EXPONENTIAL DECAY OF SEMIGROUPS FOR SECOND ORDER NON-SELFADJOINT LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The Cauchy problem for second order linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0$$

in Hilbert space H with a sectorial operator A and an accretive operator D is studied. Sufficient conditions for exponential decay of the solutions are obtained.

Many linearized equations of mechanics and mathematical physics can be reduced to a linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0, \tag{0.1}$$

where $u(t)$ is a vector-valued function in an appropriate (finite or infinite dimensional) Hilbert space H, D and A are linear (bounded or unbounded) operators on H. Properties of the differential equation (0.1) are closely connected with spectral properties of a quadric pencil

$$L(\lambda) = \lambda^2 + \lambda D + A, \quad \lambda \in \mathbb{C}$$

which is obtained by substituting exponential functions $u(t) = \exp(\lambda t)x$, $x \in H$ into (0.1). In many applications A is a self-adjoint positive definite operator, D is a self-adjoint positive definite or an accretive operator (see definition in section 1). In this case the differential equation (0.1) and spectral properties of the related quadric pencil $L(\lambda)$ are well-studied, see [2, 6, 7, 8, 10, 11, 12, 13, 15] and references therein. It was obtained a localization of the pencil's spectrum, sufficient conditions of the completeness of eigen- and adjoint vectors of the pencil $L(\lambda)$ and it was proved, that all solutions of (0.1) exponentially decay. The exponential decay means, that the total energy exponentially decreases and corresponding mechanical system is stable. In paper [16] was studied spectral properties of the pencil $L(\lambda)$ for a self-adjoint non-positive definite operator A and an accretive operator D.

But some models of continuous mechanics are reduced to differential equation (0.1) with sectorial operator A, see [1, 9, 17] and references therein. In this cases methods, developed for self-adjoint operator A, cannot be applied.

The aim of this paper is the study of a Cauchy problem for second-order linear differential equation (0.1) in a Hilbert space H with initial conditions

$$u(0) = u_0 \quad u'(0) = u_1. \tag{0.2}$$

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The shiffness operator A is assumed to be a sectorial operator, the damping operator D is assumed to be an accretive operator.

By $\mathcal{L}(H', H'')$ denote a space of bounded operators acting from a Hilbert space H' to a Hilbert space H''. $\mathcal{L}(H) = \mathcal{L}(H, H)$ is an algebra of bounded operators acting on Hilbert space H.

1. Preliminary results

First let us recall some definitions [4, 14].

Definition 1.1. Linear operator B with dense domain $\mathcal{D}(B)$ is called *accretive* if $\operatorname{Re}(Bx, x) \geq 0$ for all $x \in \mathcal{D}(B)$ and *m-accretive*, if the range of operator $B + \omega I$ is dense in H for some $\omega > 0$.

An accretive operator B is m-accretive iff B has not accretive extensions [14]. For m-accretive operator

$$\rho(B) \supset \{\lambda \in \mathbb{C} \;:\; \operatorname{Re}\lambda < 0\}.$$

Definition 1.2. An accretive operator B is called *sectorial* or *ω-accretive* if for some $\omega \in [0, \pi/2)$

$$\big|\operatorname{Im}(Bx, x)\big| \leq \tan(\omega) \operatorname{Re}(Bx, x) \quad x \in \mathcal{D}(B).$$

If a sectorial operator has not sectorial extensions, then it's called *m-sectorial* or *m-ω-accretive.*

The sectorial property means that the numerical range of the operator B belongs to a sector

$$\{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq \tan(\omega) \operatorname{Re} z\}.$$

For a sectorial operator B there exist [14] a self-adjoint non-negative operator T_B and a self-adjoint operator $S_B \in \mathcal{L}(H)$, $\|S_B\| \leq \tan(\omega)$ such that

$$\operatorname{Re}(Bx, x) = (T_B^{1/2} x, T_B^{1/2} x), \quad B \subset T_B^{1/2}(I + iS_B) T_B^{1/2}$$

and $B = T_B^{1/2}(I + iS_B) T_B^{1/2}$ iff B is m-sectorial.

