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THE SECOND PINCHING THEOREM FOR HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A SPHERE *

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Abstract

We generalize the second pinching theorem for minimal hypersurfaces in a sphere due to Peng-Terng, Wei-Xu, Zhang, and Ding-Xin to the case of hypersurfaces with small constant mean curvature. Let M^n be a compact hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}$. Denote by S the squared norm of the second fundamental form of M. We prove that there exist two positive constants $\gamma(n)$ and $\delta(n)$ depending only on n such that if $|H| \leq \gamma(n)$ and $\beta(n,H) \leq S \leq \beta(n,H) + \delta(n)$, then $S \equiv \beta(n,H)$ and M is one of the following cases: (i) $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}}),\ 1 \leq k \leq n-1$; (ii) $\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. Here $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$ and $\mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2}$.

1. Introduction

Let M^n be an n-dimensional compact hypersurface with constant mean curvature H in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. Denote by S the squared length of the second fundamental form of M and R its scalar curvature. Then $R = n(n-1) + n^2H^2 - S$. When $H = 0$, the famous pinching theorem due to Simons, Lawson, and Chern, do Carmo and Kobayashi ([2], [9], [13]) says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$, i.e., M must be the great sphere $\mathbb{S}^n$ or the Clifford torus $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}}),\ 1 \leq k \leq n-1$. Further discussions have been carried out by many other authors (see [7], [10], [14], [17], [18], [23], etc.). In 1970's, Chern proposed the following conjectures.

Chern Conjecture I. *Let M be a compact minimal hypersurface with constant scalar curvature in $\mathbb{S}^{n+1}$. Then the possible values form a discrete set. In particular, if $n \leq S \leq 2n$, then $S = n$, or $S = 2n$.*

Chern Conjecture II. *Let M be a compact minimal hypersurface in $\mathbb{S}^{n+1}$. If $n \leq S \leq 2n$, then $S \equiv n$, or $S \equiv 2n$.*

In 1983, Peng and Terng made breakthrough on the Chern conjectures I and II. They [11] proved that if M is a compact minimal hypersurface with constant scalar curvature in

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the unit sphere $\mathbb{S}^{n+1}$, and if $n \le S \le n + \frac{1}{12n}$, then $S = n$. Moreover, Peng and Terng [12] proved that if M is a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$, and if $n \le 5$ and $n \le S \le n+\tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on n, then $S \equiv n$. During the past two decades, there have been some important progress on these aspects(see [1], [4], [5], [8], [15], [16], [24], etc.). In 1993, Chang [1] solved Chern Conjecture I for the case of dimension 3. In [4] and [5], Cheng, Ishikawa and Yang obtained some interesting results on the Chern conjectures.

In 2007, Suh-Yang and Wei-Xu made some progress on Chern Conjectures, respectively. Suh and Yang [15] proved that if M is a compact minimal hypersurface with constant scalar curvature in $\mathbb{S}^{n+1}$, and if $n \le S \le n + \frac{3}{7}n$, then $S = n$ and M is a minimal Clifford torus. Meanwhile, Wei and Xu [16] proved that if M is a compact minimal hypersurface in $\mathbb{S}^{n+1}$, $n = 6, 7$, and if $n \le S \le n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on n, then $S \equiv n$ and M is a minimal Clifford torus. Later, Zhang [24] extended the second pinching theorem due to Peng-Terng [12] and Wei-Xu [16] to 8-dimensional compact minimal hypersurfaces in a unit sphere. Recently Ding and Xin [8] obtained the following pinching theorem for n-dimensional minimal hypersurfaces in a sphere.

Theorem A. *Let M be an n-dimensional compact minimal hypersurface in a unit sphere $\mathbb{S}^{n+1}$, and S the squared length of the second fundamental form of M. Then there exists a positive constant $\tau(n)$ depending only on n such that if $n \le S \le n + \tau(n)$, then $S \equiv n$, i.e., M is a Clifford torus.*

The pinching phenomenon for hypersurfaces of constant mean curvature in spheres is much more complicated than the minimal hypersurface case (see [17], [19]). In [17], Xu proved the following pinching theorem for submanifolds with parallel mean curvature in a sphere.

