Isoperimetric and Sobolev inequalities on hypersurfaces in sub-Riemannian Carnot groups

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Abstract

In this paper we shall study smooth submanifolds immersed in a k-step Carnot group $\mathbb{G}$ of homogeneous dimension Q. Our main result is an isoperimetric inequality for the case of a $\mathbf{C}^2$-smooth compact hypersurface S with - or without - boundary ∂S; see Theorem 4.1. Note that S and ∂S are endowed with their homogeneous measures σ_H^{n-1}, σ_H^{n-2}, actually equivalent (up to bounded densities) to the $(Q-1)$-dimensional and $(Q-2)$-dimensional spherical Hausdorff measures with respect to a given homogeneous metric ϱ on $\mathbb{G}$. This generalizes a classical inequality, involving the mean curvature of the hypersurface, proven by Michael and Simon [43] and Allard [1], independently. In particular, from this result we deduce some related Sobolev-type inequalities; see Section 5. The strategy of the proof is inspired by the classical one. In particular, we shall begin by proving a linear inequality; see Proposition 4.7. By using this inequality we can prove a *global monotonicity formula*; see Theorem 4.9. These results allow us to study the asymptotic behavior of σ_H^{n-1}; see Section 4.4. By using blow-up results and some homogeneity arguments, we can prove local estimates of the right-hand side of the global monotonicity formula. In this way we get a *local monotonicity formula* (see Corollary 4.20) which becomes the starting point to apply the classical strategy of the proof. At this point, one concludes the proof by a Calculus Lemma (which can be proved, via a contradiction argument based on local monotonicity formula; see Lemma 4.22) and a Vitali-type covering argument. We stress however that there are many differences, due to our different geometric setting. For instance, we shall discuss a blow-up theorem which also holds for characteristic points; see Section 3.4. Another simple but fundamental result is the smooth Coarea Formula for the HS-gradient; see Section 3.1. Other tools are the horizontal divergence theorem and the 1st variation of the H-perimeter, already developed in [48], and here generalized to hypersurfaces having non-empty characteristic set. Moreover, we will need some other results and, in particular, estimates on the sizes of the characteristic sets (of S and its boundary ∂S) and blow-up estimates at the boundary; see Section 3.5. These results can be used in the study of minimal and constant horizontal mean curvature hypersurfaces; see, for instance, Corollary 4.2. Finally, we shall discuss some natural generalizations; see Section 5.1. This paper is a new (revised and improved) version of some results obtained in the unpublished manuscript [49].

Key words and phrases: Carnot groups; Sub-Riemannian Geometry; Hypersurfaces; Isoperimetric Inequality; Sobolev Inequalities; Blow-up; Coarea Formula.

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Contents

1 Introduction

In the last decades considerable efforts have been made to extend to the general setting of metric spaces the methods of Analysis and Geometric Measure Theory. This philosophy, in a sense already contained in Federer's treatise [22], has been pursued, among other authors, by Ambrosio [2], Ambrosio and Kirchheim [3], Capogna, Danielli and Garofalo [8], Cheeger [11], Cheeger and Kleiner [12], David and Semmes [20], De Giorgi [21], Gromov [31], Franchi, Gallot and Wheeden [25], Franchi and Lanconelli [26], Franchi, Serapioni and Serra Cassano [27, 28], Garofalo and Nhieu [29], Heinonen and Koskela [32], Korany and Riemann [36], Pansu [51, 52], but the list is far from being complete.

In this respect, *sub-Riemannian* or *Carnot-Carathéodory* geometries have become a subject of great interest also because of their connections with many different areas of Mathematics and Physics, such as PDE's, Calculus of Variations, Control Theory, Mechanics and Theoretical Computer Science. For references, comments and other perspectives, we refer the reader to Montgomery's book [50] and the surveys by Gromov, [31], and Vershik and Gershkovich, [61]. We also mention, specifically for sub-Riemannian geometry, [59] and [53]. More recently, the so-called Visual Geometry has also received new impulses from this field; see [56], [15] and references therein.

The setting of the sub-Riemannian geometry is that of a smooth manifold N, endowed with a smooth non-integrable distribution $H \subset TN$ of h-planes, or *horizontal subbundle* ($h \leq \dim N$). Such a distribution is endowed with a positive definite metric g_H, defined only on the subbundle H. The manifold N is said to be a *Carnot-Carathéodory space* or *CC-space* when one introduces the so-called *CC-metric* d_{CC} (see Definition 2.2). With respect to such a metric, the only paths on the manifold which have finite length are tangent to the distribution H and therefore called *horizontal*. Roughly speaking, in connecting two points we are only allowed to follow horizontal paths joining them.

Throughout this paper we shall extensively study hypersurfaces immersed in Carnot groups which, for this reason, form the underlying ambient space. A *k-step Carnot group* $(\mathbb{G}, \bullet)$ is an

n-dimensional, connected, simply connected, nilpotent and stratified Lie group (with respect to the group multiplication $\bullet$) whose Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ satisfies:

$$\mathfrak{g} = H_1 \oplus ... \oplus H_k, \quad [H_1, H_{i-1}] = H_i \quad (i = 2, ..., k), \quad H_{k+1} = \{0\}.$$

The horizontal bundle H is generated by a frame $\underline{X_H} := \{X_1, ..., X_h\}$ of left-invariant vector fields. The horizontal frame can be completed to a global left-invariant frame $\underline{X} := \{X_1, ..., X_n\}$ for $T\mathbb{G}$. Note that the standard basis $\{\mathrm{e}_i : i = 1, ..., n\}$ of $\mathbb{R}^n$ can be relabelled to be *graded* or *adapted to the stratification*. Any Carnot group $\mathbb{G}$ on $\mathbb{R}^n$ is endowed with a one-parameter family of dilations (adapted to the grading) that makes it a *homogeneous group* with homogeneous dimension $Q := \sum_{i=1}^{k} i\, h_i$, in the sense of Stein's definition; see [58]. Note that Q coincides with the *Hausdorff dimension* of $(\mathbb{G}, d_{CC})$ as a metric space. Carnot groups are of special interest for many reasons and, in particular, because they constitute a wide class of examples of sub-Riemannian geometries. Note that, by a well-know result due to Mitchell [45] (see also Montgomery's book [50]), the *Gromov-Hausdorff tangent cone* at any regular point of a sub-Riemannian manifold is a suitable Carnot group. This motivates the interest towards the study of Carnot groups which play for sub-Riemannian geometries an analogous role to that of Euclidean spaces in Riemannian geometry. The initial development of Analysis in this setting was motivated by some works published in the first eighties. Among others, we cite the paper by Fefferman and Phong [24] about the so-called "subelliptic estimates" and that of Franchi and Lanconelli [26], where a Hölder regularity theorem was proven for a class of degenerate elliptic operators in divergence form. Meanwhile, the beginning of Geometric Measure Theory was perhaps an intrinsic isoperimetric inequality proven by Pansu in his Thesis [51], for the case of the *Heisenberg group* $\mathbb{H}^1$. For more results about isoperimetric inequalities on Lie groups and Carnot-Carathéodory spaces, see also [60], [31], [53], [29], [8], [25], [32]. For results on these topics, and for more detailed bibliographic references, we shall refer the reader to [2], [8], [27, 28], [19], [30], [29], [37, 38], [47, 48], [35]. We also quote [13], [30], [54], [55], [9] for some results about minimal and constant mean-curvature hypersurfaces immersed in the Heisenberg group.

In this paper we shall try to give some contributions to the study of both analytic and differential-geometric properties of hypersurfaces immersed in Carnot groups, endowed with the so-called *H-perimeter measure* σ_H^{n-1}; see Definition 2.13. To this aim we will preliminarily study some technical tools and among other things we shall extend to hypersurfaces with non-empty characteristic set, the horizontal Divergence Theorem and the 1st-variation of σ_H^{n-1}, proved in [46, 48] for the non-characteristic case; see Section 3.2 and Section 3.3. We shall discuss a blow-up theorem, which also holds for characteristic points, and a horizontal Coarea Formula for smooth functions on hypersurfaces; see Section 3.4 and Section 3.1. Together with those of Section 3.5, these results are used in Section 4 to investigate the validity in this context of an isoperimetric inequality, proved by Michael and Simon in [43] for a general setting including Riemannian geometries and, independently, by Allard in [1] for varifolds; see below for a more precise statement. In Section 5, we shall deduce some related Sobolev inequalities, following a classical pattern by Federer-Fleming [23] and Mazja [42]. Finally in Section 5.1 we will discuss an immediate generalization to the natural BV space for functions defined on any $\mathbf{C}^2$-smooth hypersurface. Very recently, some similar results in this direction have also been obtained by Danielli, Garofalo and Nhieu in [19], where a monotonicity estimate for the H-perimeter has been proven for graphical strips in the Heisenberg group $\mathbb{H}^1$.

Now we would like to make a short comment about the Isoperimetric Inequality for compact hypersurfaces immersed in the Euclidean space $\mathbb{R}^n$.

Theorem 1.1 (Euclidean Isoperimetric Inequality for $S \subset \mathbb{R}^n$)**.** *Let $S \subset \mathbb{R}^n\,(n > 2)$ be a $\mathbf{C}^2$-smooth compact hypersurface with -or without- piecewise $\mathbf{C}^1$-boundary. Then*

$$\left(\sigma_R^{n-1}(S)\right)^{\frac{n-2}{n-1}} \leq C_{Isop}\left(\int_S |\mathcal{H}_R|\, \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S)\right)$$

where $C_{Isop} > 0$ is a dimensional constant.

In the previous statement, $\mathcal{H}_{R}$ is the mean curvature of S and σ_{R}^{n-1}, σ_{R}^{n-2} are the Riemannian measures on S and ∂S, respectively. The first step in the proof is a linear isoperimetric inequality. More precisely, one has

$$\sigma_{R}^{n-1}(S) \leq r\left(\int_{S} |\mathcal{H}_{R}|\, \sigma_{R}^{n-1} + \sigma_{R}^{n-2}(\partial S)\right),$$

where r is the radius of a Euclidean ball $B(x, r)$ containing S. From this linear inequality and Coarea Formula, one gets the so-called *monotonicity inequality*, which says that, at every interior point $x \in \mathrm{Int}S$, one has

$$-\frac{d}{dt}\frac{\sigma_{R}^{n-1}(S_t)}{t^{n-1}} \leq \frac{1}{t^{n-1}}\left(\int_{S_t} |\mathcal{H}_{R}|\, \sigma_{R}^{n-1} + \sigma_{R}^{n-2}(\partial S \cap B(x,t))\right)$$

for $\mathcal{L}^1$-a.e. $t > 0$, where $S_t = S \cap B(x, t)$. Note that every interior point of a $\mathbf{C}^2$-smooth hypersurface S turns out to be a *density-point*[1]

By the monotonicity inequality, via a contradiction argument, one deduces a calculus lemma which, together with a standard Vitali-type covering theorem, allows to achieve the proof.

The importance of the monotonicity estimate can also be understood through one of its immediate consequences, that is an asymptotic exponential estimate, i.e.

$$\sigma_{R}^{n-1}(S_t) \geq \omega_{n-1}\, t^{n-1} e^{-\mathcal{H}^0 t}$$

as $t \to 0^+$, where $x \in \mathrm{Int}S$ and $\mathcal{H}^0$ is any constant such that $|\mathcal{H}_{R}| \leq \mathcal{H}^0$. Note that for minimal hypersurfaces this implies that

$$\sigma_{R}^{n-1}(S_t) \geq \omega_{n-1}\, t^{n-1}$$

as $t \to 0^+$. A great part of this paper is concerned about the generalization of these results to hypersurfaces immersed in Carnot groups.

Section 2 is largely devoted to introduce the subject of Carnot groups and the study of hypersurfaces (and, more generally, submanifolds) immersed in Carnot groups. In particular, we shall describe the main geometric structures useful in this setting from many points of views, including basic facts about stratified and homogeneous Lie groups, Riemannian and sub-Riemannian geometries, intrinsic measures and connections.

Now let us give a quick overview of some basic facts.

If $S \subset \mathbb{G}$ is a hypersurface of class $\mathbf{C}^1$, then $x \in S$ is a *characteristic point* if $H_x \subset T_x S$. If S is non-characteristic, the *unit H-normal* along S is given by $\nu_{_H} := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}$, where ν is the Riemannian unit normal of S. By making use of the *contraction operator* $\lrcorner$ on differential forms[2] we may define a $(Q-1)$-homogeneous measure $\sigma_{H}^{n-1} \in \bigwedge^{n-1}(T^*S)$ by

$$\sigma_{H}^{n-1} \llcorner S := (\nu_{_H} \lrcorner\, \sigma_{R}^{n})|_S.$$

Note that $\sigma_{R}^{n} := \bigwedge_{i=1}^{n} \omega_i \in \bigwedge^n(T^*\mathbb{G})$ is the Riemannian (left-invariant) volume form on $\mathbb{G}$ which is built by wedging together the dual basis $\underline{\omega} = \{\omega_1, ..., \omega_n\}$ of $T^*\mathbb{G}$, where $\omega_i = X_i^*$ for every $i = 1, ..., n$. Analogously, we may define a $(Q-2)$-homogeneous measure σ_{H}^{n-2} on any $(n-2)$-dimensional smooth submanifold N of $\mathbb{G}$. So the only difference is that $\nu_{_H}$ "becomes" (i.e. it must be replaced by) a unit horizontal 2-vector $\nu_{_H} = \nu_{_H}^1 \wedge \nu_{_H}^2 \in \bigwedge^2(H)$; see Definition 2.21. We

[1] By definition, $x \in \mathrm{Int}S$ is a *density-point* if $\lim_{t \searrow 0^+} \frac{\sigma_{R}^{n-1}(S_t)}{t^{n-1}} = \omega_{n-1}$ where ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

[2] Remind that $\lrcorner : \bigwedge^k(T^*\mathbb{G}) \to \bigwedge^{k-1}(T^*\mathbb{G})$ is defined, for $X \in T\mathbb{G}$ and $\alpha \in \bigwedge^k(T^*\mathbb{G})$, by

$$(X \lrcorner\, \alpha)(Y_1, ..., Y_{k-1}) := \alpha(X, Y_1, ..., Y_{k-1});$$

see [34], [22].

remark that σ_H^{n-1} and σ_H^{n-2} are actually equivalent (up to bounded densities functions called *metric factors*) to the $(Q-1)$-dimensional and $(Q-2)$-dimensional spherical Hausdorff measures $\mathcal{S}_\varrho^{Q-1}$ and $\mathcal{S}_\varrho^{Q-2}$, respectively, associated to any homogeneous distance ϱ on $\mathbb{G}$; see Section 3.4 and Section 3.5.

In Section 3.2 we recall the horizontal Divergence Theorem and a related integration by parts formula for smooth hypersurfaces, with piecewise smooth boundary. Clearly we are assuming that S and ∂S are endowed with the homogeneous measures σ_H^{n-1} and σ_H^{n-2}, respectively; see Theorem 3.3 and Corollary 3.4. Moreover, in Section 3.3 we state the 1st variation of σ_H^{n-1}. A great part of this material can be found in [46, 48]. However, there is some novelty in the presentation given here, because the results are generalized to hypersurfaces having possibly non-empty characteristic set.

Section 3.4 contains a blow-up theorem for the horizontal perimeter σ_H^{n-1}. In other words, we shall study the limit

$$\lim_{r\to 0^+}\frac{\sigma_H^{n-1}(S\cap B_\varrho(x,r))}{r^{Q-1}},$$

where $B_\varrho(x,r)$ is a ϱ-ball of center $x\in S$ and radius r. Note that this limit is just the density of σ_H^{n-1} at $x\in S$. More precisely, after reminding the well-known blow-up procedure for non-characteristic points of a $\mathbf{C}^1$-smooth hypersurface S (see, for instance, [27, 28], [4], [37, 38]), we shall generalize it, under some regularity assumptions on S, also to the case of characteristic points of S; see Theorem 3.14. A similar result was proven in [39] for 2-step groups. Note that the characteristic set C_S of S can be seen as the set of all points at which the horizontal projection of the unit normal vanishes, i.e. $C_S=\{x\in S: |\mathcal{P}_H\nu|=0\}$. More precisely, let $x\in C_S\cap S$ and assume that, locally around x, S can be represented as a $\mathbf{C}^i$-smooth X_α-graph, for some vertical direction $X_\alpha\in V:=H^\perp$. By hypothesis the integer $i=2,...,k$, coincides with the homogeneous "order" of α; see Notation 2.3. For the sake of simplicity, let $x=0\in\mathbb{G}$ and assume that near $x=0$ one has

$$S\cap B_\varrho(x,r)\subset exp\left\{(\zeta_1,...,\zeta_{\alpha-1},\psi(\zeta),\zeta_{\alpha+1},...,\zeta_n)\in\mathfrak{g}\,:\,\zeta=(\zeta_1,...,\zeta_{\alpha-1},0,\zeta_{\alpha+1},...,\zeta_n)\in \mathrm{e}_\alpha^\perp\right\},$$

where $\psi:\mathrm{e}_\alpha^\perp\subset\mathfrak{g}\longrightarrow\mathbb{R}$ is a $\mathbf{C}^i$-function satisfying

$$\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0)=0\qquad\text{whenever}\quad \mathrm{ord}(j_1)+...+\mathrm{ord}(j_l)<i.$$

Then, we shall show that[3]

$$\sigma_H^{n-1}(S\cap B_\varrho(x,r))\sim\kappa_\varrho(C_S(x))r^{Q-1}\qquad\text{for}\qquad r\to 0^+,$$

where the constant $\kappa_\varrho(C_S(x))$ can be explicitly computed by integrating σ_H^{n-1} along a polynomial hypersurface of homogeneous order $i=\mathrm{ord}(\alpha)$; see Theorem 3.14, Case (2).

In Section 3.1 we shall state and discuss another important tool, i.e. the Coarea Formula for the *HS*-gradient, that is an equivalent for smooth functions of the classical *Fleming-Rischel* formula. More precisely, let $S\subset\mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let $\varphi\in\mathbf{C}^1(S)$. Then

$$\int_S |grad_{HS}\,\varphi(x)|\,\sigma_H^{n-1}(x)=\int_{\mathbb{R}}\sigma_H^{n-2}(\varphi^{-1}[s]\cap S)\,ds.$$

In Section 3.5 there are some other important results quoted from the literature.

As already said, Section 4 contains the main result of this paper, i.e. an isoperimetric inequality for compact hypersurfaces with - or without- boundary, depending on the horizontal mean curvature $\mathcal{H}_H$ of the hypersurface, which generalizes to Carnot groups a classical inequality by Michael and Simon [43] and Allard [1]. We now state our main result.

[3] Henceforth, the symbol $\sim$ will mean "asymptotic".

Theorem 1.2 (Isoperimetric Inequality)**.** *Let $\mathbb{G}$ be a k-step Carnot group and let us fix a homogeneous metric ϱ on $\mathbb{G}$ as in Definition 2.5. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface with boundary ∂S -at least- piecewise $\mathbf{C}^2$-smooth. Let $\mathcal{H}_H$ denote the horizontal mean curvature of S. Then there exists a positive constant C_{Isop}, only dependent on $\mathbb{G}$ and on the homogeneous metric ϱ, such that*

$$\left(\sigma_H^{n-1}(S)\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop}\left(\int_S |\mathcal{H}_H|\, \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S)\right).$$

The proof of this result is heavily inspired from the classical one, for which we refer the reader to the book by Burago and Zalgaller [7]. A similar strategy can also be used in proving isoperimetric and Sobolev inequalities in abstract metric setting such as weighted Riemannian manifolds and graphs; see [14]. Nevertheless, we have to say that there are many non-trivial modifications to be done, due to the sub-Riemannian setting.

Roughly speaking, the starting point will be again a linear inequality; see Proposition 4.7. This one is used to obtain a *global monotonicity formula* for the H-perimeter; see Theorem 4.9. As in the Euclidean/Riemannian case, the monotonicity inequality is an ordinary differential inequality, expressing the local behavior of the first derivative of the quotient

$$\frac{\sigma_H^{n-1}(S \cap B_\varrho(x,t))}{t^{Q-1}}$$

for $t \searrow 0^+$, whenever $x \in \mathrm{Int} S$; see Section 4.1. We will also discuss the characteristic case.

Then, in Section 4.2 we shall prove local estimates dependent on blow-up results. Roughly speaking, these estimates require, in the general case, a certain amount of regularity at the boundary. They constitute a key-point in the proof of the Isoperimetric Inequality, since they allow to make more intrinsic the right-hand side of the global monotonicity inequality (20).

Section 4.3 is then devoted to the proof our main result. In Section 4.4 we shall discuss a straightforward application of the monotonicity estimate. More precisely, let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and assume that the horizontal mean curvature $\mathcal{H}_H$ is bounded by a positive constant $\mathcal{H}_H^0$. Then, for every $x \in \mathrm{Int}(S \setminus C_S)$, we shall show that

$$\sigma_H^{n-1}(S_t) \geq \kappa_\varrho(\nu_H(x))\, t^{Q-1} e^{-t\,\mathcal{H}_H^0}$$

for $t \searrow 0^+$, where $\kappa_\varrho(\nu_H(x))$ denotes the "density" of σ_H^{n-1} at x, the so-called *metric factor*; see Corollary 4.25. We shall also consider the more general case in which $x \in C_S$; see Corollary 4.26. In Section 5 we shall discuss the equivalent Sobolev-type inequalities which can be deduced by the previous isoperimetric inequality, following a well-known and classical argument by Federer-Fleming [23] and Mazja [42]; see Theorem 5.1. Some corollaries will be proven, and among others, we shall show the following:

Theorem 1.3. *Let $\mathbb{G}$ be a k-step Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth closed hypersurface. Then, for every $\psi \in \mathbf{C}_0^\infty(S)$ one has*

$$\left(\int_S |\psi|^{\frac{Q-1}{Q-2}}\, \sigma_H^{n-1}\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop}\int_S \left(|\psi|\, |\mathcal{H}_H| + |grad_{HS}\,\psi|\right)\, \sigma_H^{n-1}.$$

Note that C_{Isop} is the same constant appearing in Theorem 1.2. Finally, in Section 5.1 we shall discuss and state the equivalent versions of our main results in the BV_{HS} setting. More precisely, we shall discuss how a natural notion of *HS-variation* can be given for functions supported on a $\mathbf{C}^2$-smooth hypersurface S. Indeed, starting from the horizontal divergence Theorem 3.3, it becomes natural to mimic the original Euclidean definition of variation and so defining the space of functions having bounded *HS*-variation.

2 Carnot groups, submanifolds and measures

2.1 Sub-Riemannian Geometry of Carnot groups

In this section we will introduce the definitions and the main features concerning the sub-Riemannian geometry of Carnot groups. References for this large subject can be found, for instance, in [8], [29], [31], [37], [45], [50], [51, 52, 53], [59]. Let N be a $\mathbf{C}^{\infty}$-smooth connected n-dimensional manifold and let $H \subset TN$ be an h-dimensional smooth subbundle of TN. For any $x \in N$, let T_x^k denote the vector subspace of T_xN spanned by a local basis of smooth vector fields $X_1(x), ..., X_h(x)$ for H around x, together with all commutators of these vector fields of order $\leq k$. The subbundle H is called *generic* if, for all $x \in N$, $\dim T_x^k$ is independent of the point x and *horizontal* if $T_x^k = TN$, for some $k \in \mathbb{N}$. The pair (N, H) is a *k-step CC-space* if is generic and horizontal and if $k = \inf\{r : T_x^r = TN\}$. In this case

$$0 = T^0 \subset H = T^1 \subset T^2 \subset ... \subset T^k = TN$$

is a strictly increasing filtration of *subbundles* of constant dimensions $n_i := \dim T^i \, (i = 1, ..., k)$. Setting $(H_i)_x := T_x^i \setminus T_x^{i-1}$, then $\mathtt{gr}(T_xN) = \oplus_{i=1}^k (H_k)_x$ is the associated *graded Lie algebra* at $x \in N$, with respect to the Lie product $[\cdot, \cdot]$. We set $h_i := \dim H_i = n_i - n_{i-1}$ $(n_0 = h_0 = 0)$ and, for simplicity, $h := h_1 = \dim H$. The k-vector $\overline{h} = (h, h_2, ..., h_k)$ is the *growth vector* of H.

