

# CAPILLARY FLOATING AND THE BILLIARD BALL PROBLEM

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ABSTRACT. We establish a connection between capillary floating in neutral equilibrium and the billiard ball problem. This allows us to reduce the question of floating in neutral equilibrium at any orientation with a prescribed contact angle for infinite homogeneous cylinders to a question about billiard caustics for their orthogonal cross-sections. We solve the billiard problem. As an application, we characterize the possible contact angles and exhibit an infinite family of real analytic non-round cylinders that float in neutral equilibrium at any orientation with constant contact angles.

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## 1. INTRODUCTION: FLOATING IN NEUTRAL EQUILIBRIUM AND THE BILLIARD BALL PROBLEM

The mathematical theory of capillarity goes back to 1806. In his famous treatise on celestial mechanics [24] Laplace discussed a broad range of problems related to surface tension at fluid interfaces, among them a theory of capillary floating. One of the major open problems in this subject is to determine configurations at which a particular body will float on a liquid surface. In [24] Laplace characterized some special cases of capillary floating which was an astonishing achievement for his time. There are several physical phenomena that need to be taken into account: The mass distribution in the body, the gravity, the surface tension, etc. This leads to a highly nonlinear free boundary problem with nonlinear boundary conditions. Further specializing and simplifying the physical assumptions, we arrive at the concept of *floating in neutral equilibrium with zero gravity*. See [10]. In what follows we will simply speak of floating in neutral equilibrium. The relevant mathematical conditions involve geometry of the body surface, its space orientation, and the *contact angle* between the body and the liquid surface.

On the outset, the problem of body floating is three dimensional. In the special case when the body is an infinite homogeneous cylinder, it reduces to a two dimensional problem involving the cross-section of the cylinder. We will assume that it is a bounded, convex, planar domain, say  $\Omega \subset \mathbb{R}^2$ . The three-dimensional floating problem for the cylinder translates into properties of  $\Omega$  which are naturally interpreted as the conditions for capillary floating in two dimensions. The essential requirements for neutral equilibrium are then imposed by the basic laws of physics. In what follows we will be mostly concerned with the two-dimensional floating in neutral equilibrium. This problem can be reformulated in terms of the geometry of  $\Omega$ . Figure 1 illustrates the concept of two-dimensional floating in neutral equilibrium.

The first work connecting two-dimensional capillary floating with convex geometry appears to be [27]. The main result in [27] says that any regular, convex, bounded, planar domain will float at a given contact angle in at least four distinct orientations. The proof crucially uses the four vertex theorem. The problem was further studied by R. Finn. In [10] he showed that the mathematical assumptions in [27] follow from basic physical laws. Finn then asked which convex, smooth domains  $\Omega$  can float at a prescribed contact angle  $\gamma$  in every rotational orientation. He pointed out that for  $\gamma = \pi/2$  this happens if and only

if  $\Omega$  is a domain of constant width. See [10, 11] for this and related material.

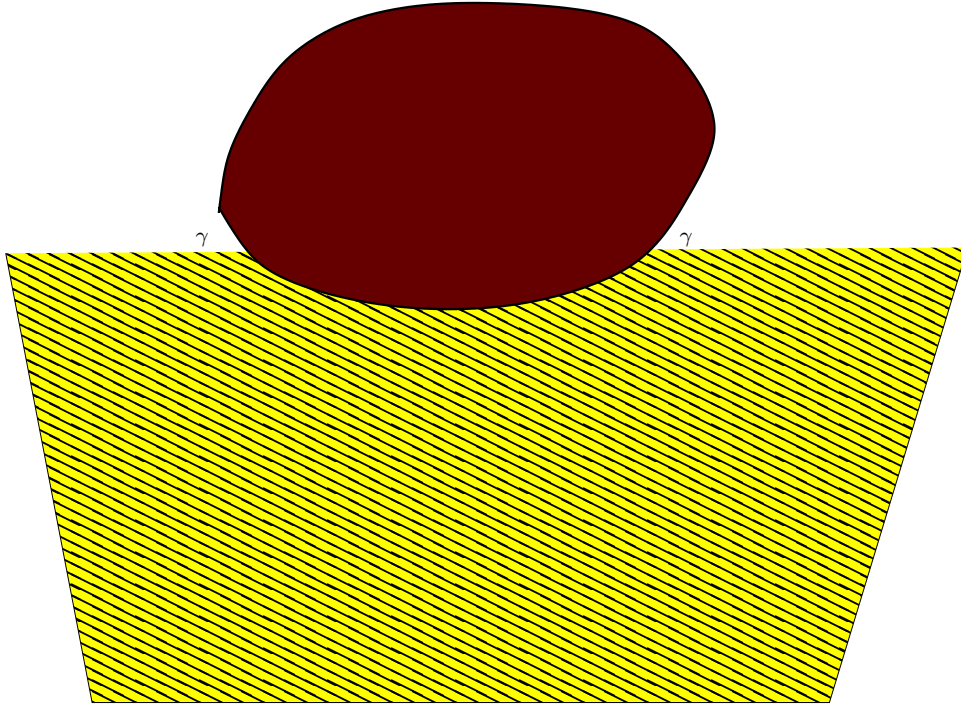


FIGURE 1. Floating in neutral equilibrium;  $\gamma$  is the contact angle.

The geometry of a convex planar domain  $\Omega$  is crucial for the billiard ball problem championed by G.D. Birkhoff in the early 20th century [5]. It can also be viewed as a highly specialized and simplified case of a physical situation. The billiard ball is a point that travels with the unit speed inside  $\Omega$  reflecting at the boundary  $\partial\Omega$  according to the law of equal angles. Disregarding the motion of the ball between collisions with the boundary, we reduce the billiard ball problem to the study of the *billiard map* on  $\Omega$ . Invariant curves of this map provide a crucial insight into the billiard dynamics. Let  $s$  be the arc length variable on  $\partial\Omega$ , and let  $\theta$  be the outgoing angle. See Figure 3. Beginning with Birkhoff [5], invariant curves of the form  $\theta = h(s)$  played an important role in the literature on billiard dynamics.

The functions  $\theta = h(s)$  that yield invariant curves have been extensively studied [25, 9, 21, 18]. The present work is based on the following observation relating the floating problem and the billiard ball problem

for the planar domain  $\Omega$ . The cylinder with the cross-section  $\Omega$  floats in neutral equilibrium at any orientation with the contact angle  $\gamma$  if and only if the billiard table  $\Omega$  admits the invariant curve  $\theta = h(s)$  with the constant function  $h(s) \equiv \pi - \gamma$ . For the reasons that we will explain in section 2, we call these invariant curves the *constant angle caustics* for  $\Omega$ . The floating problem thus becomes the following billiard problem: i) Find the regular, convex billiard tables that admit constant angle caustics; ii) Determine the corresponding angles. This work provides a fair amount of information on this subject. Before describing our results, we will further elaborate on the capillary floating in three dimensions.

The conjecture that the round ball is the only body to float in equilibrium at any orientation is usually ascribed to S. Ulam. See [2, 28]. Various authors have mathematically reformulated this question in different, albeit related ways. The interpretation of Finn et al takes the tension of the liquid surface into account [10, 12, 13], while the other interpretations disregard it [31]. Whatever the interpretation, the conditions of floating in neutral equilibrium are much more restrictive for three dimensions than in the special case of two dimensions. Thus, Finn and Sloss [12] show that the only three-dimensional body to float in neutral equilibrium in any orientation at a constant contact angle is the round ball. Using a different interpretation of the concept of floating, P. Varkony [31] finds a counterexample to the Ulam conjecture. It is clear that the concept of floating in neutral equilibrium generates challenging questions about the geometry of surfaces in  $\mathbb{R}^3$ . In the present work we relate the two-dimensional floating to the geometry of convex bounded domains in  $\mathbb{R}^2$ .

