \cap \sup e = e$ and there is some $\alpha_i^{\beta_j} \in \sup f \setminus \sup e$.

Notice the following property of T_d:

For $d \in C_i$ and $e \in M_i \cap (P_{\omega_1}\alpha_i^\eta \setminus X_i)$, $e \subseteq d$ if and only if $(e \cap \alpha_i^{\beta_j} : j < \omega)$ is an infinite branch of T_d.

Let us identify an infinite branch of T_d by the corresponding subset. By the above property for any $e \in M_i \cap (P_{\omega_1}\alpha_i^\eta \setminus X_i)$ eventually all d in C_i have e has an infinite branch of T_d.

Observe that $T_d \in M_i$ is a tree of height ω and if d is in C_0, we have that π^* induces in the natural way an isomorphism of $(T_d \cap M_0, <_d)$ with
$(T_{\pi^*(d)} \cap M_1, <_{\pi^*(d)})$, let us call again π^* this isomorphism..

Now T_d and $T_{\pi^*(d)}$ are trees of size at most $2^{\aleph_0} \leq \kappa$. Moreover if η is the order type of d, T_d and $T_{\pi^*(d)}$ are both isomorphic to the unique tree $T_0 \subseteq M_i \cap P_{\omega_1}\kappa$ contained in $P_{\omega_1}\eta$ which is uniquely defined by the collapse π_d of d to its order type. So let us denote by $\pi_d : T_d \to T_0$ and $\pi_{\pi^*(d)} : T_{\pi^*(d)} \to T_0$ these uniquely defined isomorphisms living repectively in M_0 and M_1. On the other hand remark that $T_0 \in M_0 \cap M_1$.

So there is an injection ϕ in $M_0 \cap M_1$ between the infinite branches of T_0 and κ. Let, for an infinite branch x of T_d ($T_{\pi^*(d)}$), $\phi_d(x) = \alpha$ iff $\phi(\pi_d[x]) = \alpha$ ($\phi_{\pi^*(d)}(x) = \alpha$ iff $\phi(\pi_{\pi^*(d)}(x)) = \alpha$).

Thus the map $\pi_d^* : M_0 \cap P_{\omega_1}d \to M_1 \cap P_{\omega_1}\pi^*(d)$ which maps $e \mapsto e^*$ iff $\phi_d(e) = \phi_{\pi^*(d)}(e^*)$ induces a unique natural isomorphism of the set of infinite branches $[T_d]$ of T_d with the set of infinite branches of $T_{\pi^*(d)}$.

Recall that $\xi = \mathrm{otp}(M_i \cap \alpha_i^\eta)$ and C is the transitive collapse of C_i induced by the collapse π_i of $M_i \cap \alpha$. Consider the directed structure $(\{(\eta, d) : \eta < M_0 \cap \kappa), d \in C\}, \leq)$ with $(\eta, d) \leq (\gamma, e)$ iff $d \subseteq e$ and $\phi^{-1}_{\pi_i^{-1}[d]}(\eta) = \phi^{-1}_{\pi_i^{-1}[e]}(\gamma)$.

Let us call sets of the form $\{(\alpha_d, d) : d \in E\}$ points iff $E \subseteq C_i$ is upward closed and for all $d \subseteq e \in E$ $(\alpha_d, d) \leq (\alpha_e, e)$.

All our efforts amount to the following of which we omit a rigorous proof:

Fact 5.2. *Any $e \in M_i \cap P_{\omega_1}\alpha_i^\eta$ determines the point $p(e) = \{(\phi_d(e), \pi_i[d]) : d \in C_i, e \subseteq d\}$ and conversely any point $\{(\alpha_d, d) : d \in E\}$ uniquely determines sets $e_i \in M_i$ such that $(\alpha_d, d) \in p(e_i)$ for all $(\alpha_d, d) \in p$.*

Now we can extend π^* to a full isomorphism of the structures $(M_i \cap P_{\omega_1}\alpha_i^\eta, M_i \cap \alpha_i^\eta, \in)$ mapping e to the unique e^* such that $p(e) = p(e^*)$.

Finally we can extend π^* to π_α by the usual trick employed in the previous cases.