Theorem B. *Let M be an n-dimensional compact submanifold with parallel mean curvature vector ($H \ne 0$) in an $(n + p)$-dimensional unit sphere $\mathbb{S}^{n+p}$. If $S \le \alpha(n, H)$, then either M is pseudo-umbilical, or $S \equiv \alpha(n, H)$ and M is the isoparametric hypersurface $\mathbb{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbb{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ in a great sphere $\mathbb{S}^{n+1}$. In particular, if M is a compact hypersurface with constant mean curvature $H(\ne 0)$ in $\mathbb{S}^{n+1}$, then M is either a totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$, or a Clifford hypersurface $\mathbb{S}^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times \mathbb{S}^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$. Here $\alpha(n, H) = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}$ and $\lambda = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.*

In [20], Xu and Tian generalized Suh-Yang's pinching theorem [15] to the case where M is a compact hypersurface with constant scalar curvature and small constant mean curvature in $\mathbb{S}^{n+1}$. The following second pinching theorem for hypersurfaces with small constant mean curvature was proved for $n \le 7$ by Cheng-He-Li [3] and Xu-Zhao [21] respectively, and for $n = 8$ by Xu [22].

Theorem C. *Let M be an n-dimensional compact hypersurface with constant mean curvature $H(\ne 0)$ in a unit sphere $\mathbb{S}^{n+1}$, $n \le 8$. There exist two positive constants $\gamma_0(n)$ and $\delta_0(n)$ depending only on n such that if $|H| \le \gamma_0(n)$, and $\beta(n, H) \le S < \beta(n, H) + \delta_0(n)$, then $S \equiv \beta(n, H)$ and $M = \mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. Here $\beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 +$*

$\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ *and* $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$.

In this paper, we prove the second pinching theorem for n-dimensional hypersurfaces with constant mean curvature, which is a generalization of Theorems A and C.

Main Theorem. *Let M be an n-dimensional compact hypersurface with constant mean curvature H in a unit sphere $\mathbb{S}^{n+1}$. There exist two positive constants $\gamma(n)$ and $\delta(n)$ depending only on n such that if $|H| \le \gamma(n)$, and $\beta(n,H) \le S \le \beta(n,H)+\delta(n)$, then $S \equiv \beta(n,H)$ and M is one of the following cases: (i) $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \le k \le n-1$; (ii) $\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. Here $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ and $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$.*

2. Preliminaries

Let M^n be an n-dimensional compact hypersurface with constant mean curvature in a unit sphere $\mathbb{S}^{n+1}$. We shall make use of the following convention on the range of indices.

$$1 \le A, B, C, \ldots, \le n+1, \quad 1 \le i, j, k, \ldots, \le n.$$

For an arbitrary fixed point $x \in M \subset \mathbb{S}^{n+1}$, we choose an orthonormal local frame field $\{e_A\}$ in $\mathbb{S}^{n+1}$ such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of $\mathbb{S}^{n+1}$. Restricting to M, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j,\ h_{ij} = h_{ji}. \tag{1}$$

Let h be the second fundamental form of M. Denote by R, H and S the scalar curvature, mean curvature and squared length of the second fundamental form of M, respectively. Then we have

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \tag{2}$$

$$S = \sum_{i,j} h_{ij}^2,\ \ H = \frac{1}{n}\sum_i h_{ii}, \tag{3}$$

$$R = n(n-1) + n^2H^2 - S. \tag{4}$$

We choose e_{n+1} such that $H = \frac{1}{n}\sum_i h_{ii} \ge 0$. Denote by h_{ijk}, h_{ijkl} and h_{ijklm} the first, second and third covariant derivatives of the second fundamental tensor h_{ij}, respectively. Then we have

$$\nabla h = \sum_{i,j,k} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k,\ \ h_{ijk} = h_{ikj}, \tag{5}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{6}$$

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{7}$$

At each fixed point $x \in M$, we take orthonormal frames $\{e_i\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ for all i, j. Then $\sum_i \lambda_i = nH$ and $\sum_i \lambda_i^2 = S$. By a direct computation, we have

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^2, \tag{8}$$

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla h|^2 &= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 \\
&\quad + \sum_{i,j,k,l,m}(6h_{ijk}h_{ilm}h_{jl}h_{km} - 3h_{ijk}h_{ijl}h_{km}h_{ml}) + 3nH\sum_{i,j,k,l} h_{ijk}h_{jlk}h_{li} \\
&= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 + 3(2B-A) + 3nHC,
\end{aligned} \tag{9}$$

where

$$f_k = \sum_i \lambda_i^k,\ A = \sum_{i,j,k} h_{ijk}^2\lambda_i^2,\ B = \sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_j,\ C = \sum_{i,j,k} h_{ijk}^2\lambda_i.$$

