OBSTRUCTIONS TO POSITIVE CURVATURE

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There are few known examples of manifolds with positive sectional curvature in Riemannian geometry. Until recently, they were all homogeneous spaces [Be, Wa, AW] and biquotients [E1, E2, Ba], i.e., quotients of compact Lie groups G, equipped with a biinvariant metric, by a free isometric "two sided" action of a subgroup $H \subset G \times G$. See [Zi1] for a survey of the known examples. New methods for constructing examples with positive curvature have been proposed in [PW] on the Gromoll-Meyer exotic 7-sphere and in [GVZ] on a 7-manifold homeomorphic but not diffeomorphic to $\mathrm{T}_1\mathrm{S}^4$, see also [De] for a different approach.

The example in [GVZ] arose from a systematic study of *cohomogeneity one* manifolds, i.e., manifolds with an isometric action whose orbit space is one dimensional, or equivalently the principal orbits have codimension one. A classification of positively curved cohomogeneity one manifolds was carried out in even dimensions in [V1, V2] and in odd dimensions an exhaustive description was given in [GWZ] of all simply connected cohomogeneity one manifolds that can possibly support an invariant metric with positive curvature. In addition to some of the known examples of positive curvature which admit isometric cohomogeneity one actions, two infinite families, P_k^7, Q_k^7 and one exceptional manifold R^7, all of dimension seven, appeared as the only possible new candidates, See Section 3 for a more detailed description. They all support an almost effective action by $\mathrm{S}^3 \times \mathrm{S}^3$ with finite principal isotropy group. Here P_1^7 is the 7-sphere and Q_1^7 is the normal homogeneous positively curved Aloff-Wallach space. The manifold P_2^7 is the new example of positive curvature in [GVZ].

The exceptional manifold R^7 is very similar to the family Q_k, as far as the group action is concerned and the topological properties of the manifold itself. There is also such a companion to the P_k^7 family, which is the 7-dimensional Berger space with a natural cohomogeneity one action which is quite similar to those for P_k^7. Since the Berger space has an invariant metric with positive curvature, it was expected that R^7 does so as well. We will show though that this is not the case.

Theorem A. *The manifold R^7 does not carry a cohomogeneity one metric with positive sectional curvature invariant under* $\mathrm{S}^3 \times \mathrm{S}^3$.

The proof relies on a new convexity property of Jacobi fields in non-negative curvature that holds without the presence of a group action. To describe it, consider a geodesic c on a Riemannian manifold M^{n+1}. An n-dimensional family of Jacobi fields V is called self adjoint if $\langle X', Y\rangle = \langle X, Y'\rangle$ for all $X, Y \in V$. A point $c(t_0)$ is called regular if $\operatorname{span}\{X(t_0) \mid X \in V\}$ is n-dimensional. One associates to V a Jacobi operator $J_t\colon T_{c(t_0)} \to T_{c(t)}$ with $J_t(u) = X(t)$ where $X \in V$ with $X(t_0) = u$. If V is self adjoint, then $J' = S \circ J$ with S self adjoint. At regular points S satisfies the Riccati equation $S' + S^2 + R = 0$ where R is the curvature along the geodesic. We study a new self adjoint operator $L_t := J_t^* \circ J_t\colon T_{c(t_0)} \to T_{c(t_0)}$ where J^* is the adjoint. We show that at a regular point it satisfies the differential equation

$$J(L^{-1})''J^* = 6S^2 + 2R.$$

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Thus, if $R \geq 0$ (respectively $R > 0$), then $\langle L_t^{-1} u, u\rangle$ is a convex (resp. strictly convex) function along c for any $u \in T_{c(t_0)}$. This gives rise to obstructions and rigidity properties. For example:

Theorem B. *Let M^{n+1} be a manifold with non-negative sectional curvature and V a self adjoint family of Jacobi fields along the geodesic $c\colon [t_1, t_2] \to M$. Assume there exists an $X \in V$ such that*

(a) *$X(t_i) \neq 0$ and $|X|'(t_i) = 0$ for $i = 1, 2$,*
(b) *There exists a basis $X, Y_2, \dots Y_n$ of V such that $\langle X, Y_k\rangle_{c(t_i)} = 0$ for $i = 1, 2$; $k = 2, \dots n$,*
(c) *If $Y \in V$ with $Y(t) = 0$ for some $t \in (t_1, t_2)$, then $\langle X, Y\rangle_{c(t_i)} = 0$ for $i = 1, 2$,*

Then X is a parallel Jacobi field along c.

In particular, we allow interior points of the geodesic to be singular. Notice that all 3 conditions are necessary for X to be parallel since in a self adjoint family of Jacobi fields, $\langle X, Y\rangle' = \langle X, Y'\rangle = \langle X', Y\rangle = 0$ for all $X, Y \in V$ with X parallel. Furthermore, (a) and (c) alone are not sufficient since there are Jacobi fields of constant length which are not parallel. If there are no interior singular points, (b) is the only global condition and relates the Jacobi fields at t_1 and t_2 since it can be equivalently formulated as follows: If $Y \in V$ with $\langle X, Y\rangle_{c(t_1)} = 0$ then $\langle X, Y\rangle_{c(t_2)} = 0$ and vice versa.