This completes the proof of the theorem. $\square$

5.1 Faithful models

In this section assume θ is inaccessible in W. The above characterization of isomorphism types for δ-guessing, δ-internally club models is not completely satisfactory since it could be the case that two such models $M_0, M_1 \prec W_\theta$ have the same isomorphism type, are such that $\kappa_{M_0} = \kappa_{M_1} = \kappa$ and $M_0 \cap \kappa_M = M_1 \cap \kappa_M$ but for some cardinal $\lambda \in M_0 \cap M_1 \setminus \kappa$, $\chi_{M_0}(\lambda) = \chi_{M_1}(\lambda)$ and $\chi_{M_0} \restriction \lambda \neq \chi_{M_1} \restriction \lambda$. We shall show that for 0-guessing models this cannot be the case, thus we would like that this rigidity property of 0-guessing models holds also for arbitrary guessing models. We shall see that in models of MM there is a stationary set of $\aleph_1$-guessing models which have this rigidity property. Let

$$\mathcal{G}^\delta_\kappa = \{M \prec W_\theta : M \text{ is a } \delta\text{-guessing model and } \kappa_M = \kappa\}$$

For S stationary subset of $P(W_\theta)$, let $T(S) = \{\chi_M \restriction \gamma : M \in S,\ \gamma \in Card \cap M\}$.

Theorem 5.3. *The following holds:*

1. *$T(\mathcal{G}^0_\kappa)$ is a tree of functions ordered by end extension.*
2. *Assume* MM. *Then there is S stationary subset of $\mathcal{G}^{\aleph_1}_{\aleph_2} \cap \mathcal{IC}^{\aleph_1}$ such that $T(S)$ is a tree of functions ordered by end extension.*

We need the following definition. Given a set of ordinals S such that S is a stationary subset of $\sup(S)$ let:

$$P^*(S) = \{T \subseteq S : T \text{ is stationary in } \sup(S)\}.$$

Definition 5.4. $M \prec W_\theta$ is an *S-faithful model* if for all $T \in P(S) \cap M$, T reflects on $\sup(M \cap S)$) iff $T \in P^*(S)$.

$M \prec W_\theta$ is a *λ-faithful model* if M is $E^{\aleph_0}_\lambda$-faithful.

$M \prec W_\theta$ is a *faithful model* if M is $E^{\aleph_0}_\lambda$-faithful for all regular $\lambda \in M$.

The following lemma motivates the definition of faithful models:

Lemma 5.5. *Assume $M_0, M_1 \prec W_\theta$ are λ-faithful models for some regular $\lambda \in M_0 \cap M_1$ and $\chi_{M_0}(\lambda) = \chi_{M_1}(\lambda)$. Then $\chi_{M_0} \restriction \lambda = \chi_{M_1} \restriction \lambda$.*

Proof. Let $\{S_\alpha : \alpha < \lambda\} \in M_0 \cap M_1$ be a partition of $E^{\aleph_0}_\lambda$ in stationary sets, then:

$$\alpha \in M_i \text{ iff } M_i \models S_\alpha \text{ is stationary iff } S_\alpha \text{ reflects on } \chi_{M_i}(\lambda)$$

Thus

$$M_i \cap \lambda = \{\alpha : S_\alpha \text{ reflects on } \chi_{M_i}(\lambda)\}$$

and we are done. □

Lemma 5.6. *If $M \prec W_\theta$ is a 0-guessing model then M is a faithful model.*

Proof. This follows from the fact that M is isomorphic to W_γ for some γ. □

By the two lemmas the first part of the theorem is proved. To prove the second part of the theorem we proceed as follows:

Proof. Let in W

$$X = \bigcup\{P^*(E^{\aleph_0}_\lambda) : \lambda < \theta \text{ is regular}\}$$

Fix also in W a family $\{S_\alpha : \alpha < \omega_1\}$ of disjoint stationary subsets of ω_1 such that $\min S_\alpha \geq \alpha$ for all α and $\{S_\alpha : \alpha < \omega_1\}$ is a maximal antichain on $P(\omega_1)/\mathsf{NS}_{\omega_1}$.