Definition 2.1. *$\underline{X} = \{X_1, ..., X_n\}$ is a graded frame for N if $\{X_{i_j}(x) : n_{j-1} < i_j \leq n_j\}$ is a basis for ${H_j}_x$ for all $x \in N$ and all $j \in \{1, ..., k\}$.*

Definition 2.2. *A sub-Riemannian metric $g_H = \langle \cdot, \cdot \rangle_H$ on N is a symmetric positive bilinear form on H. If (N, H) is a CC-space, the CC-distance $d_{CC}(x, y)$ between $x, y \in N$ is defined by*

$$d_{CC}(x, y) := \inf \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_H} \, dt,$$

where the infimum is taken over all piecewise-smooth horizontal paths γ joining x to y.

In fact, Chow's Theorem implies that d_{CC} is a true metric on N and that any two points can be joined with at least one horizontal path. The topology induced on N by the CC-metric is equivalent to the standard manifold topology; see [31], [50].

The general setting introduced above is the starting point of sub-Riemannian geometry. A nice and very large class of examples of these geometries is represented by *Carnot groups* which, for many reasons, play in sub-Riemannian geometry an analogous role to that of Euclidean spaces in Riemannian geometry. Below we will introduce their main features. For the geometry of Lie groups we refer the reader to Helgason's book [34] and Milnor's paper [44], while, specifically for sub-Riemannian geometry, to Gromov, [31], Pansu, [51, 53], and Montgomery, [50].

A *k-step Carnot group* $(\mathbb{G}, \bullet)$ is an n-dimensional, connected, simply connected, nilpotent and stratified Lie group (with respect to the multiplication $\bullet$) whose Lie algebra $\mathfrak{g} (\cong \mathbb{R}^n)$ satisfies:

$$\mathfrak{g} = H_1 \oplus ... \oplus H_k, \quad [H_1, H_{i-1}] = H_i \quad (i = 2, ..., k), \quad H_{k+1} = \{0\}.$$

We denote by 0 the identity on $\mathbb{G}$ and so $\mathfrak{g} \cong T_0\mathbb{G}$. The smooth subbundle H_1 of the tangent bundle $T\mathbb{G}$ is said to be *horizontal* and henceforth denoted by H. We set $V := H_2 \oplus ... \oplus H_k$ and call V the *vertical subbundle* of $T\mathbb{G}$. As for CC-spaces, we set $h_i = \dim H_i$, $i = 1, ..., k$. Moreover $n_l := h + ... + h_l$, $h = h_1$ and $n_k = n$. We assume that H is generated by a frame $\underline{X_H} := \{X_1, ..., X_h\}$ of left-invariant vector fields. This one can be completed to a global graded and left-invariant frame $\underline{X} := \{X_i : i = 1, ..., n\}$ in a way that $H_l = \mathrm{span}_{\mathbb{R}}\{X_i : n_{l-1} < i \leq n_l\}$. The standard basis $\{\mathrm{e}_i : i = 1, ..., n\}$ of $\mathbb{R}^n \cong \mathfrak{g}$ can be relabelled to be graded or *adapted to the stratification*. Any left-invariant vector field of $\underline{X}$ is given by $X_i(x) = L_{x*}\mathrm{e}_i \, (i = 1, ..., n)$, where L_{x*} denotes the differential of the left-translation at x.

Notation 2.3. *We shall set* $I_H := \{1, ..., h\}$*,* $I_{H_2} := \{n_1 + 1, ..., n_2\}$*,..., and* $I_V := \{h + 1, ..., n\}$*. Unless otherwise specified, we will use Latin letters* $i, j, k, ...,$ *for indices belonging to* I_H *and Greek letters* $\alpha, \beta, \gamma, ...,$ *for indices belonging to* I_V*. The function* $\mathrm{ord} : \{1, ..., n\} \longrightarrow \{1, ..., k\}$ *is defined by* $\mathrm{ord}(a) := i$ *whenever* $n_{i-1} < a \leq n_i$ *for some* $i = 1, ..., k$*.*

We shall use the so-called *exponential coordinates of 1st kind* and so $\mathbb{G}$ will be identified with its Lie algebra $\mathfrak{g}$, via the (Lie group) exponential map $exp : \mathfrak{g} \longrightarrow \mathbb{G}$.

As for any nilpotent Lie group, the *Baker-Campbell-Hausdorff formula* uniquely determines the group multiplication $\bullet$ of $\mathbb{G}$, from the "structure" of its own Lie algebra $\mathfrak{g}$. Using exponential coordinates, the group multiplication $\bullet$ on $\mathbb{G}$ turns out to be polynomial and explicitly computable; see [16]. Moreover, $0 = exp\,(0, ..., 0)$ and the inverse of $x = exp\,(x_1, ..., x_n) \in \mathbb{G}$ is just $x^{-1} = exp\,(-x_1, ..., -x_n)$.

If H is endowed with a metric $g_H = \langle \cdot, \cdot \rangle_H$, we say that $\mathbb{G}$ has a *sub-Riemannian structure.* It is always possible to define a left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on $\mathbb{G}$ such that $\underline{X}$ is *orthonormal* and $g_{|H} = g_H$. If we fix a Euclidean metric on $\mathfrak{g} = T_0\mathbb{G}$ (which makes $\{\mathrm{e}_i : i = 1, ..., n\}$ an orthnormal basis), this metric naturally extends to the whole tangent bundle, by means of left-translations.

Since Chow's Theorem holds true for Carnot groups, the *Carnot-Carathéodory distance* d_{CC} associated with g_H can be defined. The pair $(\mathbb{G}, d_{CC})$ turns out to be a complete metric space on which every couple of points can be joined by - at least - one d_{CC}-geodesic.

Carnot groups are *homogeneous groups*, in the sense that they admit a 1-parameter group of automorphisms $\delta_t : \mathbb{G} \longrightarrow \mathbb{G}$ $(t \geq 0)$ defined by

$$\delta_t x := exp\left(\sum_{j, i_j} t^j\, x_{i_j} \mathrm{e}_{i_j}\right),$$

where $x = exp\left(\sum_{j, i_j} x_{i_j} \mathrm{e}_{i_j}\right) \in \mathbb{G}$. The *homogeneous dimension* of $\mathbb{G}$ is the integer

$$Q := \sum_{i=1}^{k} i\, h_i$$

coinciding with the *Hausdorff dimension* of $(\mathbb{G}, d_{CC})$ as a metric space; see [45], [50], [31].

Definition 2.4. *A continuous distance* $\varrho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}_+$ *is called homogenous if*

(i) $\varrho(x, y) = \varrho(z \bullet x, z \bullet y)$ *for every* $x,\ y,\ z \in \mathbb{G}$*;*

(ii) $\varrho(\delta_t x, \delta_t y) = t\varrho(x, y)$ *for all* $t \geq 0$*.*

The CC-distance d_{CC} is an example of homogeneous distance. Another interesting example can be found in [28]. On every Carnot group there exists a smooth, subadditive and homogeneous norm; see [33]. In other words there exists a function $\| \cdot \|_\varrho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}_+ \cup \{0\}$ such that:

(i) $\|x \bullet y\|_\varrho \leq \|x\|_\varrho + \|y\|_\varrho$;

(ii) $\|\delta_t x\|_\varrho = t\|x\|_\varrho \quad (t \geq 0)$;

(iii) $\|x\|_\varrho = 0 \Leftrightarrow x = 0$;

(iv) $\|x\|_\varrho = \|x^{-1}\|_\varrho$;

(v) $\| \cdot \|_\varrho$ is continuous and smooth on $\mathbb{G} \setminus \{0\}$.

For instance, a homogeneous norm ϱ which is smooth on $\mathbb{G} \setminus \{0\}$, can be defined by

$$\|x\|_\varrho := (|x_H|^\lambda + |x_{H_2}|^{\lambda/2} + |x_{H_3}|^{\lambda/3} + ... + |x_{H_k}|^{\lambda/k})^{1/\lambda},$$

where λ is a positive number evenly divisible by i, for $i = 1, ..., k$. Here $|x_{H_i}|$ denotes the Euclidean norm of the projection of x onto the i-th layer H_i of the stratification of $\mathfrak{g}$ $(i = 1, ..., k)$.

For later purposes we will need the following:

Definition 2.5. *Let $\varrho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}_+$ be a homogeneous distance such that*

(i) *ϱ is piecewise $\mathbf{C}^1$-smooth;*

(ii) *$|grad_H \varrho| \leq 1$ at each regular point of ϱ;*

(iii) *$|x_H| \leq \varrho(x)$, where $\varrho(x) = \varrho(0, x) = \|x\|_\varrho$. Furthermore, we shall assume that there exist $c_i \in \mathbb{R}_+$ such that*

$$|x_{H_i}| \leq c_i \varrho^i(x) \qquad i = 2, ..., k.$$

Example 2.6. *It can be proved that the CC-distance d_{CC} satisfies all the previous assumptions. Another example can be found for the case of the Heisenberg group $\mathbb{H}^n$; see Example 2.11. Indeed, the Korany norm, defined by*

$$\|x\|_\varrho := \varrho(x) = \sqrt[4]{|x_H|^4 + 16t^2} \qquad (x = \exp(x_H, t) \in \mathbb{H}^n),$$

turns out to be homogeneous and $\mathbf{C}^\infty$-smooth out of the identity $0 \in \mathbb{H}^n$. By direct computation, one can show that ϱ satisfies (ii) and (iii) of Definition 2.5.

Having a Riemannian metric, we may define the left-invariant co-frame $\underline{\omega} := \{\omega_i : i = 1, ..., n\}$ dual to $\underline{X}$. In particular, the *left-invariant 1-forms* [4] ω_i are uniquely determined by the condition:

$$\omega_i(X_j) = \langle X_i, X_j \rangle = \delta_i^j \qquad (i, j = 1, ..., n)$$

where δ_i^j denotes the Kronecker delta. Remind that the *structural constants* of the Lie algebra $\mathfrak{g}$ associated with the left invariant frame $\underline{X}$ are defined by

$$C^{\mathfrak{g}}{}^r_{ij} := \langle [X_i, X_j], X_r \rangle \qquad \text{for } i, j, r = 1, ..., n.$$

They satisfy

(i) $C^{\mathfrak{g}}{}^r_{ij} + C^{\mathfrak{g}}{}^r_{ji} = 0$, (skew-symmetry)

(ii) $\sum_{j=1}^n C^{\mathfrak{g}}{}^i_{jl} C^{\mathfrak{g}}{}^j_{rm} + C^{\mathfrak{g}}{}^i_{jm} C^{\mathfrak{g}}{}^j_{lr} + C^{\mathfrak{g}}{}^i_{jr} C^{\mathfrak{g}}{}^j_{ml} = 0$ (Jacobi's identity).

The stratification hypothesis on the Lie algebra implies that

$$X_i \in H_l,\ X_j \in H_m \Longrightarrow [X_i, X_j] \in H_{l+m}.$$

Definition 2.7 (Matrices of structural constants)**.** *We shall set*

(i) $C_H^\alpha := [C^{\mathfrak{g}}{}^\alpha_{ij}]_{i,j \in I_H} \in \mathcal{M}_{h \times h}(\mathbb{R}) \qquad (\alpha \in I_{H_2})$;

(ii) $C^\alpha := [C^{\mathfrak{g}}{}^\alpha_{ij}]_{i,j=1,...,n} \in \mathcal{M}_{n \times n}(\mathbb{R}) \qquad (\alpha \in I_V)$.

The linear operators associated with these matrices will be denoted in the same way.

[4] That is, $L_p^* \omega_I = \omega_I$ for every $p \in \mathbb{G}$.

Definition 2.8. *We shall denote by* ∇ *the unique left-invariant Levi-Civita connection on* $\mathbb{G}$ *associated with* g*. Moreover, if* $X, Y \in \mathfrak{X}(H) := \mathbf{C}^\infty(\mathbb{G}, H)$*, we shall set*

$$\nabla^H_X Y := \mathcal{P}_H(\nabla_X Y).$$

Remark 2.9. *We stress that* ∇^H *is a partial connection, called horizontal* H*-connection; see [48] and references therein. Using Definition 2.8 and the properties of the structural constants of the Levi-Civita connection, we get that* ∇^H *is flat, i.e.* $\nabla^H_{X_i} X_j = 0$ *for every* $i, j \in I_H$*. Moreover* ∇^H *is compatible with the sub-Riemannian metric* g_H*, i.e.*

$$X\langle Y, Z\rangle = \langle \nabla^H_X Y, Z\rangle + \langle Y, \nabla^H_X Z\rangle \qquad \text{for all} \quad X, Y, Z \in \mathfrak{X}(H).$$

This follows immediately from the very definition of ∇^H *and the corresponding properties of the Levi-Civita connection* ∇ *on* $\mathbb{G}$*. Finally,* ∇^H *is torsion-free, i.e.*

$$\nabla^H_X Y - \nabla^H_Y X - \mathcal{P}_H[X, Y] = 0 \qquad \text{for all} \quad X, Y \in \mathfrak{X}(H).$$

For the global left-invariant frame $\underline{X} = \{X_1, ..., X_n\}$ *it turns out that*

$$\nabla_{X_i} X_j = \frac{1}{2}\sum_{r=1}^{n}\left({C^{\mathfrak{g}}}^r_{ij} - {C^{\mathfrak{g}}}^i_{jr} + {C^{\mathfrak{g}}}^j_{ri}\right) X_r \qquad (i,\, j = 1, ..., n).$$

Definition 2.10. *If* $\psi \in \mathbf{C}^\infty(\mathbb{G})$ *we define the horizontal gradient of* ψ *as the unique horizontal vector field* $\mathit{grad}_H\,\psi$ *such that*

$$\langle \mathit{grad}_H\,\psi, X\rangle = d\psi(X) = X\psi \qquad \text{for all} \quad X \in \mathfrak{X}(H).$$

The horizontal divergence of $X \in \mathfrak{X}(H)$*,* $\mathit{div}_H X$*, is defined, at each point* $x \in \mathbb{G}$*, by*

$$\mathit{div}_H X(x) := \mathrm{Trace}\big(Y \longrightarrow \nabla^H_Y X\big)(x) \quad (Y \in H_x).$$

Example 2.11 (Heisenberg group $\mathbb{H}^n$)**.** *Let* $\mathfrak{h}_n := T_0\mathbb{H}^n = \mathbb{R}^{2n+1}$ *denote the Lie algebra of the Heisenberg group* $\mathbb{H}^n$*, perhaps the most important 2-step example. Its Lie algebra* $\mathfrak{h}_n$ *is defined by the rules*

$$[\mathrm{e}_i, \mathrm{e}_{i+1}] = \mathrm{e}_{2n+1}$$

for $i = 2k+1$*,* $k = 0, ..., n-1$*, where all other commutators vanish. One has* $\mathfrak{h}_n = H \oplus \mathbb{R}\mathrm{e}_{2n+1}$*, where* $H = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}_i : i = 1, ..., 2n\}$*. The second layer of the grading is the center of* $\mathfrak{h}_n$*. These rules determine the group law* $\bullet$ *via the Baker-Campbell-Hausdorff formula. For every* $x = \exp\left(\sum_{i=1}^{2n+1} x_i X_i\right)$*,* $y = \exp\left(\sum_{i=1}^{2n+1} y_i X_i\right) \in \mathbb{H}^n$ *one has*

$$x \bullet y = \exp\Big(x_1 + y_1, ..., x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \tfrac{1}{2}\textstyle\sum_{k=1}^{n}(x_{2k-1}y_{2k} - x_{2k}y_{2k-1})\Big).$$

We also stress that

$$C^{2n+1}_H := \left|\begin{array}{ccccc} 0 & 1 & 0 & 0 & \cdot \\ -1 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 1 & \cdot \\ 0 & 0 & -1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|.$$

2.2 Hypersurfaces, homogeneous measures and geometric structures

In the sequel $\mathcal{H}^m_\varrho$ and $\mathcal{S}^m_\varrho$ will denote, respectively, the Hausdorff measure and the spherical Hausdorff measure associated with a homogeneous distance ϱ on $\mathbb{G}$[5]. In the case $\varrho = d_{CC}$ we use the notation $\mathcal{H}^m_{CC}$ and $\mathcal{S}^m_{CC}$.

The Riemannian left-invariant volume form on $\mathbb{G}$ is defined as $\sigma^n_{\scriptscriptstyle R} := \bigwedge_{i=1}^n \omega_i \in \bigwedge^n(T^*\mathbb{G})$. The measure $\sigma^n_{\scriptscriptstyle R}$ is the Haar measure of $\mathbb{G}$ and equals the push-forward of the n-dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathfrak{g} \cong \mathbb{R}^n$.

In the study of hypersurfaces of Carnot groups we need the notion of *characteristic point.*

Definition 2.12. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^1$-smooth hypersurface. Then we say that $x \in S$ is a characteristic point of S if $\dim H_x = \dim(H_x \cap T_x S)$ or, equivalently, if $H_x \subset T_x S$. The characteristic set of S is denoted by C_S. One has*

$$C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_x S)\}.$$

So a hypersurface $S \subset \mathbb{G}$, oriented by its unit normal vector ν, is *non-characteristic* if, and only if, the horizontal subbundle H is *transversal* to S.

We stress that the $(Q-1)$-dimensional CC-Hausdorff measure of the characteristic set C_S vanishes, i.e. $\mathcal{H}^{Q-1}_{CC}(C_S) = 0$; see [37]. The $(n-1)$-dimensional *Riemannian measure* along S is defined by

$$\sigma^{n-1}_{\scriptscriptstyle R} \llcorner S := (\nu \lrcorner \sigma^n_{\scriptscriptstyle R})|_S,$$

where $\lrcorner$ denotes the "contraction" operator on differential forms; see footnote 2. Just as in [46, 48], [13], [35], [55], since we are studying smooth hypersurfaces, instead of the variational definition of the H-perimeter measure *à la De Giorgi* (see, for instance, [27, 28], [29], [46] and bibliographies therein) we shall define an $(n-1)$-differential form which, by integration, coincides with the usual variational H-perimeter measure.

Definition 2.13 ($\sigma^{n-1}_{\scriptscriptstyle H}$-measure on hypersurfaces)**.** *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^1$-smooth non-characteristic hypersurface and denote by ν its unit normal vector. We call unit H-normal along S, the normalized projection of ν onto H, i.e.*

$$\nu_{\scriptscriptstyle H} := \frac{\mathcal{P}_{\scriptscriptstyle H}\,\nu}{|\mathcal{P}_{\scriptscriptstyle H}\,\nu|}.$$

We define the $(n-1)$-dimensional measure $\sigma^{n-1}_{\scriptscriptstyle H}$ along S to be the measure associated with the $(n-1)$-differential form $\sigma^{n-1}_{\scriptscriptstyle H} \in \Lambda^{n-1}(T^*S)$ defined as the contraction of the volume form $\sigma^n_{\scriptscriptstyle R}$ by the horizontal unit normal $\nu_{\scriptscriptstyle H}$, i.e. $\sigma^{n-1}_{\scriptscriptstyle H} \llcorner S := (\nu_{\scriptscriptstyle H} \lrcorner \sigma^n_{\scriptscriptstyle R})|_S$.

If we allow S to have characteristic points, we may extend the definition of $\sigma^{n-1}_{\scriptscriptstyle H}$ by setting $\sigma^{n-1}_{\scriptscriptstyle H} \llcorner C_S = 0$. For every $\mathbf{C}^1$-smooth hypersurface $S \subset \mathbb{G}$, it turns out that

$$\sigma^{n-1}_{\scriptscriptstyle H} \llcorner S = |\mathcal{P}_{\scriptscriptstyle H}\,\nu|\, \sigma^{n-1}_{\scriptscriptstyle R} \llcorner S$$

[5]We remind that:

(i) $\mathcal{H}^m_\varrho(S) = \lim_{\delta\to 0^+} \mathcal{H}^m_{\varrho,\delta}(S)$ where, up to a constant multiple,

$$\mathcal{H}^m_{\varrho,\delta}(S) = \inf\Big\{\sum_i \Big(\mathrm{diam}_\varrho(C_i)\Big)^m :\ S \subset \bigcup_i C_i;\ \mathrm{diam}_\varrho(C_i) < \delta\Big\}$$

and the infimum is taken with respect to any non-empty family of closed subsets $\{C_i\}_i \subset \mathbb{G}$;

(ii) $\mathcal{S}^m_\varrho(S) = \lim_{\delta\to 0^+} \mathcal{S}^m_{\varrho,\delta}(S)$ where, up to a constant multiple,

$$\mathcal{S}^m_{\varrho,\delta}(S) = \inf\Big\{\sum_i \Big(\mathrm{diam}_\varrho(B_i)\Big)^m :\ S \subset \bigcup_i B_i;\ \mathrm{diam}_\varrho(B_i) < \delta\Big\}$$

and the infimum is taken with respect to closed ϱ-balls B_i.

and that $C_S = \{x \in S : |\mathcal{P}_H \nu| = 0\}$. It is also important to remark that

$$\sigma_H^{n-1}(S \cap B) = k_\varrho(\nu_H)\, \mathcal{S}_\varrho^{Q-1} \llcorner (S \cap B),$$

for all $B \in \mathcal{B}or(\mathbb{G})$, where the (bounded) density-function $k_\varrho(\nu_H)$, called *metric factor*, depends on ν_H and on the fixed homogeneous metric ϱ on $\mathbb{G}$; see [37]. Later we shall discuss these aspects in Section 3.4.

Definition 2.14. *For every $x \in S \setminus C_S$ let $H_xS = H_x \cap T_xS$ be the horizontal tangent space at x. Obviously $H_x = H_xS \oplus \nu_H(x)$. We then define in the usual way the subbundles HS and $\nu_H S$, called, respectively, horizontal tangent bundle and horizontal normal bundle of S. One has*

$$H = HS \oplus \nu_H S.$$

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface. We stress that if ∇^{TS} is the connection induced on TS from the Levi-Civita connection ∇ on $T\mathbb{G}$[6], then ∇^{TS} induces a partial connection ∇^{HS} on $HS \subset TS$ defined by[7]

$$\nabla_X^{HS} Y := \mathcal{P}_{HS}(\nabla_X^{TS} Y) \qquad \text{for every } X, Y \in \mathfrak{X}(HS) := \mathbf{C}^\infty(S, HS).$$

Starting from the orthogonal decomposition $H = HS \oplus \nu_H S$, we could also define ∇^{HS} by making use of the classical definition of "connection on submanifolds"; see [10]. It turns out that

$$\nabla_X^{HS} Y = \nabla_X^H Y - \langle \nabla_X^H Y, \nu_H \rangle\, \nu_H \qquad \text{for every } X, Y \in \mathfrak{X}(HS).$$

Definition 2.15. *We call HS- gradient of $\psi \in \mathbf{C}^\infty(S)$ the unique horizontal tangent vector field $grad_{HS}\psi$ satisfying*

$$\langle grad_{HS}\psi, X\rangle = d\psi(X) = X\psi \qquad \textit{for all} \quad X \in HS.$$

We denote by div_{HS} the divergence operator on HS, i.e. if $X \in HS$ and $x \in S$, then

$$div_{HS} X(x) := \mathrm{Trace}\big(Y \longrightarrow \nabla_Y^{HS} X\big)(x) \quad (Y \in H_xS).$$

The horizontal 2nd fundamental form of S is the map given by

$$B_H(X, Y) := \langle \nabla_X^H Y, \nu_H \rangle \qquad \textit{for every } X, Y \in \mathfrak{X}(HS).$$

The horizontal mean curvature $\mathcal{H}_H$ is the trace of B_H, i.e.

$$\mathcal{H}_H := \mathrm{Tr} B_H = -div_H \nu_H.$$

Definition 2.16. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurace oriented by ν. We shall set*

(i) $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|} \qquad (\alpha \in I_V)$;

(ii) $\varpi := \sum_{\alpha \in I_V} \varpi_\alpha X_\alpha$;

(iii) $C_H := \sum_{\alpha \in I_{H_2}} \varpi_\alpha C_H^\alpha$.