The differential equation is reminiscent of the transverse Jacobi field equation due to B.Wilking [Wi], although they do not seem to be directly related. One can view Theorem B as a local version of a rigidity Theorem in non-negative curvature in [Wi], which holds along an infinite geodesic in M. The convexity property is also reminiscent of another observation by B.Wilking. It says that in the presence of a cohomogeneity one group action, the inverse of the homogeneous metric along the orbits is convex.

In positive curvature we can view Theorem B as an obstruction. This is what we will use to prove Theorem A. For a cohomogeneity one manifold there is a natural family V of self adjoint Jacobi fields coming from the group action, which we will consider along a minimal geodesic between the two singular orbits. Condition (b) will follow by studying the metric tangent to the singular orbits and equivariance properties of their second fundamental form will be used to satisfy (a). It turns out that for R^7 there exists a unique vector $X \in V$ with the properties required in Theorem B.

The differential equation and its applications also hold if we consider Jacobi fields only in a subbundle invariant under parallel translation. This arises frequently in the presence of an isometric group action. For example, if the group action is polar, it gives rise to a self adjoint family of Jacobi fields in the parallel subbundle orthogonal to the section. Thus the analogue of Theorem B holds here as well.

In Section 1 we recall properties of the Riccati equation and derive the differential equation for L and its implication for convexity properties. In Section 2 we discuss rigidity and prove Theorem B. In Section 3 we describe the geometry of R^7 and prove Theorem A.

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1. Convexity

In this section we present a new convexity result about Jacobi fields, and first recall some standard notation, see e.g. [E3],[EH].

Let c be a geodesic in a Riemannian manifold M^{n+1} defined on an interval $t_1 \leq t \leq t_2$ and $\dot{c}^\perp$ the orthogonal complement of $\dot{c}(t) \subset T_{c(t)}M$. We identify $\dot{c}^\perp$ via parallel translation with a fixed

n-dimensional vector space $E := \dot c(t_0)^\perp \subset T_{c(t_0)}M$. Thus a vector field X along c, orthogonal to $\dot c$, becomes, via parallel translation, a curve in E and covariant derivative $\nabla_{\dot c}X$ becomes ordinary derivative X' in E.

Let V be an n-dimensional vector space of Jacobi fields along c orthogonal to $\dot c$. Along the geodesic we have that $\langle X', Y\rangle - \langle X, Y'\rangle$ is constant for any $X, Y \in V$. If this constant is 0, V is called *self adjoint*, i.e.

$$\langle X', Y\rangle = \langle X, Y'\rangle, \quad \text{for all } X, Y \in V \tag{1.1}$$

We call t regular if $X(t),\ X \in V$ span E and singular otherwise. The singular points are isolated and we will assume from now on that t_0 is regular. We can then equivalently describe the set of Jacobi fields V by a (smooth) family of linear maps

$$J_t\colon E \to E \quad \text{where} \quad J_t(u) = X(t) \text{ for } X \in V \text{ with } X(t_0) = u \tag{1.2}$$

and thus t is regular if and only if J_t is invertible. At regular points t one defines the Riccati operator:

$$S_t\colon E \to E \quad \text{where} \quad S_t(u) = X'(t) \text{ for } X \in V \text{ with } X(t) = u, \ \text{ i.e. } J_t' = S_t J_t \tag{1.3}$$

Thus condition 1.1 is equivalent to S_t being self adjoint. J_t satisfies the Riccati equation, which reduces the differential equation for Jacobi fields into two uncoupled first order equations:

$$A'' + RA = 0 \quad \text{if and only if} \quad S' + S^2 + R = 0 \quad \text{and} \quad A' = SA \tag{1.4}$$

where $R = R_t\colon E \to E$ is the parallel translate of the self adjoint curvature operator $R(\,\cdot\,, \dot c), \dot c :$ $\dot c(t)^\perp \to \dot c(t)^\perp$. We call any solution A_t of (1.4) with S self adjoint a *self adjoint Jacobi tensor*. Notice that if A_t is self adjoint, then $A_t \circ F$, for any fixed linear isomorphism F, is also a self adjoint Jacobi tensor, in fact with the same tensor S.

The solutions $A_t = J_t$, defined in terms of a fixed V as above, are special since $A(t_0) = \mathrm{Id}$. Notice that if we choose a different regular point t_0^*, then we obtain another self adjoint Jacobi tensor A^* with $A^*(t_0^*) = \mathrm{Id}$ but they just differ by a linear isomorphism $F : E \to E$. Indeed, define F by $Fu = J(t_0)$ where $J \in V$ with $J(t_0^*) = u$. Then (1.2) implies that $A_t^*(u) = J(t) = A_t(F(u))$, i.e. $A^* = A \circ F$.