We will now briefly describe our results and the structure of the paper. In section 2 we review the concept of the *billiard map*. Let  $\Omega \subset \mathbb{R}^2$  be a bounded, strictly convex domain with the smooth boundary  $\partial\Omega$ . The *phase space*  $Z = Z(\Omega)$  for the billiard map on  $\Omega$  consists of rays<sup>1</sup> intersecting  $\Omega$ . Let  $0 \leq \theta, \theta_1 \leq \pi$  be the two angles that a ray  $l \in Z$  forms at the points of intersection  $s, s_1 \in \partial\Omega$ . See Figure 2. Let  $F : Z \rightarrow Z$  be the billiard map. The ray  $l_1 = F(l)$  is obtained by reflecting  $l$  at  $s_1$  about  $\partial\Omega$ , as if  $\partial\Omega$  was a perfect mirror. The domain  $\Omega$  floats in neutral equilibrium at any orientation with a constant contact angle if and only if there exists  $0 < \delta < \pi$  such that for any  $l \in Z$  satisfying  $\theta(l) = \delta$  we have  $\theta_1(l) = \delta$ .

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<sup>1</sup>I. e., oriented straight lines.

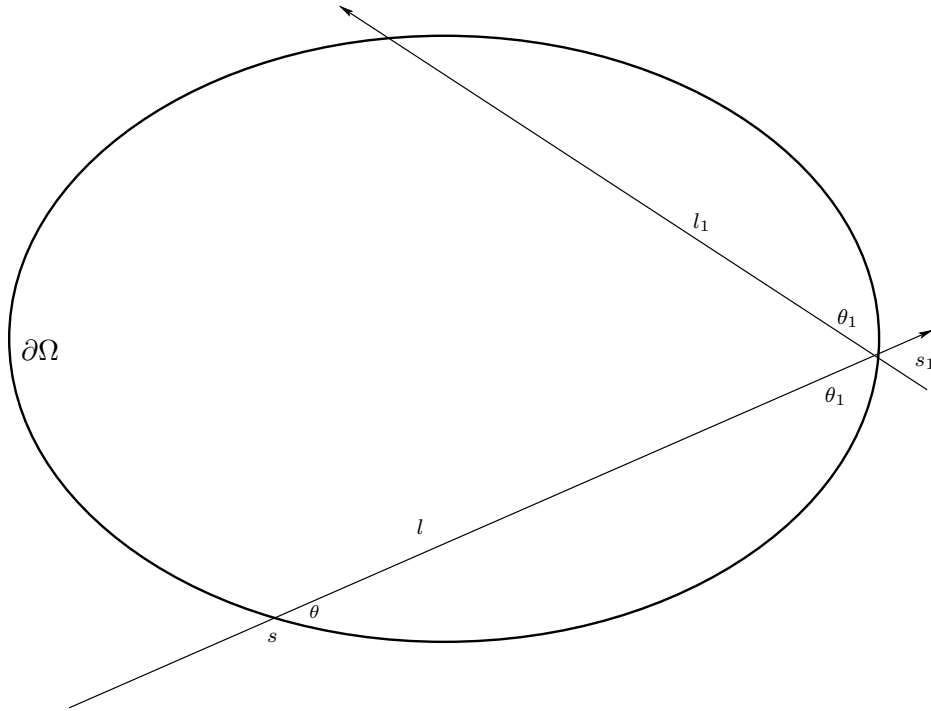


FIGURE 2. The billiard phase space as a space of rays intersecting the billiard table.

In section 2 we study this geometric condition from the viewpoint of the billiard map. Let  $\rho(s)$  be the radius of curvature for  $\partial\Omega$ ; let  $c_k, k \in \mathbb{Z}$ , be its Fourier coefficients. Note that  $\partial\Omega$  is circular if and only if  $c_k = 0$  for all nonzero  $k$ . Theorem 1 says that a noncircular domain  $\Omega$  has the above property if and only if the following two conditions hold: i) There exists  $n > 1$  such that the pair  $n, \delta$  satisfies the trigonometric equation (6); ii) The coefficients  $c_k$  vanish if the pair  $k, \delta$  does not satisfy equation (6).

In section 3 and section 5 we study equations (6) and obtain several applications. The value  $\delta = \pi/2$  is special in that the pair  $n, \pi/2$  satisfies equation (6) with any odd  $n$ . The corresponding regions  $\Omega$  are the domains of constant width; we briefly review their geometry in section 6. The symmetry  $\delta' = \pi - \delta$  allows us to reduce the study of solutions of equations (6) to the range  $0 < \delta < \pi/2$ . Restricted to this interval, equations (6) are equivalent to  $\tan nx = n \tan x$ . In section 4 we obtain fairly detailed qualitative information about solutions of these equations. Let  $B_n \subset (0, \pi/2)$  denote the set of solutions. We

show that  $B_n$  has roughly  $n/2$  elements; it is  $(\pi/n)$ -dense in the interval  $(0, \pi/2)$ . For every  $n > 3$  we exhibit a one-parameter family  $\Omega_{n,\tau}$  of noncircular, real analytic domains that float in every orientation at the contact angles  $\gamma \in B_n$  and  $\gamma \in \pi - B_n$ . See equations (11), (12) and Corollary 3.

The classification of domains that float in neutral equilibrium at constant contact angles hinges on the information about the solutions to  $\tan nx = n \tan x$ . In particular, we need to know whether the sets  $B_n \cap B_m$  can have nonempty intersections for  $m \neq n$ . In section 7 and section 8 we reduce these questions to a study of roots of an infinite chain of polynomials that are closely related to Chebyshev polynomials. This reveals a number-theoretic aspect of capillary floating. Let  $S_n, n \geq 1$ , be the polynomials. In section 8.1 we study the roots of  $S_n$  and obtain some information about them. However, the question whether  $S_m, S_n$  have nontrivial common roots for  $m \neq n$  remains unresolved. It is the lack of this information that prevented the author from publishing his findings immediately after the 1993 PennState Dynamics Workshop [14]. The book [29] contains a brief report on these findings.

Based on substantial partial evidence, we formulate three conjectures about the roots of  $S_n$ . Conjectures 1 and 2 are equivalent. In section 9, assuming that these conjectures hold, we derive consequences for the billiard and for the floating problem. Theorems 2, 3, 4 completely describe billiard tables with constant angle caustics. Theorem 5 gives a classification of regular planar domains that float in neutral equilibrium in any orientation at constant contact angles.

The present work can be viewed as one of many examples of fruitful relationships between the billiard and other mathematical subjects. We refer the reader to [22, 23, 20, 29, 16, 30] for other examples of this nature. The billiard framework offers a variety of open problems that often bear on fundamental and elementary mathematical concepts [17]. The author hopes that the present work will help to advertise the subject in the mathematical fluid mechanics community. The author is grateful to Bob Finn for bringing the subject of capillary floating to his attention and for making several comments on the present work. It was partially supported by the MNiSzW grant N N201 384834.

2. THE BIRKHOFF BILLIARD: GENERAL CAUSTICS VERSUS  
CONSTANT ANGLE CAUSTICS

The billiard in the sense of G.D. Birkhoff plays on a compact, convex domain  $\Omega \subset \mathbb{R}^2$ . We will assume that the boundary  $\partial\Omega$  is twice continuously differentiable. Let  $0 \leq s \leq |\partial\Omega|$  be an arc length parameter. Then the *curvature*  $\kappa(s)$  is a continuous, nonnegative function on  $\partial\Omega$ . We will assume throughout the paper that  $\Omega$  is *strictly convex in the sense of differential geometry*:  $\kappa > 0$ . In what follows we refer to such  $\Omega$  as *regular billiard tables*, or *regular convex domains*.

The elements of the *phase space of the billiard map* are the inward pointing unit vectors  $v$  based on  $\partial\Omega$ . Let  $0 \leq \theta \leq \pi$  be the angle between  $v$  and the positively oriented  $\partial\Omega$ . The coordinates  $0 \leq s \leq |\partial\Omega|, 0 \leq \theta \leq \pi$  induce a diffeomorphism of the phase space  $Z$  and the cylinder  $(\mathbb{R}/|\partial\Omega|\mathbb{Z}) \times [0, \pi]$ .

The phase point  $(s, \theta) \in Z$  corresponds to the billiard ball located at  $s \in \partial\Omega$ , which is about to shoot of in the direction that makes angle  $\theta$  with  $\partial\Omega$ . This shot lands at  $s_1 \in \partial\Omega$ . Let  $0 \leq \theta_1 \leq \pi$  be the other angle of the chord  $[s, s_1]$ . The ball bounces elastically at the boundary and is set to shoot of again. The *law of equal angles* yields that the new vector  $v_1$  makes angle  $\theta_1$  with  $\partial\Omega$ . The transformation  $F : Z \rightarrow Z$  given by  $F(s, \theta) = (s_1, \theta_1)$  is the *billiard ball map* for  $\Omega$ . Figure 3 illustrates the discussion.