In particular, from (i) it follows that $\frac{\nu}{|\mathcal{P}_H \nu|} = \nu_H + \varpi$. We stress that the horizontal 2nd fundamental form $B_H(X, Y)$ is a $\mathbf{C}^\infty(S)$-bilinear form in X and Y. In general, B_H *is not symmetric* and so it is a sum of two matrices, one symmetric and the other skew-symmetric, i.e. $B_H = S_H + A_H$, where the skew-symmetric matrix A_H satisfies $A_H = \frac{1}{2} C_H\big|_{HS}$; see [48]. Moreover, the following identities hold true:

$$B_H(Y, X) - B_H(X, Y) = \langle \mathcal{P}_H[Y, X], \nu_H \rangle = \langle [X, Y], \varpi \rangle = -\langle C_H X, Y \rangle$$

for every X, $Y \in \mathfrak{X}(HS)$.

[6]Therefore, ∇^{TS} is the Levi-Civita connection on S; see [10].

[7]The map $\mathcal{P}_{HS} : TS \longrightarrow HS$ denotes the orthogonal projection of TS onto HS.

Remark 2.17 (Induced stratification on TS; see [31]). *The stratification of* $\mathfrak{g}$ *induces a "natural" decomposition of the tangent space of any smooth submanifold of* $\mathbb{G}$*. Let us analyze the case of a hypersurface* $S \subset \mathbb{G}$*. So let us intersect, at each point* $x \in S$*, the tangent spaces* T_xS *with* $T^i_x = \oplus_{j=1}^{i}(H_j)_x$*. We shall set* $T^iS := TS \cap T^i\mathbb{G}$*,* $n'_i := \dim T^iS$*,* $H_iS := T^iS \setminus T^{i-1}S$ *and, for simplicity,* $HS = H_1S$*. It follows that* $TS := \oplus_{i=1}^{k} H_iS$ *and that* $\sum_{i=1}^{k} n'_i = n-1$*. We also set* $VS := \oplus_{i=2}^{k} H_iS$*. It turns out that the Hausdorff dimension of a smooth hypersurface* S *is* $Q-1 = \sum_{i=1}^{k} i\, n'_i$*; see [31], [53], [28], [37, 41], [35]. If the horizontal tangent bundle* HS *is generic and horizontal, then the couple* (S, HS) *is actually a* k*-step CC-space; see Section 2.1.*

Example 2.18. *Let us consider the case of a smooth hypersurface* $S \subset \mathbb{H}^n$*. If* $n = 1$*, then the horizontal tangent bundle* HS *of* S *cannot be a 2-step CC-space because* HS *is 1-dimensional. Nevertheless, if* $n > 1$*, this is no longer true, since along any non-characteristic domain* $\mathcal{U} \subseteq S$*,* HS *turns out to be generic and horizontal.*

Definition 2.19. *We say that a* $(n-2)$*-dimensional submanifold* N *of* $\mathbb{G}$ *is* H*-regular or non-characteristic at* $x \in N$ *if there exist two linearly independent vectors* $\nu^1_{_H}, \nu^2_{_H} \in H_x$ *transversal to* N *at* x*. Without loss of generality, these vectors can be taken orthonormal at that point. The horizontal tangent space at* x *is defined by*

$$H_xN := H_x \cap T_xN.$$

When this condition is independent of $x \in N$*, we say that* N *is* H*-regular or non-characteristic. In this case, we define the associated vector bundles* $HN(\subset TN)$ *and* $\nu_{_H}N$*, called, respectively, horizontal tangent bundle and horizontal normal bundle. One has*

$$H := HN \oplus \nu_{_H}N, \qquad \nu_{_H}N = \mathbb{R}\nu^1_{_H} \oplus \mathbb{R}\nu^2_{_H}.$$

Definition 2.20 (Characteristic set of N). *The characteristic set* C_N *of a* $\mathbf{C}^1$*-smooth* $(n-2)$*-dimensional submanifold* N *of* $\mathbb{G}$ *is defined by*

$$C_N := \{x \in N : \dim H_x - \dim(H_x \cap T_xN) \leq 1\}.$$

This definition of C_N has been used in [37], where it was shown that the $(Q-2)$-dimensional Hausdorff measure (with respect to any homogeneous metric ϱ on $\mathbb{G}$) of a $\mathbf{C}^1$-smooth submanifold $N \subset \mathbb{G}$ vanishes, i.e. $\mathcal{H}^{Q-2}_{\varrho}(C_N) = 0$.

Definition 2.21 ($\sigma^{n-2}_{_H}$-measure). *Let* $N \subset \mathbb{G}$ *be a* $(n-2)$*-dimensional,* H*-regular submanifold and let* $\nu^1_{_H}, \nu^2_{_H} \in \nu_{_H}N$ *as in Definition 2.19. Set* $\nu_{_H} := \nu^1_{_H} \wedge \nu^2_{_H} \in \bigwedge^2(T\mathbb{G}|_N)$ *and define the homogeneous measure* $\sigma^{n-2}_{_H}$ *along* N *by*

$$\sigma^{n-2}_{_H} \llcorner S := (\nu_{_H} \lrcorner\, \sigma^n_{_R})|_S.$$

In other words, $\sigma^{n-2}_{_H}$ *is the* $(Q-2)$*-homogeneous measure defined by contraction*[8] *of the top-dimensional volume form* $\sigma^n_{_R}$ *by the horizontal 2-vector* $\nu_{_H} = \nu^1_{_H} \wedge \nu^2_{_H}$*.*

As in the case of the H-perimeter, $\sigma^{n-2}_{_H}$ can explicitly be represented by using the $(n-2)$-dimensional Riemannian measure $\sigma^{n-2}_{_R}$ along N. More precisely, for every $x \in N$, let $\nu_1, \nu_2 \in \nu_xN$, where νN denotes the Riemannian normal bundle along N. We also assume that they are orthonormal at that point. In other words, the (decomposable) 2-vector $\nu_1 \wedge \nu_2 \in \bigwedge^2(T_x\mathbb{G})$ is a unit 2-normal vector along N at x. We may assume that $\nu_1 \wedge \nu_2 \in \bigwedge^2(T\mathbb{G}|_N)$ is the unit 2-vector field which determines the orientation of N. By standard Linear Algebra, we get

$$\nu_{_H} = \frac{\mathcal{P}_{_H}\nu_1 \wedge \mathcal{P}_{_H}\nu_2}{|\mathcal{P}_{_H}\nu_1 \wedge \mathcal{P}_{_H}\nu_2|}.$$

[8] For the most general definition of $\lrcorner$, see [22], Ch.1.

If $C_N \neq \emptyset$ we extend the definition of σ_H^{n-2} by setting $\sigma_H^{n-2} \llcorner C_N = 0$. So for every $\mathbf{C}^1$-smooth $(n-2)$-dimensional submanifold $N \subset \mathbb{G}$, it turns out that

$$\sigma_H^{n-2} = |\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2| \, \sigma_R^{n-2}$$

and one has $C_N = \{x \in N : |\mathcal{P}_H \nu_1 \wedge \mathcal{P}_H \nu_2| = 0\}$. The measure σ_H^{n-2} is $(Q-2)$-homogeneous, with respect to Carnot dilations $\{\delta_t\}_{t>0}$, i.e. $\delta_t^* \sigma_H^{n-2} = t^{Q-2} \sigma_H^{n-2}$. It can be shown that σ_H^{n-2} is equivalent, up to a bounded density-function, to the $(Q-2)$-dimensional Hausdorff measure associated to a homogeneous distance ϱ on $\mathbb{G}$; see [41].

3 Preliminary tools

3.1 Coarea Formula for the *HS*-gradient

Theorem 3.1. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let $\varphi \in \mathbf{C}^1(S)$. Then*

$$\int_S |grad_{HS} \varphi(x)| \, \sigma_H^{n-1}(x) = \int_{\mathbb{R}} \sigma_H^{n-2}(\varphi^{-1}[s] \cap S) \, ds \tag{1}$$

and

$$\int_S \psi(x) |grad_{HS} \varphi(x)| \, \sigma_H^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \psi(y) \, \sigma_H^{n-2}(y)$$

for every $\psi \in L^1(S, \sigma_H^{n-1})$.

Proof. The theorem can be deduced by using the Riemannian Coarea Formula. Indeed, let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and $\varphi \in \mathbf{C}^1(S)$. Then

$$\int_S \phi(x) |grad_{TS} \varphi(x)| \sigma_R^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \sigma_R^{n-2}(y)$$

for every $\psi \in L^1(S, \sigma_R^{n-1})$; see [7], [22]. Choosing

$$\phi = \psi \frac{|grad_{HS} \varphi|}{|grad_{TS} \varphi|} |\mathcal{P}_H \nu|,$$

for some $\psi \in L^1(S, \sigma_H^{n-1})$, yields

$$\int_S \phi |grad_{TS} \varphi| \, \sigma_R^{n-1} = \int_S \psi \frac{|grad_{HS} \varphi|}{|grad_{TS} \varphi|} \underbrace{|\mathcal{P}_H \nu| \, \sigma_R^{n-1}}_{=\sigma_H^{n-1}} = \int_S |grad_{HS} \varphi| \, \sigma_H^{n-1}.$$

Along $\varphi^{-1}[s]$ it turns out that $\eta = \frac{grad_{TS} \varphi}{|grad_{TS} \varphi|}$. Therefore $|\mathcal{P}_{HS} \eta| = \frac{|grad_{HS} \varphi|}{|grad_{TS} \varphi|}$ and it follows that

$$\begin{aligned} \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \, \sigma_R^{n-2} &= \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \psi \frac{|grad_{HS} \varphi|}{|grad_{TS} \varphi|} |\mathcal{P}_H \nu| \, \sigma_R^{n-2} \\ &= \int_{\varphi^{-1}[s] \cap S} \psi \, \underbrace{|\mathcal{P}_{HS} \eta| |\mathcal{P}_H \nu| \, \sigma_R^{n-2}}_{=\sigma_H^{n-2}} \\ &= \int_{\varphi^{-1}[s] \cap S} \psi \, \sigma_H^{n-2}. \end{aligned}$$

□

3.2 Horizontal Diverge Theorem and Integration by Parts

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let $\mathcal{U} \subset S$ be a relatively compact open set with $\mathbf{C}^1$-smooth boundary (or, smooth enough for Stokes' Theorem). If $X \in \mathfrak{X}(S)$, by definition of σ_H^{n-1}, using the "infinitesimal" *Riemannian Divergence Formula* (see [57]), one gets

$$\begin{aligned} d(X \lrcorner\, \sigma_H^{n-1})|_{\mathcal{U}} &= d(|\mathcal{P}_H \nu|\, X \lrcorner\, \sigma^{n-1}) = div_{TS}(|\mathcal{P}_H \nu|\, X)\, \sigma_R^{n-1} \\ &= \left(div_{TS} X + \left\langle X, \frac{grad_{TS}|\mathcal{P}_H \nu|}{|\mathcal{P}_H \nu|} \right\rangle \right) \sigma_H^{n-1} \llcorner\, \mathcal{U}, \end{aligned} \tag{2}$$

where $grad_{TS}$ and div_{TS} are, respectively, the tangential gradient and the tangential divergence operators. By integrating (2) and using Stokes' formula one gets the integration by parts formula for the H-perimeter measure in the case of general vector fields. In this sub-Riemannian setting, there are however more intrinsic tools.

At this regard, let us discuss the horizontal integration by parts formulas for hypersurfaces immersed in a k-step Carnot group $\mathbb{G}$; see [46, 48] or [18].

Remark 3.2 (Homogeneous measure σ_H^{n-2} along ∂S)**.** *Assume that ∂S is a $(n-2)$-dimensional manifold, oriented by the unit normal vector η and denote by σ_R^{n-2} the Riemannian measure on ∂S, which can be written out as*

$$\sigma_R^{n-2} \llcorner\, \partial S = (\eta \lrcorner\, \sigma_R^{n-1})|_{\partial S}.$$

So if $X \in \mathfrak{X}(S)$, then

$$(X \lrcorner\, \sigma_H^{n-1})|_{\partial S} = \langle X, \eta \rangle |\mathcal{P}_H \nu|\, \sigma_R^{n-2} \llcorner\, \partial S.$$

Denote by $C_{\partial S}$ the characteristic set of ∂S, which turns out to be given by

$$C_{\partial S} = \{p \in \partial S : |\mathcal{P}_H \nu|\, |\mathcal{P}_{HS}\, \eta| = 0\}.$$

Using Definition 2.21 yields that

$$\sigma_H^{n-2} \llcorner\, \partial S = \left(\frac{\mathcal{P}_{HS}\, \eta}{|\mathcal{P}_{HS}\, \eta|} \lrcorner\, \sigma_H^{n-1} \right) \bigg|_{\partial S},$$

or, equivalently

$$\sigma_H^{n-2} \llcorner\, \partial S = |\mathcal{P}_H \nu|\, |\mathcal{P}_{HS}\, \eta|\, \sigma_R^{n-2} \llcorner\, \partial S.$$

Setting

$$\eta_{HS} := \frac{\mathcal{P}_{HS}\, \eta}{|\mathcal{P}_{HS}\, \eta|},$$

we say that η_{HS} is the unit horizontal normal along ∂S. One has

$$(X \lrcorner\, \sigma_H^{n-1})|_{\partial S} = \langle X, \eta_{HS} \rangle\, \sigma_H^{n-2} \llcorner\, \partial S \qquad \textit{for all} \quad X \in \mathbf{C}^1(S, HS).$$

Theorem 3.3 (Horizontal Divergence Theorem)**.** *Let $\mathbb{G}$ be a k-step Carnot group. Let $S \subset \mathbb{G}$ be an immersed hypersurface and $\mathcal{U} \subset S \setminus C_S$ be a non-characteristic relatively compact open set. Assume that $\partial\mathcal{U}$ is a smooth, $(n-2)$-dimensional manifold oriented by its unit normal vector η. Then, for every $X \in \mathbf{C}^1(S, HS)$ one has*

$$\int_{\mathcal{U}} (div_{HS} X + \langle C_H \nu_H, X \rangle)\, \sigma_H^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta_{HS} \rangle\, \sigma_H^{n-2}.$$

Corollary 3.4 (Horizontal Integration by Parts)**.** *Under the hypotheses Theorem 3.3, for every $X \in \mathbf{C}^1(S, H)$ one has*

$$\int_{\mathcal{U}} (div_{HS} X + \langle C_H \nu_H, X \rangle)\, \sigma_H^{n-1} = - \int_{\mathcal{U}} \mathcal{H}_H \langle X, \nu_H \rangle\, \sigma_H^{n-1} + \int_{\partial\mathcal{U}} \langle X, \eta_{HS} \rangle\, \sigma_H^{n-2}.$$

The proof of these results can be found in [48]. From Corollary 3.4 we get the next:

Corollary 3.5 (Integral Minkowsky-type formula)**.** *Under the hypotheses of Theorem 3.3, let* $x_H := \sum_{i\in I_H} x_i X_i$ *be the "horizontal position vector". Then*

$$\int_{\mathcal{U}} \left((h-1) + \mathcal{H}_H \langle x_H, \nu_H\rangle + \langle C_H \nu_H, x_H\rangle\right) \sigma_H^{n-1} = \int_{\partial\mathcal{U}} \langle x_H, \eta_{HS}\rangle \sigma_{HS}^{n-2}. \tag{3}$$

Proof. Apply Corollary 3.5 to the horizontal position vector field $x_H = \sum_{i\in I_H} x_i X_i$. □

Remark 3.6. *Let $S \subset \mathbb{G}$ be a compact $\mathbf{C}^2$-smooth hypersurface with boundary and let $\{\mathcal{U}_\epsilon\}_{\epsilon>0}$ be a family of open subsets of S with piecewise $\mathbf{C}^2$-smooth boundaries such that:*

(i) $C_S \subset \mathcal{U}_\epsilon$ *for every* $\epsilon > 0$*;*

(ii) $\sigma_R^{n-1}(\mathcal{U}_\epsilon) \longrightarrow 0$ *for* $\epsilon \to 0^+$*;*

(iii) $\int_{\mathcal{U}_\epsilon} |\mathcal{P}_H \nu|\, \sigma_R^{n-2} \longrightarrow 0$ *for* $\epsilon \to 0^+$*.*

Note that (iii) implies that $\sigma_H^{n-2}(\partial\mathcal{U}_\epsilon) \to 0$ as $\epsilon \to 0^+$. By means of the family $\{\mathcal{U}_\epsilon\}_{\epsilon>0}$ we may extend the previous formulae to hypersurfaces having non-empty characteristic set. Indeed, by applying Theorem 3.3 to $S \setminus \mathcal{U}_\epsilon$, we get that

$$\int_{S\setminus\mathcal{U}_\epsilon} \left(div_{HS} X + \langle C_H \nu_H, X\rangle\right) \sigma_H^{n-1} = \int_{\partial S} \langle X, \eta_{HS}\rangle \sigma_H^{n-2} - \int_{\partial\mathcal{U}_\epsilon} \langle X, \eta_{HS}\rangle \sigma_H^{n-2}. \tag{4}$$

Since C_S is a null set for the σ_H^{n-1}-measure, letting $\epsilon \to 0^+$ yields

$$\lim_{\epsilon\to 0^+} \int_{S\setminus\mathcal{U}_\epsilon} \left(div_{HS} X + \langle C_H \nu_H, X\rangle\right) \sigma_H^{n-1} = \int_S \left(div_{HS} X + \langle C_H \nu_H, X\rangle\right) \sigma_H^{n-1}.$$

By using (iii), the third integral in (4) *vanishes. It follows that Theorem 3.3 and Corollary 3.4 hold true for hypersurfaces having non-empty characteristic set.*

Remark 3.7. *Let $B_\varrho(0,t)$ be the ϱ-ball of radius t centered at $0 \in \mathbb{G}$ and circumscribed about S. Using* (3) *and $|\langle x_H, \nu_H\rangle| \le |x_H| \le \|x\|_\varrho$, yields*

$$(h-1)\,\sigma_H^{n-1}(S) \le t \left(\int_S \left(|\mathcal{H}_H| + |C_H \nu_H|\right) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S)\right). \tag{5}$$

If S is minimal, i.e. $\mathcal{H}_H = 0$, it follows that

$$(h-1)\,\sigma_H^{n-1}(S) \le t \left(\int_S |C_H \nu_H|\, \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S)\right).$$

Set $S_t := S \cap B_\varrho(x,t)$ for $x \in \mathrm{Int}(S \setminus C_S)$. Starting from (5) *it is possible to prove the validity of a horizontal monotonicity inequality. More precisely, one has*

$$-\frac{d}{dt}\frac{\sigma_H^{n-1}(S_t)}{t^{h-1}} \le \frac{1}{t^{h-1}}\left(\int_{S_t} \left(|\mathcal{H}_H| + |C_H \nu_H|\right) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S \cap B_\varrho(x,t))\right)$$

for $\mathcal{L}^1$-a.e. $t > 0$; see [49]. The proof of this inequality is mainly based on the Coarea Formula; see Theorem 3.1. The strategy of the proof is the same as in the classical setting. Nevertheless, from the last inequality we cannot deduce the "right" isoperimetric inequality. Note indeed that the power $(h-1)$ is not the right one, which should be $(Q-1)$; see also Section 4.1.

3.3 1st variation of σ_H^{n-1} up to C_S

We recall the 1st variation formula of σ_H^{n-1} along the lines of [48], but with a further analysis of the characteristic case; see also [18], [46], [13], [55], [35].

Let $\mathbb{G}$ be a k-step Carnot group and let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface oriented by its unit normal vector ν. Moreover, let $\mathcal{U} \subset S \setminus C_S$ be a non-characteristic relatively compact open set and assume that $\partial\mathcal{U}$ is a $(n-2)$-dimensional $\mathbf{C}^1$-smooth submanifold oriented by its outward unit normal vector η.

Definition 3.8. *Let $\imath : \mathcal{U} \to \mathbb{G}$ denote the inclusion of $\mathcal{U}$ in $\mathbb{G}$ and let $\vartheta :]-\epsilon, \epsilon[\times \mathcal{U} \to \mathbb{G}$ be a smooth map. We say that ϑ is a smooth variation of $\imath$ if*

(i) *every $\vartheta_t := \vartheta(t, \cdot) : \mathcal{U} \to \mathbb{G}$ is an immersion;*

(ii) *$\vartheta_0 = \imath$.*

The variation vector of ϑ is given by $W := \frac{\partial\vartheta}{\partial t}\big|_{t=0} = \vartheta_* \frac{\partial}{\partial t}\big|_{t=0}$.

For any $t \in]-\epsilon, \epsilon[$ let ν^t be the unit normal vector along $\mathcal{U}_t := \vartheta_t(\mathcal{U})$ and let $\left(\sigma_H^{n-1}\right)_t$ be the Riemannian measure on $\mathcal{U}_t$. Let us define the differential $(n-1)$-form $\left(\sigma_H^{n-1}\right)_t$ along $\mathcal{U}_t$, by

$$\left(\sigma_H^{n-1}\right)_t \llcorner \mathcal{U}_t = (\nu_H^t \lrcorner \sigma_R^n) \llcorner \mathcal{U}_t \in \Lambda^{n-1}(T\mathcal{U}_t) \qquad t \in (-\epsilon, \epsilon)$$

where $\nu_H^t := \frac{\mathcal{P}_H \nu^t}{|\mathcal{P}_H \nu^t|}$. Moreover set $\Gamma(t) := \vartheta_t^* \left(\sigma_H^{n-1}\right)_t \in \Lambda^{n-1}(T\mathcal{U}),\ t \in]-\epsilon, \epsilon[$. The 1st variation $I_{\mathcal{U}}(W, \sigma_H^{n-1})$ of σ_H^{n-1} is given by

$$I_{\mathcal{U}}(W, \sigma_H^{n-1}) = \frac{d}{dt}\left(\int_{\mathcal{U}} \Gamma(t)\right)\bigg|_{t=0} = \int_{\mathcal{U}} \dot{\Gamma}(0).$$

Theorem 3.9 (1st variation of σ_H^{n-1})**.** *Under the previous assumptions, we have*

$$I_{\mathcal{U}}(W, \sigma_H^{n-1}) = -\int_{\mathcal{U}} \mathcal{H}_H \left\langle W, \frac{\nu}{|\mathcal{P}_H \nu|}\right\rangle \sigma_H^{n-1} + \int_{\partial\mathcal{U}} \langle W, \eta\rangle\, |\mathcal{P}_H \nu|\, \sigma_R^{n-2}. \tag{6}$$

For a proof, see [48]. It is clear that if W is horizontal, then (6) becomes more "intrinsic".

Theorem 3.10 (Horizontal 1st variation of σ_H^{n-1})**.** *Under the previous assumptions, let W be horizontal. Then*

$$I_{\mathcal{U}}(W, \sigma_H^{n-1}) = -\int_{\mathcal{U}} \mathcal{H}_H \langle W, \nu_H\rangle\, \sigma_H^{n-1} + \int_{\partial\mathcal{U}} \langle W, \eta_{HS}\rangle\, \sigma_H^{n-2}. \tag{7}$$

Proof. Use Theorem 3.9 and Remark 3.2. □

Therefore, in the case of horizontal variations, remembering Corollary 3.4, we get that

$$I_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \left(div_{HS} W + \langle C_H \nu_H, W\rangle\right) \sigma_H^{n-1}.$$

We stress that the horizontal 1st variation formula (7) is the sum of two terms, the first of whose only depends on the horizontal normal component of W, while the second one, only depends on its horizontal tangential component.