Throughout this paper we will assume, that

(A) Operator $A : \mathcal{D}(A) \subset H \to H$ is m-sectorial and for some positive a_0

$$\operatorname{Re}(Ax, x) \geq a_0(x, x) \quad x \in \mathcal{D}(A).$$

Since A is m-sectorial there exist a self-adjoint positive definite operator T and a self-adjoint $S \in \mathcal{L}(H)$, such that

$$\operatorname{Re}(Ax, x) = (T^{1/2} x, T^{1/2} x) \geq a_0(x, x), \quad x \in \mathcal{D}(A)$$
$$A = T^{1/2}(I + iS) T^{1/2}.$$

The operator A is invertible and

$$A^{-1} = T^{-1/2}(I + iS)^{-1} T^{-1/2}.$$

By H_s ($s \in \mathbb{R}$) denote a collection of Hilbert spaces generated by a self-adjoint operator $T^{1/2}$:

- for $s \geq 0$ $H_s = \mathcal{D}(T^{s/2})$ endowed with a norm $\|x\|_s = \|T^{s/2} x\|$;
- for $s < 0$ H_s is a closure of H with respect to the norm $\|\cdot\|_s$.

Obviously $H_0 = H$. The operator $T^{1/2}$ can be considered now as an unitary operator mapping H_s on H_{s-1}. A is a bounded operator $A \in \mathcal{L}(H_2, H_0)$ and it can be extended to a bounded operator $\tilde{A} \in \mathcal{L}(H_1, H_{-1})$. The inverse operator A^{-1} can be extended to a bounded operator $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$.

By $(\cdot,\cdot)_{-1,1}$ denote a duality pairing on $H_{-1} \times H_1$. Note, that for all $x \in H_{-1}$ and $y \in H_1$ we have

$$\Big|(x,y)_{-1,1}\Big| \le \|x\|_{-1} \cdot \|y\|_1$$

and $(x,y)_{-1,1} = (x,y)$ if $x \in H$. Further,

$$\operatorname{Re}(\tilde{A}x,x)_{-1,1} = (Tx,x)_{-1,1} = (T^{1/2}x, T^{1/2}x) = \|x\|_1^2, \quad x \in H_1 = \mathcal{D}(T^{1/2}).$$

Denote $\tilde{S} = T^{1/2}ST^{1/2} \in \mathcal{L}(H_1, H_{-1})$. Then, for the operator $\tilde{A}$ we have a representation $\tilde{A} = T + i\tilde{S}$ and

$$\operatorname{Im}(\tilde{A}x,x)_{-1,1} = (\tilde{S}x,x)_{-1,1} \quad x \in H_1.$$

Also $(\tilde{S}x,y)_{-1,1} = \overline{(\tilde{S}y,x)}_{-1,1}$ for all $x, y \in H_1$.

Following paper [11] we will assume

(B) D is a bounded operator $D \in \mathcal{L}(H_1, H_{-1})$, and

$$\beta = \inf_{x \in H_1, x \neq 0} \frac{\operatorname{Re}(Dx,x)_{-1,1}}{\|x\|^2} > 0. \tag{1.1}$$

Operator $T^{-1/2}$ is an unitary operator mapping H_s on H_{s+1}, therefore an operator $D' = T^{-1/2}DT^{-1/2}$, acting on H, is bounded. Let

$$D_1 = \frac{1}{2}T^{1/2}\Big(D' + (D')^*\Big)T^{1/2} \quad D_2 = \frac{1}{2i}T^{1/2}\Big(D' - (D')^*\Big)T^{1/2},$$

Obviously $D_1, D_2 \in \mathcal{L}(H_1, H_{-1})$, $D = D_1 + iD_2$ and for all $x \in H_1$

$$\operatorname{Re}(Dx,x)_{-1,1} = (D_1x,x)_{-1,1} \ge \beta\|x\|^2, \quad \operatorname{Im}(Dx,x)_{-1,1} = (D_2x,x)_{-1,1}.$$

Also $(D_jx,y)_{-1,1} = \overline{(D_jy,x)}_{-1,1}$ for all $x, y \in H_1$ $(j = 1, 2)$.