In view of our assumptions on  $\Omega$ , the billiard map is of class  $C^1$ . Let  $l(s, s_1)$  denote the length of the chord  $[s, s_1]$ . The differential of the billiard map is given by the following expressions [18]:

$$\frac{\partial s_1}{\partial s} = \frac{\kappa(s)l(s, s_1) - \sin \theta}{\sin \theta_1}, \quad \frac{\partial s_1}{\partial \theta} = \frac{l(s, s_1)}{\sin \theta_1}, \quad \frac{\partial \theta_1}{\partial \theta} = \frac{\kappa(s_1)l(s, s_1) - \sin \theta_1}{\sin \theta_1},$$

and

$$\frac{\partial \theta_1}{\partial s} = \frac{\kappa(s)\kappa(s_1)l(s, s_1) - \kappa(s)\sin \theta_1 - \kappa(s_1)\sin \theta}{\sin \theta_1}.$$

The billiard ball map is an *area preserving twist map*. The classical results of Birkhoff on the dynamics of the billiard map received a “second life” in the theory of area preserving twist maps. See the accounts in [3], [26] and [23]. We are concerned with a particular aspect of the billiard ball map: The *invariant circles*.

**Definition 1.** *Let  $\Omega$  be a regular billiard table. An invariant circle for the billiard map on  $\Omega$  is a closed curve  $\Gamma \subset Z$  which is homotopic to a boundary component of  $Z$  and is invariant under the billiard map.*

By a theorem of Birkhoff, any invariant circle  $\Gamma$  is the graph of a lipshitz function:  $\theta = h_\Gamma(s)$ . Thus, for every base point  $s \in \partial\Omega$  there

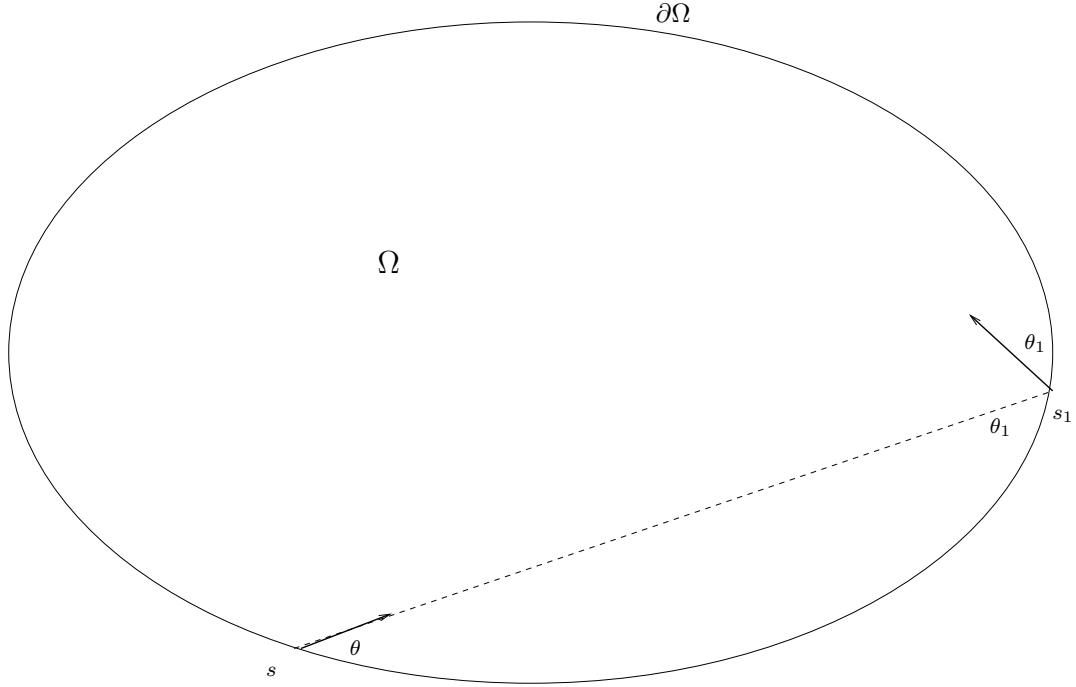


FIGURE 3. The billiard map for a regular convex domain.

is a unique angle  $\theta = h_\Gamma(s)$  such that the ball shooting from  $s$  in the direction  $\theta$  will “stay” on the invariant circle  $\Gamma$ . For a typical  $\Gamma$  the function  $h_\Gamma$  is not constant. See [3, 15, 18, 21, 19, 25]. We will study invariant circles such that  $h_\Gamma$  is constant. Both boundary components of  $Z$  are trivial invariant circles of that type. We will consider only nontrivial invariant circles in what follows. To simplify the terminology, we will often call them the *invariant curves*. This is justified, since we will not study other invariant curves.

**Definition 2.** Let  $\Gamma \subset Z$  be an invariant circle, and let  $\theta = h_\Gamma(s)$  the corresponding lipshitz function. If  $h_\Gamma = \text{const}$ , we will say that  $\Gamma$  is a constant angle invariant circle. A constant angle invariant circle is determined by that angle, say  $0 < \delta < \pi$ . We will denote it by  $\Gamma_\delta$ . See Figure 4.

It is instructive to think of the phase space  $Z$  as the space of oriented lines (i. e., *rays*) intersecting  $\Omega$ , or, alternatively, as the space of directed chords in  $\Omega$ . In this representation, an invariant circle  $\Gamma$  is a



one-parameter family of rays. Its *envelope*  $\gamma \subset \mathbb{R}^2$  is the *caustic of  $\Omega$*  corresponding to the invariant circle  $\Gamma$ .<sup>2</sup>

Let  $\Gamma'$  be the family obtained from  $\Gamma$  by reversing the directions of rays. Then  $\Gamma'$  is an invariant circle as well. This is a consequence of the well known fact that the direction reversing involution  $\sigma : Z \rightarrow Z$  conjugates the billiard map with its inverse:  $\sigma F \sigma = F^{-1}$ .<sup>3</sup> It is clear that  $\Gamma$  and  $\Gamma'$  have the same evolute; hence, the correspondence between invariant circles and the caustics is 2-to-1. The geometry of caustics for regular billiard tables offers challenging open questions. See [9, 18] and [19] for this material. Since invariant circles are determined by their caustics essentially uniquely, in what follows we identify them; in particular, we will speak of *general caustics* and of *constant angle caustics*.

**Remark 1.** The reader should keep in mind that the two invariant circles, say  $\Gamma$  and  $\Gamma' = \sigma(\Gamma)$  corresponding to the caustic  $\gamma$  are distinct subsets of the phase space  $Z$ . Let  $0 < r(\Gamma) < 1$  be the *rotation number* of the invariant circle. Then  $r(\Gamma') = 1 - r(\Gamma)$ .

Since  $\partial\Omega$  is strictly convex, we parameterize it by the direction  $0 \leq \alpha \leq 2\pi$  of the tangent ray to  $s \in \partial\Omega$ . Thus,  $s = s(\alpha)$ . The derivative  $\rho(\alpha) = ds/d\alpha$  is the *radius of curvature* function for  $\Omega$ . Set  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ . Then the billiard map is a diffeomorphism of  $\mathbb{T} \times [0, \pi]$ ; we will use the notation  $F(\alpha, \theta) = (\alpha_1, \theta_1)$ .