Let $\mathbb{C}$ be Cohen forcing. In $W[G]$ where G is W-generic for $\mathbb{C}$ we define the poset $\mathbb{P}$ as follows.

A condition $p \in \mathbb{P}$ is a pair (f_p, ϕ_p) such that:

- $f_p : \alpha + 1 \to W \cap (P_{\omega_1} W_\theta)^{W[G]}$ is a continuos map.
- $\phi_p : \alpha + 1 \to X$ is such that for all $\eta < \xi \leq \alpha$:

$$\xi \in S_\eta \text{ iff } \sup(f_p(\xi) \cap \sup \phi_p(\eta)) \in \phi_p(\eta).$$

$p \leq q$ if f_p extends f_q and ϕ_p extends ϕ_q. We omit the proof of the following:

Lemma 5.7. *The poset $\mathbb{R} = \mathbb{C} * \dot{\mathbb{P}}$ is stationary set preserving and has the ω_1-covering and ω_1-approximation properties.*

By MM in W, there are stationarily many $N \prec H_{(2^\theta)^+}$ of size $\aleph_1$ which have a generic filter for the poset $\mathbb{R} * \mathbb{Q}_\mathbb{R}$, where $\mathbb{Q}_\mathbb{R}$ is the CCC-poset in $W^\mathbb{R}$ used in the proof of theorem 4.3. For any such N we can check the following properties of $M = N \cap W_\theta$:

$$M \prec W_\theta \text{ is an } \aleph_1\text{-guessing faithful model which is internally club.}$$

This completes the proof of the second part of the theorem. □

6 Applications of guessing models

We show that the failure of the weakest forms of square principle and the singular cardinal hypothesis are simple byproduct of the existence of guessing models. In particular the first application yields that the existence of a guessing models has very large cardinal strength.

6.1 The failure of square principles

Recall the following definitions:

Definition 6.1. A sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \mathrm{Lim} \cap E \cap \lambda \rangle$ is called a $\square_E(\kappa, \lambda)$-*sequence* if it satisfies the following properties.

(i) $0 < |\mathcal{C}_\alpha| < \kappa$ for all $\alpha \in \mathrm{Lim} \cap E \cap \lambda$,

(ii) $C \subset \alpha$ is club for all $\alpha \in \mathrm{Lim} \cap E \cap \lambda$ and $C \in \mathcal{C}_\alpha$,

(iii) $C \cap \beta \in \mathcal{C}_\beta$ for all $\alpha \in \mathrm{Lim} \cap E \cap \lambda$, $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{Lim}\, C$,

(iv) there is no club $D \subset \lambda$ such that $D \cap \delta \in \mathcal{C}_\delta$ for all $\delta \in \mathrm{Lim}\, D \cap E \cap \lambda$.

We say that $\square_E(\kappa, \lambda)$ holds if there exists a $\square_E(\kappa, \lambda)$-sequence. $\square(\kappa, \lambda)$ stands for $\square_\lambda(\kappa, \lambda)$.

Note that $\square_{\tau, <\kappa}$ implies $\square(\kappa, \tau^+)$ and that $\square(\lambda)$ is $\square(2, \lambda)$.

The theorem below is just a rephrasing using the notion of guessing models of the results on the failure of square principles Weiss obtained assuming his ineffability property for thin lists (see [7]).

Theorem 6.2. *Suppose there is a δ-guessing model $M \prec H_\theta$ for some $\delta < \kappa_M$. Then for every $\lambda < \theta$ such that* $\mathrm{cf}\,\lambda \geq \kappa_M$, $\square_{\mathrm{cof}(<\kappa_M)}(\kappa_M, \lambda)$ *fails.*