Using a similar method as in [11], we obtain

$$h_{ijij} = h_{jiji} + t_{ij}, \tag{10}$$

$$|\nabla^2 h|^2 \geq \frac{3}{4}\sum_{i\neq j} t_{ij}^2 = \frac{3}{4}\sum_{i,j} t_{ij}^2, \tag{11}$$

and

$$3(A-2B) \leq aS|\nabla h|^2, \tag{12}$$

where $t_{ij} = (\lambda_i - \lambda_j)(1+\lambda_i\lambda_j)$ and $a = \frac{\sqrt{17}+1}{2}$. From (11), we have

$$|\nabla^2 h|^2 \geq \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3]. \tag{13}$$

By a computation, we obtain

$$\begin{aligned}
\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} &= \frac{1}{3}\sum_k \lambda_k (f_3)_{kk} \\
&= \sum_k \lambda_k\Big(\sum_i h_{iikk}\lambda_i^2 + 2\sum_{i,j} h_{ijk}^2\lambda_i\Big) \\
&= \sum_{i,k} h_{iikk}\lambda_k\lambda_i^2 + 2\sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_k \\
&= \sum_{i,k}[h_{kkii} + (\lambda_i-\lambda_k)(1+\lambda_i\lambda_k)]\lambda_k\lambda_i^2 + 2B \\
&= \sum_i\Big(\frac{S_{ii}}{2} - \sum_{j,k} h_{ijk}^2\Big)\lambda_i^2 + \sum_{i,k}\lambda_i^2\lambda_k(\lambda_i-\lambda_k)(1+\lambda_i\lambda_k) + 2B \\
&= \sum_{i,j,k}\frac{h_{ik}h_{kj}}{2}S_{ij} + nHf_3 - S^2 - f_3^2 + Sf_4 - (A-2B).
\end{aligned} \tag{14}$$

Since $\int_M \sum_{i,j} h_{ij}(f_3)_{ij}dM = 0$, we drive the following integral formula.

$$\begin{aligned}\int_M (A-2B)dM &= \int_M (nHf_3 - S^2 - f_3^2 + Sf_4 + \sum_{i,j,k} \frac{h_{ik}h_{kj}}{2} S_{ij})dM \\ &= \int_M (nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} (h_{ik}h_{kj})_j \frac{S_i}{2})dM \\ &= \int_M (nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj} \frac{S_i}{2} - \sum_{i,j,k} h_{ik}h_{kjj} \frac{S_i}{2})dM \\ &= \int_M (nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj} \frac{S_i}{2})dM \\ &= \int_M (nHf_3 - S^2 - f_3^2 + Sf_4 - \frac{|\nabla S|^2}{4})dM. \end{aligned} \tag{15}$$

3. Proof of Main Theorem

The key to the proof of Main Theorem is to establish some integral equalities and inequalities on the second fundamental form of M and its covariant derivatives by the parameter method.

To simplify the computation, we introduce the tracefree second fundamental form $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. If $h_{ij} = \lambda_i \delta_{ij}$, then $\phi_{ij} = \mu_i \delta_{ij}$, where $\mu_i = \lambda_i - H$. Putting $\Phi = |\phi|^2$ and $\bar{f}_k = \sum_i \mu_i^k$, we get $\Phi = S - nH^2$, $f_3 = \bar{f}_3 + 3H\Phi + nH^3$ and $f_4 = \bar{f}_4 + 4H\bar{f}_3 + 6H^2\Phi + nH^4$. From (8), we obtain

$$\begin{aligned}\frac{1}{2}\Delta\Phi &= S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^2 \\ &= -\Phi^2 + n\Phi + nH\bar{f}_3 + nH^2\Phi + |\nabla\phi|^2 \\ &= -F(\Phi) + |\nabla\phi|^2, \end{aligned} \tag{16}$$

where $F(\Phi) = \Phi^2 - n\Phi - nH^2\Phi - nH\bar{f}_3$. Therefore, we have

$$|\nabla\Phi|^2 = \frac{1}{2}\Delta(\Phi)^2 - \Phi\Delta\Phi = \frac{1}{2}\Delta(\Phi)^2 + 2\Phi F(\Phi) - 2\Phi|\nabla\phi|^2, \tag{17}$$

and

$$\int_M F(\Phi)dM = \int_M |\nabla\phi|^2 dM. \tag{18}$$

Lemma 1.(See [17]) *Let a_1, a_2, ..., a_n be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = a$. Then*

$$|\sum_i a_i^3| \leq \frac{n-2}{\sqrt{n(n-1)}} a^{\frac{3}{2}},$$

and the equality holds if and only if at least $n-1$ numbers of a_i's are same with each other.