2. Main result

Definition 2.1. A vector-valued function $u(t) \in H_1$ is called a solution of the differential equation (0.1) if $u'(t) \in H_1$, $u''(t) \in H$, $Du'(t) + \tilde{A}u(t) \in H$ and

$$u''(t) + Du'(t) + \tilde{A}u(t) = 0 \tag{2.1}$$

If $u(t)$ is a solution of (2.1), then a vector-function

$$\mathbf{x}(t) = \begin{pmatrix} u'(t) \\ u(t) \end{pmatrix}$$

(formally) satisfies a first-order differential equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{2.2}$$

with a block operator matrix

$$\mathbf{A} = \begin{pmatrix} -D & -\tilde{A} \\ I & 0 \end{pmatrix}.$$

From mechanical viewpoint it is most natural to consider the equation (2.2) in an "energy" space $\mathcal{H} = H \times H_1$ with a dense domain of the operator $\mathbf{A}$ [6, 7, 11, 16]

$$\mathcal{D}(\mathbf{A}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, x_1, x_2 \in H_1, \ Dx_1 + \tilde{A}x_2 \in H \right\}.$$

An inverse of $\mathbf{A}$ is formally defined by a block operator matrix

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & I \\ -\tilde{A}^{-1} & -\tilde{A}^{-1}D \end{pmatrix}.$$

Let $\mathbf{y} = (y_1, y_2)^\top \in \mathcal{H} = H \times H_1$, then

$$\mathbf{A}^{-1}\mathbf{y} = \begin{pmatrix} y_2 \\ -\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since $\tilde{A}^{-1} \in \mathcal{L}(H_{-1}, H_1)$ and $D \in \mathcal{L}(H_1, H_{-1})$, then $\tilde{A}^{-1}D \in \mathcal{L}(H_1, H_1)$. Therefore $-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2 \in H_1$ and $\mathbf{A}^{-1}\mathbf{y} \in H_1 \times H_1$. Moreover,

$$Dx_1 + \tilde{A}x_2 = Dy_2 + \tilde{A}\left(-\tilde{A}^{-1}y_1 - \tilde{A}^{-1}Dy_2\right) = -y_1 \in H.$$

Thus $\mathbf{A}^{-1}\mathbf{y} \in \mathcal{D}(\mathbf{A})$. Since $I \in \mathcal{L}(H_1, H)$ the operator $\mathbf{A}^{-1}$ is bounded and therefore the operator $\mathbf{A}$ is closed and $0 \in \rho(\mathbf{A})$.

Let $(\mathbf{x}, \mathbf{y})_{\mathcal{H}}$ be a natural scalar product on $\mathcal{H} = H \times H_1$ and $\|\mathbf{x}\|^2_{\mathcal{H}} = (\mathbf{x}, \mathbf{y})_{\mathcal{H}}$.

If operator A is self-adjoint, the spectral properties of operator $\mathbf{A}$ are well-studied: $-\mathbf{A}$ is an m-accretive operator in the Hilbert space $\mathcal{H} = H \times H_1$ (see [2, 6, 7, 8, 10, 11] and references therein) and, consequently, $\mathbf{A}$ is a generator of a C_0-semigroup. Thus, differential equation (2.2) (and equation (2.1)) is correctly solvable in the space $\mathcal{H}$ for all $\mathbf{x}(0) = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$. Moreover, in this case operator $\mathbf{A}$ is a generator of a contraction semigroup [7]. It implies, that all solutions of (2.2) (and (2.1)) exponentially decay, i.e. for some $C, \omega > 0$

$$\|\mathbf{x}(t)\|_{\mathcal{H}} \leq C\exp(-\omega t)\|\mathbf{x}(0)\|_{\mathcal{H}} \quad t \geq 0.$$

For non-selfadjoint A operator $(-\mathbf{A})$ is not longer accretive in the space $\mathcal{H}$ with respect to the standard scalar product. But, under some assumptions, one can define a new scalar product on $\mathcal{H}$, which is topologically equivalent to the given one, such that an operator $(-\mathbf{A} - qI)$ (for some $q \geq 0$) is m-accretive and therefore the operator $\mathbf{A}$ is a generator of a C_0-semigroup on $\mathcal{H}$. If $q > 0$, then $\mathbf{A}$ is a generator of a contraction semigroup and all solutions of (2.2) exponentially decay.

Let $k \in (0, \beta)$ (β is defined by (1.1)). Consider on the space $\mathcal{H}$ a sesquilinear form

$$\begin{aligned}[] &[\mathbf{x}, \mathbf{y}]_{\mathcal{H}} = \\ &\quad (T^{1/2}x_2, T^{1/2}y_2) + k(D_1x_2, y_2)_{-1,1} - k^2(x_2, y_2) + (x_1 + kx_2, y_1 + ky_2), \\ &\qquad\qquad \mathbf{x} = (x_1, x_2)^\top, \ \mathbf{y} = (y_1, y_2)^\top \in \mathcal{H}. \end{aligned}$$