The previous formulae provide the 1st variation of σ_H^{n-1} on regular non-characteristic subsets of S containing $\mathrm{spt}(W)$. In the following remark we explain how one can extend the previous results to include the case in which the hypersurface has a possibly non-empty characteristic set C_S. A similar remark in the case of the Heisenberg group $\mathbb{H}^1$ was done in a recent work by Ritoré and Rosales [55]; see also [35].

Remark 3.11 (1st variation: case $C_S \neq \emptyset$)**.** *Let $S \subset \mathbb{G}$ be a compact $\mathbf{C}^2$-smooth hypersurface and let $W \in \mathbf{C}^1(S, T\mathbb{G})$ be the variation vector field of ϑ_t. Note that $|\mathcal{P}_H \nu|$ vanishes along C_S. Furthermore, $|\mathcal{P}_H \nu|$ is Lipschitz continuous at C_S and of class $\mathbf{C}^1$ out of C_S. Let $\{\mathcal{U}_\epsilon\}_{\epsilon>0}$ be a family of open neighborhoods of C_S; see Remark 3.6. For every $\epsilon > 0$ one computes*

$$I_S(W, \sigma_H^{n-1}) = I_{S\setminus\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) + I_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}). \tag{8}$$

The first addend is given by Theorem 3.9 and one has

$$I_{S\setminus\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) = -\int_{S\setminus\mathcal{U}_\epsilon} \mathcal{H}_H \left\langle W, \frac{\nu}{|\mathcal{P}_H \nu|} \right\rangle \sigma_H^{n-1} + \int_{\partial S} \langle W, \eta\rangle\, |\mathcal{P}_H \nu|\, \sigma_R^{n-2} + \int_{\partial\mathcal{U}_\epsilon} \langle W, \eta^-\rangle\, |\mathcal{P}_H \nu|\, \sigma_R^{n-2},$$

where η^- denotes the outward unit normal along $\partial\mathcal{U}_\epsilon$. The second addend in (8) *is given by*

$$I_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) = \frac{d}{dt}\left(\int_{\mathcal{U}_\epsilon} \left(\sigma_H^{n-1}\right)_t\right)\bigg|_{t=0} = \int_{\mathcal{U}_\epsilon} \frac{d}{dt}\left(\sigma_H^{n-1}\right)_t\bigg|_{t=0}.$$

Note that

$$\frac{d}{dt}\left(\sigma_H^{n-1}\right)_t\bigg|_{t=0} = \frac{d}{dt}|\mathcal{P}_H \nu^t|\bigg|_{t=0} \sigma_R^{n-1} + |\mathcal{P}_H \nu| \frac{d}{dt}\left(\sigma_H^{n-1}\right)_t\bigg|_{t=0}.$$

Now the first addend is bounded, since the function $|\mathcal{P}_H \nu^t|$ is Lipschitz along $\vartheta_t S$, while the second one, up to the bounded function $|\mathcal{P}_H \nu|$, is just the $(n-1)$-form which expresses the "infinitesimal" Riemannian 1st variation formula of σ_R^{n-1}; see [57]. Note that term can be written by means of a Lie derivative; see [46, 48]. More precisely, it turns out that

$$\frac{d}{dt}\left(\sigma_H^{n-1}\right)_t\bigg|_{t=0} = \imath^*_{\mathcal{U}_\epsilon} \mathcal{L}_W \left(\sigma_H^{n-1}\right)_t.$$

From this formula, Cartan's identity and a simple computation, it follows that

$$I_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) = \underbrace{\int_{\mathcal{U}_\epsilon} \left(\frac{d}{dt}|\mathcal{P}_H \nu^t|\bigg|_{t=0} - |\mathcal{P}_H \nu|\mathcal{H}_R\right) \left\langle W, \frac{\nu}{|\mathcal{P}_H \nu|}\right\rangle \sigma_H^{n-1}}_{I^{Int.}_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1})} + \underbrace{\int_{\partial\mathcal{U}_\epsilon} \langle W, \eta^+\rangle |\mathcal{P}_H \nu|\, \sigma_R^{n-2}}_{I^{Bound.}_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1})},$$

where η^+ denotes the inward unit normal along $\partial\mathcal{U}_\epsilon$. Since $\mathcal{H}_R$ is bounded [9]*, by using (ii) of Remark 3.6, we get that $I^{Int.}_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) \to 0$ as $\epsilon \to 0^+$. Moreover, since*[10] *$\eta^+ = -\eta^-$ along $\partial\mathcal{U}_\epsilon$, we have*

$$I_S(W, \sigma_H^{n-1}) = -\int_{S\setminus\mathcal{U}_\epsilon} \mathcal{H}_H \left\langle W, \frac{\nu}{|\mathcal{P}_H \nu|}\right\rangle \sigma_H^{n-1} + I^{Int.}_{\mathcal{U}_\epsilon}(W, \sigma_H^{n-1}) + \int_{\partial S} \langle W, \eta\rangle\, |\mathcal{P}_H \nu|\, \sigma_R^{n-2}$$

for every $\epsilon > 0$. By letting $\epsilon \to 0^+$, we get that

$$I_S(W, \sigma_H^{n-1}) = -\int_S \mathcal{H}_H \left\langle W, \frac{\nu}{|\mathcal{P}_H \nu|}\right\rangle \sigma_H^{n-1} + \int_{\partial S} \langle W, \eta\rangle\, |\mathcal{P}_H \nu|\, \sigma_R^{n-2},$$

which generalizes (6) *to the characteristic case; compare with [55], [35].*

[9] Since S is of class $\mathbf{C}^2$, the Riemannian mean curvature $\mathcal{H}_R$ is continuous along S.
[10] We stress that $\partial\mathcal{U}_\epsilon$ is the common boundary of $\mathcal{U}_\epsilon$ and $S \setminus \mathcal{U}_\epsilon$.

Remark 3.12. *By arguing as in the previous Remark 3.11, we also get that*

$$-\mathcal{H}_H = \frac{d}{dt}|\mathcal{P}_H \nu^t|\Big|_{t=0} - |\mathcal{P}_H \nu|\mathcal{H}_R,$$

at each non-characteristic point $x \in S \setminus C_S$. We stress that the right-hand side of this identity is well-defined for every $x \in S$, even if $x \in C_S$, and it is locally bounded whenever S is any close hypersurface of class $\mathbf{C}^2$.

The previous Remark 3.11 enables us to state the following:

Corollary 3.13 (1st variation of σ_H^{n-1}). *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface having possibly non-empty characteristic set C_S. Then, the 1st variation formula (6) holds true.*

3.4 Blow-up of the horizontal perimeter σ_H^{n-1} up to C_S

Let $S \subset \mathbb{G}$ be a smooth hypersurface. In this section we shall discuss the behavior of the horizontal perimeter σ_H^{n-1} at any point $x \in \mathrm{Int}(S)$. More precisely, we shall study the limit

$$\kappa_\varrho(\nu_H(x)) := \lim_{r \to 0^+} \frac{\sigma_H^{n-1}(S \cap B_\varrho(x,r))}{r^{Q-1}}, \tag{9}$$

where $B_\varrho(x,r)$ is the ϱ-ball of center x and radius r. The point $x \in \mathrm{Int}S$ is *not necessarily non-characteristic*. For a very similar analysis, we refer the reader to [37, 41, 40] and to [39], for what concerns the characteristic case in the setting of 2-step Carnot groups; see also [4, 5], [27, 28].

Theorem 3.14. *Let $\mathbb{G}$ be a k-step Carnot group.*

Case (i) *Let S be a $\mathbf{C}^1$-smooth hypersurface and $x \in \mathrm{Int}(S \setminus C_S)$; then*

$$\sigma_H^{n-1}(S \cap B_\varrho(x,r)) \sim \kappa_\varrho(\nu_H(x))\, r^{Q-1} \qquad \textit{for} \quad r \to 0^+, \tag{10}$$

where the constant $\kappa_\varrho(\nu_H(x))$ is called metric factor and is given by

$$\kappa_\varrho(\nu_H(x)) = \sigma_H^{n-1}\left(\mathcal{I}(\nu_H(x)) \cap B_\varrho(x,1)\right),$$

where $\mathcal{I}(\nu_H(x))$ denotes the vertical hyperplane [11] *orthogonal to $\nu_H(x)$.*

Case (ii) *Let $x \in \mathrm{Int}(S \cap C_S)$ and let $\alpha \in I_V$, $\mathrm{ord}(\alpha) = i$ be such that S can be represented, locally around x, as the exponential image of an X_α-graph of class $\mathbf{C}^i$. Without loss of generality, we may assume that $x = 0 \in \mathbb{G}$. In such case, one has*

$$S \cap B_\varrho(x,r) \subset \exp\left\{(\zeta_1, ..., \zeta_{\alpha-1}, \psi(\zeta), \zeta_{\alpha+1}, ..., \zeta_n\} : \zeta := (\zeta_1, ..., \zeta_{\alpha-1}, 0, \zeta_{\alpha+1}, ..., \zeta_n) \in \mathrm{e}_\alpha^\perp\right\},$$

where $\psi : \mathrm{e}_\alpha^\perp \cong \mathbb{R}^{n-1} \to \mathbb{R}$ is a function of class $\mathbf{C}^i$. If ψ satisfies

$$\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0) = 0 \qquad \textit{whenever} \quad \mathrm{ord}(j_1) + ... + \mathrm{ord}(j_l) < i, \tag{11}$$

then

$$\sigma_H^{n-1}(S \cap B_\varrho(x,r)) \sim \kappa_\varrho(C_S(x))\, r^{Q-1} \qquad \textit{for} \quad r \to 0^+ \tag{12}$$

[11] Note that $\mathcal{I}(\nu_H(x))$ corresponds to an ideal of the Lie algebra $\mathfrak{g}$. We also remark that the H-perimeter on a vertical hyperplane equals the Euclidean-Hausdorff measure $\mathcal{H}_{Eu}^{n-1}$ on the hyperplane.

where the constant $\kappa_\varrho(C_S(x))$ *can be computed by integrating* σ_H^{n-1} *along a polynomial hypersurface which is the graph of the Taylor's expansion up to order* $i = \mathrm{ord}(\alpha)$ *of* ψ *at* $\zeta = 0 \in \mathrm{e}_\alpha^\perp$. *More precisely, it turns out that*

$$\kappa_\varrho(C_S(x)) = \sigma_H^{n-1}(S_\infty \cap B_\varrho(x,1)),$$

where the limit-set S_∞ *is given by*

$$S_\infty = \left\{ \left(\zeta_1, ..., \zeta_{\alpha-1}, \widetilde{\psi}(\zeta), \zeta_{\alpha+1}, ..., \zeta_n\right) \,:\, \zeta \in \mathrm{e}_\alpha^\perp \right\}$$

and

$$\widetilde{\psi}(\zeta) = \sum_{\substack{j_1 \\ \mathrm{ord}(j_1)=i}} \frac{\partial \psi}{\partial \zeta_{j_1}}(0)\, \zeta_{j_1} + \ldots + \sum_{\substack{j_1,\ldots,j_l \\ \mathrm{ord}(j_1)+\ldots+\mathrm{ord}(j_l)=i}} \frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0)\, \zeta_{j_1} \cdot ... \cdot \zeta_{j_l}.$$

If (11) *does not hold, then* S_∞ *degenerates into a subset of the vertical* X_α*-line. Therefore, it turns out that* $\kappa_\varrho(C_S(x)) = 0$ *and we have*

$$\lim_{t\to 0^+} \frac{\sigma_H^{n-1}(S \cap B_\varrho(x,r))}{r^{Q-1}} = 0.$$

Remark 3.15. *The rescaled hypersurfaces* $\delta_{\frac{1}{r}} S$ *locally converge to a limit-set* S_∞, *i.e.*

$$\delta_{\frac{1}{r}} S \longrightarrow S_\infty \qquad \textit{for} \quad r \to 0^+,$$

where the convergence is understood with respect the Hausdorff convergence of sets; see also [41, 39]. If $x \in \mathrm{Int}(S\setminus C_S)$ *then the limit-set* S_∞ *coincides with the vertical hyperplane* $\mathcal{I}(\nu_H(x))$. *Otherwise* S_∞ *is the polynomial hypersurface described in Theorem 3.14, Case (ii).*

Remark 3.16 (Order of $x \in C_S$)**.** *Assume* S *to be smooth enough near its characteristic set* C_S, *say of class* $\mathbf{C}^k$. *Then there must exist a minimum* $i = \mathrm{ord}(\alpha)$ *such that* (11) *holds true. The integer* $\mathrm{ord}(x) = Q - \mathrm{ord}(\alpha)$ *is called the order of the characteristic point* $x \in C_S$.

Proof of Theorem 3.14. Let us preliminarily note that the limit (9) can be computed, without loss of generality, at the identity $0 \in \mathbb{G}$, just by left-translating S. Indeed, one has

$$\sigma_H^{n-1}\left(S \cap B_\varrho(x,r)\right) = \sigma_H^{n-1}\left(x^{-1} \bullet (S \cap B_\varrho(x,r))\right) = \sigma_H^{n-1}\left(\left(x^{-1} \bullet S\right) \cap B_\varrho(0,r)\right)$$

for any $x \in \mathrm{Int}S$, where the second equality follows from the additivity of the group law $\bullet$.

Notation 3.17. *Throughout this proof, we shall set:*

(i) $S_r(x) := S \cap B_\varrho(x,r)$;

(ii) $\widetilde{S} := x^{-1} \bullet S$;

(iii) $\widetilde{S}_r := x^{-1} \bullet S_r(x) = \widetilde{S} \cap B_\varrho(0,r)$.

By using the homogeneity of ϱ and the invariance of σ_H^{n-1} under positive Carnot dilations[12], it follows that

$$\sigma_H^{n-1}(\widetilde{S}_r) = r^{Q-1}\sigma_H^{n-1}\left(\delta_{1/r}\widetilde{S} \cap B_\varrho(0,1)\right)$$

[12]This means that $\delta_t^*\sigma_H^{n-1} = t^{Q-1}\sigma_H^{n-1}$, $t \in \mathbb{R}_+$; see Section 2.1.

for all $r \geq 0$. Therefore

$$\frac{\sigma_H^{n-1}(\widetilde{S}_r)}{r^{Q-1}} = \sigma_H^{n-1}\left(\delta_{1/r}\widetilde{S} \cap B_\varrho(0,1)\right)$$

and hence we have to compute the limit

$$\lim_{r \to 0^+} \sigma_H^{n-1}\left(\delta_{1/r}\widetilde{S} \cap B_\varrho(0,1)\right). \tag{13}$$

We begin by studying the non-characteristic case; see also [39, 40].

Case (1). Blow-up for non-characteristic points. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^1$ and let $x \in \mathrm{Int}S$ be non-characteristic.* In such a case the hypersurface S is oriented at x by the horizontal unit normal vector $\nu_H(x)$, i.e. $\nu_H(x)$ is transversal to S at x. Thus, at least locally around x, we may think of S as the (exponential image of a) $\mathbf{C}^1$-graph with respect to the horizontal direction $\nu_H(x)$. Moreover, at the level of the Lie algebra $\mathfrak{g} \cong T_0\mathbb{G}$, we can find an orthonormal change of coordinates such that

$$\mathrm{e}_1 = X_1(0) = (L_{x^{-1}})_*\nu_H(x).$$

With no loss of generality, by the Implicit Function Theorem we can write $\widetilde{S}_r = x^{-1} \bullet S_r(x)$, for some (small enough) $r > 0$, as the exponential image in $\mathbb{G}$ of a $\mathbf{C}^1$-graph[13]

$$\Psi = \{(\psi(\xi), \xi) \,:\, \xi \in \mathbb{R}^{n-1}\} \subset \mathfrak{g},$$

where $\psi : \mathrm{e}_1^\perp \cong \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a $\mathbf{C}^1$-function satisfying:

(i) $\psi(0) = 0$;

(ii) $\partial\psi/\partial\xi_j(0) = 0$ for every $j = 2, ..., h\,(= \mathrm{dim}H)$,

where $\xi \in \mathrm{e}_1^\perp \cong \mathbb{R}^{n-1}$. In this way $\widetilde{S}_r = exp\,\Psi \cap B_\varrho(0,r)$, for all (small enough) $r > 0$. Clearly, this remark can be used to compute (13). So let us us fix a positive r_0 satisfying the previous assumptions and let $0 \leq r \leq r_0$. Then

$$\delta_{1/r}\widetilde{S} \cap B_\varrho(0,1) = exp\left(\widehat{\delta}_{1/r}\Psi\right) \cap B_\varrho(0,1), \tag{14}$$

where $\{\widehat{\delta}_t\}_{t \geq 0}$ are the induced dilations on $\mathfrak{g}$, i.e. $\delta_t = exp \circ \widehat{\delta}_t$ for $t \in \mathbb{R}_+$. Henceforth, we shall often consider the restriction of $\widehat{\delta}_t$ to the hyperplane $\mathrm{e}_1^\perp \cong \mathbb{R}^{n-1}$. For this reason and with a slight abuse of notation, instead of $(\widehat{\delta}_t)\big|_{\mathrm{e}_1^\perp}(\xi)$ we shall simply write $\widehat{\delta}_t\xi$. Moreover, we shall assume $\mathbb{R}^{n-1} = \mathbb{R}^{h-1} \oplus \mathbb{R}^{n-h}$. Note that the induced dilations $\{\widehat{\delta}_t\}_{t \geq 0}$ make $\mathrm{e}_1^\perp \cong \mathbb{R}^{n-1}$ a *graded vector space* whose grading respects that of $\mathfrak{g}$. We have

$$\widehat{\delta}_{1/r}\Psi = \widehat{\delta}_{1/r}\left\{(\psi(\xi), \xi) \,:\, \xi \in \mathbb{R}^{n-1}\right\} = \left\{\left(\frac{\psi(\xi)}{r}, \widehat{\delta}_{1/r}\xi\right) \,:\, \xi \in \mathbb{R}^{n-1}\right\}.$$

By using the change of variables $\zeta := \widehat{\delta}_{1/r}\xi$, we get that

$$\widehat{\delta}_{1/r}\Psi = \left\{\left(\frac{\psi\left(\widehat{\delta}_r\zeta\right)}{r}, \zeta\right) \,:\, \zeta \in \mathbb{R}^{n-1}\right\}.$$

[13] Actually, since the argument is local, ψ can be defined just on a suitable neighborhood of $0 \in \mathrm{e}_1^\perp \cong \mathbb{R}^{n-1}$.

By hypothesis $\psi \in \mathbf{C}^1(U_0)$, where U_0 is a suitable open neighborhood of $0 \in \mathbb{R}^{n-1}$. Using a Taylor's expansion of ψ at $0 \in \mathbb{R}^{n-1}$ and the assumptions (i) and (ii), yields

$$\begin{aligned}\psi(\xi) &= \psi(0) + \langle grad_{\mathbb{R}^{n-1}}\psi(0), \xi\rangle_{\mathbb{R}^{n-1}} + \mathrm{o}(\|\xi\|_{\mathbb{R}^{n-1}})\\ &= \langle grad_{\mathbb{R}^{n-h}}\psi(0), \xi_{\mathbb{R}^{n-h}}\rangle_{\mathbb{R}^{n-h}} + \mathrm{o}(\|\xi\|_{\mathbb{R}^{n-1}}),\end{aligned}$$

for $\xi \to 0 \in \mathbb{R}^{n-1}$. Note that $\widehat{\delta}_r\zeta \to 0 \in \mathbb{R}^{n-1}$ as $r \to 0^+$. By applying into ψ the previous change of variables we get

$$\psi\left(\widehat{\delta}_r\zeta\right) = \left\langle grad_{\mathbb{R}^{n-h}}\psi(0), \widehat{\delta}_r\left(\zeta_{\mathbb{R}^{n-h}}\right)\right\rangle_{\mathbb{R}^{n-h}} + \mathrm{o}\left(r\right)$$

as $r \to 0^+$. Since $\left\langle grad_{\mathbb{R}^{n-h}}\psi(0), \widehat{\delta}_r\left(\zeta_{\mathbb{R}^{n-h}}\right)\right\rangle_{\mathbb{R}^{n-h}} = \mathrm{o}(r)$ as $r \to 0^+$, we easily get that the limit-set (obtained by blowing-up $\widetilde{S}$ at the non-characteristic point 0) is given by

$$\Psi_\infty = \lim_{r\to 0^+} \widehat{\delta}_{1/r}\Psi = exp\left(\mathrm{e}_1^\perp\right) = \mathcal{I}(X_1(0)). \tag{15}$$

We stress that $\mathcal{I}(X_1(0))$ is the vertical hyperplane through the identity $0 \in \mathbb{G}$ and orthogonal to $X_1(0)$. Thus we have shown that (13) can be computed by means of (14) and (15). More precisely

$$\lim_{r\to 0^+} \sigma_H^{n-1}\left(\delta_{1/r}\widetilde{S} \cap B_\varrho(0,1)\right) = \sigma_H^{n-1}\left(\mathcal{I}(X_1(0)) \cap B_\varrho(0,1)\right)$$

By remembering the change of variables, it follows that $S_\infty = \mathcal{I}(\nu_H(x))$ and that

$$\kappa_\varrho(\nu_H(x)) = \lim_{r\to 0^+} \frac{\sigma_H^{n-1}(S \cap B_\varrho(x,r))}{r^{Q-1}} = \sigma_H^{n-1}\left(\mathcal{I}(\nu_H(x)) \cap B_\varrho(x,1)\right),$$

which was to be proven.

Case (2). Blow-up at the characteristic set. *We are now assuming that $S \subset \mathbb{G}$ is a $\mathbf{C}^i$-smooth hypersurface ($i \geq 2$) and that $x \in \mathrm{Int}(S \cap C_S)$.* Near x the hypersurface S is then oriented by some vertical vector. Hence, at least locally around x, we may think of S as the (exponential image of a) $\mathbf{C}^i$-graph with respect to a some vertical direction X_α transversal to S at x, i.e. $\langle X_\alpha, \nu\rangle \neq 0$ at x, where ν is a unit normal vector to S. Note that X_α is a vertical left-invariant vector field of the fixed left-invariant frame $\underline{X} = \{X_1, ..., X_n\}$ on $\mathbb{G}$ and $\alpha \in I_V = \{h+1, ..., n\}$ is any "vertical" index; see Notation 2.3. *Furthermore, we are assuming that*

$$\mathrm{ord}(\alpha) := i \qquad \text{for } i = 2, .., k.$$

To the sake of simplicity, as in the non-characteristic case, we left-translate the hypersurface in such a way that x will coincide with the identity $0 \in \mathbb{G}$. To this end, it suffices to replace S by $\widetilde{S} = x^{-1} \bullet S$. At the level of the Lie algebra $\mathfrak{g}$, we consider the hyperplane $\mathrm{e}_\alpha^\perp$ through the origin $0 \in \mathfrak{g} \cong \mathbb{R}^n$ and orthogonal to $\mathrm{e}_\alpha = X_\alpha(0)$. Clearly $\mathrm{e}_\alpha^\perp$ is the "natural" domain of a graph along the direction e_α. By the classical Implicit Function Theorem, for some (small enough) $r > 0$, we may write $\widetilde{S}_r = x^{-1} \bullet S_r(x)$ as the exponential image in $\mathbb{G}$ of a $\mathbf{C}^i$-graph. We have

$$\Psi = \left\{\left(\xi_1, ..., \xi_{\alpha-1} \underbrace{, \psi(\xi),}_{\alpha-th\,place} \xi_{\alpha+1}, ..., \xi_n\right) : \xi := (\xi_1, ..., \xi_{\alpha-1}, 0, \xi_{\alpha+1}, ..., \xi_n) \in \mathrm{e}_\alpha^\perp \cong \mathbb{R}^{n-1}\right\}$$

where $\psi : \mathrm{e}_\alpha^\perp \cong \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a $\mathbf{C}^i$-smooth function satisfying:

(j) $\psi(0)=0$;

(jj) $\partial\psi/\partial\xi_j(0)=0$ for every $j=1,...,h\,(=\dim H)$.