**Proposition 1.** *Let  $\Omega \subset \mathbb{R}^2$  be a billiard table, and let  $\rho(\alpha), 0 \leq \alpha \leq 2\pi$ , be its radius of curvature. Then  $\Omega$  has the constant angle caustic  $\Gamma_\delta$  iff the function  $\rho(\cdot)$  satisfies the identity*

$$(1) \quad \int_{\alpha-\delta}^{\alpha+\delta} \rho(\xi) \sin(\alpha - \xi) d\xi = 0.$$

*Proof.* Set  $P = P(\alpha) = (x(\alpha), y(\alpha))$  and let  $P_1 = P(\alpha_1)$ . Let  $O$  be the intersection point of the tangent lines at  $\alpha$  and  $\alpha_1$ . From the triangle  $POP_1$  we have  $\alpha_1 = \alpha + 2\delta$ . See Figure 5. As is well known

$$(2) \quad x'(\alpha) = \rho(\alpha) \cos \alpha, \quad y'(\alpha) = \rho(\alpha) \sin \alpha.$$

Thus

$$x(\alpha + 2\delta) - x(\alpha) = \int_{\alpha}^{\alpha+2\delta} \rho(\xi) \cos \xi d\xi,$$

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<sup>2</sup>The term “caustic” is widely used in the geometric optics, mechanics, and the geometric theory of singularities; in different contexts it stands for different, although related things. We refer the reader to [1, 4, 8] for many variations of this concept.

<sup>3</sup>It is often called the *billiard involution*.

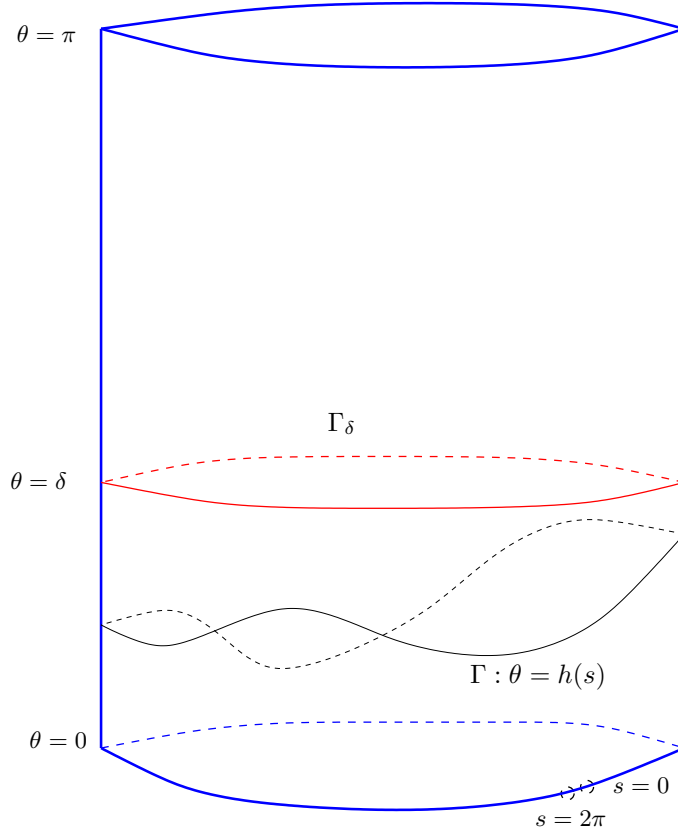


FIGURE 4. Billiard map phase space with a general invariant circle and a constant angle invariant circle.

$$y(\alpha + 2\delta) - y(\alpha) = \int_{\alpha}^{\alpha+2\delta} \rho(\xi) \sin \xi d\xi.$$

The direction of the chord  $[PP_1]$  is  $\alpha + \delta$ . We introduce the new variable  $\beta = \alpha + \delta$ . Thus, the slope of  $[PP_1]$  is  $\tan \beta$ . Computing the slope from the coordinates of points  $P$  and  $P_1$ , we obtain

$$(3) \quad \frac{\int_{\beta-\delta}^{\beta+\delta} \rho(\xi) \sin \xi d\xi}{\int_{\beta-\delta}^{\beta+\delta} \rho(\xi) \cos \xi d\xi} = \tan \beta.$$

Equation (3) is an identity that holds for any  $\beta \in \mathbb{T}$ . Performing elementary trigonometric manipulations in equation (3), and renaming the independent variable by  $\alpha$  again, we obtain the claim.  $\blacksquare$

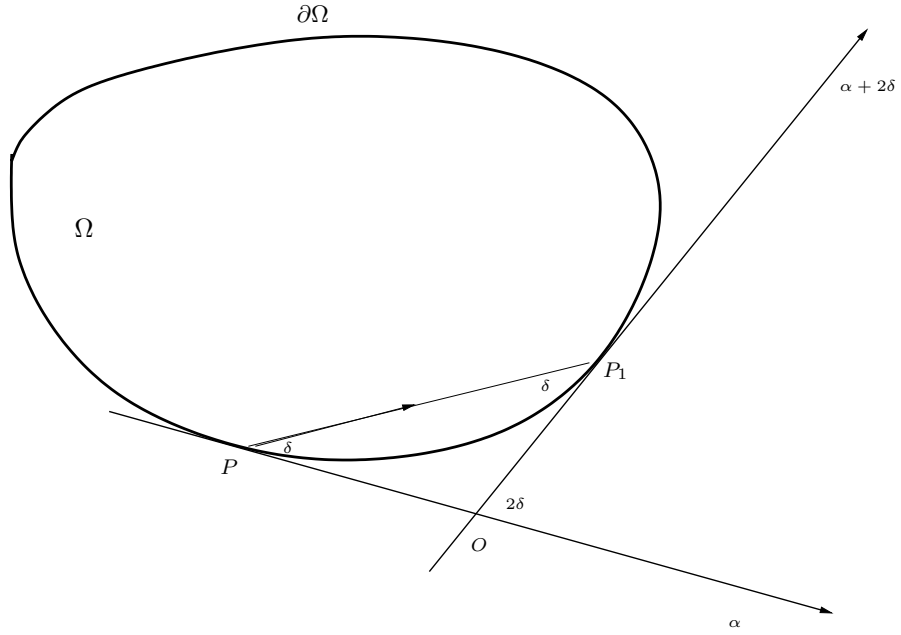


FIGURE 5. The billiard map restricted to a constant angle invariant circle.

We will briefly review basic facts from harmonic analysis on the circle. The reader may find proofs of the statements below in most analysis textbooks.

If  $g$  is a distribution on  $\mathbb{T}$ , its Fourier transform is defined by  $\hat{g}(n) = \int_{\mathbb{T}} g(\alpha) e^{-in\alpha} d\alpha$  for  $n \in \mathbb{Z}$ . The radius of curvature has a Fourier expansion

$$(4) \quad \rho(\alpha) = \sum_{n \in \mathbb{Z}} c_n e^{in\alpha}$$

where  $c_n = \hat{\rho}(n)/2\pi$  are the Fourier coefficients. The Fourier coefficients of a real function satisfy  $c_{-n} = \bar{c}_n$ . Equation (4) is equivalent to the trigonometric expansion

$$\rho(\alpha) = a_0 + \sum_{n \geq 1} a_n \cos n\alpha + b_n \sin n\alpha$$

whose coefficients are real. The coefficients in these equations are related by  $a_0 = c_0$  and  $a_n = 2\Re(c_n)$ ,  $b_n = -2\Im(c_n)$  for  $n > 0$ .

Denote by  $x + y$  the group operation on  $\mathbb{T}$ . Let  $k(\cdot)$  be a function or a distribution on  $\mathbb{T}$ . The operator of convolution with  $k$  is defined by

$$(5) \quad (K\rho)(x) = \int_{\mathbb{T}} \rho(x - \xi)k(\xi)d\xi = \int_{\mathbb{T}} \rho(\xi)k(x - \xi)d\xi.$$

The standard notation for convolution operators is  $K(\rho) = \rho * k = k * \rho$ .

Let  $F_n$  be the complex line in the space of functions on  $\mathbb{T}$  spanned by  $e^{inx}$ . We view equation (4) as the orthogonal decomposition by the subspaces  $F_n$ ,  $n \in \mathbb{Z}$ . Convolution operators preserve this decomposition. The restriction  $K|_{F_n}$  is the operator of multiplication by  $\hat{k}(n)$ . The above discussion yields the following statement which is crucial for Theorem 1.