Obviously, $[\mathbf{x}, \mathbf{y}] = \overline{[\mathbf{y}, \mathbf{x}]}$ and

$$[\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \|x_2\|_1^2 + k(D_1x_2, x_2)_{-1,1} + \|x_1\|^2 + 2k\,\mathrm{Re}(x_1, x_2).$$

Since $(D_1x, x)_{-1,1} = \mathrm{Re}(Dx, x)_{-1,1} \geq \beta\|x\|^2$ and

$$2|\,\mathrm{Re}(x_1, x_2)| \leq 2|(x_1, x_2)| \leq 2\|x_1\| \cdot \|x_2\| \leq \frac{\|x_1\|^2}{\beta} + \beta\|x_2\|^2,$$

then

$$[\mathbf{x},\mathbf{x}]_{\mathcal{H}} \geq \|x_2\|_1^2 + k\Big((D_1x,x)_{-1,1} - \beta\|x_2\|^2\Big) + \left(1-\frac{k}{\beta}\right)\|x_1\|^2 \geq \|x_2\|_1^2 + \left(1-\frac{k}{\beta}\right)\|x_1\|^2.$$

Inequatlities[1] $|(D_1x,x)_{-1,1}| \leq \|D_1x\|_{-1}\cdot\|x\|_1 \leq \|D_1\|\cdot\|x\|_1^2$ and $\|x\|_1^2 \geq a_0\|x\|^2$ imply

$$[\mathbf{x},\mathbf{x}]_{\mathcal{H}} \leq \Big(1+k\|D_1\|\Big)\|x_2\|_1^2 + k\beta\|x_2\|^2 + \left(1+\frac{k}{\beta}\right)\|x_1\|^2 \leq \left(1+k\|D_1\|+\frac{k\beta}{a_0}\right)\|x_2\|_1^2 + \left(1+\frac{k}{\beta}\right)\|x_1\|^2.$$

Thus,

$$\left(1-\frac{k}{\beta}\right)\|\mathbf{x}\|_{\mathcal{H}}^2 \leq [\mathbf{x},\mathbf{x}]_{\mathcal{H}} \leq \text{const}\,\|\mathbf{x}\|_{\mathcal{H}}^2$$

and $[\cdot,\cdot]_{\mathcal{H}}$ is a scalar product on $\mathcal{H}$, which is topologically equivalent to the given one. Denote $|\mathbf{x}|_{\mathcal{H}}^2 = [\mathbf{x},\mathbf{x}]_{\mathcal{H}}$.

Theorem 2.2. *Let the assumptions (A) and (B) hold and for some $k\in(0,\beta)$ and $m\in(0,1]$*

$$\omega_1 = \inf_{x\in H_1, x\neq 0} \frac{\frac{1}{k}(D_1x,x)_{-1,1} - \|x\|^2 - \frac{1}{4m}\big\|(\frac{1}{k}\tilde{S}-D_2)x\big\|_{-1}}{\|x\|^2} \geq 0. \tag{2.3}$$

Then the operator $\mathbf{A}$ is a generator of a C_0-semigroup $\mathcal{T}(t)=\exp\{t\mathbf{A}\}$ $(t\geq 0)$ and

$$\big\|\mathcal{T}(t)\big\|_{\mathcal{H}} \leq \text{const}\cdot\exp(-tk\theta)$$

where

$$\theta = \min\left\{\frac{\omega_1}{2}, \frac{1-m}{\omega_2}\right\} \geq 0$$

and[2]

$$\omega_2 = \sup_{x\in H_1, x\neq 0} \frac{\|x\|_1^2 + k(D_1x,x)_{-1,1} + k^2\|x\|^2}{\|x\|_1^2} \tag{2.4}$$

Proof. For $\mathbf{x}=(x_1,x_2)^\top \in \mathcal{D}(\mathbf{A})$ let us consider a quadric form

$$\begin{aligned}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} = {} & (T^{1/2}x_1, T^{1/2}x_2) + k(D_1x_1,x_2)_{-1,1} - k^2(x_1,x_2) + \\ & (-Dx_1 - \tilde{A}x_2 + kx_1, x_1 + kx_2) = \\ & (Tx_1,x_2)_{-1,1} + k(D_1x_1,x_2)_{-1,1} - (Dx_1,x_1)_{-1,1} \\ & - (\tilde{A}x_2,x_1)_{-1,1} + k(x_1,x_1) - k(Dx_1,x_2)_{-1,1} - k(\tilde{A}x_2,x_2)_{-1,1} = \\ & -(Dx_1,x_1)_{-1,1} + k(x_1,x_1) - k(\tilde{A}x_2,x_2)_{-1,1} - ik(D_2x_1,x_2)_{-1,1} + \\ & (Tx_1,x_2)_{-1,1} - (Tx_2,x_1)_{-1,1} - i(\tilde{S}x_2,x_1)_{-1,1}\end{aligned}$$