From now on let A be a self adjoint Jacobi tensor. Thus for any $v \in E$, $A_t v$ is a Jacobi field, t is regular if and only if A_t is invertible, and singular points are isolated. Furthermore, A_t is self adjoint if and only if

$$\langle A_t' v, A_t w\rangle = \langle A_t v, A_t' w\rangle \text{ for all } t \text{ and } v, w \in E. \tag{1.5}$$

When clear from context we simply use $A = A_t,\ S = S_t$.

We will study a new self adjoint operator, defined for all t, by:

$$L_t = A_t^* A_t\colon E \to E \quad \text{where} \quad \langle A_t^* v, w\rangle = \langle v, A_t w\rangle \quad \text{for all } v, w \in E \tag{1.6}$$

Thus we have

$$\langle Lv, w\rangle = \langle Av, Aw\rangle, \quad \text{for all } v, w \in E. \tag{1.7}$$

and hence at a regular point L determines A up to an orthogonal transformation F.

Remark. In the case where A_t arises from a family of Jacobi fields V as in (1.2), we can view $\langle Lv, w\rangle$ as the metric tensor along c in the following sense. Regard, via parallel translation, $A_t\colon \dot c(t_0)^\perp \to \dot c(t)^\perp$ so that $A_t v$ is a Jacobi field along c with $A_0 v = v$. Then (1.7) says that at a regular point $\langle L_t v, w\rangle$ is the pullback of the metric at $c(t)$ to $c(t_0)$ via the linear isomorphism A_t.

If we change A by a linear isomorphism F to $\bar A = A \circ F$ (e.g. choosing a different initial point t_0), we obtain a new operator $\bar L$ such that $\bar L = F^* L F$ and hence $\langle \bar L u, u\rangle = \langle L(Fu), Fu\rangle$. Since F is an isomorphism, properties of L are shared by properties of $\bar L$.

Our main tool is the following differential equation for L^{-1}:

PROPOSITION 1.8. *Let A be a self adjoint Jacobi tensor and $S = A'A^{-1}$, $L = A^*\, A$. Then at a regular point we have*
(a) $A(L^{-1})'A^* = -2S$.
(b) $A(L^{-1})''A^* = 6S^2 + 2R$.

Proof. For part (a) we observe that $A^{*\prime} = A'^*$ and hence

$$\begin{aligned}(L^{-1})' &= (A^{-1}(A^*)^{-1})' = (A^{-1})'(A^*)^{-1} + A^{-1}((A^*)^{-1})' \\ &= (-A^{-1}A'A^{-1})(A^*)^{-1} + A^{-1}(-(A^*)^{-1}A'^*(A^*)^{-1}) \\ &= -A^{-1}S(A^*)^{-1} - A^{-1}(A'A^{-1})^*(A^*)^{-1} \\ &= -2A^{-1}S(A^*)^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}(L^{-1})'' &= -2(A^{-1}S(A^*)^{-1})' \\ &= -2(-A^{-1}A'A^{-1})S(A^*)^{-1} - 2A^{-1}S'(A^*)^{-1} - 2A^{-1}S(-A^{-1}A'A^{-1})^* \\ &= 2A^{-1}S^2(A^*)^{-1} - 2A^{-1}(-S^2 - R)(A^*)^{-1} + 2A^{-1}S(A^{-1}S)^* \\ &= 6A^{-1}S^2(A^*)^{-1} + 2A^{-1}R(A^*)^{-1}.\end{aligned}$$

□

This implies convexity properties in non-negative curvature.

COROLLARY 1.9. *Let A be a self adjoint Jacobi tensor. If $R \geq 0$ (resp. $R > 0$), then at a regular point we have*
(a) *L^{-1} is convex, i.e. $(L^{-1})''$ is positive semi-definite (resp. positive definite).*
(b) *For any $v \in E$, $F(t) = \langle L_t^{-1}v, v\rangle$ is a convex (resp. strictly convex) function.*

Proof. Since $S^2 \geq 0$, Proposition 1.8 (b) implies that $\langle A(L^{-1})''A^*v, v\rangle = \langle (L^{-1})''A^*v, A^*v\rangle \geq 0$ for all v. At a regular point A has no kernel and hence A^* is onto. This implies (a), and (a) clearly implies (b). □

Notice on the other hand that in general $\langle Lv, v\rangle$ is neither convex not concave since $\langle Lv, v\rangle'' = 2|A'v|^2 - 2\langle R(Av), Av\rangle$.

One of the issues is differentiability of L^{-1} at singular points. For this we have

PROPOSITION 1.10. *If t^* is a singular point, then $F(t) = \langle L_t^{-1}v, v\rangle$ has a smooth extension at $t = t^*$ if and only if v is orthogonal to* $\ker L_{t^*}$.