**Lemma 1.** *Let  $k(\cdot)$  be a distribution on  $\mathbb{T}$ , and let  $\hat{k}(n)$ ,  $n \in \mathbb{Z}$ , be its Fourier transform. Let  $K$  be the operator of convolution with the distribution  $k(\cdot)$ . Let  $\rho(\cdot)$  be a function on  $\mathbb{T}$ , and let  $c_n$ ,  $n \in \mathbb{Z}$ , be its Fourier coefficients.*

*Then  $K\rho = 0$  iff  $\hat{k}(n)c_n = 0$  for all  $n \in \mathbb{Z}$ .*

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^2$  be a regular, noncircular billiard table. Let  $\rho(\cdot)$  be the radius of curvature of  $\partial\Omega$ , and let  $c_n$ ,  $n = 1, 2, \dots$  be its Fourier coefficients. Then  $\Omega$  has the constant angle caustic  $\Gamma_\delta$  iff the following conditions hold:*

i) *There exist  $n > 1$  such that*

$$(6) \quad \frac{\sin(n-1)\delta}{n-1} = \frac{\sin(n+1)\delta}{n+1};$$

ii) *We have  $c_k = 0$  for all  $k > 1$  such that equation (6) is not satisfied.*

iii) *We have  $c_n \neq 0$  for at least one  $n > 1$  such that equation (6) is satisfied.*

*Proof.* By Proposition 1,  $\Gamma_\delta$  is a caustic for  $\Omega$  iff  $\rho(\cdot)$  belongs to the zero space of the convolution with the function

$$(7) \quad k(x) = (\sin x) 1_{[-\delta, \delta]}.$$

The function  $k(\cdot)$  is odd, hence  $\hat{k}(0) = 0$ . By a straightforward computation, for  $n > 1$  we have

$$i \cdot \hat{k}(n) = \frac{\sin(n-1)\delta}{n-1} - \frac{\sin(n+1)\delta}{n+1}.$$

It is well known that for any billiard table  $\Omega$  we have  $c_1 = 0$ . By Lemma 1,  $\Gamma_\delta$  is a caustic for  $\Omega$  iff  $c_m \hat{k}(m) = 0$  for all  $m$ . By the

preceding discussion,  $\Gamma_\delta$  is a caustic iff

$$c_m \left[ \frac{\sin(m-1)\delta}{m-1} - \frac{\sin(m+1)\delta}{m+1} \right] = 0$$

for  $m > 1$ . Since  $\Omega$  is not circular, at least one coefficient, say  $c_n$ , does not vanish. But  $\Gamma_\delta$  being a caustic, equation (6) holds for all  $n > 1$  such that  $c_n \neq 0$ . ■

### 3. CONSTANT ANGLE CAUSTICS AND A CHAIN OF TRIGONOMETRIC EQUATIONS

By Theorem 1, the description of billiard tables with constant angle caustics hinges on solving equation (6). In this section we will reduce equation (6) to a chain of trigonometric equations involving the function  $\tan(\cdot)$ .

Recall that  $\delta \in A$  if there exists  $n > 1$  such that equation (6) holds. Let  $A_n \subset A$  be the set of  $\delta \in (0, \pi)$  such that the pair  $\delta, n$  satisfies equation (6). Thus

$$A = \cup_{n=2}^{\infty} A_n.$$

**Lemma 2.** *Let  $n > 1$ . Then the following claims hold.*

- i) *We have  $\frac{\pi}{2} \in A_n$  iff  $n$  is odd.*
- ii) *Set  $\tilde{A}_n = A_n \setminus \{\frac{\pi}{2}\}$ . Then  $\tilde{A}_n$  is the set of solutions in  $(0, \pi)$  of the equation  $\tan n\delta = n \tan \delta$ .*

*Proof.* Set  $\delta = \frac{\pi}{2}$ . If  $n$  is odd, then both sides in equation (6) vanish, hence  $\frac{\pi}{2} \in A_n$ . If  $n$  is even, then the numerators in equation (6) are  $\pm 1$ , and their signs are opposite. Thus,  $\frac{\pi}{2} \notin A_n$ , proving claim i).

Let  $\delta \in A_n$ . Arguing as above, we establish that  $\sin(n+1)\delta = 0$  iff  $n$  is odd and  $\delta = \frac{\pi}{2}$ . Hence for  $\delta \in \tilde{A}_n$  we have  $\sin(n-1)\delta, \sin(n+1)\delta \neq 0$ . Therefore,  $\tilde{A}_n$  is the set of  $\delta \in (0, \pi)$  satisfying

$$\frac{\sin(n-1)\delta}{\sin(n+1)\delta} = \frac{n-1}{n+1}.$$

We rewrite this as

$$(8) \quad \frac{\sin n\delta \cos \delta - \cos n\delta \sin \delta}{\sin n\delta \cos \delta + \cos n\delta \sin \delta} = \frac{n-1}{n+1}.$$

If  $\cos n\delta = 0$ , then the left hand side in equation (8) is 1, which is impossible. Thus,  $\cos n\delta \neq 0$ . Dividing the numerators and the denominators in equation (8) by  $\cos \delta \cos n\delta$ , we obtain

$$\frac{\tan n\delta - \tan \delta}{\tan n\delta + \tan \delta} = \frac{n-1}{n+1}.$$

Claim 2 follows. ■

For  $X \subset \mathbb{R}$  and  $a \in \mathbb{R}$  let  $\{a - X\} = \{a - x : x \in X\}$ . Set  $\tilde{A} = A \setminus \{\frac{\pi}{2}\}$ . Then

$$(9) \quad \tilde{A} = \cup_{n=2}^{\infty} \tilde{A}_n.$$

Set  $B_n = A_n \cap (0, \pi/2)$  and  $B = A \cap (0, \pi/2)$ . Lemma 2 and the preceding discussion imply the following.

**Proposition 2.** *Let  $n > 1$ . Then for  $n$  even,  $A_n = B_n \cup \{\pi - B_n\}$  and for  $n$  odd,  $A_n = B_n \cup \{\pi - B_n\} \cup \{\frac{\pi}{2}\}$ . Moreover,  $B_n$  is the set of solutions in  $(0, \pi/2)$  of the equation*

$$(10) \quad \tan nx = n \tan x.$$

#### 4. ANALYSIS OF TRIGONOMETRIC EQUATIONS

In this section we begin to analyze solutions of the chain of equations (10) in the interval  $(0, \pi/2)$ .

**Proposition 3.** 1. *Let  $n > 1$  be even. Then  $B_n$  consists of  $\frac{n}{2} - 1$  points  $\xi_k^{(n)}$ , where*

$$\frac{2k}{2n}\pi < \xi_k^{(n)} < \frac{(2k+1)}{2n}\pi : k = 1, \dots, \frac{n}{2} - 1.$$

2. *Let  $n > 1$  be odd. Then  $B_n$  consists of  $\frac{n-1}{2} - 1$  points  $\xi_k^{(n)}$ , where*

$$\frac{2k}{2n}\pi < \xi_k^{(n)} < \frac{(2k+1)}{2n}\pi : k = 1, \dots, \frac{n-1}{2} - 1.$$

*Proof.* The graph of the function  $y = \tan nx$  on  $(0, \pi/2)$  is the disjoint union of  $n$  connected curves; we will call them *branches*. A branch is defined on the interval  $\frac{k}{2n}\pi < x < \frac{k+1}{2n}\pi : 0 \leq k \leq n-1$ . Set  $I_k^{(n)} = (\frac{k}{2n}\pi, \frac{k+1}{2n}\pi)$ . Each branch extends by continuity to one of the endpoints of  $I_k^{(n)}$ . These endpoints don't enter in our analysis, and we ignore them in what follows. We say that a branch is *positive* (resp. *negative*) if it belongs to the upper (resp. lower) halfplane.

Positive branches correspond to  $I_k^{(n)}$  with  $k$  even. Thus, there are  $n/2$  (resp.  $(n-1)/2$ ) positive branches if  $n$  is even (resp. odd). We observe that each point in  $B_n$  belongs to the intersection of the graph of  $y = n \tan x$  on  $(0, \pi/2)$  with a positive branch; this intersection contains at most one point. See Figure 6 and Figure 7.