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[1] $\|D_1\|$ is a norm of operator $D_1\in\mathcal{L}(H_1,H_{-1})$, i.e. $\|D_1\| = \sup_{x\in H_1,x\neq 0}\|D_1x\|_{-1}/\|x\|_1$

[2] Obviously, $\omega_2 \leq 1 + k\|D_1\| + k^2/a_0$

We used decompositions $\tilde{A} = T + i\tilde{S}$ and $D = D_1 + iD_2$. Consequently,

$$\begin{aligned}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} = -(D_1x_1,x_1)_{-1,1} + k(x_1,x_1) - k(Tx_2,x_2)_{-1,1} - \\ \operatorname{Re}\Big(ik(D_2x_1,x_2)_{-1,1} + i(\tilde{S}x_2,x_1)_{-1,1}\Big) = \\ -(D_1x_1,x_1)_{-1,1} + k\|x_1\|^2 - k\|x_2\|_1^2 - \\ \operatorname{Im}\Big((\tilde{S}x_1,x_2)_{-1,1} - k(D_2x_1,x_2)_{-1,1}\Big)\end{aligned}$$

and

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \operatorname{Im}\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1,x_2\right)_{-1,1}.$$

Since

$$\begin{aligned}\left|\left(\left(\frac{1}{k}\tilde{S} - D_2\right)x_1,x_2\right)_{-1,1}\right| \le \left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1} \cdot \|x_2\|_1 \le \\ \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + m\|x_2\|_1^2,\end{aligned}$$

then

$$\begin{aligned}-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \ge \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 - \frac{1}{4m}\left\|\left(\frac{1}{k}\tilde{S} - D_2\right)x_1\right\|_{-1}^2 + \\ (1-m)\|x_2\|_1^2 \ge \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2.\end{aligned}$$

Further, an inequality

$$2k|\operatorname{Re}(x_1,x_2)| \le 2|(x_1,kx_2)| \le 2\|x_1\|\cdot\|kx_2\| \le \|x_1\|^2 + k^2\|x_2\|^2$$

implies

$$[\mathbf{x},\mathbf{x}]_{\mathcal{H}} \le 2\|x_1\|^2 + \|x_2\|_1^2 + k(D_1x_2,x_2)_{-1,1} + k^2\|x_2\|^2 \le 2\|x_1\|^2 + \omega_2\|x_2\|_1^2. \tag{2.5}$$

Thus

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \ge \omega_1\|x_1\|^2 + (1-m)\|x_2\|_1^2 \ge \theta(2\|x_1\|^2 + \omega_2\|x_2\|_1^2) \ge \theta[\mathbf{x},\mathbf{x}]_{\mathcal{H}}$$

and an operator $(-\mathbf{A} - k\theta I)$ is accretive. Moreover, the operator $(-\mathbf{A} - k\theta I)$ is m-accretive (since $0 \in \rho(\mathbf{A})$) and[3]

$$\rho(-\mathbf{A} - k\theta I) \subset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0\} \Rightarrow \rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < k\theta\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$, $t \ge 0$ and

$$\big|\mathcal{T}(t)\big|_{\mathcal{H}} \le \exp(-k\theta t), \quad t \ge 0.$$

On the space $\mathcal{H}$ norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent and the inequality

$$\big\|\mathcal{T}(t)\big\|_{\mathcal{H}} \le \operatorname{const}\cdot\exp(-k\theta t), \quad t \ge 0$$

holds for some positive constant. □

[3]Obviously, the operator $(-\mathbf{A})$ is m-accretive as well.