From Lemma 1, we get

$$\begin{aligned} F(\Phi) &\geq \Phi^2 - n\Phi - nH^2\Phi - \frac{n(n-2)H\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \\ &= \Phi\Big[\Phi - \frac{n(n-2)H\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1+H^2)\Big] \\ &\geq 0, \end{aligned} \tag{19}$$

provided

$$\Phi \geq \beta_0(n,H) := n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} - nH^2.$$

Moreover, $F(\Phi) = 0$ if and only if $\Phi = \beta_0(n,H)$.

Set

$$G = \sum_{i,j}(\lambda_i - \lambda_j)^2(1 + \lambda_i\lambda_j)^2.$$

Then we have

$$G = 2[Sf_4 - f_3^2 - S^2 - S(S-n) + 2nHf_3 - n^2H^2]. \tag{20}$$

This together with (8) and (15) implies

$$\frac{1}{2}\int_M GdM = \int_M [(A - 2B) - |\nabla h|^2 + \frac{1}{4}|\nabla S|^2]dM. \tag{21}$$

Lemma 2. *Let M be an $n(\geq 4)$-dimensional compact hypersurface with constant mean curvature in $\mathbb{S}^{n+1}$. If $S \geq \beta(n,H)$, then we have*

$$3(A - 2B) \leq 2S|\nabla h|^2 + C_1(n)|\nabla h|^2 G^{\frac{1}{3}},$$

where $C_1(n) = (\sqrt{17} - 3)[6(\sqrt{17} + 1)]^{-\frac{1}{3}}(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n})^{-\frac{2}{3}}$.

Proof. We derive the estimate above at each fixed point $x \in M$. If $\lambda_j^2 - 4\lambda_i\lambda_j \leq 2S$ for all $i \neq j$, then we get the desired estimate immediately. Otherwise, we assume that there exist $i \neq j$, such that $\lambda_j^2 - 4\lambda_i\lambda_j = tS > 2S$.

We get

$$S \geq \lambda_i^2 + \lambda_j^2 = (\frac{tS - \lambda_j^2}{4\lambda_j})^2 + \lambda_j^2. \tag{22}$$

Then

$$\lambda_j^2 \leq \frac{1}{17}(t + 8 + 4\sqrt{4 + t - t^2})S, \quad 2 < t \leq \frac{\sqrt{17}+1}{2}, \tag{23}$$

which implies

$$-\lambda_i\lambda_j \geq \frac{1}{17}(4t - 2 - \sqrt{4 + t - t^2})S \geq 0.26S > \frac{S}{n} \geq 1. \tag{24}$$

On the other hand, we have

$$(\lambda_i - \lambda_j)^2 = (\frac{\lambda_j}{2} + \lambda_i)^2 + \frac{3}{4}(\lambda_j^2 - 4\lambda_i\lambda_j) \geq \frac{3t}{4}S. \tag{25}$$

By the definition of G, we get

$$\begin{aligned} G &\geq 2(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2 \\ &\geq \frac{3t}{2}S(1+\lambda_i\lambda_j)^2 \\ &\geq \frac{3t}{2}S(-\lambda_i\lambda_j-\frac{S}{n})^2 \\ &\geq \frac{3t}{2}\Big[\frac{1}{17}(4t-2-\sqrt{4+t-t^2})-\frac{1}{n}\Big]^2 S^3. \end{aligned} \tag{26}$$

We define an auxiliary function

$$\zeta(t)=\frac{t}{(t-2)^3}\Big[\frac{1}{17}(4t-2-\sqrt{4+t-t^2})-\frac{1}{n}\Big]^2, \quad 2<t\leq\frac{\sqrt{17}+1}{2}.$$