Comparing the asymptotics of  $n \tan x$  and  $\tan nx$  as  $x \rightarrow 0+$ , we see that the first branch, which corresponds to  $k = 0$ , does not yield an intersection point. When  $n$  is even, all other positive branches intersect the graph  $n \tan x$ . This proves claim 1. Let now  $n$  be odd. Then both the last branch and the graph of  $y = n \tan x$  are asymptotic

to the vertical line  $x = \pi/2$ . Comparing the asymptotics of  $n \tan x$  and  $\tan nx$  as  $x \rightarrow \frac{\pi}{2}-$ , we see that the curves do not intersect. This proves claim 2. We leave details to the reader. ■

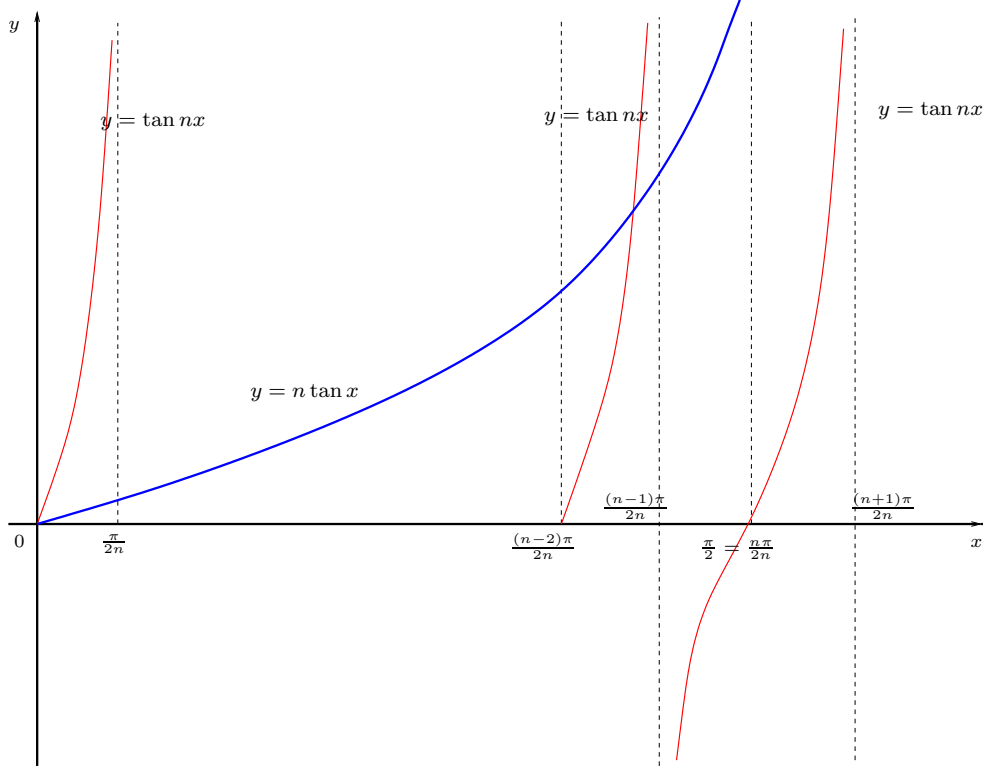


FIGURE 6. The graphs of functions  $y = \tan nx$  and  $y = n \tan x$  for  $n$  even.

We will state an immediate consequence of Proposition 3. Recall that  $A_n \subset (0, \pi)$  is the set of numbers satisfying equation (6) and that  $A = \cup_{n>1} A_n$ .

**Corollary 1.** *We have  $|A_n| = n - 2$ . The sets  $A_n$  are  $2\pi/n$  dense in  $(0, \pi)$ .*

*Proof.* By Lemma 2 and Proposition 2,  $|A_n| = 2|B_n|$  if  $n$  is even and  $|A_n| = 2|B_n| + 1$  if  $n$  is odd. The first claim now follows from Proposition 3. The above propositions imply that the distances between consecutive points of  $A_n$  are at most  $\pi/n$ . Besides, the distances from  $A_n$  and the endpoints of  $(0, \pi)$  are at most  $2\pi/n$ . ■

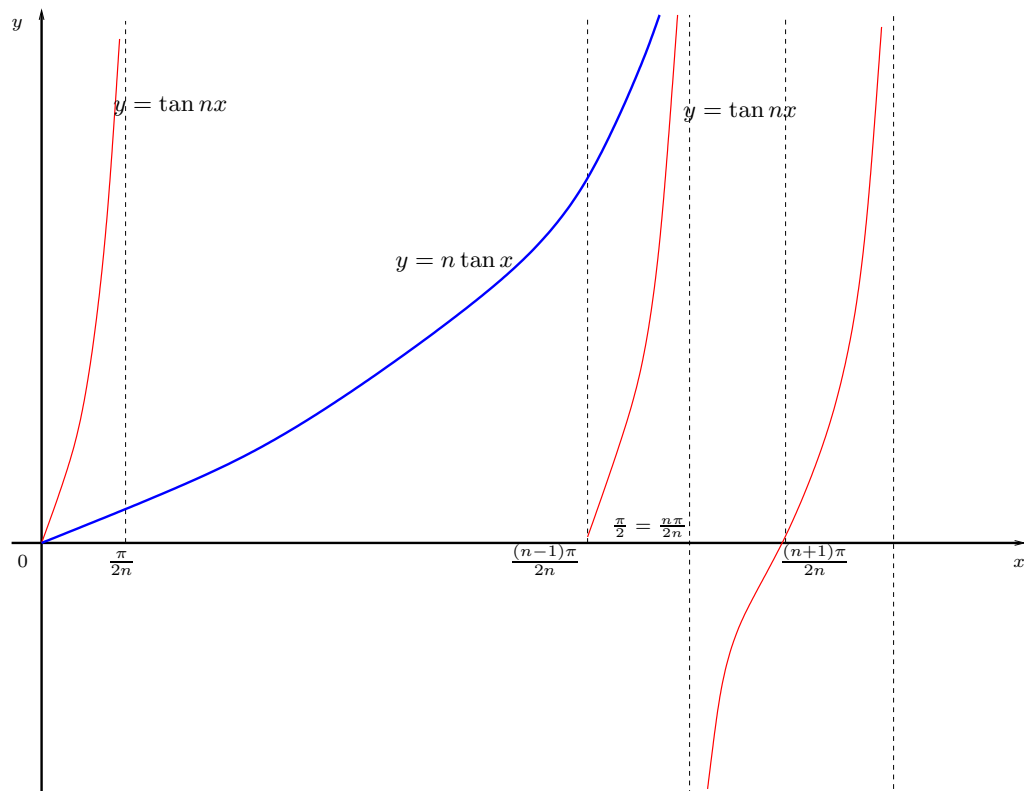


FIGURE 7. The graphs of functions  $y = \tan nx$  and  $y = n \tan x$  for  $n$  odd.

## 5. IMMEDIATE IMPLICATIONS

The results of section 2, section 3, and section 4 have immediate consequences for the billiard and for the floating. We begin with the former. We will say that the billiard tables  $\Omega_1, \Omega_2$  are *conformally equivalent* if there is a conformal mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $F(\Omega_1) = \Omega_2$ . For instance, all discs in  $\mathbb{R}^2$  are conformally equivalent.

**Corollary 2.** *There is a dense countable set  $\tilde{A} \subset (0, \pi) \setminus \{\pi/2\}$  such that the following holds.*

1. *For any  $\delta \in \tilde{A}$  there is  $n > 1$  and a real analytic 1-parameter family  $\Omega_{n,\tau}$ ,  $0 \leq \tau < 1$ , of conformally inequivalent, regular billiard tables having the constant angle caustic  $\Gamma_\delta$ . The curves  $\partial\Omega_{n,\tau}$  are real analytic;  $\partial\Omega_{n,0}$  is the unit circle.*
2. *A regular billiard table  $\Omega$  has the caustic  $\Gamma_{\pi/2}$  iff  $\partial\Omega$  is a curve of constant width.*



3. Let  $0 < \delta < \pi$  belong to the complement of  $\tilde{A} \cup \{\pi/2\}$  in  $(0, \pi)$ . If a regular billiard table  $\Omega$  has the constant angle caustic  $\Gamma_\delta$ , then  $\Omega$  is circular.