Corollary 2.3. *Under the conditions of the theorem 2.2 for all* $\mathbf{x}_0 = (u_1, u_0)^\top \in \mathcal{D}(\mathbf{A})$ *vector-function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first order differential equation (2.2)*.* $u(t)$ *satisfies the second-order differential equation* (2.1) *with the initial conditions* (0.2) *and an inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \le \text{const} \cdot \exp\{-2k\theta t\}\Big(\|u_0\|_1^2 + \|u_1\|^2\Big)$$

holds for all $t \ge 0$*.*

Consider now a more strong assumption on the operator D:

(C) $D \in \mathcal{L}(H_1, H_{-1})$ and

$$\delta = \inf_{x \in H_1, x \neq 0} \frac{\operatorname{Re}(Dx, x)_{-1,1}}{\|x\|_1^2} > 0.$$

It is easy to show that the assumption (C) implies (B) and $\beta > a_0\delta$.

By $\|\tilde{S}\|$ and $\|D_2\|$ denote norms of the bounded operators $\tilde{S} \in \mathcal{L}(H_1, H_{-1})$ and $D_2 \in \mathcal{L}(H_1, H_{-1})$. Then for all $x \in H_1$

$$\|\tilde{S}x\|_{-1} \le \|\tilde{S}\| \cdot \|x\|_1, \quad \|D_2 x\|_{-1} \le \|D_2\| \cdot \|x\|_1$$

Theorem 2.4. *Let the assumptions (A) and (C) are fulfilled and for some* $k \in (0, \beta)$ *and some* $p, q > 0$ *with* $p + q \le 1$

$$\omega_1' = a_0 \left(\frac{\delta}{k} - \frac{1}{4pk^2}\|\tilde{S}\|^2 - \frac{1}{4q}\|D_2\|^2 \right) \ge 1$$

Then the operator $\mathbf{A}$ *is a generator of a* C_0*-semigroup* $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \ge 0)$ *and*

$$\big\|\mathcal{T}(t)\big\|_{\mathcal{H}} \le \text{const} \cdot \exp(-tk\theta')$$

where

$$\theta' = \min\left\{ \frac{\omega_1' - 1}{2}, \frac{1 - p - q}{\omega_2} \right\} \ge 0$$

and ω_2 *is defined by* (2.4)*.*

Proof. Consider on Hilbert space $\mathcal{H} = H \times H_1$ the scalar product $[\mathbf{x}, \mathbf{y}]_{\mathcal{H}}$. Then

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x}, \mathbf{x}]_{\mathcal{H}} = \frac{1}{k}(D_1 x_1, x_1)_{-1,1} - \|x_1\|^2 + \|x_2\|_1^2 + \frac{1}{k}\operatorname{Im}(\tilde{S}x_1, x_2)_{-1,1} - \operatorname{Im}(D_2 x_1, x_2)_{-1,1}$$

(see the proof of the theorem 2.2). Since

$$|\operatorname{Im}(D_2 x_1, x_2)_{-1,1}| \le |(D_2 x_1, x_2)_{-1,1}| \le \|D_2 x_1\|_{-1} \cdot \|x_2\|_1 \le \frac{1}{4q}\|D_2 x_1\|_{-1}^2 + q\|x_2\|_1^2 \le \frac{1}{4q}\|D_2\|^2 \cdot \|x_1\|_1^2 + q\|x_2\|_1^2$$

$$\frac{1}{k}|\operatorname{Im}(\tilde{S}x_1, x_2)_{-1,1}| \le |(\frac{1}{k}\tilde{S}x_1, x_2)_{-1,1}| \le \left\|\frac{1}{k}\tilde{S}x_1\right\|_{-1} \cdot \|x_2\|_1 \le \frac{1}{4p}\left\|\frac{1}{k}\tilde{S}x_1\right\|_{-1}^2 + p\|x_2\|_1^2 \le \frac{1}{4pk^2}\|\tilde{S}\|^2 \cdot \|x_1\|_1^2 + p\|x_2\|_1^2$$

and taking into account $(D_1x,x)_{-1,1} \geq \delta\|x\|_1^2$ and $\|x\|_1^2 \geq a_0\|x\|^2$ we obtain

$$
\begin{aligned}
-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \geq \frac{1}{k}(D_1x_1,x_1)_{-1,1} - \|x_1\|^2 - \\
\frac{\|\tilde{S}\|^2}{4pk^2}\cdot\|x_1\|_1^2 - \frac{\|D_2\|^2}{4q}\cdot\|x_1\|_1^2 + (1-p-q)\|x_2\|_1^2 \geq \\
\left(\frac{\delta}{k} - \frac{\|\tilde{S}\|^2}{4pk^2} - \frac{\|D_2\|^2}{4q}\right)\|x_1\|_1^2 - \|x_1\|^2 + (1-p-q)\|x_2\|_1^2 \geq \\
(\omega_1'-1)\|x_1\|^2 + (1-p-q)\|x_2\|_1^2.
\end{aligned}
$$