Then we have

$$\begin{aligned} \zeta(t) &\geq \frac{t}{(t-2)^3}\Big[\frac{1}{17}(4t-2-\sqrt{2})-\frac{1}{n}\Big]^2 \\ &\geq \inf_{2<t\leq\frac{\sqrt{17}+1}{2}} \frac{t}{(t-2)^3}\Big[\frac{1}{17}(4t-2-\sqrt{2})-\frac{1}{n}\Big]^2 \\ &= \frac{4(\sqrt{17}+1)}{(\sqrt{17}-3)^3}\Big(\frac{2}{\sqrt{17}}-\frac{\sqrt{2}}{17}-\frac{1}{n}\Big)^2. \end{aligned} \tag{27}$$

Hence

$$\begin{aligned} (\lambda_j^2-4\lambda_i\lambda_j-2S)^3 &= (t-2)^3S^3 \\ &\leq \frac{2G}{3\zeta(t)} \\ &\leq \frac{(\sqrt{17}-3)^3}{6(\sqrt{17}+1)}(\frac{2}{\sqrt{17}}-\frac{\sqrt{2}}{17}-\frac{1}{n})^{-2}G \\ &= (C_1(n)G^{\frac{1}{3}})^3. \end{aligned} \tag{28}$$

This implies

$$\begin{aligned} 3(A-2B) &\leq \sum_{i,j,k\ distinct}[2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2]h_{ijk}^2+3\sum_{i\neq j}(\lambda_j^2-4\lambda_i\lambda_j)h_{iij}^2 \\ &\leq 2S\sum_{i,j,k\ distinct}h_{ijk}^2+3\sum_{i\neq j}h_{iij}^2(2S+C_1(n)G^{\frac{1}{3}}) \\ &\leq 2S|\nabla h|^2+C_1(n)|\nabla h|^2G^{\frac{1}{3}}. \end{aligned} \tag{29}$$

Proof of Main Theorem.(i) When $H=0$, the assertion follows from Theorem A.
(ii) When $H\neq 0$, the assertion for lower dimensional cases ($n\leq 8$) was verified in [3], [21]

and [22]. We consider the case for $n \geq 4$. From (10) and (11), we see that $G = \sum_{i,j} t_{ij}^2$ and $|\nabla^2 h|^2 \geq \frac{3}{4}G$. Let $0 < \theta < 1$, we have

$$\int_M |\nabla^2 h|^2 dM \geq \Big[\frac{3(1-\theta)}{4} + \frac{3\theta}{4}\Big] \int_M G dM. \tag{30}$$

From (9), (21), Lemma 2 and Young's inequality, we drive the following inequality.

$$\begin{aligned}
\frac{3(1-\theta)}{4}\int_M G dM &\leq \int_M \Big[(S-2n-3)|\nabla h|^2 + \frac{3}{2}|\nabla S|^2 + 3(A-2B) - 3nHC - \frac{3\theta}{4}G\Big] dM \\
&= \int_M (S-2n-3+\frac{3\theta}{2})|\nabla h|^2 dM + (3-\frac{3\theta}{2})\int_M (A-2B) dM \\
&\quad +(\frac{3}{2}-\frac{3\theta}{8})\int_M |\nabla S|^2 dM - 3nH\int_M C dM \\
&\leq \int_M (S-2n-3+\frac{3\theta}{2})|\nabla h|^2 dM + (1-\frac{\theta}{2})\int_M (2S|\nabla h|^2 \\
&\quad +C_1(n)|\nabla h|^2 G^{\frac{1}{3}}) dM + (\frac{3}{2}-\frac{3\theta}{8})\int_M |\nabla S|^2 dM - 3nH\int_M C dM \\
&\leq \int_M \Big[(3-\theta)S - 2n - 3 + \frac{3\theta}{2}\Big]|\nabla h|^2 dM + \frac{3(1-\theta)}{4}\int_M G dM \\
&\quad +C_2(n,\theta)\int_M |\nabla h|^3 dM + (\frac{3}{2}-\frac{3\theta}{8})\int_M |\nabla S|^2 dM \\
&\quad -3nH\int_M C dM,
\end{aligned} \tag{31}$$

where $C_2(n,\theta) = \frac{4}{9}C_1(n)^{\frac{3}{2}}(1-\frac{\theta}{2})^{\frac{3}{2}}(1-\theta)^{-\frac{1}{2}}$.