*Proof.* Let  $\tilde{A}_n$  and  $\tilde{A}$  be as in equation (9). Then  $\delta \in \tilde{A}$  iff there exists  $n > 1$  such that  $\tan n\delta = n \tan \delta$ . Let  $a, b \in \mathbb{R}$  be arbitrary. Set

$$\rho(\alpha) = 1 + a \cos n\alpha + b \sin n\alpha.$$

By elementary trigonometry, there exists  $\alpha_0$  depending on  $a, b, n$  such that  $\rho(\alpha) = 1 + \sqrt{a^2 + b^2} \sin n(\alpha + \alpha_0)$ . This is the radius of curvature of a regular billiard table iff  $a^2 + b^2 < 1$ . Different values of  $\alpha_0$  correspond to isometric billiard tables. Set  $\rho_{n,\tau}(\alpha) = 1 + \tau \sin n\alpha$ .

Integrating equation (2), we obtain

$$(11) \quad x_{n,\tau}(\alpha) = \xi_0 + \sin \alpha + \frac{\tau}{2(n-1)} \cos(n-1)\alpha - \frac{\tau}{2(n+1)} \cos(n+1)\alpha,$$

$$(12) \quad y_{n,\tau}(\alpha) = \eta_0 - \cos \alpha + \frac{\tau}{2(n-1)} \sin(n-1)\alpha - \frac{\tau}{2(n+1)} \sin(n+1)\alpha.$$

Fixing the constants  $\xi_0, \eta_0$ , we obtain a real analytic family  $\Omega_{n,\tau}$ . This proves claim 1. Claim 2 will follow from the discussion in the beginning of section 6. Claim 3 is immediate from Theorem 1.  $\blacksquare$

Set  $\rho(\alpha) = c + a \cos n\alpha + b \sin n\alpha$ . This formula, provided  $0 \leq \sqrt{a^2 + b^2} < c$ , yields a 3-parameter family of functions that serve as radii of curvature for billiard tables  $\Omega$  having the caustic  $\Gamma_\delta$ . By equations (11), (12), we have a 5-parameter family of these domains. However, the conformal equivalence eats up 4 of the parameters. We can view equations (11), (12) as a deformation  $\Omega_{n,\tau}$  of the circular table.

The following is the counterpart of Corollary 2 for the floating in neutral equilibrium. Its claims are the reformulations of the corresponding claims in Corollary 2; we do not repeat the proof.

**Corollary 3.** *There is a dense countable set  $\tilde{A} \subset (0, \pi) \setminus \{\pi/2\}$  such that the following holds.*

1. *For any  $\delta \in \tilde{A}$  there is  $n > 1$  and a real analytic 1-parameter family  $\Omega_{n,\tau}$ ,  $0 < \tau < 1$ , of conformally inequivalent planar domains with real analytic boundaries that float in neutral equilibrium at any orientation with the contact angle  $\pi - \delta$ . The domain  $\Omega_{n,0}$  is the unit circle.*
2. *A regular convex domain floats in neutral equilibrium at any orientation with the contact angle  $\pi/2$  if and only if its boundary is a curve of constant width.*

3. Let  $0 < \delta < \pi$  belong to the complement of  $\tilde{A} \cup \{\pi/2\}$  in  $(0, \pi)$ . If a regular convex domain floats in neutral equilibrium at any orientation with the contact angle  $\delta$ , then it is a disc.

Recall that  $B_n \subset (0, \pi/2)$  is the set of solutions to equation (10). To continue our study of floating in neutral equilibrium, we need further number theoretic information about these sets. Below we formulate questions about floating and/or billiard whose answers depend on this information.

**Question 1.** Let  $\delta \in B$ . Describe the set of billiard tables  $\Omega$  such that  $\Gamma_\delta$  is a caustic. Equivalently, describe the set of cross-sections of cylinders that float in neutral equilibrium with the contact angle  $\pi - \delta$  at any orientation.

**Question 2.** Let  $\delta \in B$ . Let  $\Omega$  be a billiard table with the caustic  $\Gamma_\delta$ . Let  $\varphi : \Gamma_\delta \rightarrow \Gamma_\delta$  be the restriction of the billiard map on  $\Omega$  to  $\Gamma_\delta$ . Can  $\varphi$  be periodic?

In order to answer Question 1, we need to investigate the intersections  $B_n \cap B_m$  for  $m \neq n$ . In particular, we need to know for what pairs  $m \neq n$  the set  $B_n \cap B_m$  is nonempty. The answer to Question 2 depends on whether  $\delta \in \pi\mathbb{Q}$  or not. In particular, if  $B \cap \pi\mathbb{Q}$  is empty, the answer is negative. In section 7 and the following sections we will study the sets of solutions to equation (10) and equation (6). The solution  $\delta = \pi/2$  of the latter is special. The corresponding planar domains have been studied by geometers from an independent viewpoint. We briefly review this in the next section.

## 6. THE CAUSTICS $\Gamma_{\pi/2}$ AND CURVES OF CONSTANT WIDTH

To illustrate the preceding discussion, we will now study the question: Which billiard tables have the caustic  $\Gamma_{\pi/2}$ ? Let  $\Omega$  be a regular billiard table. Then  $\Gamma_{\pi/2}$  is a caustic iff any chord which is perpendicular to  $\partial\Omega$  at one of its ends, is also perpendicular to  $\partial\Omega$  at the other end. The values of the angle parameter at these points are  $\alpha, \alpha + \pi$ . The length  $d(\alpha) = |P(\alpha)P(\alpha + \pi)|$  is the *width of  $\Omega$  in the direction  $\alpha + \pi/2$* . The chord  $[P(\alpha)P(\alpha + \pi)]$  is perpendicular to  $\partial\Omega$  iff  $\alpha + \pi/2$  is a critical point for the function  $d(\cdot)$ . Therefore,  $\Gamma_{\pi/2}$  is a caustic iff  $d(\alpha) = \text{const}$ . These curves are known in geometry as the *curves of constant width* [6]. Thus, a regular billiard table  $\Omega$  has the caustic  $\Gamma_{\pi/2}$  iff  $\Omega$  is a domain of constant width.

We point out that the analysis below assumes that  $\partial\Omega$  is twice continuously differentiable. In particular, it is not valid for domains of constant width with corners. The boundary of the famous example of

such a domain, the Reuleaux triangle [6], consists of three circular arcs of the same radius; it has corners at the endpoints of the arcs. See Figure 8. The Reuleaux triangle is not a regular billiard table.

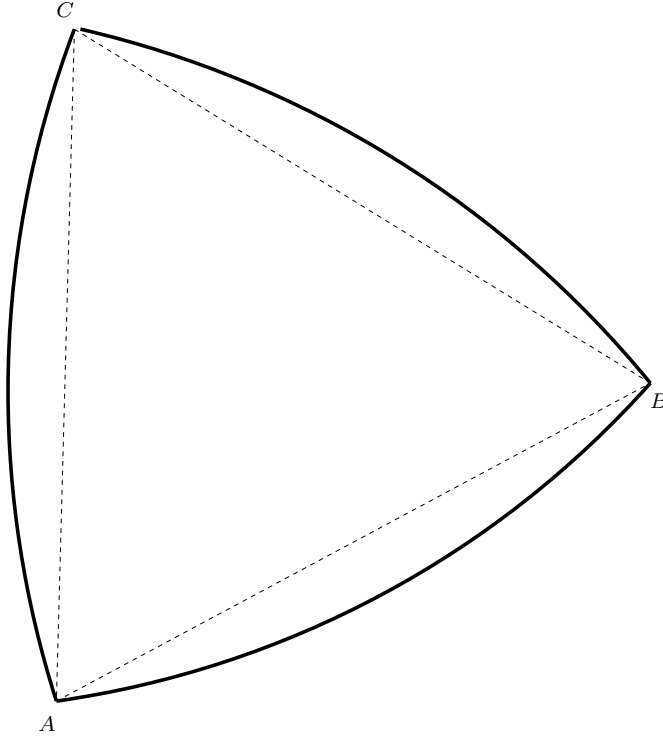


FIGURE 8. Reuleaux triangle: A domain of constant width with corners.