Proof. For simplicity assume $t^* = 0$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of E such that $\{e_{k+1}, \ldots, e_n\}$ is a basis of $\ker L_0$ and let

$$g_{ij}(t) = \langle L_t e_i, e_j\rangle = \langle A_t e_i, A_t e_j\rangle$$

be the matrix of L with respect to the basis e_i, defined on an interval $I = (-\epsilon, \epsilon)$ in which 0 is the only singular point. The power series expansion of g_{ij} at $t = 0$ is given by

$$g_{ij}(t) = \langle Ae_i, Ae_j\rangle + 2\langle Ae_i, A'e_j\rangle t + (\langle Ae_i, A''e_j\rangle + \langle A'e_i, A'e_j\rangle)t^2 + o(t^2).$$

For $t = 0$ and $i > k$ we have $Ae_i = 0$ and hence

$$g_{ij} =< A'e_i, A'e_j > t^2 + o(t^2) \text{ for } i > k \text{ or } j > k$$

Thus, close to $t = 0$, we may represent $G = (g_{ij})$ as a block matrix

$$G \simeq \begin{pmatrix} X & t^2 Y \\ t^2 Y & t^2 Z \end{pmatrix}$$

Here X is non-singular at $t = 0$ since otherwise there exists a linear combination $e = \sum_{i=1}^{i=k} e_i$ such that $0 = Xe = \sum_i \langle Ae_i, Ae_j\rangle a_i e_i = \langle Ae, Ae_j\rangle = \langle Le, e_j\rangle$ for all $j = 1, \dots, k$ and hence $e \in \ker L_0$ which implies $e = 0$. Similarly, Z is non-singular since otherwise there exists a linear combination $e = \sum_{i=k+1}^{i=n} e_i$ such that $0 = Ze = \frac{1}{t^2}\langle Le, e_j\rangle$ for all $j = k+1, \dots, n$ and $t \neq 0$ and hence $e \in (\ker L_0)^\perp$ which again implies $e = 0$. There is a general formula for the inverse of a block matrix, which implies that for $t \neq 0$ we have

$$G^{-1} \simeq \begin{pmatrix} (X - t^2 Y Z^{-1} Y)^{-1} & -X^{-1} Y (Z - t^2 Y X^{-1} Y)^{-1} \\ -X^{-1} Y (Z - t^2 Y X^{-1} Y)^{-1} & \frac{1}{t^2}(Z - t^2 Y X^{-1} Y)^{-1} \end{pmatrix}.$$

Thus $\langle L_t^{-1} v, v\rangle = G^{-1} v \cdot v$ is smooth at $t = 0$ if and only if v is a linear combination of $e_1, \dots, e_k$. □

2. Rigidity

We now use the results in the previous section to prove the existence of parallel Jacobi fields in non-negative curvature, i.e. vectors $w \in E$ with $A_t' w = 0$. We allow endpoints and interior points of the geodesic to be singular.

Proposition 2.1. *Let A be a self adjoint Jacobi tensor along the geodesic $c\colon [t_1, t_2] \to M$ with $L = A^* A$. If $R \geq 0$ and if there exists a non-zero vector $v \in E$ such that*

(a) *$\langle (L^{-1})'_{t_i} v, v\rangle = 0$ for $i = 1, 2$,*
(b) *v is orthogonal to $\ker L_t$ for all $t_1 \leq t \leq t_2$,*

then $w := L_t^{-1} v$ is constant and $A_t' w = 0$ for all t.

Proof. Let $F(t) = \langle L_t^{-1} v, v\rangle$. By Proposition 1.10, assumption (b) implies that F is smooth for all $t_1 \leq t \leq t_2$. Assumption (a) says that $F'(t_i) = 0$ and by Corollary 1.9, F is convex and hence constant.

Let I be a connected component of the regular set of $[t_1, t_2]$. Since F is constant $\langle (L^{-1})'' v, v\rangle = 0$ and by convexity we have, for all ϵ, w:

$$0 \leq \langle (L^{-1})''(v + \epsilon w), v + \epsilon w\rangle = 2\epsilon \langle (L^{-1})'' v, w\rangle + \epsilon^2 \langle (L^{-1})'' w, w\rangle \text{ for all } w \in E.$$

and thus

$$(L^{-1})'' v = 0.$$

Using (1.8) (b) this implies $A^{-1} S^2 (A^*)^{-1} v = 0$ in I and hence

$$0 = \langle A^{-1} S^2 (A^*)^{-1} v, v\rangle = \langle S^2 (A^*)^{-1} v, (A^*)^{-1} v\rangle = \langle S(A^*)^{-1} v, S(A^*)^{-1} v\rangle$$

and thus

$$S(A^*)^{-1} v = 0.$$

Using (1.8) (a) we conclude that

$$(L^{-1})' v = -2 A^{-1} S (A^*)^{-1} v = 0$$

so that $(L^{-1}) v$ is constant in I. Taking the limit at boundary points of I we can conclude that $L^{-1} v$ is constant everywhere and we set $w = L^{-1} v$. Thus, for any $z \in E$,

$$\langle Aw, Az\rangle = \langle Lw, z\rangle = \langle v, z\rangle$$

does not depend on t. Since the Jacobi tensor is self-adjoint, we have

$$0 = \langle Aw, Az\rangle' = 2\langle A'w, Az\rangle.$$

The points in I consist of regular points and hence the image of A spans E which implies $A_t'w = 0$ for $t \in I$. Taking limits again at the boundary points of I finishes the proof. □