Using (2.5) we finally have

$$-\frac{1}{k}\operatorname{Re}[\mathbf{A}\mathbf{x},\mathbf{x}]_{\mathcal{H}} \geq \theta'[\mathbf{x},\mathbf{x}]_{\mathcal{H}}.$$

Thus an operator $(-\mathbf{A} - k\theta' I)$ in m-accretive (since $0 \in \rho(\mathbf{A})$) and

$$\rho(-\mathbf{A}) \supset \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda < k\theta'\}.$$

Therefore, the operator $\mathbf{A}$ is a generator of a C_0-semigroup [4, 5] $\mathcal{T}(t) = \exp\{t\mathbf{A}\}$ $(t \geq 0)$ and

$$\big|\mathcal{T}(t)\big|_{\mathcal{H}} \leq \exp(-k\theta' t), \quad t \geq 0.$$

Since the norms $|\mathbf{x}|_{\mathcal{H}}$ and $\|\mathbf{x}\|_{\mathcal{H}}$ are equivalent then we have an inequality

$$\big\|\mathcal{T}(t)\big\|_{\mathcal{H}} \leq \mathrm{const}\cdot\exp(-k\theta' t), \quad t \geq 0$$

for some positive constant. □

Corollary 2.5. *Under the conditions of the theorem 2.4 for all* $\mathbf{x}_0 = (u_1,u_0)^\top \in \mathcal{D}(\mathbf{A})$ *a vector-valued function*

$$\mathbf{x}(t) = \begin{pmatrix} w(t) \\ u(t) \end{pmatrix} = \mathcal{T}(t)\mathbf{x}_0 \in \mathcal{D}(\mathbf{A})$$

satisfies the first order differential equation (2.2). $u(t)$ *satisfies the second-order differential equation* (2.1) *with an initial conditions* (0.2) *and the inequality*

$$\|u(t)\|_1^2 + \|u'(t)\|^2 \leq \mathrm{const}\cdot\exp\{-2k\theta' t\}\Big(\|u_0\|_1^2 + \|u_1\|^2\Big)$$

holds for all $t \geq 0$.

3. Related spectral problem

Let us consider a quadric pencil associated with the differential equation (0.1)

$$L(\lambda) = \lambda^2 I + \lambda D + A \quad \lambda \in \mathbb{C}.$$

Since $D : H_1 \to H_{-1}$ it is more naturally to consider an extension of pencil

$$\tilde{L}(\lambda) = \lambda^2 I + \lambda D + \tilde{A}$$

mapping H_1 to H_{-1}. Moreover, $\tilde{L}(\lambda) \in \mathcal{L}(H_1, H_{-1})$ for all $\lambda \in \mathbb{C}$.

Definition 3.1. The resolvent set of the pencil $\tilde{L}(\lambda)$ is defined as

$$\rho(\tilde{L}) = \{\lambda \in \mathbb{C}\ :\ \exists \tilde{L}^{-1}(\lambda) \in \mathcal{L}(H_{-1}, H_1)\}$$

The spectrum of the pencil is $\sigma(\tilde{L}) = \mathbb{C}\backslash\rho(\tilde{L})$.

In [7, 16] it was proved that $\sigma(\tilde{L}) = \sigma(\mathbf{A})$ and for $\lambda \neq 0$

$$(\mathbf{A} - \lambda I)^{-1} = \begin{pmatrix} \lambda^{-1}\left(\tilde{L}^{-1}(\lambda)\tilde{A} - I\right) & -\tilde{L}^{-1}(\lambda) \\ \tilde{L}^{-1}(\lambda)\tilde{A} & -\lambda\tilde{L}^{-1}(\lambda) \end{pmatrix}$$

This result allows to obtain a localization of the pencil's spectrum in a half-plane.

Proposition 3.2. ***1.*** *Under the conditions of the theorem 2.2 the spectrum of the pencil $\tilde{L}(\lambda)$ belongs to a half-plane*

$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta\}.$$

2. *Under the conditions of the theorem 2.4 the spectrum of the pencil $\tilde{L}(\lambda)$ belongs to a half-plane*

$$\sigma(\tilde{L}) \subseteq \{\mathrm{Re} \leq -k\theta'\}.$$

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