Proof. Assume not. Since M is a δ-guessing model, M is closed under countable suprema, thus $\gamma = \sup(M \cap \lambda)$ has uncountable cofinality. Pick a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \lambda,\ \mathrm{cf}(\alpha) < \kappa_M \rangle \in M$ witnessing $\square_{\mathrm{cof}(<\kappa_M)}(\kappa_M, \lambda)$. Since $|\mathcal{C}_\xi| < \kappa_M$ for all $\xi < \lambda$, $\mathcal{C}_\xi \subseteq M$ for all $\xi \in M$. Pick $C \in \mathcal{C}_\gamma$. Then $C \cap \xi \in \mathcal{C}_\xi \subseteq M$ for all $\xi \in M$ which are limit points of C. Since M is closed under countable suprema, there are club many such ξ of countable cofinality in M. Now given $z \in M \cap P_\delta \lambda$, find $\xi \in C \cap M$ above $\sup(z)$ and $D \in \mathcal{C}_\xi$ such that $C \cap \xi = D$. Then $C \cap z = D \cap z \in M$ since $z, D \in M$. Thus C is (δ, M)-approximated. Since M is a δ-guessing model, there is $E \in M$ be such that $C \cap M = E$. Then $M \models$ *E is a club subset of λ* and for all $\xi \in M$ limit points of E, $E \cap \xi \cap M = C \cap \xi \cap M = D \cap M$ for some $D \in C_\xi$. This shows that M models that E is a counterexample to $\langle C_\alpha \mid \alpha \in \lambda \rangle$ being a $\square_{\mathrm{cof}(<\kappa_M)}(\kappa_M, \lambda)$-sequence. $\square$

6.2 A proof of SCH

We give a proof of SCH assuming there are $\aleph_1$-internally unbounded, $\aleph_1$-guessing models M. This assumption is known to hold in all consistent cases.

We recall the following definition from [5]:

Definition 6.3. Suppose λ is a cardinal with $\mathrm{cf}\,\lambda = \omega$. $\mathcal{D} = \langle D(n, \alpha) \mid n < \omega,\ \alpha < \lambda^+ \rangle$ is called a *strong covering matrix on λ^+* if

(i) $\bigcup_{n<\omega} D(n, \alpha) = \alpha$ for all $\alpha < \lambda^+$,

(ii) $D(m, \alpha) \subset D(n, \alpha)$ for all $\alpha < \lambda^+$ and $m < n < \omega$,

(iii) for all $\alpha < \alpha' < \lambda^+$ there is $n < \omega$ such that $D(m, \alpha) \subset D(m, \alpha')$ for all $m \geq n$,

(iv) for all $x \in P_{\omega_1}\lambda^+$ there is $\gamma_x < \lambda^+$ such that for all $\alpha \geq \gamma_x$ there is $n < \omega$ such that $D(m, \alpha) \cap x = D(m, \gamma_x) \cap x$ for all $m \geq n$,

(v) $|D(n, \alpha)| < \lambda$ for all $\alpha < \lambda^+$ and $n < \omega$.

The following simple facts are proved in the cited paper [5]:

Fact 6.4. Assume $\lambda > 2^{\aleph_0}$ has countable cofinality. Then there is a strong covering matrix $\mathcal{D}$ on λ^+.

Fact 6.5. Assume that for all $\lambda > 2^{\aleph_0}$ of countable cofinality, there is a strong covering matrix $\mathcal{D}$ on λ^+ and A unbounded subset of λ^+ such that $P_{\omega_1}A$ is covered by $\mathcal{D}$. Then SCH holds.

Lemma 6.6. *Suppose λ is a cardinal with* $\mathrm{cf}\,\lambda = \omega$ *and $\mathcal{D}$ is a strong covering matrix on λ^+. Let θ be sufficiently large. Suppose $M \in P_{\omega_2}H_\theta$ is an $\aleph_1$-internally unbounded model and $\mathcal{D} \in M$. Then there is $n < \omega$ such that $D(m, \sup(M \cap \lambda^+)) \cap x \in M$ for all $x \in P_{\omega_1}\lambda^+ \cap M$ and $m \geq n$.*

Proof. Assume not and for each n pick $x_n \in M \cap P_{\omega_1}\lambda^+$ such that $D(n, \sup(M \cap \lambda^+)) \cap x_n \notin M$. Let $x \in M$ be a countable set containing all the x_n. x exists since M is $\aleph_1$-internally unbounded. Now $\gamma_x \in M$ by elementarity and thus there is n_0 such that $D(n, \sup(M \cap \lambda^+)) \cap x = D(n, \gamma_x) \cap x \in M$ for all $n \geq n_0$. This means that $D(n, \sup(M \cap \lambda^+)) \cap x_n = (D(n, \sup(M \cap \lambda^+)) \cap x) \cap x_n \in M$ since $D(n, \sup(M \cap \lambda^+)) \cap x \in M$ and $x_n \in M$. This is the desired contradiction. □