Thus we get that $\widetilde{S}_r = exp\,\Psi\cap B_\varrho(0,r)$, for every (small enough) $r>0$. Clearly we may apply the previous considerations to compute (13) and, by arguing as in the non-characteristic case, we can use (14). So let us compute

$$\begin{aligned}\widehat{\delta}_{1/r}\Psi &= \widehat{\delta}_{1/r}\left\{(\xi_1,...,\xi_{\alpha-1},\psi(\xi),\xi_{\alpha+1},...,\xi_n)\,:\,\xi\in \mathrm{e}_\alpha^\perp\right\}\\ &= \left\{\left(\frac{\xi_1}{r},...,\frac{\xi_{\alpha-1}}{r^{\mathrm{ord}(\alpha-1)}},\frac{\psi(\xi)}{r^i},\frac{\xi_{\alpha+1}}{r^{\mathrm{ord}(\alpha+1)}},...,\frac{\xi_n}{r^k}\right)\,:\,\xi\in\mathrm{e}_\alpha^\perp\right\}.\end{aligned}$$

By setting

$$\zeta:=\widehat{\delta}_{1/r}\xi=\left(\frac{\xi_1}{r},...,\frac{\xi_{\alpha-1}}{r^{\mathrm{ord}(\alpha-1)}},0,\frac{\xi_{\alpha+1}}{r^{\mathrm{ord}(\alpha+1)}},...,\frac{\xi_n}{r^k}\right),$$

where $\zeta=(\zeta_1,...,\zeta_{\alpha-1},0,\zeta_{\alpha+1},...,\zeta_n)\in\mathrm{e}_\alpha^\perp$, we therefore get that

$$\widehat{\delta}_{1/r}\Psi=\left\{\left(\zeta_1,...,\zeta_{\alpha-1},\frac{\psi\left(\widehat{\delta}_r\zeta\right)}{r^i},\zeta_{\alpha+1},...,\zeta_n\right)\,:\,\zeta\in\mathrm{e}_\alpha^\perp\right\}.$$

By hypothesis $\psi\in\mathbf{C}^i(U_0)$, where U_0 is an open neighborhood of $0\in\mathrm{e}_\alpha^\perp\cong\mathbb{R}^{n-1}$. Obviously, one has $\widehat{\delta}_r\zeta\to 0$ as $r\to 0^+$. So we have to study the limit

$$\widetilde{\psi}(\zeta):=\lim_{r\to 0^+}\frac{\psi\left(\widehat{\delta}_r\zeta\right)}{r^i}, \tag{16}$$

whenever exists. The first remark is that if this limit equals $+\infty$, we have

$$\lim_{r\to 0^+}\frac{\sigma_H^{n-1}(\widetilde{S}_r)}{r^{Q-1}}=\lim_{r\to 0^+}\sigma_H^{n-1}\left(exp\,\left(\widehat{\delta}_{1/r}\Psi\right)\cap B_\varrho(0,1)\right)=0,$$

since $\widehat{\delta}_{1/r}\Psi\cap B_\varrho(0,1)$ degenerates into a subset of the X_α-line as long as $r\to 0^+$.

Making use of a Taylor's expansion of ψ together with (j) and (jj), yields

$$\begin{aligned}\psi\left(\widehat{\delta}_r\zeta\right) &= \psi(0)+\sum_{j_1} r^{\mathrm{ord}(j_1)}\frac{\partial\psi}{\partial\zeta_{j_1}}(0)\,\zeta_{j_1}+\sum_{j_1,j_2} r^{\mathrm{ord}(j_1)+\mathrm{ord}(j_2)}\frac{\partial^{(2)}\psi}{\partial\zeta_{j_1}\partial\zeta_{j_2}}(0)\,\zeta_{j_1}\zeta_{j_2}\\ &\quad+...+\sum_{j_1,...,j_i} r^{\mathrm{ord}(j_1)+...+\mathrm{ord}(j_i)}\frac{\partial^{(i)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_i}}(0)\,\zeta_{j_1}\cdot...\cdot\zeta_{j_i}+\mathrm{o}\left(r^i\right)\\ &= \sum_{j_1} r^{\mathrm{ord}(j_1)}\frac{\partial\psi}{\partial\zeta_{j_1}}(0)\,\zeta_{j_1}+\sum_{j_1,j_2} r^{\mathrm{ord}(j_1)+\mathrm{ord}(j_2)}\frac{\partial^{(2)}\psi}{\partial\zeta_{j_1}\partial\zeta_{j_2}}(0)\,\zeta_{j_1}\zeta_{j_2}\\ &\quad+...+\sum_{j_1,...,j_i} r^{\mathrm{ord}(j_1)+...+\mathrm{ord}(j_l)}\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_i}}(0)\,\zeta_{j_1}\cdot...\cdot\zeta_{j_l}+\mathrm{o}\left(r^i\right)\end{aligned}$$

as $r\to 0^+$. Therefore

$$\begin{aligned}\frac{\psi\left(\widehat{\delta}_r\zeta\right)}{r^i} &= \sum_{j_1} r^{\mathrm{ord}(j_1)-i}\frac{\partial\psi}{\partial\zeta_{j_1}}(0)\,\zeta_{j_1}+\sum_{j_1,j_2} r^{\mathrm{ord}(j_1)+\mathrm{ord}(j_2)-i}\frac{\partial^{(2)}\psi}{\partial\zeta_{j_1}\partial\zeta_{j_2}}(0)\,\zeta_{j_1}\zeta_{j_2}\\ &\quad+...+\sum_{j_1,...,j_l} r^{\mathrm{ord}(j_1)+...+\mathrm{ord}(j_l)-i}\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0)\,\zeta_{j_1}\cdot...\cdot\zeta_{j_l}+\mathrm{o}\,(1)\end{aligned}$$

as $r \to 0^+$. By hypothesis

$$\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0) = 0 \qquad \text{whenever} \qquad \mathrm{ord}(j_1) + ... + \mathrm{ord}(j_l) < i.$$

This shows that (16) exists. Moreover, setting

$$\Psi_\infty = \lim_{r\to 0^+} \widehat{\delta}_{1/r}\Psi = \left\{ \left(\zeta_1, ..., \zeta_{\alpha-1}, \widetilde{\psi}(\zeta), \zeta_{\alpha+1}, ..., \zeta_n\right) \,:\, \zeta \in \mathrm{e}_\alpha^\perp \right\},$$

where $\widetilde{\psi}$ is the polynomial function of homogeneous order $i = \mathrm{ord}(\alpha)$ given by

$$\widetilde{\psi}(\zeta) = \sum_{\substack{j_1 \\ \mathrm{ord}(j_1)=i}} \frac{\partial\psi}{\partial\zeta_{j_1}}(0)\,\zeta_{j_1} + \ldots + \sum_{\substack{j_1,...,j_l \\ \mathrm{ord}(j_1)+...+\mathrm{ord}(j_l)=i}} \frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0)\,\zeta_{j_1}\cdot ...\cdot\zeta_{j_l},$$

yields $S_\infty = x \bullet \Psi_\infty$ and the thesis easily follows. □

Remark 3.18. *The metric factor $k_\varrho(\nu_H)$ turns out to be constant for instance by assuming that ϱ be symmetric on all layers; see, for instance, [41]. Anyway, it is uniformly bounded by two positive constants K_1 and K_2. This can be easily deduced by making use of the so-called ball-box metric*[14] *and by a homogeneity argument. Indeed, for any given ϱ-ball $B_\varrho(x, r)$, there exist two boxes* $\mathrm{Box}(x, r_1)$, $\mathrm{Box}(x, r_2)$ $(r_1 \leq r \leq r_2)$ *such that*

$$\mathrm{Box}(x, r_1) \subseteq B_\varrho(x, r) \subseteq \mathrm{Box}(x, r_2).$$

Remind that

$$k_\varrho(\nu_H(x)) = \sigma_H^{n-1}(\mathcal{I}(\nu_H(x)) \cap B_\varrho(x, 1)) = \mathcal{H}_{Eu}^{n-1}(\mathcal{I}(\nu_H(x)) \cap B_\varrho(x, 1)).$$

where $\mathcal{I}(\nu_H(x)$ denotes the vertical hyperplane orthogonal to $\nu_H(x)$. So let us fix r_1, r_2 in a way that $0 < r_1 \leq 1 \leq r_2$ and

$$\mathrm{Box}(x, r_1) \subseteq B_\varrho(x, 1) \subseteq \mathrm{Box}(x, r_2).$$

Since $\delta_t\mathrm{Box}(x, 1/2) = \mathrm{Box}(x, t/2)$ *for every $t \geq 0$, by a simple computation*[15] *we get that*

$$(2r_1)^{Q-1} \leq k_\varrho(\nu_H(x)) \leq \sqrt{n-1}\,(2r_2)^{Q-1}.$$

In particular, we may put $K_1 := (2r_1)^{Q-1}$ *and* $K_2 := \sqrt{n-1}\,(2r_2)^{Q-1}$.

[14]By definition one has

$$\mathrm{Box}(x, r) = \left\{ y = exp\left(\sum_{i=1,...,k} y_{H_i}\right) \in \mathbb{G} \,:\, \|y_{H_i} - x_{H_i}\|_\infty \leq r^i \right\},$$

where $y_{H_i} = \sum_{j_i \in I_{H_i}} y_{j_i}\mathrm{e}_{j_i}$ and $\|y_{H_i}\|_\infty$ is the sup-norm on the i-th layer of $\mathfrak{g}$; see, for instance, [31], [50].

[15]The unit box $\mathrm{Box}(x, 1/2)$ is the left-translated at x of $\mathrm{Box}(0, 1/2)$ and so, by left-invariance of σ_H^{n-1}, the computation can be done at $0 \in \mathbb{G}$. Since $\mathrm{Box}(0, 1/2)$ is the unit hypercube of $\mathbb{R}^n \cong \mathfrak{g}$, it remains to show how we can estimate the σ_H^{n-1}-measure of the intersection of $\mathrm{Box}(0, 1/2)$ with a generic vertical hyperplane through the origin $0 \in \mathbb{R}^n$. This can be done as follows: If $\mathcal{I}(X)$ denotes the vertical hyperplane through the origin of $\mathbb{R}^n$ and orthogonal to $X \in H$, we get that

$$1 \leq \mathcal{H}_{Eu}^{n-1}(\mathrm{Box}(0, 1/2) \cap \mathcal{I}(X)) \leq \sqrt{n-1},$$

where we notice that $\sqrt{n-1}$ is just the diameter of any face of the unit hypercube of $\mathbb{R}^n$. Therefore

$$\left(\delta_{2r_1}\mathrm{Box}(0, 1/2) \cap \mathcal{I}(X)\right) \subseteq \left(B_\varrho(0, 1) \cap \mathcal{I}(X)\right) \subseteq \left(\delta_{2r_2}\mathrm{Box}(x, 1/2) \cap \mathcal{I}(X)\right)$$

and so

$$\begin{aligned} (2r_1)^{Q-1} &\leq (2r_1)^{Q-1}\mathcal{H}_{Eu}^{n-1}(\mathrm{Box}(0, 1/2) \cap \mathcal{I}(X)) \leq \mathcal{H}_{Eu}^{n-1}(B_\varrho(0, 1) \cap \mathcal{I}(X)) \\ &= \kappa_\varrho(X) \leq (2r_2)^{Q-1}\mathcal{H}_{Eu}^{n-1}(\mathrm{Box}(0, 1/2) \cap \mathcal{I}(X)) \leq \sqrt{n-1}(2r_2)^{Q-1}. \end{aligned}$$

Therefore, for every homogeneous metric ϱ on $\mathbb{G}$ one can choose two positive constants K_1, K_2 independent of S such that

$$K_1 \leq \kappa_\varrho(\nu_{_H}(x)) \leq K_2 \tag{17}$$

for every $x \in S \setminus C_S$.

3.5 Other important tools

In this section we collect some useful results which will be used later on. As a first thing, we apply to our setting a recent result by Balogh, Pintea and Rohner (see [6]) about the size of horizontal tangencies to non-involutive distributions.

Theorem 3.19 (Generalized Derridj's Theorem; see Theorem 4.5 in [6])**.** *Let $\mathbb{G}$ be a k-step Carnot group.*

(i) *If $S \subset \mathbb{G}$ is a hypersurface of class $\mathbf{C}^2$, then the Euclidean-Hausdorff dimension of the caracteristic set C_S of S satisfies*

$$\dim_{\mathrm{Eu-Hau}}(C_N) \leq n-2.$$

(ii) *If $V = H^\perp \subset T\mathbb{G}$ satisfies $\dim V \geq 2$ and if $N \subset \mathbb{G}$ is a $(n-2)$-dimensional submanifold of class $\mathbf{C}^2$, then the Euclidean-Hausdorff dimension of the caracteristic set C_N of N satisfies*

$$\dim_{\mathrm{Eu-Hau}}(C_N) \leq n-3.$$

Remark 3.20. *The previous $\mathbf{C}^2$-smoothness condition is sharp, see [6]. Moreover, we stress that $\dim V = 1$ just in the following cases:*

(i) *Heisenberg groups $\mathbb{H}^n$;*

(ii) *2-step Carnot groups $\mathbb{G}$ having 1-dimensional center T and Lie algebra $\mathfrak{g}$ such that:*

$$\mathfrak{g} = H \oplus T, \qquad H \cong \mathbb{R}^h = \mathrm{span}_\mathbb{R}\{\mathrm{e}_1, ..., \mathrm{e}_h\}, \qquad \mathrm{e}_n = T$$

with bracket-relations:

$$[\mathrm{e}_i, \mathrm{e}_j] = C^{\mathfrak{g}\,n}_{ij} \quad i,j = 1..., h, \qquad [\mathrm{e}_i, \mathrm{e}_n] = 0 \qquad \textit{for } i = 1..., h.$$

In the case of Heisenberg groups $\mathbb{H}^n$, $n > 1$, by applying Frobenious' Theorem it follows that

$$\dim_{\mathrm{Eu-Hau}}(C_N) \leq n$$

where $n = \frac{\dim H}{2}$; see also [6]. On the contrary, in the first Heisenberg group $\mathbb{H}^1$, 1-dimensional curves can be either horizontal or transversal to H. For the general case (ii), by applying Frobenious' Theorem it follows that there exist horizontal submanifolds of dimension at most $\frac{\widehat{h}}{2}$, where $\widehat{h}$ is the greatest number of commutative-pairs of the left-invariant basis $\{\mathrm{e}_1, ..., \mathrm{e}_h\}$ of $H \cong \mathbb{R}^h$. This implies that

$$\dim_{\mathrm{Eu-Hau}}(C_N) \leq \frac{\widehat{h}}{2}.$$

Note that $\dim_{\mathrm{top}}\mathbb{G} = h+1$, where $h = \dim H$. Clearly $N \subset \mathbb{G}$ can be a horizontal submanifold if, and only if, $\frac{\widehat{h}}{2} = n-2$. So there must exist $(n-2)$ commutative pairs of left-invariant vector fields among $(n-1)$ left-invariant vector fields of any basis of H. But this can happen only *if there exists one, and only one, non-commutative pair. We stress that the matrix C^n_H of the structural constants of $\mathfrak{g}$, up to a linear change of basis, has the following simple form:*

$$C^n_H = \begin{pmatrix} 0 & 1 & \mathbf{0}_{h-2} \\ -1 & 0 & \mathbf{0}_{h-2} \\ \mathbf{0}^{h-2} & \mathbf{0}^{h-2} & \mathbf{0}^{h-2}_{h-2} \end{pmatrix} \in \mathcal{M}_{h\times h}(\mathbb{R}),$$

where $\mathbf{0}_{h-2}$ is a $(n-2)$-row vector, $\mathbf{0}^{h-2}$ is a $(n-2)$-column vector and $\mathbf{0}^{h-2}_{h-2}$ is a square matrix of order $(h-2)$.

In the case that $N \subset \mathbb{G}$ is a $(n-2)$-dimensional submanifold of class $\mathbf{C}^2$, we may apply some general blow-up theorems by Magnani and Vittone [41] and Magnani [39]. For later purposes, we record some consequences of their results in the next:

Theorem 3.21 (Blow-up for $(n-2)$-dimensional submanifolds; see [41])**.** *Let $N \subset \mathbb{G}$ be a $(n-2)$-dimensional submanifold of class $\mathbf{C}^{1,1}$ and let $x \in N$ be non-characteristic. Then*

$$\delta_{\frac{1}{r}}(x^{-1} \bullet N) \cap B_\varrho(0,1) \longrightarrow \mathcal{I}^2(\nu_H(x)) \cap B_\varrho(0,1)$$

as long as $r \to 0^+$, where $\mathcal{I}^2(\nu_H(x))$ denotes the $(n-2)$-dimensional subgroup of $\mathbb{G}$ defined by

$$\mathcal{I}^2(\nu_H(x)) := \{y \in \mathbb{G} : y = \exp(Y),\ Y \wedge \nu_H(x) = 0\}$$

where $\nu_H = \nu^1_H \wedge \nu^2_H$ is the unit horizontal normal 2-vector that determines the orientation of N. We stress that the convergence is understood with respect to the Hausdorff distance of sets. Moreover, if $\nu = \nu_1 \wedge \nu_2$ denotes the unit normal 2-vector field orienting N, it turns out that

$$\lim_{r\to 0^+} \frac{\sigma^{n-2}_R(N \cap B_\varrho(x,r))}{r^{Q-2}} = \frac{\kappa(\nu_H(x))}{|\mathcal{P}_H \nu(x)|},$$

where

$$\kappa(\nu_H(x)) := \sigma^{n-2}_H(B_\varrho(0,1) \cap \mathcal{I}^2(\nu_H(x)))$$

is a strictly positive and bounded density-function, called metric factor. Finally, if we have $\mathcal{H}^{Q-2}_\varrho(C_N) = 0$, then the following representation formula holds

$$\sigma^{n-2}_H(N) = \int_N \kappa(\nu_H(x))\, d\mathcal{S}^{Q-2}_\varrho.$$

For the 2-step case there is a more precise statement. Indeed, in this case any $x \in N$ can have only two different "orders"[16], that are $(Q-2)$ and $(Q-3)$; see Definition 2.6 in [39].

Theorem 3.22 (2-step case; see [39])**.** *Let $\mathbb{G}$ be 2-step Carnot group and let $N \subset \mathbb{G}$ be a $(n-2)$-dimensional submanifold of class $\mathbf{C}^{1,1}$. Then, for every $x \in N$ there exists a neighborhood $U_x \subset \mathbb{G}$ of x and there exist positive constants C_1, C_2 and r_0 depending on $U_x \cap N$ such that*

$$C_1 r^{\mathrm{ord}(x)} \leq \sigma^{n-2}_R(N \cap B_\varrho(z,r)) \leq C_2 r^{\mathrm{ord}(x)}$$

for every $z \in N \cap U_x$ with $\mathrm{ord}(z) = \mathrm{ord}(x)$ and every $r < r_0$. Moreover $\mathcal{H}^{Q-2}_\varrho(C_N) = 0$ and

$$\sigma^{n-2}_H(N) = \int_N \kappa(\nu_H)\, d\mathcal{S}^{Q-2}_\varrho.$$

If N is of class $\mathbf{C}^2$, for every $x \in N$ the rescaled sets $\delta_{\frac{1}{r}}(x^{-1} \bullet N)$ locally converge, with respect to the Hausdorff distance of sets, to an algebraic variety N_∞ which is the graph of a homogeneous polynomial function.

[16] Roughly speaking, the *order* of a point $x \in N$ is $(Q-2)$ if x is non-characteristic and $(Q-3)$ otherwise.

Actually, if the order of $x \in N$ is $(Q-3)$, the homogeneous order of this polynomial function must be 2.

We end this section by remembering a classical fact. In his treatise [22], Federer proved an important result which allows to represent a regular measure μ of an abstract metric space (X, ϱ) in terms of the intrinsic spherical Hausdorff measure $\mathcal{S}^q_\varrho$ of the space; see Theorem 2.10.17 in [22]. A simplified version of his result reads as follows:

Lemma 3.23. *Let (X, ϱ) be a locally compact, separable metric space and let μ be a regular measure on X. If $A \subset X$, $k > 0$ and*

$$\limsup_{r \to 0^+} \frac{\mu(A \cap B_\varrho(x,r))}{r^q} \leq t$$

whenever $x \in A$, then $\mu(A) \leq k\, \mathcal{S}^q_\varrho(A)$.

4 Isoperimetric Inequality on hypersurfaces

The main result of this paper is the following:

Theorem 4.1 (Isoperimetric Inequality)**.** *Let $\mathbb{G}$ be a k-step Carnot group and let us fix a homogeneous metric ϱ on $\mathbb{G}$ as in Definition 2.5. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface with boundary ∂S -at least- piecewise $\mathbf{C}^2$-smooth. Let $\mathcal{H}_H$ denote the horizontal mean curvature of S. Then there exists a positive constant C_{Isop}, only dependent on $\mathbb{G}$ and on the homogeneous metric ϱ, such that*

$$\left(\sigma_H^{n-1}(S)\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop} \left(\int_S |\mathcal{H}_H|\, \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right). \qquad (18)$$

The next sections are devoted to prove this theorem. Furthermore, in Section 5 we shall show some related Sobolev-type inequalities. In particular some generalizations will be discussed at Section 5.1.

Nevertheless, we would like to state an immediate but interesting corollary of this theorem, which holds true in some special cases. Among them the Heisenberg group $\mathbb{H}^1$ is the more important one; see Remark 4.15 and footnote 19.

Corollary 4.2. *Let $\mathbb{G}$ be a 2-step Carnot group and assume*[17] *that its horizontal bundle $H \subset T\mathbb{G}$ is of codimension 1. Furthermore, let $S \subset \mathbb{G}$ be a compact hypersurface of class $\mathbf{C}^2$ with smooth boundary ∂S. If ∂S is horizontal, then S cannot be H-minimal.*

Note that if ∂S is horizontal this means that $\partial S = C_{\partial S}$.

Proof. Under these assumptions one has $\sigma_H^{n-2}(\partial S) = 0$. If $\mathcal{H}_H = 0$ along S, the right-hand side of (18) vanishes identically. □

4.1 Linear isoperimetric inequality and Global Monotonicity formula

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth compact hypersurface with boundary ∂S smooth enough for the validity of the Riemannian Divergence Theorem. As usual ν denotes the unit normal along S and $\varpi = \frac{\mathcal{P}_V \nu}{|\mathcal{P}_H \nu|}$. We shall set

$$\varpi_{H_i} := \mathcal{P}_{H_i} \varpi = \sum_{\alpha \in I_{H_i}} \varpi_\alpha X_\alpha$$

for $i = 2, ..., k$. We have $\frac{\nu}{|\mathcal{P}_H \nu|} = \nu_H + \sum_{i=2}^k \varpi_{H_i}$.

[17] In this case, it can happen that there exist $(n-2)$-dimensional horizontal submanifolds; see Section 3.5.

Definition 4.3. *Let η be the unit normal vector η along ∂S*[18]*. In the sequel, we shall set*

(i) $\chi := \frac{\mathcal{P}_{VS}\,\eta}{|\mathcal{P}_{HS}\,\eta|}$;

(ii) $\chi_{H_iS} := \mathcal{P}_{H_iS}\,\chi, \qquad i = 2, ..., k;$

see Remark 2.17. Using this notation and the very definition of η_{HS} yields $\chi = \sum_{i=2}^{k} \chi_{H_iS}$ and $\frac{\eta}{|\mathcal{P}_{HS}\,\eta|} = \eta_{HS} + \chi$; see Remark 3.2.