Let $\epsilon > 0$, from (16), we get

$$\begin{aligned}
\int_M |\nabla h|^3 dM &= \int_M |\nabla \phi|^3 dM \\
&= \int_M |\nabla \phi|(F(\Phi) + \frac{1}{2}\Delta\Phi) dM \\
&= \int_M F(\Phi)|\nabla\phi| dM - \frac{1}{2}\int_M \nabla|\nabla\phi|\cdot\nabla\Phi dM \\
&\leq \int_M F(\Phi)|\nabla\phi| dM + \epsilon\int_M |\nabla^2\phi|^2 dM + \frac{1}{16\epsilon}\int_M |\nabla\Phi|^2 dM.
\end{aligned} \tag{32}$$

Since

$$|C| \leq \sqrt{S}|\nabla h|^2, \tag{33}$$

we have

$$\begin{aligned}
0 &\leq \int_M [(3+3\sqrt{n}H-\theta)(\Phi+nH^2) - 2n - 3 + \frac{3\theta}{2}]|\nabla\phi|^2 dM \\
&\quad +C_2(n,\theta)[\int_M F(\Phi)|\nabla\phi| dM + \epsilon\int_M |\nabla^2\phi|^2 dM + \frac{1}{16\epsilon}\int_M |\nabla\Phi|^2 dM] \\
&\quad +(\frac{3}{2}-\frac{3\theta}{8})\int_M |\nabla\Phi|^2 dM.
\end{aligned} \tag{34}$$

Substituting (12) and (33) into (9), we have

$$\begin{aligned}\int_M |\nabla^2\phi|^2 dM &= \int_M |\nabla^2 h|^2 dM \\ &\leq \int_M [(S-2n-3)|\nabla h|^2 + \frac{3}{2}|\nabla S|^2 + aS|\nabla h|^2 - 3nHC] dM \\ &\leq \int_M [(a+1+3\sqrt{n}H)S - 2n - 3]|\nabla\phi|^2 dM + \frac{3}{2}\int_M |\nabla S|^2 dM. \quad (35)\end{aligned}$$

Combining (16) and (17), we have

$$\begin{aligned}\int_M \frac{1}{2}|\nabla\Phi|^2 dM &= \int_M \Phi F(\Phi) dM - \int_M \Phi|\nabla\phi|^2 dM + \beta_0(n,H)\int_M |\nabla\phi|^2 dM \\ &\quad -\beta_0(n,H)\int_M F(\Phi) dM \\ &= \int_M (\Phi - \beta_0(n,H))F(\Phi) dM + \int_M (\beta_0(n,H) - \Phi)|\nabla\phi|^2 dM. \quad (36)\end{aligned}$$

Hence

$$\begin{aligned}0 \leq & \int_M \Big\{\Big[3 + 3\sqrt{n}H - \theta + \epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\Big](\Phi - \beta_0(n,H)) \\ & +\beta(n,H)\Big[3 + 3\sqrt{n}H - \theta + \epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\Big] \\ & -2\Big(\frac{3}{2} - \frac{3\theta}{8} + \frac{C_2(n,\theta)}{16\epsilon} + \frac{3\epsilon C_2(n,\theta)}{2}\Big)(\Phi - \beta_0(n,H)) \\ & -2n - 3 + \frac{3\theta}{2} - \epsilon C_2(n,\theta)(2n+3)\Big\}|\nabla\phi|^2 dM \\ & +2\Big(\frac{3}{2} - \frac{3\theta}{8} + \frac{C_2(n,\theta)}{16\epsilon} + \frac{3\epsilon C_2(n,\theta)}{2}\Big)\int_M (\Phi - \beta_0(n,H))F(\Phi) dM \\ & +C_2(n,\theta)\int_M F(\Phi)|\nabla\phi| dM \\ = & \int_M \Big\{D(n,H)\Big[3 + 3\sqrt{n}H - \theta + \epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\Big] \\ & +(1-\theta)n - 3 + \frac{3\theta}{2} + 3n^{\frac{3}{2}}H + \epsilon C_2(n,\theta)(an + 3n^{\frac{3}{2}}H - n - 3)\Big\}|\nabla\phi|^2 dM \\ & -\Big(\frac{\theta}{4} + \frac{C_2(n,\theta)}{8\epsilon} - 3\sqrt{n}H + \epsilon C_2(n,\theta)(2 - a - 3\sqrt{n}H)\Big)\int_M (\Phi - \beta_0(n,H))|\nabla\phi|^2 dM \\ & +\Big(3 - \frac{3\theta}{4} + \frac{C_2(n,\theta)}{8\epsilon} + 3\epsilon C_2(n,\theta)\Big)\int_M (\Phi - \beta_0(n,H))F(\Phi) dM \\ & +C_2(n,\theta)\int_M F(\Phi)|\nabla\phi| dM, \quad (37)\end{aligned}$$