If one assumes that v is an eigenvector of L at the endpoints, one obtains a simple formulation in terms of the Jacobi tensor A:

Corollary 2.2. *Let M^{n+1} be a manifold with non-negative sectional curvature and A a self adjoint Jacobi tensor along the geodesic $c\colon [t_1, t_2] \to M$. If there exists a vector $v \in E$ such that*

(a) *$A_t v \neq 0$ and $|A_t v|' = 0$ for $t = t_i$,*
(b) *$A_t(v^\perp) \subset (A_t v)^\perp$, for $t = t_i$,*
(c) *v is orthogonal to* $\ker A_t$ *for all $t_1 < t < t_2$,*

then $A_t' v = 0$ for all t.

Proof. First observe that $\ker L_t = \ker A_t$ since $L_t z = 0$ is equivalent to $\langle Lz, x\rangle = \langle Az, Ax\rangle$ for all $x \in E$. Assumption (b) says that if $\langle v, z\rangle = 0$ then $\langle Av, Az\rangle = \langle Lv, z\rangle = 0$ and hence $L_{t_i} v = \alpha_i v$ and $\alpha_i \neq 0$ since $A_{t_i} v \neq 0$. Since L is self adjoint, v is orthogonal to $\ker A_{t_i}$. Together with assumption (c) this implies that $\langle L_t^{-1} v, v\rangle$ is differentiable for all $t \in [t_1, t_2]$.

For $t = t_i$ we have

$$\langle (L^{-1})' v, v\rangle = -\langle L^{-1} L' L^{-1} v, v\rangle = -\langle L' L^{-1} v, L^{-1} v\rangle = -\frac{1}{\alpha_i^2}\langle L' v, v\rangle.$$

Furthermore,

$$\langle L' v, v\rangle = \langle (A^{*\prime} A + A^* A')v, v\rangle = \langle Av, A'v\rangle + \langle A'v, Av\rangle = 2\langle A'v, Av\rangle = (|Av|^2)' = 0.$$

Thus $\langle (L^{-1})'_{t_i} v, v\rangle = 0$ and the claim follows from Proposition 2.1. □

We are now ready for Theorem B.

Proof of Theorem B. Recall that V defines a Jacobi tensor J_t after we choose a regular point t_0. Let $v \in E$ be such that $X(t_0) = v$ and thus $X(t) = J_t v$. Then (b) implies

$$\text{given } w \in E, \quad \langle J_{t_1}(v), J_{t_1}(w)\rangle = 0 \iff \langle J_{t_2}(v), J_{t_2}(w)\rangle = 0.$$

Since $J_{t_i}(v) \neq 0$ there exists a $(n-1)$-dimensional subspace W of E such that

$$\langle J_{t_i}(v), J_{t_i}(w)\rangle = 0 \iff w \in W.$$

By (c), if z is such that $J_t(z) = 0$ for some t in (t_1, t_2), then $z \in W$ i.e. the span of the kernels of J_t is a subspace of W. Let $F : E \to E$ be a linear isomorphism such that $F(v) = v$ and $F(v^\perp) = W$. Then $A_t = J_t \cdot F$ is a self-adjoint Jacobi operator and we claim that the assumptions of Corollary 2.2 are satisfied. (a) is clear since F does not depend on t. (b) is satisfied thanks to the definition of F. Indeed, if $\langle v, z\rangle = 0$, then $F(z) \in W$ and thus $\langle A_{t_i} v, A_{t_i} z\rangle = \langle J_{t_i} v, J_{t_i}(Fz)\rangle = 0$ which says that $A_{t_i}(v^\perp) \subset (A_{t_i} v)^\perp$. Furthermore, since $\ker(A_t) = F^{-1}(\ker J_t)$ and $\ker(J_t) \subset W$, we have $\ker(A_t) \subset v^\perp$ and (c) follows. □

Theorem B can be viewed as a local version of a result by B.Wilking [Wi], which states:

Proposition 2.3 (Wilking). *Let M be a manifold with non-negative sectional curvature and $c\colon (-\infty, \infty) \to M$ a geodesic in M. If V a self adjoint family of Jacobi fields along c, one has an orthogonal splitting*

$$E = \operatorname{span}\{J \in V \mid J(t) = 0 \textit{ for some } t \in \mathbb{R}\} \oplus \{J \in V \mid J \textit{ is parallel for all } t \in \mathbb{R}\}$$

Well known self adjoint Jacobi operators are the ones related to the two Rauch comparison theorems. For the first one, $A(0) = 0,\ A'(0) = \mathrm{Id}$, and for the second one $A(0) = \mathrm{Id},\ A'(0) = 0$. In the first case, one can choose a regular point $t_1 = \epsilon$ and let $\epsilon \to 0$. In both cases, Corollary B does not seem to follow from standard Jacobi field estimates since c can have interior conjugate or focal points.