Definition 4.4. *Fix a point $x \in \mathbb{G}$ and consider the Carnot homothety centered at x, i.e. $\vartheta^x(t, y) := x \bullet \delta_t(x^{-1} \bullet y)$. The variational vector field of $\vartheta^x_t(y) := \vartheta^x(t, y)$ at $t = 1$ is given by*

$$Z_x := \frac{\partial \vartheta^x_t}{\partial t}\Big|_{t=1}.$$

Definition 4.5. *Let $\mathbb{G}$ be a k-step Carnot group and $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface with boundary ∂S, at least, piecewise $\mathbf{C}^1$-smooth. Moreover, let $S_r := S \cap B_\varrho(x, r)$, where $B_\varrho(x, r)$ is the open ϱ-ball centered at $x \in \mathbb{G}$ and of radius $r > 0$. We shall set*

$$\begin{aligned}
\mathcal{A}(r) &:= \int_{S_r} |\mathcal{H}_H| \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\varpi_{H_i}|\right) \sigma_H^{n-1},\\
\mathcal{B}_0(r) &:= \int_{\partial S_r} \frac{1}{\varrho_x} \left|\left\langle Z_x, \frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right| \sigma_H^{n-2},\\
\mathcal{B}(r) &:= \int_{\partial S_r} \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_iS}|\right) \sigma_H^{n-2},\\
\mathcal{B}_1(r) &:= \int_{\partial B_\varrho(x,r)\cap S} \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_iS}|\right) \sigma_H^{n-2},\\
\mathcal{B}_2(r) &:= \int_{\partial S \cap B_\varrho(x,r)} \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_iS}|\right) \sigma_H^{n-2},
\end{aligned}$$

where $\varrho_x(y) := \varrho(x, y)$ for $y \in S$, i.e. ϱ_x denotes the ϱ-distance from the fixed point $x \in \mathbb{G}$.

Remark 4.6. *By Cauchy-Schwartz inequality it follows that $\mathcal{B}_0(r) \leq \mathcal{B}(r)$ for every $r > 0$.*

In the sequel we shall apply the 1st variation of σ_H^{n-1} (see Theorem 3.9 and Corollary 3.13), with a "special" choice of the variational vector field. More precisely, let us fix a point $x \in \mathbb{G}$ and consider the Carnot homothety $\vartheta^x_t(y) := x \bullet \delta_t(x^{-1} \bullet y)$ centered at x. Without loss of generality, by using group translations, we may choose $x = 0 \in \mathbb{G}$. One has

$$\vartheta^0(t, y) := \exp\left(t y_H, t^2 y_{H_2}, t^3 y_{H_3}, ..., t^i y_{H_i}, ..., t^k y_{H_k}\right) \qquad \text{for every } t \in \mathbb{R},$$

where $y_{H_i} = \sum_{j_i \in I_{H_i}} y_{j_i} \mathrm{e}_{j_i}$ and *exp* denotes the Carnot exponential mapping; see Section 2.1. Thus the variational vector field related to $\vartheta^0_t(y) := \vartheta^0(t, y) = \delta_t y$, at $t = 1$, is just

$$Z_0 := \frac{\partial \vartheta^0_t}{\partial t}\Big|_{t=1} = \frac{\partial \delta_t}{\partial t}\Big|_{t=1} = y_H + 2y_{H_2} + ... + k y_{H_k}.$$

As it is well known, by invariance of σ_H^{n-1} under Carnot dilations, one has

$$\frac{d}{dt}\delta_t^* \sigma_H^{n-1}\Big|_{t=1} = (Q - 1)\, \sigma_H^{n-1}(S).$$

[18] Note that, at each point $x \in \partial S$, $\eta(x) \in T_xS$

Furthermore, by using the 1st variation formula (see Corollary 3.13), one gets

$$(Q-1)\,\sigma_{H}^{n-1}(S)=-\int_{S}\mathcal{H}_{H}\left\langle Z_0,\frac{\nu}{|\mathcal{P}_{H}\,\nu_{H}|}\right\rangle\sigma_{H}^{n-1}+\int_{\partial S}\left\langle Z_0,\frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\underbrace{|\mathcal{P}_{H}\,\nu_{H}|\,|\mathcal{P}_{HS}\,\eta|\,\sigma_{R}^{n-2}}_{=\sigma_{H}^{n-2}}.$$

Note that

$$\left\langle Z_0,\frac{\nu}{|\mathcal{P}_{H}\,\nu_{H}|}\right\rangle=\langle Z_0,(\nu_{H}+\varpi)\rangle=\langle y_{H},\nu_{H}\rangle+\sum_{i=2}^{k}\langle y_{H_i},\varpi_{H_i}\rangle.$$

Analogously

$$\left\langle Z_0,\frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle=\langle Z_0,(\eta_{HS}+\chi)\rangle=\langle y_{H},\eta_{HS}\rangle+\sum_{i=2}^{k}\langle y_{H_i},\chi_{H_iS}\rangle.$$

By Cauchy-Schwartz inequality we immediately get the following estimates:

$$\left|\left\langle Z_0,\frac{\nu}{|\mathcal{P}_{H}\,\nu_{H}|}\right\rangle\right|\leq|y_{H}|+\sum_{i=2}^{k}i\,|y_{H_i}||\varpi_{H_i}|,$$

$$\left|\left\langle Z_0,\frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right|\leq|y_{H}|+\sum_{i=2}^{k}i\,|y_{H_i}||\chi_{H_iS}|.$$

According with Definition 2.5, let $c_i\in\mathbb{R}_+$ be constants such that $|y_{H_i}|\leq c_i\varrho^i(y)$ for $i=2,...,k$. Using the previous estimates together with these assumptions on ϱ yields:

$$\left|\left\langle Z_0,\frac{\nu}{|\mathcal{P}_{H}\,\nu_{H}|}\right\rangle\right|\leq\varrho\left(1+\sum_{i=2}^{k}i\,c_i\varrho^{i-1}|\varpi_{H_i}|\right),\qquad\left|\left\langle Z_0,\frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right|\leq\varrho\left(1+\sum_{i=2}^{k}i\,c_i\varrho^{i-1}|\chi_{H_iS}|\right).$$

Proposition 4.7. *Let $S\subset\mathbb{G}$ be a $\mathbf{C}^2$-smooth compact hypersurface with piecewise $\mathbf{C}^1$-smooth boundary ∂S. Let r be the radius of a ϱ-ball centered at $x\in\mathbb{G}$ and circumscribed about S. Then*

$$(Q-1)\,\sigma_{H}^{n-1}(S)\leq r\left(\mathcal{A}(r)+\mathcal{B}_0(r)\right)\leq r\left(\mathcal{A}(r)+\mathcal{B}(r)\right).$$

Proof. Immediate by the previous discussion and the invariance of σ_{H}^{n-1} under left-translations. □

Remark 4.8. *The proof of the monotonicity inequality will follow from the next inequality:*

$$\int_{\partial B_\varrho(x,r)\cap S}\frac{1}{\varrho_x}\left|\left\langle Z_x,\frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right|\sigma_{H}^{n-2}\leq\frac{d}{dr}\sigma_{H}^{n-1}(S_r)$$

for $\mathcal{L}^1$-a.e. $r>0$. Roughly speaking, in the classical setting this inequality follows from the Coarea Formula together with a key-property: the Euclidean metric satisfies the Ikonal equation. *We observe that if we followed the classical pattern, then we would assume that:*

- There exists a smooth homogeneous norm $\varrho:\mathbb{G}\times\mathbb{G}\longrightarrow\mathbb{R}_+$ such that:

$$\frac{|\langle Z_x,grad_{TS}\,\varrho_x\rangle|}{\varrho_x}\leq 1\qquad(19)$$

  for every $x,y\in S$.

Clearly (19) *turns out to be trivially true in the Euclidean setting. Indeed,* $\varrho_x(y) = |y - x|$, $Z_x(y) = y - x$ *and* $grad_{\mathbb{R}^n}|y - x| = \frac{y-x}{|y-x|}$. *Therefore,*

$$\frac{|\langle Z_x(y), grad_{TS}\,\varrho_x(y)\rangle|}{\varrho_x(y)} = 1 - \left\langle \frac{y - x}{|y - x|}, \mathrm{n_e} \right\rangle^2 \le 1,$$

where $\mathrm{n_e}$ *is the Euclidean unit normal of* S. *Moreover* (19) *would be "natural" in the Riemannian setting and at this regard we quote the paper by Chung, Grigor'jan and Yau [14], where this hypothesis is the starting points of a general theory about isoperimetric inequalities on weighted Riemannian manifolds and graphs. Unlike [49], here we will not follow this approach but rather a much more direct computation.*

By using Proposition 4.7 we may prove a global monotonicity formula for the H-perimeter σ_H^{n-1}. Henceforth, we shall set set $S_t := S \cap B_\varrho(x, t)$, for $t > 0$.

Theorem 4.9 (Global Monotonicity of σ_H^{n-1})**.** *Let* $S \subset \mathbb{G}$ *be a* $\mathbf{C}^2$*-smooth hypersurface. For every* $x \in \mathrm{Int}S$ *the following ordinary differential inequality holds*

$$-\frac{d}{dt}\frac{\sigma_H^{n-1}(S_t)}{t^{Q-1}} \le \frac{\mathcal{A}(t) + \mathcal{B}_2(t)}{t^{Q-1}} \tag{20}$$

for $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}_+$.

Proof. By applying Sard's Theorem we get that S_t is a $\mathbf{C}^2$-smooth manifold with boundary for $\mathcal{L}^1$-a.e. $t > 0$. From the first inequality in Proposition 4.7 we have

$$(Q - 1)\,\sigma_H^{n-1}(S_t) \le t\,(\mathcal{A}(t) + \mathcal{B}_0(t))$$

for $\mathcal{L}^1$-a.e. $t > 0$, where t is the radius of a ϱ-ball centered at $x \in \mathrm{Int}S$. Since

$$\partial S_t = \{\partial B_\varrho(x, t) \cap S\} \cup \{\partial S \cap B_\varrho(x, t)\}$$

we get that

$$(Q - 1)\,\sigma_H^{n-1}(S_t) \le t\,\left(\mathcal{A}(t) + \mathcal{B}_0^1(t) + \mathcal{B}_0^2(t)\right),$$

where we have set

$$\begin{aligned}
\mathcal{B}_0^1(t) &:= \int_{\partial B_\varrho(x,t)\cap S} \frac{1}{\varrho_x}\left|\left\langle Z_x, \frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right| \sigma_H^{n-2},\\
\mathcal{B}_0^2(t) &:= \int_{\partial S\cap B_\varrho(x,t)} \frac{1}{\varrho_x}\left|\left\langle Z_x, \frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right| \sigma_H^{n-2}.
\end{aligned}$$

Exactly as in Proposition 4.7 and Remark 4.6, the second integral $\mathcal{B}_0^2(t)$ can be estimated by Cauchy-Schwartz inequality and we get that

$$\mathcal{B}_0^2(t) \le \mathcal{B}_2(t).$$

However the crucial point of this proof is the estimate of the first integral $\mathcal{B}_0^1(t)$. To this end we have to use Coarea Formula. By Definition 2.5 and the explicit form of the variational vector field Z_x, we get that

$$\frac{1}{\varrho_x}\left|\left\langle Z_x, \frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle\right| \le 1 + \frac{2c_2\,\varrho_x(1 + O(\varrho_x))}{|\mathcal{P}_{HS}\,\eta|}$$

as long as $y \to x$ or, equivalently, $\varrho_x \to 0^+$. For any $h > 0$, let us compute

$$\begin{aligned}
\int_t^{t+h} \mathcal{B}_0^1(s)\, ds &= \int_t^{t+h} ds \int_{\partial B_\varrho(x,s)\cap S} \frac{1}{\varrho_x} \left| \left\langle Z_x, \frac{\eta}{|\mathcal{P}_{HS}\eta|} \right\rangle \right| \sigma_H^{n-2} \\
&\leq \int_t^{t+h} ds \int_{\partial B_\varrho(x,s)\cap S} \left(1 + \frac{2c_2\, \varrho_x (1 + O(\varrho_x))}{|\mathcal{P}_{HS}\eta|} \right) \sigma_H^{n-2} \\
&\leq \int_t^{t+h} \sigma_H^{n-2}(\partial B_\varrho(x,s)\cap S)\, ds + 2c_2 \int_t^{t+h} s\,(1+O(s))\; ds \int_{\partial B_\varrho(x,s)\cap S} \frac{1}{|\mathcal{P}_{HS}\eta|}\, \sigma_H^{n-2} \\
&\leq \int_t^{t+h} \sigma_H^{n-2}(\partial B_\varrho(x,s)\cap S)\, ds + 2c_2\, h\,(1+O(h)) \int_t^{t+h} ds \int_{\partial B_\varrho(x,s)\cap S} \frac{1}{|\mathcal{P}_{HS}\eta|}\, \sigma_H^{n-2} \\
&\leq \int_{S_{t+h}\setminus S_t} |grad_{HS}\, \varrho_x|\, \sigma_H^{n-1} + 2c_2\, h\,(1+O(h))\, \sigma_H^{n-1}(S_{t+h}\setminus S_t)
\end{aligned}$$

as $h \to 0^+$. We stress that the last inequality follows from Coarea formula (1) and the fact that $\eta_{HS} = \frac{grad_{HS}\, \varrho_x}{|grad_{HS}\, \varrho_x|}$ along $\partial B_\varrho(x,s)\cap S$ for $\mathcal{L}^1$-a.e. $s \in]t, t+h[$. By Definition 2.5 we have

$$|grad_{HS}\, \varrho| \leq |grad_H\, \varrho| \leq 1.$$

Therefore

$$\frac{\int_t^{t+h} \mathcal{B}_0^1(s)\, ds}{h} \leq \frac{\sigma_H^{n-1}(S_{t+h}\setminus S_t)}{h}\,(1+o(1))$$

as long as $h \to 0^+$. Hence

$$\mathcal{B}_0^1(t) \leq \frac{d}{dt}\, \sigma_H^{n-1}(S_t)$$

for $\mathcal{L}^1$-a.e. $t > 0$. Therefore

$$(Q-1)\, \sigma_H^{n-1}(S_t) \leq t \left(\mathcal{A}(t) + \mathcal{B}_2(t) + \frac{d}{dt}\, \sigma_H^{n-1}(S_t) \right)$$

which is easily seen to be equivalent to (20). □

In the sequel, we shall shaw that the right-hand side of previous global monotonicity formula can be made more intrinsic whenever the radius t of the ϱ-ball $B_\varrho(x,t)$ goes to 0^+. Taking into account the results of Section 3.5, in order to estimate $\mathcal{B}_2(t)$ we shall assume more regularity on the boundary.

4.2 Local estimates dependent on blow-up results

This section is devoted to show how estimating the integrals $\mathcal{A}(t)$ and $\mathcal{B}_2(t)$ which appear in the right-hand side of the global monotonicity formula (20).

Estimate of $\mathcal{A}(t)$.

Lemma 4.10. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^k$-smooth hypersurface. Let $x \in \mathrm{Int}S$ and $S_t = S \cap B_\varrho(x,t)$ for $t > 0$. Then there exists a constant $b_\varrho > 0$, only dependent on ϱ and $\mathbb{G}$, such that*

$$\lim_{t\to 0^+} \frac{\int_{S_t} |\varpi_{H_i}|\, \sigma_H^{n-1}}{t^{Q-i}} \leq h_i\, b_\varrho \tag{21}$$

for every $i = 2, ..., k$, where $h_i = \mathrm{dim}H_i$.

Remark 4.11. *As we shall see below, if S is just of class $\mathbf{C}^2$, then (21) holds true for every $x \in S \setminus C_S$. Moreover, if $x \in C_S$ has order $(Q-i)$, $i = 2, ..., k$, the same claim holds true if S is of class $\mathbf{C}^i$.*

Proof of Lemma 4.10. For any $\alpha = h+1, ..., n$, we have

$$(X_\alpha \lrcorner \sigma_{\scriptscriptstyle R}^n)\big|_S = \left(\langle X_\alpha, \nu\rangle \, \sigma_{\scriptscriptstyle R}^{n-1}\right)\big|_S = (*\omega_\alpha)|_S,$$

where $*$ denotes the Hodge star operation on $T^*\mathbb{G}$; see [34]. Moreover

$$\delta_t^*(*\omega_\alpha) = t^{Q-\mathrm{ord}(\alpha)}(*\omega_\alpha)$$

for every $t > 0$. So we get that

$$\int_{S_t} |\varpi_{H_i}| \, \sigma_{\scriptscriptstyle H}^{n-1} = \int_{S_t} |\mathcal{P}_{H_i}\nu| \, \sigma_{\scriptscriptstyle R}^{n-1} \leq \sum_{\mathrm{ord}(\alpha)=i} \int_{S_t} |X_\alpha \lrcorner \sigma_{\scriptscriptstyle R}^n| = \sum_{\mathrm{ord}(\alpha)=i} t^{Q-i} \int_{\vartheta^x_{1/t}S\cap B_\varrho(x,1)} |(*\omega_\alpha)\circ \vartheta^x_t| \, .$$

Now since

$$\int_{\vartheta^x_{1/t}S\cap B_\varrho(x,1)} |(*\omega_\alpha)\circ \vartheta^x_t| \leq \sigma_{\scriptscriptstyle R}^{n-1}\left(\vartheta^x_{1/t}S \cap B_\varrho(x,1)\right),$$

by using Theorem 3.14 we may pass to the limit as $t \to 0^+$ the right-hand side. More precisely, if $x \in \mathrm{Int}(S \setminus C_S)$ the rescaled hypersurfaces $\vartheta^x_{1/t}S$ converge to the vertical hyperplane $\mathcal{I}(\nu_{\scriptscriptstyle H}(x))$ as $t \to 0^+$. Otherwise we may assume that $x \in \mathrm{Int}(S \cap C_S)$ has order $(Q-i)$, for some $i = 2, ..., k$. In this case the limit-set is a polynomial hypersurface of homogeneous order i passing through x; see Remark 3.15. We remind that the convergence is understood with respect to the Hausdorff distance of sets. So let us set

$$b_1 := \sup_{X\in H,\,|X|=1} \sigma_{\scriptscriptstyle R}^{n-1}(\mathcal{I}(X) \cap B_\varrho(0,1)), \tag{22}$$

where $\mathcal{I}(X)$ denotes the vertical hyperplane through $0 \in \mathbb{G}$ and orthogonal to X. Furthermore, in order to study the characteristic case, we may define another useful constant, i.e.

$$b_2 := \sup_{\Psi\in\mathcal{P}ol_0^k} \sigma_{\scriptscriptstyle R}^{n-1}(\Psi \cap B_\varrho(0,1)), \tag{23}$$

where $\mathcal{P}ol_0^k$ denotes the class of all graphs of polynomial functions passing through $0 \in \mathbb{G}$ and of homogeneous order $\leq k$. Setting

$$b_\varrho := \max\{b_1, b_2\} \tag{24}$$

and using the left-invariance of $\sigma_{\scriptscriptstyle R}^{n-1}$, yields

$$\lim_{t\to 0^+} \sigma_{\scriptscriptstyle R}^{n-1}\left(\vartheta^x_{1/t}S \cap B_\varrho(x,1)\right) \leq b_\varrho.$$

Therefore

$$\frac{\int_{S_t} |\varpi_{H_i}| \, \sigma_{\scriptscriptstyle H}^{n-1}}{t^{Q-i}} \leq h_i \, \sigma_{\scriptscriptstyle R}^{n-1}\left(\vartheta^x_{1/t}S \cap B_\varrho(x,1)\right) \leq h_i \, b_\varrho.$$

This achieves the proof of (21). ☐

Let $S \subset \mathbb{G}$ be of class $\mathbf{C}^2$, let $x \in \mathrm{Int}(S \setminus C_S)$ and $\mathcal{A}(t)$ as in Definition 4.5. By applying Theorem 3.19, we get that

$$\dim_{\mathrm{Eu-Hau}}(C_S) \leq n-2.$$

In particular $\sigma_{\scriptscriptstyle R}^{n-1}$-a.e. interior point of S is non-characteristic.

Lemma 4.12. *Under the previous assumptions, one has*

$$\mathcal{A}(t) \leq \left(\int_{S_t} |\mathcal{H}_H|\, \sigma_H^{n-1} \right) (1 + o(1)) \tag{25}$$

as long as $t \to 0^+$, where $S_t = S \cap B_\varrho(x, t)$.

Proof. We have

$$\begin{aligned} \mathcal{A}(t) &= \int_{S_t} |\mathcal{H}_H| \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\varpi_{H_i}| \right) \sigma_H^{n-1} \\ &\leq \int_{S_t} |\mathcal{H}_H|\, \sigma_H^{n-1} + \|\mathcal{H}_H\|_{L^\infty(S_t)} \sum_{i=2}^{k} \int_{S_t} i\, c_i \varrho_x^{i-1} |\varpi_{H_i}|\, \sigma_H^{n-1} \\ &\leq \int_{S_t} |\mathcal{H}_H|\, \sigma_H^{n-1} + \|\mathcal{H}_H\|_{L^\infty(S_t)} \int_{S_t} \frac{2c_2 \varrho_x (1 + o(1))}{|\mathcal{P}_H \nu|} \sigma_H^{n-1} \end{aligned}$$

as long as $t \to 0^+$. Indeed note that $\varrho_x(y) = \varrho(x, y) \to 0^+$ as $t \to 0^+$. Since $\frac{1}{|\mathcal{P}_H \nu|}$ is continous near $x \in \mathrm{Int}(S \setminus C_S)$, by standard results in Measure Theory we easily ge that

$$\lim_{t \to 0^+} \frac{\int_{S_t} \frac{1}{|\mathcal{P}_H \nu|} \sigma_H^{n-1}}{\sigma_H^{n-1}(S_t)} = \frac{1}{|\mathcal{P}_H \nu(x)|}.$$

Therefore

$$\int_{S_t} \frac{2c_2 \varrho_x (1 + o(1))}{|\mathcal{P}_H \nu|} \sigma_H^{n-1} \leq \frac{2c_2 t (1 + o(1))}{|\mathcal{P}_H \nu(x)|}$$

as $t \to 0^+$ and so

$$\lim_{t \to 0^+} \frac{\int_{S_t} \frac{2c_2 \varrho_x (1+o(1))}{|\mathcal{P}_H \nu|} \sigma_H^{n-1}}{\sigma_H^{n-1}(S_t)} = 0.$$

Since $\mathcal{H}_H$ turns out to be continuous near every non-characteristic point, then $\|\mathcal{H}_H\|_{L^\infty(S_t)}$ is bounded and (25) easily follows. □

Actually, a similar result holds true even if $x \in \mathrm{Int}(S \cap C_S)$, at least whenever S is smooth enough near C_S. In the sequel, we shall make heavy use of Theorem 3.14, Case (2).

Lemma 4.13. *Let $x \in \mathrm{Int}(S \cap C_S)$, be an interior characteristic point of S of order $(Q - i)$ for some $i = 2, ..., k$ and assume that, there exists $\alpha = h + 1, ..., n$, $\mathrm{ord}(\alpha) = i$, such that S can be represented, locally around x, as the X_α-graph of a $\mathbf{C}^i$-smooth function for which (11) holds true. Then there exists a constant $d_\varrho > 0$, only dependent on ϱ and $\mathbb{G}$, such that*

$$\mathcal{A}(t) \leq \|\mathcal{H}_H\|_{L^\infty(S)} (1 + d_\varrho)\, \sigma_H^{n-1}(S_t)$$

as long as $t \to 0^+$.

Proof. Using Lemma 4.10 yields

$$\frac{\sum_{i=2}^{k} \int_{S_t} i\, c_i \varrho_x^{i-1} |\varpi_{H_i}|\, \sigma_H^{n-1}}{t^{Q-1}} \leq \sum_{i=2}^{k} i\, c_i h_i\, b_\varrho$$

as $t \to 0^+$, where b_ϱ is the constant defined by (24). Set now $d_\varrho := \sum_{i=2}^{k} i\, c_i h_i\, b_\varrho$. By arguing as in the proof of Lemma 4.12, the proof easily follows. □

Estimate of $\mathcal{B}_2(t)$.