where $\beta(n,H) = \beta_0(n,H) + nH^2$ and $D(n,H) = \beta(n,H) - n$.

Note that

$$\frac{\theta}{4} + \frac{C_2(n,\theta)}{8\epsilon} - 3\sqrt{n}H + \epsilon C_2(n,\theta)(2 - a - 3\sqrt{n}H) \geq 0, \quad (38)$$

for all $\epsilon \in (0, \epsilon_1]$, where ϵ_1 is some positive constant. When $\beta(n,H) \le S \le \beta(n,H) + \epsilon^2$, we obtain

$$\begin{aligned} 0 \le & \int_M [(1-\theta)n - 3 + \frac{3\theta}{2} + 3n^{\frac{3}{2}}H + D(n,H)(3 + 3\sqrt{n}H - \theta) + O(\epsilon,\theta,H)]|\nabla\phi|^2 dM \\ & + C_2(n,\theta)\int_M F(\Phi)|\nabla\phi| dM, \end{aligned} \tag{39}$$

where

$$\begin{aligned} O(\epsilon,\theta,H) &= \epsilon D(n,H)C_2(n,\theta)(a + 1 + 3\sqrt{n}H) + \epsilon C_2(n,\theta)(an + 3n^{\frac{3}{2}}H - n - 3) \\ & + \epsilon^2(3 - \frac{3\theta}{4} + \frac{C_2(n,\theta)}{8\epsilon} + 3\epsilon C_2(n,\theta)). \end{aligned}$$

On the other hand, we have

$$C_2(n,\theta)\int_M F(\Phi)|\nabla\phi| dM \le \frac{3}{8}\int_M F(\Phi) dM + \frac{2C_2(n,\theta)^2}{3}\int_M F(\Phi)|\nabla\phi|^2 dM. \tag{40}$$

Using Lemma 1, we drive an upper bound for $F(\Phi)$.

$$\begin{aligned} F(\Phi) &\le \Phi^2 - n\Phi - nH^2\Phi + \frac{n(n-2)H\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \\ &= \Phi\Big[\Phi + \frac{n(n-2)H\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1+H^2)\Big] \\ &= \frac{\Phi(\Phi^{\frac{1}{2}} + \beta_0(n,H)^{\frac{1}{2}})(\Phi - \alpha_0(n,H))}{\Phi^{\frac{1}{2}} + \alpha_0(n,H)^{\frac{1}{2}}}, \end{aligned} \tag{41}$$

where $\alpha_0(n,H) = \left[\frac{-n(n-2)H + n\sqrt{n^2H^2 + 4n - 4}}{2\sqrt{n(n-1)}}\right]^2$.

When $\delta(n) \le \epsilon^2$ and $\epsilon \le 1$, we choose positive constant $\gamma_1(n)$ such that $n \le \Phi \le 2n$ and $x_1 \le 2\sqrt{n}$ for all $H \le \gamma_1(n)$. We obtain

$$F(\Phi) \le 8n(\Phi - \alpha_0(n,H)) \le 8n\Big(\epsilon^2 + \frac{n(n-2)}{(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\Big). \tag{42}$$

Let $\theta = \theta(n) = 1 - \frac{1}{8n}$. We choose positive constants $\gamma_2(n)$ and $\gamma_3(n)$ such that $3n^{\frac{3}{2}}H + D(n,H)(3 + 3\sqrt{n}H) \le \frac{1}{8}$ for all $H \le \gamma_2(n)$, and $\frac{16n^2(n-2)}{(n-1)}\sqrt{n^2\gamma_3(n)^4 + 4(n-1)\gamma_3(n)^2} \le \frac{9}{16C_2(n,\theta(n))^2}$.