The results in this Section have obvious generalizations if there exists a subbundle $W \subset \dot{c}^{\perp}$ invariant under parallel translation. This arises naturally in the presence of group actions, for example polar manifolds.

3. Proof of Theorem A

We now use the results in the previous section to prove that the manifold R cannot have an invariant metric with positive curvature. Since there cannot be any parallel Jacobi fields, we use Theorem B to obtain a contradiction. We start by describing the manifold R more explicitly.

A cohomogeneity one manifold is the union of two homogeneous disc bundles. Given compact Lie groups H, K^-, K^+ and G with inclusions $H \subset K^{\pm} \subset G$ satisfying $K^{\pm}/H = \mathbb{S}^{\ell_{\pm}}$, the transitive action of $K^{\pm}$ on $\mathbb{S}^{\ell_{\pm}}$ extends to a linear action on the disc $\mathbb{D}^{\ell_{\pm}+1}$. We can thus define $M = G \times_{K^-} \mathbb{D}^{\ell_- +1} \cup G \times_{K^+} \mathbb{D}^{\ell_+ +1}$ glued along the boundary $\partial(G \times_{K^{\pm}} \mathbb{D}^{\ell_{\pm}+1}) = G \times_{K^{\pm}} K^{\pm}/H = G/H$ via the identity. G acts on M on each half via left action in the first component. This action has principal isotropy group H and singular isotropy groups $K^{\pm}$. One possible description of a cohomogeneity one manifold is thus simply in terms of the Lie groups $H \subset \{K^-, K^+\} \subset G$.

The candidates for positive curvature in [GWZ] are cohomogeneity one under an action of $\mathrm{S}^3 \times \mathrm{S}^3$. The group diagram for P_k is

$$\Delta Q \subset \{(e^{it}, e^{it}) \cdot H\ ,\ (e^{j(1+2k)t}, e^{j(1-2k)t}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

and for Q_k it is

$$\{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{it}, e^{it}) \cdot H\ ,\ (e^{jkt}, e^{j(k+1)t}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

whereas for R we have

$$\{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{3it}, e^{it}) \cdot H\ ,\ (e^{jt}, e^{2jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3\, .$$

The isolated manifold R can thus be viewed as associated to the family Q_k, and indeed shares many of its properties. The positively curved Berger space B^7 with its cohomogeneity one action can be viewed as associated to the family P_k since its group diagram is

$$\Delta Q \subset \{(e^{3it}, e^{it}) \cdot H\ ,\ (e^{jt}, e^{3jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

We now describe the geometry of a general cohomogeneity one action. A G invariant metric is determined by its restriction to a geodesic c normal to all orbits. At the points $c(t)$ which are regular with respect to the action of G, the isotropy is constant and we denote it by H. In terms of a fixed biinvariant inner product Q on the Lie algebra $\mathfrak{g}$ and corresponding Q-orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ we identify, at regular points, $\dot{c}^{\perp} \subset T_{c(t)}M$ with $\mathfrak{h}^{\perp}$ via action fields: $X \in \mathfrak{h}^{\perp} \to X^*(c(t))$.

Since G acts by isometries, $X^*,\ X \in \mathfrak{g}$ is a Killing field on M and hence the restriction to a geodesic is a Jacobi field. This gives rise to an $(n-1)$-dimensional family of Jacobi fields along c defined by $V := \{X^*(c(t)) \mid X \in \mathfrak{h}^{\perp}\}$. The self adjoint shape operator S_t of the regular hypersurface orbit G/H at $c(t)$ satisfies $\nabla_{\dot{c}(t)} X^* = \nabla_{X^*} \dot{c} = S_t(X^*(c(t)))$, i.e. $X' = S_t(X),\ X \in \mathfrak{h}^{\perp}$. Hence V is self adjoint.

A singular point of V is a point $c(t_0)$ such that there exists an $X^* \in V$ with $X^*(c(t_0)) = 0$, i.e. the isotropy group $K_{c(t_0)}$ satisfies $\dim K_{c(t_0)} > \dim H$ and is thus a singular point of the action. For simplicity set $K := K_{c(t_0)}$ and define a Q-orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p} \text{ and thus } \mathfrak{h}^\perp = \mathfrak{p} \oplus \mathfrak{m}.$$

Here $\mathfrak{m}$ can be viewed as the tangent space to the singular orbit G/K at $c(t_0)$ and $\mathfrak{p}$ as the tangent space to K/H, which, since the action is cohomogeneity one, must be a sphere. Thus $D := \dot{c}(t_0) \oplus \mathfrak{p}$ is a the slice of the action, i.e. the normal space to the orbit G/K at $c(t_0)$ and hence $\langle \mathfrak{k}, \mathfrak{p} \rangle_{c(t_0)} = 0$. K acts via the isotropy action $\mathrm{Ad}(K)_{|\mathfrak{m}}$ of G/K on $\mathfrak{m}$ and via the slice representation on D. The second fundamental form of the singular orbit can be viewed as a linear map $B \colon D \to S^2(\mathfrak{m})$, $N \to S_N$. Since K acts by isometries, B is equivariant with respect to the slice representation of K on D and the action on $S^2(\mathfrak{m})$ induced by its isotropy representation on $\mathfrak{m}$. An $\mathrm{Ad}(K)$ invariant irreducible splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ inducing a splitting of $S^2(\mathfrak{m})$ into irreducible summands. The action of K on D is special since it is one of the few transitive actions on a sphere, and in particular irreducible. If for some i, D is not a subrepresentation of $S^2(\mathfrak{m}_i)$, this implies that $S_{\dot{c}(t_0)|\mathfrak{m}_i} = 0$, i.e. $\langle X', X \rangle_{c(t_0)} = 0$ for $X \in \mathfrak{m}_i$.