Warning 4.14. *From now on we shall assume that ∂S be -at least piecewise- $\mathbf{C}^2$-smooth.*

Remark 4.15. *Since ∂S is assumed to be piecewise $\mathbf{C}^2$-smooth, we may apply Theorem 3.19. In particular, if $\dim V \geq 2$ it follows that $\dim_{\mathrm{Eu-Hau}}(C_{\partial S}) \leq n-3$ and $\sigma_{\mathcal{R}}^{n-2}$-a.e. $x \in \partial S$ is non-characteristic. The same holds true for the Heisenberg groups $\mathbb{H}^n$, $n > 1$, and for those 2-step Carnot groups $\mathbb{G}$, described at (ii) of Remark 3.20, which satisfy the condition $\frac{\widehat{h}}{2} < n-2$. Nevertheless, in the remaining cases[19], by using Theorem 3.22 we get that for any $x \in \partial S$, there exist an open neighborhood $U_x \subset \mathbb{G}$ and positive constants C_1, C_2 and r_0 dependent on $U_x \cap \partial S$, such that*

$$C_1 r^{\mathrm{ord}(x)} \leq \sigma_{\mathcal{R}}^{n-2}(\partial S \cap B_\varrho(z,r)) \leq C_2 r^{\mathrm{ord}(x)}$$

for every $z \in \partial S \cap U_x$ with $\mathrm{ord}(z) = \mathrm{ord}(x)$ and every $r < r_0$. Note that in this case, the order $\mathrm{ord}(x)$ of $x \in \partial S$ can be $(Q-2)$, if the point x is non-characteristic, or $(Q-3)$ otherwise. Furthermore for every smooth point $x \in \partial S$ the rescaled sets $\delta_{\frac{1}{r}}(x^{-1} \bullet \partial S)$ locally converge, with respect to the Hausdorff distance of sets, to the $(n-2)$-dimensional plane $\mathcal{I}^2(\nu_{_H}(x))$, if $x \in \partial S \setminus C_{\partial S}$. Otherwise, the limit set ∂S_∞ is an algebraic variety and, more precisely, the 2-graph of a polynomial function of homogeneous order 2.

Remark 4.16. *Since we have to estimate $\mathcal{B}_2(t)$ for t small, it is clear that $\varrho(x, \partial S)$ must be comparable with t, where $\varrho(x, \partial S)$ denotes the ϱ-distance from x and ∂S.*

The key-point is the following one:

Lemma 4.17. *Assume that $\dim_{\mathrm{Eu-Hau}}(C_{\partial S}) \leq n-3$. Then*

$$\mathcal{B}_2(t) \leq \sigma_{_H}^{n-2}(\partial S \cap B_\varrho(x,t))\,(1 + o(1)) \tag{26}$$

as long as $t \to 0^+$.

Proof. Let $x_0 \in \partial S \cap B_\varrho(x,t)$ be a non-characteristic point[20]. One has

$$\partial S \cap B_\varrho(x,t) \subset \partial S \cap B_\varrho(x_0, 2t).$$

We therefore get that

$$\begin{aligned}
\mathcal{B}_2(t) &= \int_{\partial S \cap B_\varrho(x,t)} \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_i S}|\right) \sigma_{_H}^{n-2} \\
&= \sigma_{_H}^{n-2}(\partial S \cap B_\varrho(x,t)) + \int_{\partial S \cap B_\varrho(x,t)} \left(\sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_i S}|\right) \sigma_{_H}^{n-2} \\
&\leq \sigma_{_H}^{n-2}(\partial S \cap B_\varrho(x,t)) + \int_{\partial S \cap B_\varrho(x_0,2t)} \left(\sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_i S}|\right) \sigma_{_H}^{n-2}.
\end{aligned}$$

By using again standard results in Measure Theory it is not difficult to show that

$$f(x_0) = \lim_{t \to 0+} \frac{\int_{\partial S \cap B_\varrho(x_0,2t)} f\, \sigma_{_H}^{n-2}}{\sigma_{_H}^{n-2}(\partial S \cap B_\varrho(x_0,2t))}. \tag{27}$$

[19] They are, up to isomorphisms, $\mathbb{H}^1$ and those 2-step Carnot groups $\mathbb{G}$, introduced at (ii) of Remark 3.20, for which $\frac{\widehat{h}}{2} = n-2$.

[20] We stress that if x_0 is a non-characteristic point of the boundary ∂S, then $|\mathcal{P}_{HS}\,\eta(x_0)| \neq 0$.

for every $f \in \mathbf{C}(\partial S \cap U_{x_0})$, where $U_{x_0} \subset \mathbb{G}$ is an open neighborhood of x_0. So let us set

$$f(y) := \sum_{i=2}^{k} i\, c_i \varrho(x,y)^{i-1} |\chi_{H_i S}(y)|.$$

Since $x_0 \in \partial S \cap B_\varrho(x,t)$ is a non-characteristic boundary point, then $|\mathcal{P}_{HS}\eta(x_0)| \neq 0$. Hence the function f turns out to be continuous in an open neighborhood of x_0 and we may therefore apply (27). Note that

$$f(x_0) = \sum_{i=2}^{k} i\, c_i \varrho(x,x_0)^{i-1} |\chi_{H_i S}(x_0)| \leq \frac{2c_2}{|\mathcal{P}_{HS}\eta(x_0)|}\, \varrho(x,x_0)\ (1+o(1)) = O(\varrho(x,x_0))$$

as long as $x \to x_0$. By construction if $t \to 0^+$, then $\varrho(x,x_0) \to 0^+$. Therefore

$$\lim_{t\to 0+} \frac{\int_{\partial S\cap B_\varrho(x_0,2t)} \left(\sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_i S}|\right)\ \sigma_H^{n-2}}{\sigma_H^{n-2}(\partial S \cap B_\varrho(x_0,2t))} = 0.$$

By applying Theorem 3.21, we easily get that

$$\lim_{t\to 0+} \frac{\sigma_H^{n-2}(\partial S \cap B_\varrho(x_0,2t))}{\sigma_H^{n-2}(\partial S \cap B_\varrho(x_0,t))} = 2^{Q-2}.$$

It follows that

$$\lim_{t\to 0+} \frac{\int_{\partial S\cap B_\varrho(x_0,2t)} \left(\sum_{i=2}^{k} i\, c_i \varrho_x^{i-1} |\chi_{H_i S}|\right)\ \sigma_H^{n-2}}{\sigma_H^{n-2}(\partial S \cap B_\varrho(x_0,t))} = 0$$

which implies the thesis. □

Lemma 4.18. *Let $\mathbb{G}$ be 2-step Carnot group $\mathbb{G}$ and let ∂S be piecewise $\mathbf{C}^2$-smooth. Then there exists a constant $k > 0$, only dependent on ϱ and $\mathbb{G}$, such that*

$$\mathcal{B}_2(t) \leq (1+k)\, \sigma_H^{n-2}(\partial S \cap B_\varrho(x,t)) \tag{28}$$

as long as $t \to 0^+$.

Proof. We shall show that (28) turns out to be true near any characteristic point $x_0 \in \partial S$. This will be done by using Theorem 3.22; see also Remark 4.15. Let $\mathbb{G}$ be any 2-step Carnot group and assume that $x_0 \in \partial S \cap B_\varrho(x,t)$ be such that

$$\mathrm{ord}(x_0) = Q - 3,$$

see footnote 16. Note that we only need to estimate the integral

$$\int_{\partial S\cap B_\varrho(x_0,t)} 2\, c_2 \varrho_x |\chi_{H_2 S}|\, \sigma_H^{n-2}$$

as $t \to 0^+$. By Theorem 3.22, for any $x_0 \in C_{\partial S}$, the rescaled sets $\delta_{\frac{1}{r}}(x_0^{-1} \bullet \partial S)$ locally converge, with respect to the Hausdorff distance of sets, to an algebraic variety which is the graph of a polynomial function of homogeneous order 2. So let us consider the quotient

$$\frac{\int_{\partial S\cap B_\varrho(x_0,t)} \varrho(x_0,y)\, \sigma_R^{n-2}(y)}{t^{Q-2}}$$

for $t \to 0^+$. *In fact, estimating this integral by a dimensional constant is the key point in order to achieve the estimate of $\mathcal{B}_2(t)$, even near characteristic points.* This can be done as follows. Since, at this moment, we are working in a 2-step Carnot group $\mathbb{G}$, we easily see that $\delta_t^*\sigma_{\scriptscriptstyle R}^{n-2}$ splits into two homogeneous components of (homogeneous) degree $(Q-2)$ and $(Q-3)$, respectively. In other words, there exist $\mathbf{C}^1$-smooth $(n-1)$-forms, say $(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-2}$, $(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-3}$, such that

$$\delta_t^*\sigma_{\scriptscriptstyle R}^{n-2} = t^{Q-2}(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-2} + t^{Q-3}(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-3}.$$

Using this, together with the left-invariance of $\sigma_{\scriptscriptstyle R}^{n-2}$ and the 1-homogeneity of ϱ, yields

$$\begin{aligned}
\frac{\int_{\partial S\cap B_\varrho(x_0,t)}\varrho(x_0,y)\,\sigma_{\scriptscriptstyle R}^{n-2}(y)}{t^{Q-2}} &= \frac{t^{Q-2}\,(1+o(1))\int_{\delta_{\frac{1}{t}}(x_0^{-1}\bullet\partial S)\cap B_\varrho(0,1)}\varrho(y)\,(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-3}(y)}{t^{Q-2}}\\
&= \int_{\delta_{\frac{1}{t}}(x_0^{-1}\bullet\partial S)\cap B_\varrho(0,1)}\varrho\,(\sigma_{\scriptscriptstyle R}^{n-2})_{Q-3}\,(1+o(1))\\
&\leq \sigma_{\scriptscriptstyle R}^{n-2}\left(\delta_{\frac{1}{t}}(x_0^{-1}\bullet\partial S)\cap B_\varrho(0,1)\right)(1+o(1))
\end{aligned}$$

as $t \to 0^+$. Set now

$$\partial S_\infty := \lim_{t\to 0^+}\delta_{\frac{1}{t}}(x_0^{-1}\bullet\partial S),$$

where the limit is understood with respect to the Hausdorff convergence of sets. As already said, ∂S_∞ is an algebraic variety and turns out to be the 2-graph of a polynomial function of homogeneous degree 2. So let us set

$$k_1 := \sup_{\Psi\in\mathcal{P}ol_0^2}\sigma_{\scriptscriptstyle R}^{n-2}(\Psi\cap B_\varrho(0,1)), \tag{29}$$

where $\mathcal{P}ol_0^2$ denotes the family of all 2-graph of homogeneous polynomial functions of degree 2 which vanish at $0\in\mathbb{G}$. Obviously, k_1 is a finite constant which only depends on ϱ. Remind that if $t\to 0^+$, then $\varrho(x,x_0)\to 0^+$. Therefore

$$\lim_{t\to 0^+}\frac{\int_{\partial S\cap B_\varrho(x_0,t)}2\,c_2\,\varrho_x|\chi_{\scriptscriptstyle H_2S}|\,\sigma_{\scriptscriptstyle H}^{n-2}}{t^{Q-2}} \leq \underbrace{2\,c_2\,k_1}_{=:k_2}.$$

By applying Claim 4.17, we see that this estimate holds true even if $x_0\in\partial S\setminus C_{\partial S}$. By using Lemma 3.23 we therefore get that

$$\int_{\partial S\cap B_\varrho(x_0,t)}2\,c_2\,\varrho_x|\chi_{\scriptscriptstyle H_2S}|\,\sigma_{\scriptscriptstyle H}^{n-2} \leq k_2\,\mathcal{S}_\varrho^{Q-2}(\partial S\cap B_\varrho(x_0,t)).$$

By means of Theorem 3.22 we can estimate the right-hand side in terms of the measure $\sigma_{\scriptscriptstyle H}^{n-2}$. Indeed, at the characteristic set, both measures vanish. Moreover, near non-characteristic points, the measures $\sigma_{\scriptscriptstyle H}^{n-2}$ and $\mathcal{S}_\varrho^{Q-2}$ are locally equivalent, up to the metric-factor $\kappa(\nu_{\scriptscriptstyle H})$ which is a bounded density-function. So let us define the constant:

$$\kappa_\varrho := \sup_{X\in\Lambda^2(H)}\sigma_{\scriptscriptstyle H}^{n-2}(B_\varrho(0,1)\cap\mathcal{I}^2(X)\}$$

where $\Lambda^2(H) := \{X_1\wedge X_2\in\Lambda^2(T\mathbb{G}) : X_1,X_2\in H,\,|X_1\wedge X_2| = 1\}$. Clearly κ_ϱ bounds from above the metric factor $\kappa(\nu_{\scriptscriptstyle H})$. Setting $k := k_2\,\kappa_\varrho$, the thesis easily follows. $\square$

Remark 4.19 (The constant k_1 for $\mathbf{C}^1$-smooth transversal curves in $\mathbb{H}^1$)**.** *In this example one has $n = 3$ and $Q = 4$. In particular, ∂S is 1-dimensional and it can be characteristic. In such a case, the order of any characteristic point is 1. So let $\gamma_x :]-\epsilon, \epsilon[\subseteq \mathbb{R} \longrightarrow \mathbb{H}^1$ be a $\mathbf{C}^1$-smooth curve which parametrizes ∂S locally around $x \in \partial S \cap C_{\partial S}$. In this case, one can show that the limit-set at $x = \exp(x_H, t)$ is an interval of the vertical line*

$$x \bullet \exp(T) = \{\exp(y_H, s) \in \mathbb{H}^1 : x_H = y_H\}$$

over the point x. The proof can be done by using a Taylor's expansion of γ_x at 0 and Heisenberg dilations; see, for instance, [51]. As a consequence, we can show that

$$k_1 \leq \mathrm{diam}_\varrho(B_\varrho(0,1)) = 2$$

where the constant k_1 is that given by (29).

4.3 Proof of Theorem 4.1

By applying the results of Section 4.2 and Theorem 4.9 we get the following local version of the monotonicity inequality:

Corollary 4.20 (Local Monotonicity)**.** *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let ∂S be piecewise $\mathbf{C}^2$-smooth. Then there exists a constant $C_\varrho \geq 1$, only dependent on ϱ and $\mathbb{G}$, such that the following statement holds: for every $x \in \mathrm{Int}(S \setminus C_S)$ there exists $\overline{r}(x) > 0$ such that*

$$-\frac{d}{dt}\frac{\sigma_H^{n-1}(S_t)}{t^{Q-1}} \leq \frac{C_\varrho}{t^{Q-1}}\left(\int_{S_t} |\mathcal{H}_H|\,\sigma_H^{n-1} + \sigma_H^{n-2}(\partial S \cap B_\varrho(x,t))\right) \tag{30}$$

for $\mathcal{L}^1$-a.e. $t \in]0, \overline{r}(x)[$.

Proof. The corollary is an immediate consequence of Theorem 4.9, Lemma 4.12, Lemma 4.17 and Lemma 4.18. We stress that if $\dim_{\mathrm{Eu-Hau}}(C_S) \leq n-2$ (as, for instance, for every Carnot group $\mathbb{G}$ such that $\dim V \geq 2$, or for the Heisenberg groups $\mathbb{H}^n$ with $n > 1$; see Remark 4.15) we may take, for example, $C_\varrho = 2$. Otherwise, we are necessarily in a 2-step Carnot group. We may therefore apply Lemma 4.18 and take $C_\varrho = 2(1+k)$. ☐

Notation 4.21. *Let $t \geq 0$. Henceforth, shall set*

$$\mathcal{D}(t) := C_\varrho\left(\int_{S_t} |\mathcal{H}_H|\,\sigma_H^{n-1} + \sigma_H^{n-2}(\partial S \cap B_\varrho(x,t))\right).$$

Lemma 4.22. *Let $x \in \mathrm{Int}(S \setminus C_S)$ and set*[21] $r_0(x) := \min\left\{\overline{r}(x), 2\left(\frac{\sigma_H^{n-1}(S)}{k_\varrho(\nu_H(x))}\right)^{1/Q-1}\right\}$*. Then, for every $\lambda \geq 2$ there exists $r \in]0, r_0(x)]$ such that*

$$\sigma_H^{n-1}(S_{\lambda r}) \leq \lambda^{Q-1}\, r_0(x)\, \mathcal{D}(r).$$

Proof of Lemma 4.22. Fix $r \in]0, r_0(x)]$ and note that $\sigma_H^{n-1}(S_t)$ is a monotone non-decreasing function of t on $]r, r_0(x)]$. So let us write the identity

$$\sigma_H^{n-1}(S_t)/t^{Q-1} = \left\{\sigma_H^{n-1}(S_t) - \sigma_H^{n-1}\left(S_{r_0(x)}\right)\right\}/t^{Q-1} + \sigma_H^{n-1}\left(S_{r_0(x)}\right)/t^{Q-1}.$$

The first addend is an increasing function of t, while the second one is an absolutely continuous function of t. Therefore, by integrating the differential inequality (20), we get that

[21] The quantity $\overline{r}(x)$ is that in Corollary 4.20.

$$\frac{\sigma_H^{n-1}(S_r)}{r^{Q-1}} \leq \frac{\sigma_H^{n-1}\left(S_{r_0(x)}\right)}{(r_0(x))^{Q-1}} + \int_r^{r_0(x)} \mathcal{D}(t)\, t^{-(Q-1)} dt. \tag{31}$$

Therefore

$$\beta \;:=\; \sup_{r\in]0,r_0(x)]} \frac{\sigma_H^{n-1}(S_r)}{r^{Q-1}} \leq \frac{\sigma_H^{n-1}\left(S_{r_0(x)}\right)}{(r_0(x))^{Q-1}} + \int_0^{r_0(x)} \mathcal{D}(t)\, t^{-(Q-1)} dt.$$

Now we argue by contradiction. If the lemma is false, it follows that for every $r \in]0, r_0(x)]$

$$\sigma_H^{n-1}(S_{\lambda r}) > \lambda^{Q-1} r_0(x)\, \mathcal{D}(t).$$

From the last inequality we infer that

$$\begin{aligned}
\int_0^{r_0(x)} \mathcal{D}(t)\, t^{-(Q-1)} dt &\leq \frac{1}{\lambda^{Q-1} r_0(x)} \int_0^{r_0(x)} \sigma_H^{n-1}(S_{\lambda t})\, t^{-(Q-1)} dt \\
&= \frac{1}{\lambda\, r_0(x)} \int_0^{\lambda r_0(x)} \sigma_H^{n-1}(S_s)\, s^{-(Q-1)} ds \\
&= \frac{1}{\lambda\, r_0(x)} \int_0^{r_0(x)} \sigma_H^{n-1}(S_s)\, s^{-(Q-1)} ds + \frac{1}{\lambda r_0(x)} \int_{r_0(x)}^{\lambda r_0(x)} \sigma_H^{n-1}(S_s)\, s^{-(Q-1)} ds \\
&\leq \frac{\beta}{\lambda} + \frac{\lambda-1}{\lambda} \frac{\sigma_H^{n-1}(S)}{(r_0(x))^{Q-1}}.
\end{aligned}$$

Therefore, using (31) yields

$$\beta \leq \frac{\sigma_H^{n-1}\left(S_{r_0(x)}\right)}{(r_0(x))^{Q-1}} + \frac{\beta}{\lambda} + \frac{\lambda-1}{\lambda} \frac{\sigma_H^{n-1}(S)}{(r_0(x))^{Q-1}}$$

and so

$$\frac{\lambda-1}{\lambda}\beta \leq \frac{2\lambda-1}{\lambda} \left(\frac{\sigma_H^{n-1}(S)}{(r_0(x))^{Q-1}} \right) = \frac{2\lambda-1}{\lambda} \left(\frac{k_\varrho(\nu_H(x))}{2^{Q-1}} \right).$$

By its own definition, one has

$$k_\varrho(\nu_H(x)) = \lim_{r\searrow 0^+} \frac{\sigma_H^{n-1}(S_r)}{r^{Q-1}} \leq \beta.$$

Furthermore, since[22] $Q-1 \geq 3$, we get that

$$\lambda - 1 \leq \frac{2\lambda-1}{8},$$

or equivalently $\lambda \leq \frac{7}{6}$, which contradicts the hypothesis $\lambda \geq 2$.

□

The next covering lemma is well-known and can be found in [7]; see also [22].

[22] Indeed, the first non-abelian Carnot group is the Heisenberg group $\mathbb{H}^1$ for which $Q = 4$. Moreover, since the theory of Carnot groups also contains as a special case the theory of Euclidean spaces, in the previous argument we can also use the estimate $Q - 1 \geq 2$ which is relative to the case of a surface in $\mathbb{R}^3$. In such a case $Q = 3$, since the homogeneous dimension coincides with the topological one.

Lemma 4.23 (Vitali's Covering Lemma)**.** *Let (X,ϱ) be a compact metric space and let $A\subseteq X$. Moreover, let C be a covering of A by closed ϱ-balls with centers in A. We also assume that each point x of A is the center of at least one closed ϱ-ball belonging to C and that the radii of the balls of the covering C are uniformly bounded by some positive constant. Then, for every $\lambda>2$ there exists a no more than countable subset $\mathsf{C}_\lambda\subsetneq\mathsf{C}$ of pairwise non-intersecting closed balls $\overline{B}_\varrho(x_k,r_k)$, $k\in\mathbb{N}$, such that*

$$A\subset\bigcup_{k\in\mathbb{N}}B_\varrho(x_k,\lambda\, r_k).$$

Notation 4.24. *Henceforth, we shall set $r_0(S):=\sup_{x\in S\setminus C_S}r_0(x)$.*

We are now in a position to prove our main result.

Proof of Theorem 4.1. Fist we shall apply Lemma 4.22. To this aim, let $\lambda>2$ be fixed and, for every $x\in\mathrm{Int}(S\setminus C_S)$, let $r(x)\in]0,r_0(S)[$ be such that

$$\sigma_{H}^{n-1}(S_{r(x)})\leq\lambda^{Q-1}\,r_0(S)\,\mathcal{D}(r(x)).\tag{32}$$

So let us consider the covering $\mathsf{C}=\left\{\overline{B_\varrho}(x,r(x)):x\in(S\setminus C_S)\right\}$ of the (relatively compact) set $S\setminus C_S\subsetneq\mathbb{G}$. By Lemma 4.23, there exists a non more than countable subset $\mathsf{C}_\lambda\subsetneq\mathsf{C}$ of pairwise non-intersecting closed balls $\overline{B}_\varrho(x_k,r_k)$, where we have set $r_k:=r(x_k)$, $k\in\mathbb{N}$, such that

$$S\setminus C_S\subset\bigcup_{k\in\mathbb{N}}B_\varrho(x_k,\lambda\, r_k).$$

We therefore get

$$\begin{aligned}\sigma_{H}^{n-1}(S)&\leq\sum_{k\in\mathbb{N}}\sigma_{H}^{n-1}(S\cap B_\varrho(x_k,\lambda\, r_k))\\&\leq\lambda^{Q-1}\,r_0(S)\sum_{k\in\mathbb{N}}\mathcal{D}(r_k)\qquad\text{(by (32))}\\&=\lambda^{Q-1}\,r_0(S)\sum_{k\in\mathbb{N}}C_\varrho\left(\int_{S_{r_k}}|\mathcal{H}_H|\,\sigma_{H}^{n-1}+\sigma_{H}^{n-2}(\partial S\cap B_\varrho(x_k,r_k))\right)\\&\leq\lambda^{Q-1}\,r_0(S)\,C_\varrho\left(\int_S|\mathcal{H}_H|\,\sigma_{H}^{n-1}+\sigma_{H}^{n-2}(\partial S)\right).\end{aligned}$$

By letting $\lambda\searrow 2$, we get that

$$\sigma_{H}^{n-1}(S)\leq 2^{Q-1}r_0(S)\,C_\varrho\left(\int_S|\mathcal{H}_H|\,\sigma_{H}^{n-1}+\sigma_{H}^{n-2}(\partial S)\right).$$

Since

$$2^{Q-1}r_0(S)\leq 2^{Q-1}\sup_{x\in(S\setminus C_S)}2\left(\frac{\sigma_{H}^{n-1}(S)}{k_\varrho(\nu_{H}(x))}\right)^{\frac{1}{Q-1}}=2^Q\sup_{x\in(S\setminus C_S)}\frac{\left(\sigma_{H}^{n-1}(S)\right)^{\frac{1}{Q-1}}}{\left(k_\varrho(\nu_{H}(x))\right)^{\frac{1}{Q-1}}},$$

using (17) yields

$$2^{Q-1}r_0(S)\leq 2^Q\,\frac{\left(\sigma_{H}^{n-1}(S)\right)^{\frac{1}{Q-1}}}{K_1^{\frac{1}{Q-1}}}.$$

Therefore

$$\left(\sigma_{H}^{n-1}(S)\right)^{\frac{Q-2}{Q-1}} \leq \frac{2^{Q}\, C_{\varrho}}{K_1^{\frac{1}{Q-1}}}\left(\int_S |\mathcal{H}_H|\, \sigma_{H}^{n-1} + \sigma_{H}^{n-2}(\partial S)\right).$$

The proof of (18) is achieved by setting

$$C_{Isop} := \frac{2^{Q}\, C_{\varrho}}{K_1^{\frac{1}{Q-1}}},$$

where K_1 and C_ϱ have been defined in Remark 3.18 and Corollary 4.20, respectively. Note that these constants only depend on the group $\mathbb{G}$ and on the homogeneous metric ϱ. □

4.4 An application of the monotonicity formula: asymptotic behavior of σ_{H}^{n-1}

The global monotonicity formula (20) (see Theorem 4.9) can be formulated as follows:

$$\frac{d}{dt}\left[\frac{\sigma_{H}^{n-1}(S_t)}{t^{Q-1}}\, exp\left(\int_0^t \frac{\mathcal{A}(s)+\mathcal{B}_2(s)}{\sigma_{H}^{n-1}(S_s)}ds\right)\right] \geq 0 \tag{33}$$

for $\mathcal{L}^1$-a.e. $t>0$ and for every $x \in \mathrm{Int}S$. For sake of simplicity, let S be closed (and hence $\mathcal{B}_2(s)=0$, identically) and let us first restrict ourselves to consider non-characteristic points. By Theorem 3.14, Case (1), we may pass to the limit as $t \searrow 0^+$ in the previous inequality; see Section 3.4. Hence

$$\sigma_{H}^{n-1}(S_t) \geq \kappa_\varrho(\nu_H(x))\, t^{Q-1} exp\left(-\int_0^t \frac{\mathcal{A}(s)}{\sigma_{H}^{n-1}(S_s)}ds\right), \tag{34}$$

for every $x \in \mathrm{Int}(S \setminus C_S)$.