Theorem 6.7. *Suppose that there are stationarily many $\aleph_1$-guessing models $M \prec H_\theta$ which are $\aleph_1$-internally unbounded for all regular $\theta \geq \kappa$. Then* SCH *holds.*

Proof. Let λ be a cardinal with $\operatorname{cf} \lambda = \omega$. By [5] there exists a strong covering matrix on λ^+ and it suffices to show there is an unbounded $A \subset \lambda^+$ such that $P_{\omega_1}A$ is covered by $\mathcal{D}$, that is, for all $x \in P_{\omega_1}A$ there is $\alpha < \lambda^+$ and $n < \omega$ such that $x \subset D(n, \alpha)$.

Let θ be sufficiently large. Pick an $\aleph_1$-guessing model $M \prec H_\theta$ which is $\aleph_1$-internally unbounded and is also such that $\mathcal{D} \in M$. Pick a strong covering matrix $\mathcal{D} \in M$, and by proposition 2.1 we may assume $\operatorname{cf} \sup(M \cap \lambda^+) \geq \omega_1$. By Lemma 6.6 there is $n' < \omega$ such that $D(m, \sup(M \cap \lambda^+)) \cap x \in M$ for all $x \in P_{\omega_1}\lambda^+ \cap M$ and $m \geq n'$. As M is an $\aleph_1$-guessing model, this means that for all $m \geq n'$ there is $A_m \in M$ such that $D(m, \sup(M \cap \lambda^+)) = A_m \cap M$.

Since $\operatorname{cf} \sup(M \cap \lambda^+) \geq \omega_1$ and $\bigcup\{D(m, \sup(M \cap \lambda^+)) \mid m < \omega\} = \sup(M \cap \lambda^+)$ there is an $n < \omega$, $n \geq n'$, such that A_n is unbounded in $\sup(M \cap \lambda^+)$. As $A_n \in M$, this implies A_n is unbounded in λ^+.

Let $x \in M \cap P_{\omega_1}A_n$. Then $x = A_n \cap x = D(n, \sup(M \cap \lambda^+)) \cap x \subseteq D(n, \sup(M \cap \lambda^+))$. Thus H_θ models that x is covered by some $D(n, \alpha)$. Since $x \in M$, also M models it. Since this occurs for an arbitrary $x \in M \cap P_{\omega_1}X$, M models $P_{\omega_1}A_n$ is covered by $\mathcal{D}$, whence it really holds. □

7 Conclusions and open problems

We close this paper with a list of open problems and some guesses on their possible solutions:

1. Is it at all consistent that there are δ-guessing models which are not $\aleph_1$-guessing for some $\delta > \aleph_1$? It seems reasonable to expect this is the case but it is not clear what kind of forcing may achieve this.

2. Is it consistent that for a guessing model M, κ_M is the successor of a singular cardinal? I think that this shouldn't be possible.

3. Assuming PFA in W, $\mathcal{G}^{\aleph_1}_{\aleph_2} W_\theta$ is stationary for all inaccessible θ. Is it possible to build a transitive inner model V of W such that $\aleph_2$ is supercompact in V? Note that this would be the case if in V, $\mathcal{G}^{\aleph_0}_{\aleph_2} V_\theta$ is stationary for all inaccessible θ. In [6] and [7] there are several positive partial answers when we assume that W is a forcing extension of V. A possible attempt to overcome this latter assumption would be to isolate in models of MM some stationary subset T of $\mathcal{G}^{\aleph_1}_{\aleph_2} W_\theta$, and then try to argue that $\aleph_2$ is θ-supercompact in $L(\{M \cap \theta : M \in T\})$ or in some simple transitive class model of ZFC defined using $\{M \cap \theta : M \in T\}$ as a parameter to define it.

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