This is especially useful if c, $0 \le t \le L$, is a minimal geodesic from the singular orbit G/K^- to G/K^+ with tangent space $\mathfrak{m}_\pm$ and slice $D_\pm = \dot{c} \oplus \mathfrak{p}_\pm$. In this case condition (c) in Proposition B is automatically satisfied since there are no singular points for $0 < t < L$.

We now apply this to the manifold R. Since H is finite, we have $\mathfrak{m} = \mathfrak{h}^\perp = \mathfrak{g}$. Regarding S^3 as the unit quaternions, we choose the basis of $\mathfrak{g}$ given by the left invariant vector fields X_i and Y_i on $G = \mathrm{S}^3 \times \mathrm{S}^3$ corresponding to i, j and k in the Lie algebras of the first and second S^3 factor of G. Then the Jacobi fields X_i, Y_i are a basis of the self adjoint family V along a minimal geodesic from G/K^- to G/K^+.

The tangent spaces $\mathfrak{p}_\pm$ to the circles $K^\pm/H$ are given by $\mathfrak{p}_- = \mathrm{span}\{3X_1 + Y_1\}$ and $\mathfrak{p}_+ = \mathrm{span}\{X_2 + 2Y_2\}$. Thus the tangent spaces $\mathfrak{m}_\pm$ to the singular orbits split up into $K^\pm$ irreducible subspaces as follows: At $t = 0$, $\mathfrak{m}_-$ is the direct sum of $W_0 = \mathrm{span}\{-X_1 + 3Y_1\}$, $W_1 = \mathrm{span}\{X_2, X_3\}$ and $W_2 = \mathrm{span}\{Y_2, Y_3\}$. At $t = L$, $\mathfrak{m}_+$ is the direct sum of $\bar{W}_0 = \mathrm{span}\{-2X_2 + Y_2\}$, $\bar{W}_1 = \mathrm{span}\{X_1, X_3\}$ and $\bar{W}_2 = \mathrm{span}\{Y_1, Y_3\}$.

The non-trivial irreducible representations of $\mathrm{K}_0 = \mathrm{S}^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ consist of two dimensional representations given by multiplication by $e^{in\theta}$ on $\mathbb{C}$, called a weight n representation. If $\mathrm{K}_0 \subset \mathrm{S}^3 \times \mathrm{S}^3$ has slope (p_1, p_2) with $\gcd(p_1, p_2) = 1$, its action, which is given by conjugation on imaginary quaternions in each component, is trivial on W_0 and has weight $2p_i$ on W_i. If $p_1 \neq p_2$, all representations in $\mathfrak{m}$ are inequivalent and hence orthogonal by Schur's Lemma. This holds for R since $(p_1, p_2) = (3, 1)$ at $t = 0$ and $(p_1, p_2) = (1, 2)$ at $t = L$.

Furthermore, the weight is 0 and $4p_i$ on $S^2 W_i$. The action on the slice $V = \mathbb{R}^2$ has weight k if $H \cap K_0 = \mathbb{Z}_k$, since $\mathbb{Z}_k$ is necessarily the ineffective kernel. Thus if $4p_i \neq k$, the second fundamental form vanishes on $S^2 W_i$.

At $t = 0$ we have $H \cap K_0^- = \{\pm(1,1), \pm(-i, i)\}$ and hence $k = 4$, which implies that the second fundamental form vanishes on $S^2(W_1)$ (but not necessarily on $S^2(W_2)$).

At $t = L$ we have $H \cap K_0^+ = \{(1,1), (-1,1)\}$ and hence $k = 2$. Thus the second fundamental form vanishes on $S^2(\bar{W}_1)$ and $S^2(\bar{W}_2)$. Since $X_3 \in W_1 \cap \bar{W}_1$, it satisfies $\langle X_3', X_3 \rangle = 0$ at $t = 0$ and $t = L$. Furthermore, $X_3 \neq 0$ since it is orthogonal to $\mathfrak{p}_\pm$. Thus condition (a) in Theorem B is satisfied. Finally, we need to check that condition (b) is satisfied as well. But X_3 is orthogonal to X_i, Y_i, $i = 1, 2$ since the modules at the end points are irreducible, inequivalent, and orthogonal to $\mathfrak{p}_\pm$. It is orthogonal to Y_3 as well, since X_3, Y_3 are orthogonal in the Killing form of $\mathfrak{g}$, and Schur's Lemma implies that the metric is a multiple of the Killing form. Thus we can apply Theorem B to obtain a contradiction. $\square$