Take $\epsilon_2(n) = \left[\frac{n(n-2)}{(n-1)}\sqrt{n^2\gamma_3(n)^4 + 4(n-1)\gamma_3(n)^2}\right]^{\frac{1}{2}} > 0$. Combining (39), (40) and (42), we obtain

$$\int_M [-\frac{1}{2} + O(\epsilon,\theta(n),H)]|\nabla\phi|^2 dM \ge 0, \tag{43}$$

for all $H \le \gamma(n) = \min\{\gamma_1(n), \gamma_2(n), \gamma_3(n)\}$ and $\epsilon \le \min\{\epsilon_1, \epsilon_2(n)\}$.

For $\epsilon \le 1$, we have

$$O(\epsilon,\theta(n),H) \le \epsilon D(n,\gamma(n))C_2(n,\theta(n))(a + 1 + 3\sqrt{n}\gamma(n))$$

$$
\begin{aligned}
&+\epsilon C_2(n,\theta(n))(an+3n^{\frac{3}{2}}\gamma(n)) \\
&+\epsilon(3-\frac{3\theta(n)}{4}+\frac{C_2(n,\theta(n))}{8}+3C_2(n,\theta(n))) \\
:=\ &\epsilon\eta(n), \qquad (44)
\end{aligned}
$$

where $a=\frac{\sqrt{17}+1}{2}$.

For $\epsilon \le \epsilon_1(n)$, where $\epsilon_1(n)=\frac{C_2(n,\theta(n))}{8[3\sqrt{n}\gamma(n)+C_2(n,\theta(n))(a+3\sqrt{n}\gamma(n)-2)]}>0$, $a=\frac{\sqrt{17}+1}{2}$, we have

$$
\frac{C_2(n,\theta(n))}{8\epsilon}\ge 3\sqrt{n}\gamma(n)+C_2(n,\theta(n))(a+3\sqrt{n}\gamma(n)-2)-\frac{\theta(n)}{4}. \qquad (45)
$$

So

$$
\frac{\theta(n)}{4}+\frac{C_2(n,\theta(n))}{8\epsilon}-3\sqrt{n}H+\epsilon C_2(n,\theta(n))(2-a-3\sqrt{n}H)\ge 0.
$$

Taking $\delta(n)=\epsilon(n)^2$, where $\epsilon(n)=\min\{1,\epsilon_1(n),\epsilon_2(n),\epsilon_3(n)\}$ and $\epsilon_3(n)=\frac{1}{3\eta(n)}$, we have $\delta(n)>0$. From (43) and the assumption that $\beta(n,H)\le S\le\beta(n,H)+\delta(n)$, we obtain $\nabla\phi=0$. This implies $F(\Phi)=0$ and $\Phi=\beta_0(n,H)$.

By Lemma 1, we have

$$
\lambda_1=\ldots=\lambda_{n-1}=H-\sqrt{\frac{\beta(n,H)-nH^2}{n(n-1)}},
$$

$$
\lambda_n=H+\sqrt{\frac{(n-1)(\beta(n,H)-nH^2)}{n}}.
$$

Therefore M is the Clifford hypersurface

$$
\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}})\times\mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})
$$

in $\mathbb{S}^{n+1}$, where $\mu=\frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$. This completes the proof of Main Theorem.

Finally we would like to propose the following problems.

Open Problem A. *Let M be an n-dimensional compact hypersurface with constant mean curvature H in the unit sphere $\mathbb{S}^{n+1}$. Does there exist a positive constant $\delta(n)$ depending only on n such that if $\beta(n,H)\le S\le\beta(n,H)+\delta(n)$, then $S\equiv\beta(n,H)$?*

Open Problem B. *For an n-dimensional compact hypersurface M^n with constant mean curvature H in $\mathbb{S}^{n+1}$, set $\mu_k=\frac{n|H|+\sqrt{n^2H^2+4k(n-k)}}{2k}$. Suppose that $\alpha(n,H)\le S\le\beta(n,H)$. Is it possible to prove that M must be the isoparametric hypersurface $S^k(\frac{1}{\sqrt{1+\mu_k^2}})\times S^{n-k}(\frac{\mu_k}{\sqrt{1+\mu_k^2}})$, $k=1,2,\cdots,n-1$?*

When $H=0$, the rigidity theorem due to Lawson [9], Chern, do Carmo and Kobayashi [2] provides an affirmative answer for Open Problem B.

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