Corollary 4.25. *Let $\mathbb{G}$ be a k-step Carnot group and let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^2$. Assume that $\partial S \cap B_\varrho(x,t)=\emptyset$ for some $t>0$ and that $|\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty$. Then, for every $x \in \mathrm{Int}(S\setminus C_S)$, one has*

$$\sigma_{H}^{n-1}(S_t) \geq \kappa_\varrho(\nu_H(x))\, t^{Q-1} e^{-t\,\mathcal{H}_H^0} \tag{35}$$

as long as $t \to 0^+$.

Proof. We just have to bound $\int_0^t \frac{\mathcal{A}(s)}{\sigma_{H}^{n-1}(S_s)}\, ds$ from above. Using Lemma 4.12 yields

$$\int_0^t \frac{\mathcal{A}(s)}{\sigma_{H}^{n-1}(S_s)}\, ds \leq \mathcal{H}_H^0\, (1+o(1))$$

as long as $t \to 0^+$ and (35) follows from (34). □

If S is smooth enough near its characteristic set C_S, the previous result can be generalized by applying some results of Section 4.2.

Corollary 4.26. *Let $x \in \mathrm{Int}(S \cap C_S)$, be an interior characteristic point of S of order $(Q-i)$, for some $i=2,...,k$. Assume that there exists $\alpha = h+1,...,n$, $\mathrm{ord}(\alpha)=i$, such that S can be represented, locally around x, as the X_α-graph of a $\mathbf{C}^i$-smooth function satisfying (11). Assume that $\partial S \cap B_\varrho(x,t)=\emptyset$ for some $t>0$ and that $|\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty$. Then*

$$\sigma_{H}^{n-1}(S_t) \geq \kappa_\varrho(C_S(x))\, t^{Q-1} e^{-t\,\mathcal{H}_H^0\,(1+d_\varrho)} \tag{36}$$

as long as $t \to 0^+$.

We remind that $\kappa_\varrho(C_S(x))$ has been defined in Theorem 3.14, Case (2). We also stress that

$$d_\varrho = \sum_{i=2}^{k} i\, c_i h_i\, b_\varrho,$$

where b_ϱ is the constant, only depends on ϱ and $\mathbb{G}$, defined by (24).

Proof. By arguing as above, we may pass to the limit in (33) as $t \searrow 0^+$ and we get that

$$\sigma_H^{n-1}(S_t) \geq \kappa_\varrho(C_S(x))\, t^{Q-1}\, exp\left(-\int_0^t \frac{\mathcal{A}(s)}{\sigma_H^{n-1}(S_s)} ds\right).$$

By applying Lemma 4.13 we get that

$$\int_0^t \frac{\mathcal{A}(s)}{\sigma_H^{2n}(S_s)} \leq \mathcal{H}_H^0\, (1 + d_\varrho)$$

as $t \to 0^+$. This achieves the proof. □

In particular, in the case of Heisenberg groups $\mathbb{H}^n$, the following holds:

Corollary 4.27. *Let $(\mathbb{H}^n, \varrho)$ be the Heisenberg group endowed with its Korany distance; see Example 2.6. Let $S \subset \mathbb{H}^n$ be a $\mathbf{C}^2$-smooth hypersurface. Assume that $\partial S \cap B_\varrho(x,t) = \emptyset$ for some $t > 0$ and that $|\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty$. Then, for every $x \in S \cap C_S$, one has*

$$\sigma_H^{2n}(S_t) \geq \kappa_\varrho(C_S(x))\, t^{Q-1} e^{-t\mathcal{H}_H^0\,(1+b_\varrho)} \tag{37}$$

as long as $t \to 0^+$.

The constant $\kappa_\varrho(C_S(x))$ has been defined in Theorem 3.14, Case (2). Even in this case the constant b_ϱ is that defined by (24).

Proof. By arguing as for the non-characteristic case, we may pass to the limit in (33) as $t \searrow 0^+$. As above, we have

$$\sigma_H^{2n}(S_t) \geq \kappa_\varrho(C_S(x))\, t^{Q-1}\, exp\left(-\int_0^t \frac{\mathcal{A}(s)}{\sigma_H^{2n}(S_s)} ds\right),$$

as $t \searrow 0^+$, for every $x \in S \cap C_S$. By applying Lemma 4.10 we get

$$\frac{\mathcal{A}(s)}{\sigma_H^{2n}(S_s)} \leq \mathcal{H}_H^0\, (1 + 2\, c_2\, b_\varrho) = \mathcal{H}_H^0\, (1 + b_\varrho)\,,$$

for every small enough $s > 0$, since in this case $c_2 = \frac{1}{2}$. □

Example 4.28. *Consider $(\mathbb{H}^n, \varrho)$ where ϱ is the Korany distance and remind that $Q = 2n + 2$. Let $S = \{\exp(x_H, t) \in \mathbb{H}^n : t = 0\}$. One has $C_S = 0 \in \mathbb{H}^n$. Furthermore $\mathcal{H}_H = 0$, since $\nu_H = -\frac{1}{2} C_H^{2n+1} x_H$ and*

$$div_H\, \nu_H = \frac{1}{2} div_{\mathbb{R}^{2n}}(-x_2, x_1, ..., -x_{2n-1}, x_{2n}) = 0.$$

By a little computation we see that $\kappa_\varrho(C_S) = \frac{O_{2n}}{4n}$, where O_{2n-1} is the surface measure of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$. Thus (37) says that

$$\sigma_H^{2n}(S_t) \geq \frac{O_{2n}}{4n}\, t^{Q-1}.$$

In this elementary case, the claim can easily be verified by using $\sigma_H^{2n} = \frac{|x_H|}{2}\, d\mathcal{L}^{2n}$ and then spherical coordinates on $\mathbb{R}^{2n}$.

5 Sobolev-type inequalities on hypersurfaces

The isoperimetric inequality (18) is actually equivalent to a Sobolev-type inequality. The proof is analogous to that of the equivalence between the Euclidean Isoperimetric Inequality and the Sobolev one; see [7].

Theorem 5.1. *Let $\mathbb{G}$ be a k-step Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth closed hypersurface. Then*

$$\left(\int_S |\psi|^{\frac{Q-1}{Q-2}}\,\sigma_H^{n-1}\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop}\int_S \left(|\psi|\,|\mathcal{H}_H| + |grad_{HS}\psi|\right)\,\sigma_H^{n-1} \tag{38}$$

for every $\psi \in \mathbf{C}_0^\infty(S)$, where C_{Isop} is the constant appearing in Theorem 4.1.

Proof. The proof follows a classical argument; see [23], [42]. Since $|grad_{HS}\psi| \leq |grad_{HS}|\psi||$, without loss of generality, we may assume $\psi \geq 0$. Set

$$S_t := \{x \in S : \psi(x) > t\}.$$

Since ψ has compact support, the set S_t is a bounded open subset of S and, by applying Sard's Lemma, one sees that its boundary ∂S_t is smooth for $\mathcal{L}^1$-a.e. $t \geq 0$. Furthermore, $S_t = \emptyset$ for each (large enough) $t > 0$. The main tools we are going to use are, in order, *Cavalieri's principle*[23] and Coarea Formula; see Theorem 1. We start by the identity

$$\int_S |\psi|^{\frac{Q-1}{Q-2}}\sigma_H^{n-1} = \frac{Q-1}{Q-2}\int_0^{+\infty} t^{\frac{1}{Q-2}}\,\sigma_H^{n-1}(S_t)\,dt \tag{39}$$

which follows from Lemma 5.2 with $\alpha = \frac{Q-1}{Q-2}$. We also remind that, if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a positive *decreasing* function and $\alpha \geq 1$, then

$$\alpha\int_0^{+\infty} t^{\alpha-1}\varphi^\alpha\,dt \leq \left(\int_0^{+\infty}\varphi(t)\,dt\right)^\alpha.$$

Using (39) and the last inequality yields

$$\begin{aligned}
\int_S \psi^{\frac{Q-1}{Q-2}}\,\sigma_H^{n-1} &= \frac{Q-1}{Q-2}\int_0^{+\infty} t^{\frac{1}{Q-2}}\,\sigma_H^{n-1}(S_t)\,dt\\
&\leq \left[\int_0^{+\infty}\left(\sigma_H^{n-1}(S_t)\right)^{\frac{Q-2}{Q-1}}\,dt\right]^{\frac{Q-1}{Q-2}}\\
&\leq \left[\int_0^{+\infty} C_{Isop}\left(\int_{S_t}|\mathcal{H}_H|\,\sigma_H^{n-1} + \sigma_H^{n-2}(\partial S_t)\right)dt\right]^{\frac{Q-1}{Q-2}} \qquad \text{(by (18) with } S = S_t)\\
&= \left[C_{Isop}\int_S \left(|\psi|\,|\mathcal{H}_H| + |grad_{HS}\psi|\right)\,\sigma_H^{n-1}\right]^{\frac{Q-1}{Q-2}},
\end{aligned}$$

where we have used Cavalieri's principle and Coarea Formula. The thesis easily follows. □

[23]The following lemma, also known as *Cavalieri's principle*, is an easy consequence of Fubini's Theorem:

Lemma 5.2. *Let X be an abstract space, μ a measure on X, $\alpha > 0$, $\varphi \geq 0$ and $A_t = \{x \in X : \varphi > t\}$. Then*

$$\int_0^{+\infty} t^{\alpha-1}\mu(A_t)\,dt = \frac{1}{\alpha}\int_{A_0}\varphi^\alpha\,d\mu.$$

Notation 5.3. *As in the standard theory of Sobolev spaces, for any $p > 0$, we shall set*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q-1}.$$

Moreover, we will denote by p' the Hölder conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. In the sequel, any L^p norm will be understood with respect to the measure σ_H^{n-1}.

Warning 5.4. *Henceforth, we shall assume that $\mathcal{H}_H$ is globally bounded along S. Furthermore we shall set*

$$\varepsilon := \max\{\|\mathcal{H}_H\|_{L^\infty(S)}, 0\}.$$

Corollary 5.5. *Under the previous assumptions, one has*

$$\|\psi\|_{L^{p^*}(S)} \leq C_{Isop} \left(\varepsilon \|\psi\|_{L^p(S)} + c_{p^*} \|grad_{HS}\psi\|_{L^p(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$, where $c_{p^*} := p^* \frac{Q-2}{Q-1}$. Thus, there exists $C_{p^*} = C_{p^*}(\varepsilon, \varrho, \mathbb{G})$ such that

$$\|\psi\|_{L^{p^*}(S)} \leq C_{p^*} \left(\|\psi\|_{L^p(S)} + \|grad_{HS}\psi\|_{L^p(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$.

Proof. Let us apply (38) with ψ replaced by $\psi|\psi|^{t-1}$, for some $t > 0$. It follows that

$$\left(\int_S |\psi|^{t\,\frac{Q-1}{Q-2}}\, \sigma_H^{n-1}\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop} \int_S \left(\varepsilon\, |\psi|^t + t|\psi|^{t-1}|grad_{HS}\psi|\right)\, \sigma_H^{n-1}. \tag{40}$$

If we put $(t-1)p' = p^*$, one gets $p^* = t\,\frac{Q-1}{Q-2}$. Using Hölder inequality yields

$$\left(\int_S |\psi|^{p^*} \sigma_H^{n-1}\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop} \left(\int_S |\psi|^{p^*} \sigma_H^{n-1}\right)^{\frac{1}{p'}} \left(\varepsilon\, \|\psi\|_{L^p(S)} + t\, \|grad_{HS}\psi\|_{L^p(S)}\right),$$

which is equivalent to the thesis. □

Corollary 5.6. *Under the previous assumptions, let $p \in [1, Q-1[$. For all $q \in [p, p^*]$ one has*

$$\|\psi\|_{L^q(S)} \leq (1 + \varepsilon\, C_{Isop})\, \|\psi\|_{L^p(S)} + c_{p^*}\, C_{Isop}\, \|grad_{HS}\psi\|_{L^p(S)}$$

for every $\psi \in \mathbf{C}_0^\infty(S)$. In particular, there exists $C_q = C_q(\varepsilon, \varrho, \mathbb{G})$ such that

$$\|\psi\|_{L^q(S)} \leq C_q \left(\|\psi\|_{L^p(S)} + \|grad_{HS}\psi\|_{L^p(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$.

Proof. For any given $q \in [p, p^*]$ there exists $\alpha \in [0,1]$ such that $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$. Hence

$$\|\psi\|_{L^q(S)} \leq \|\psi\|^\alpha_{L^p(S)} \|\psi\|^{1-\alpha}_{L^{p^*}(S)} \leq \|\psi\|_{L^p(S)} + \|\psi\|_{L^{p^*}(S)},$$

where we have used the usual *interpolation inequality* and Young's inequality. The thesis follows from Corollary 5.5. □

Corollary 5.7 (Limit case: $p = Q-1$)**.** *Under the previous assumptions, let $p = Q-1$. For every $q \in [Q-1, +\infty[$ there exists $C_q = C_q(\varepsilon, \varrho, \mathbb{G})$ such that*

$$\|\psi\|_{L^q(S)} \leq C_q \left(\|\psi\|_{L^p(S)} + \|grad_{HS}\psi\|_{L^p(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$.

Proof. By using (40) we easily get that there exists $C_1 = C_1(\varepsilon, t, \varrho, \mathbb{G}) > 0$ such that

$$\left(\int_S |\psi|^{t\,\frac{Q-1}{Q-2}} \sigma_H^{n-1}\right)^{\frac{Q-2}{Q-1}} \leq C_1 \int_S \left(|\psi|^t + |\psi|^{t-1}|grad_{HS}\,\psi|\right)\sigma_H^{n-1}$$

for every $\psi \in \mathbf{C}_0^\infty(S)$. From now on we assume that $t \geq 1$. Using Hölder inequality with $p = Q-1$, yields

$$\|\psi\|^t_{L^{t\,\frac{Q-1}{Q-2}}(S)} \leq C_1 \left(\|\psi\|^t_{L^t(S)} + \|\psi\|^{t-1}_{L^{\frac{(t-1)(Q-1)}{Q-2}}(S)} \|\psi\|_{L^{Q-1}(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$ and $t \geq 1$. By means of Young's inequality, we get there that there exists another constant $C_2 = C_2(\varepsilon, t, \varrho, \mathbb{G})$ such that

$$\|\psi\|_{L^{t\,\frac{Q-1}{Q-2}}(S)} \leq C_2 \left(\|\psi\|_{L^t(S)} + \|\psi\|_{L^{\frac{(t-1)(Q-1)}{Q-2}}(S)} + \|grad_{HS}\,\psi\|_{L^{Q-1}(S)}\right).$$

By setting $t = Q-1$ in the last inequality we get that

$$\|\psi\|_{L^{\frac{(Q-1)^2}{Q-2}}(S)} \leq C_2\left(\|\psi\|_{L^{Q-1}(S)} + \|grad_{HS}\,\psi\|_{L^{Q-1}(S)}\right).$$

By reiterating this procedure for $t = Q, Q+1, \ldots$ one can show that for all $q \geq Q-1$ there exists $C_q = C_q(\varepsilon, \varrho, \mathbb{G})$ such that

$$\|\psi\|_{L^q(S)} \leq C_q\left(\|\psi\|_{L^{Q-1}(S)} + \|grad_{HS}\,\psi\|_{L^{Q-1}(S)}\right)$$

for every $\psi \in \mathbf{C}_0^\infty(S)$. □

5.1 Final remarks and generalizations

Since Carnot groups are endowed with natural and rich geometric structures, we may easily give the notion of *horizontal variation*. More precisely, if $U \subseteq \mathbb{G}$ is an open set and $\psi : U \to \mathbb{R}$, the H-variation of ψ in U is defined by

$$Var_H\,\psi(U) := \sup\left\{\int_U \psi\, div_H\,\phi\, \sigma_R^n : \phi \in \mathbf{C}_0^1(U, H),\ |\phi| \leq 1\right\}. \quad (41)$$

If $\psi \in \mathbf{C}_H^1(U)$, by using an integration by parts, one can show that

$$Var_H\,\psi(U) = \int_U |grad_H\,\psi|\, \sigma_R^n,$$

where we stress that $\sigma_R^n = d\mathcal{L}^n$. If $U = \mathbb{G}$ we also set $Var_H\,\psi$. Starting from (41) we may define the space of functions of *bounded horizontal variation* on U as follows:

$$BV_H(U) := \{\psi \in L^1(U) : Var_H\,\psi(U) < +\infty\}.$$

By definition, $E \subset \mathbb{G}$ is a *set of finite H-perimeter* in U if $\mathbf{1}_E \in BV_H(U)$. We also set $|\partial E|_H(U) := Var_H\,\mathbf{1}_E(U)$ and just $|\partial E|_H$, if $U = \mathbb{G}$. Note that the previous notions are based on the validity of a Divergence Theorem for horizontal vector fields on $\mathbb{G}$.

Analogous remarks can be done when we define horizontal Sobolev spaces. For the theory of (horizontal) Sobolev and BV_H spaces in Carnot groups we refer the reader to [18], [8], [27, 28], [29], [37], [47], [60] and bibliographies therein.

The results of this paper and, in particular the validity of a horizontal Divergence Theorem (see Theorem 3.3) enable us to define the *HS-variation* (i.e. horizontal tangent variation, denoted by Var_{HS}) for functions defined on any $\mathbf{C}^2$-smooth hypersurface $S \subset \mathbb{G}$. More precisely, let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth closed hypersurface and let $\mathcal{U} \subseteq S$ be any open subset of S. We denote by $\overline{\mathcal{D}}_{HS}$ the differential operator $\overline{\mathcal{D}}_{HS} : \mathfrak{X}(HS) \longrightarrow \mathbb{R}$ given by

$$\overline{\mathcal{D}}_{HS}\,\phi := div_{HS}\,\phi + \langle C_H\,\nu_{_H}, \phi\rangle \qquad \text{for every } \phi \in \mathfrak{X}(HS).$$

By the results of Section 3.2, for every $\phi \in \mathfrak{X}^1_0(H\mathcal{U}) = \mathbf{C}^1_0(\mathcal{U}, H\mathcal{U})$, the following holds:

$$\int_{\mathcal{U}} \psi\,\overline{\mathcal{D}}_{HS}\,\phi\,\sigma^{n-1}_{_H} = -\int_{\mathcal{U}} \langle grad_{HS}\,\psi, \phi\rangle\,\sigma^{n-1}_{_H}$$

whenever $\psi \in \mathbf{C}^1(\mathcal{U})$.

Definition 5.8. *The HS-variation of $\psi : \mathcal{U} \subseteq S \to \mathbb{R}$ is defined by*

$$Var_{HS}\,\psi(\mathcal{U}) := \sup\left\{\int_{\mathcal{U}} \psi\,\overline{\mathcal{D}}_{HS}\,\phi\,\sigma^{n-1}_{_H} : \phi \in \mathbf{C}^1_0(\mathcal{U}, H\mathcal{U}),\ |\phi| \leq 1\right\}. \tag{42}$$

The space of functions of bounded HS-variation on $\mathcal{U}$ is given by

$$BV_{HS}(\mathcal{U}) := \{\psi \in L^1(\mathcal{U}, \sigma^{n-1}_{_H}) : Var_{HS}\,\psi(\mathcal{U}) < +\infty\}.$$

Any subset $E \subset S$ is said to have finite HS-perimeter in $\mathcal{U}$ if $\mathbf{1}_E \in BV_{HS}(\mathcal{U})$. We denote by $|\partial E|_{HS}(\mathcal{U}) := Var_{HS}\,\mathbf{1}_E(\mathcal{U})$ the HS-perimeter of E in $\mathcal{U}$. If $\mathcal{U} = S$ we also set $|\partial E|_H$.

Starting from this definition, a complete theory of BV_{HS} spaces and of finite *HS*-perimeter sets can be developed, in particular, by adapting to this context some standard approximation tools[24]. The same observation applies for *horizontal tangent* Sobolev spaces on hypersurfaces.

The Isoperimetric Inequality (see Theorem 4.1) and the related Sobolev-type inequalities (see Theorem 5.1 and its corollaries proved throughout Section 5) can easily be generalized for the weakly-differentiable function spaces introduced above.

More precisely, we state without proof, the following:

Theorem 5.9 (Generalized Isoperimetric and Sobolev inequalities)**.** *Let $\mathbb{G}$ be a k-step Carnot group and fix a homogeneous metric ϱ on $\mathbb{G}$ just as in Definition 2.5. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth closed hypersurface and let $\mathcal{H}_H$ be its horizontal mean curvature. Then there exists a positive constant C_{Isop}, only dependent on $\mathbb{G}$ and on the homogeneous metric ϱ, such that*

$$\left(\sigma^{n-1}_{_H}(E)\right)^{\frac{Q-2}{Q-1}} \leq C_{Isop}\left(\int_E |\mathcal{H}_H|\,\sigma^{n-1}_{_H} + |\partial E|_{HS}\right) \tag{43}$$

for every set E of finite HS-perimeter in S. Furthermore, for every $\psi \in BV_{HS}(S)$ one has

$$\|\psi\|_{L^{\frac{Q-1}{Q-2}}(S)} \leq C_{Isop}\left(\int_S |\psi|\,|\mathcal{H}_H|\,\sigma^{n-1}_{_H} + Var_{HS}\,\psi\right). \tag{44}$$

[24] In particular, it is not difficult to define mollifiers on smooth submanifolds of Carnot groups. Actually this can be done by studying mollifiers on *graded vector spaces*.

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