The arguments in the proof also give a simple classification of positively curved cohomogeneity one manifolds in the case $G = \mathrm{S}^3 \times \mathrm{S}^3$ and H finite. For example, consider the cohomogeneity one manifolds which have a group diagram as the one for Q_k and R, where $k = 4$ on the left and $k = 2$ on the right, but with slopes (p_-, q_-) on the left and (p_+, q_+) on the right. The above obstruction immediately implies that $(p_-, q_-) = (1, 1)$ up to sign. On the right the second fundamental form of G/K^+ vanishes unless $|p_+ \pm q_+| = 1$, which one sees by examining all the weights on $S^2\mathfrak{m}_+$. But the singular orbit cannot be totally geodesic since the homogeneous space G/K^+ does not carry a homogeneous metric with positive curvature. Thus only the family Q_k remains.

A similar argument holds for the P_k family. But now one has as an exceptional case the Berger space with $(p_-, q_-) = (3, 1)$ and $(p_+, q_+) = (1, 3)$. Since in this case $k = 4$ on both sides, the argument only implies that B_- vanishes on W_1 and B_+ vanishes on $\bar{W}_2$, but $W_1 \cap \bar{W}_2 = 0$.

Furthermore, it follows that if H is finite and $\dim K^\pm/H = 1$, either $k = 4$ on both sides or $k = 2$ on one side and $k = 4$ on the other. One then easily shows that the only possibility for H is as in the P_k and Q_k family.

Notice though that for the remaining families P_k, Q_k such arguments do not give rise to a contradiction since at $t = 0$ the module W_1 does not have to be orthogonal to W_2, and $S^2 W_i$ is not necessarily 0 either since $k = 4$.

References

[AA] A.V. Alekseevsy and D.V. Alekseevsy, *G- manifolds with one dimensional orbit space,* Ad. in Sov. Math. **8** (1992), 1–31.

[AW] S. Aloff and N. Wallach, *An infinite family of 7–manifolds admitting positively curved Riemannian structures,* Bull. Amer. Math. Soc. **81**(1975), 93–97.

[Ba] Y.V. Bazaikin, *On a certain family of closed 13–dimensional Riemannian manifolds of positive curvature,* Sib. Math. J. **37**, No. 6 (1996), 1219-1237.

[Be] M. Berger, *Les variétés riemanniennes homogenes normales simplement connexes a courbure strictment positive*, Ann. Scuola Norm. Sup. Pisa **15** (1961), 191-240.

[De] O. Dearricott, *A 7-manifold with positive curvature*, Preprint.

[E1] J.H. Eschenburg, *New examples of manifolds with strictly positive curvature,* Inv. Math **66** (1982), 469-480.

[E2] J.H. Eschenburg, *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen,* Schriftenr. Math. Inst. Univ. Münster **32** (1984).

[E3] J.H. Eschenburg, *Comparison theorems and hypersurfaces,* Manuscripta Math. **59** (1987), 295-323.

[EH] J.H. Eschenburg and E. Heintze, *Comparison theory for Riccati equation,* Manuscripta Math. **68** (1990), 209-214.

[GWZ] K. Grove, B. Wilking and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry,* J. Diff. Geom. **78** (2008), 33–111.

[GVWZ] K.Grove, L.Verdiani, B.Wilking and W.Ziller, *Non-negative curvature obstruction in cohomogeneity one and the Kervaire spheres*, Ann. del. Scuola Norm. Sup. 5 (2006), 159-170.

[GVZ] K. Grove, L. Verdiani and W. Ziller, *An exotic T_1S^4 with positive curvature*, Preprint 2008.

[GZ1] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. **152** (2000), 331–367.

[GZ2] K. Grove and W. Ziller, *Cohomogeneity one manifolds with positive Ricci curvature*, Inv. Math. **149** (2002), 619-646.

[PW] P. Petersen and F. Wilhelm, *An exotic sphere with positive sectional curvature*, preprint.

[V1] L. Verdiani, *Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature, I,* Math. Z. **241** (2002), 329–339.

[V2] L. Verdiani, *Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature,* J. Diff. Geom. **68** (2004), 31–72.

[Wa] N. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. **96** (1972), 277-295.

[Wi] B. Wilking, *A duality theorem for Riemannian foliations in nonnegative sectional curvature*, Geom. Funct. Anal **17** (2007), 1297-1320.

[Zi1] W. Ziller, *Examples of manifolds with nonnegative sectional curvature*, in: *Metric and Comparison Geometry*, ed. J.Cheeger and K.Grove, Surv. Diff. Geom. Vol. XI, International Press (2007).

[Zi2] W. Ziller, *Geometry of positively curved cohomogeneity one manifolds*, in: *Topology and Geometric Structures on Manifolds*, in honor of Charles P.Boyer's 65th birthday, Progress in Mathematics, Birkhäuser, (2008).

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