**Corollary 4.** *Let  $\Omega$  be a regular billiard table, and let  $\rho(\cdot)$  be its radius of curvature. Then  $\Gamma_{\pi/2}$  is a caustic for  $\Omega$  iff we have the identity*

$$(13) \quad \rho(\alpha) + \rho(\alpha + \pi) = \text{const.}$$

*Proof.* Let  $c_m, m \in \mathbb{Z}$ , be the Fourier coefficients of  $\rho$ . By the proof of Theorem 1,  $\Gamma_{\pi/2}$  is a caustic iff

$$c_m \left[ \frac{\sin \frac{(m-1)\pi}{2}}{m-1} - \frac{\sin \frac{(m+1)\pi}{2}}{m+1} \right] = 0$$

for all  $m > 1$ . For  $m = 2k$  this means  $4kc_{2k}/(4k^2 - 1) = 0$ , yielding  $c_{2k} = 0$ . For odd  $m$  the equation holds for any  $c_m$ . Thus, the caustic

$\Gamma_{\pi/2}$  exists iff the radius of curvature has the Fourier expansion of the form

$$(14) \quad \rho(\alpha) = c_0 + \sum_{m \text{ odd}} c_m e^{im\alpha}.$$

Set  $\rho_0(\alpha) = \rho(\alpha) - c_0$ . Then equation 14 holds iff  $\rho_0$  is an odd function on  $\mathbb{T}$ . Equivalently,  $\rho(\alpha + \pi) + \rho(\alpha) = 2c_0$ . ■

**Remark 2.** We point out that the identity equation (14) characterizes all billiard tables  $\Omega$  with the caustic  $\Gamma_{\pi/2}$ , including the circular billiard table. By the discussion preceding Corollary 4, the width of any such  $\Omega$  is constant, and is equal to  $2c_0$ . Let  $|\partial\Omega|$  be the perimeter of  $\Omega$ . If  $\Omega$  has constant width, we denote it by  $w(\Omega)$ . By the above argument, for a curve of constant width we have

$$\rho(\alpha + \pi) + \rho(\alpha) = w(\Omega).$$

Integrating this equation and using that  $\int_{\mathbb{T}} \rho(\alpha) d\alpha = |\partial\Omega|$ , we obtain the identity

$$(15) \quad \pi \cdot w(\Omega) = |\partial\Omega|.$$

Note that we have used the regularity of  $\partial\Omega$  to derive equation (15). In fact, it is valid for arbitrary curves of constant width; it is called Barbier's theorem. Another amusing fact about domains of constant width is the Blaschke-Lebesgue theorem [6]. It says that amongst the domains of a fixed constant width the Reuleaux triangle has the smallest area. By the isoperimetric theorem, the disc has the biggest area. Let  $\Omega$  be any domain of constant width  $w$ ; let  $|\Omega|$  be the area of  $\Omega$ . By an elementary calculation

$$\frac{\pi - \sqrt{3}}{2} w^2 \leq |\Omega| \leq \frac{\pi}{4} w^2.$$

The equalities take place only for the Reuleaux triangle and the disc.

## 7. TRIGONOMETRIC EQUATIONS AND A FAMILY OF POLYNOMIALS

We will now obtain quantitative information about the solutions of equations (10).

**Lemma 3.** *Let  $n \geq 1$ . There are polynomials  $P_n, Q_n$  such that*

$$(16) \quad \tan nx = \frac{P_n(\tan x)}{Q_n(\tan x)}.$$

*Polynomials  $P_n, Q_n$  are uniquely determined by the recurrence relations*

$$(17) \quad P_{n+1}(z) = P_n(z) + zQ_n(z), \quad Q_{n+1}(z) = Q_n(z) - zP_n(z)$$

and the initial data  $P_1(z) = z$ ,  $Q_1(z) = 1$ . The polynomial  $P_n$  (resp.  $Q_n$ ) is odd (resp. even). The degree of each of the two polynomials is either  $n$  or  $n - 1$ , depending on the parity of  $n$ .

*Proof.* The formula

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

in the special case  $y = nx$  yields

$$\tan(n + 1)x = \frac{\tan nx + \tan x}{1 - \tan nx \tan x}.$$

The claims follow by induction on  $n$ . ■

**Remark 3.** Polynomials  $P_n, Q_n$  can be expressed in terms of the Chebyshev polynomials of the first and the second kind. We will not pursue this approach here.

**Proposition 4.** *The polynomials in equation (16) satisfy*

$$(18) \quad -2P_n(z) = i^{n+1}(z - i)^n + (-i)^{n+1}(z + i)^n$$

and

$$(19) \quad 2Q_n(z) = i^n(z - i)^n + (-i)^n(z + i)^n.$$

*Proof.* We rewrite equation (17) as

$$\begin{bmatrix} P_{n+1}(z) \\ Q_{n+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & z \\ -z & 1 \end{bmatrix} \begin{bmatrix} P_n(z) \\ Q_n(z) \end{bmatrix}.$$

The claims follow by the elementary algebra. ■

**Corollary 5.** *For  $n > 1$  set*

$$(20) \quad R_n(z) = -\frac{1}{2}i^n [(nz + i)(z - i)^n + (-1)^n(nz - i)(z + i)^n].$$

*Let  $0 < x < \pi/2$ , and set  $z = \tan x$ . Then  $x \in B_n$  iff  $z$  is a positive root of the polynomial  $R_n$ .*

*Proof.* By Proposition 2, equation (10), and Lemma 3,  $x \in B_n$  iff  $\tan x = z > 0$  satisfies  $P_n(z) - nzQ_n(z) = 0$ . By equations (18) and (19),  $P_n(z) - nzQ_n(z) = R_n(z)$ . ■

## 8. NUMBER THEORETIC CONJECTURES AND IMPLICATIONS

We have reduced our investigation of equations (10) to a study of roots of the polynomials  $R_n$ . We now continue to study these polynomials, and bring in some number theory.

### 8.1. Polynomials and fractional linear transformations.

We will investigate the roots of  $R_n$ . The following lemma summarizes the immediate properties of these polynomials.

**Lemma 4.** *Let  $n \geq 1$ . Then the following holds:*

- i) *The polynomials  $R_n$  are real, odd polynomials;*
- ii) *The degree of  $R_n$  is equal to  $n + 1$  for  $n$  even, and to  $n$  for  $n$  odd;*
- iii) *The highest coefficient of  $R_n$  is  $2n$  for  $n$  even and  $\pm 1$  for  $n$  odd;*
- iv) *We have  $R_n(z) = O(z^3)$ ;*
- v) *The roots of  $R_n$  are real and simple, except for the zero root, which has multiplicity three.*

*Proof.* Claims i) - iv) follow either from  $R_n(z) = P_n(z) - nzQ_n(z)$  or directly from equation (20). We will prove claim v). Suppose  $n$  is even; set  $n = 2k$ . Then  $\deg(R_n) = 2k + 1$ . By iv),  $R_n$  has at most  $2k - 2$  nonzero roots, counted with multiplicities. By claim 1 in Proposition 3 and Corollary 5,  $R_n$  has  $k - 1$  distinct positive roots. By i),  $R_n$  has  $k - 1$  distinct negative roots, hence the claim. The case of odd  $n$  is similar, and we leave it to the reader.  $\blacksquare$

Let  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be a nondegenerate matrix. We will use the notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z = \frac{az + b}{cz + d}.$$

Let  $A, A'$  be nondegenerate matrices. We will write  $A \sim A'$  to mean that  $A'A^{-1}$  is a scalar matrix. Then  $A \sim A'$  holds iff  $A \circ z \equiv A' \circ z$ .

**Proposition 5.** *Let  $n > 1$ . There is a 1-to-1 correspondence, preserving the multiplicities, between the nonzero roots of  $R_n$  and the roots of the equation*

$$(21) \quad \zeta^n = (-1)^{n+1} \begin{bmatrix} n+1 & n-1 \\ n-1 & n+1 \end{bmatrix} \circ \zeta,$$

other than  $\zeta = \pm 1$ .

*Proof.* By equation (20), we have  $R_n(z) = 0$  iff

$$(22) \quad \left( \frac{z-i}{z+i} \right)^n = (-1)^{n+1} \frac{nz-i}{nz+i}.$$

We recall a few well known facts. The fractional linear transformations

$$\zeta = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \circ z, \quad z = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \circ \zeta.$$

are inverse to each other; they induce a diffeomorphism of  $\mathbb{R} \cup \infty$  onto the unit circle which sends the natural orientation of the real axis to the counter clockwise orientation of the unit circle.

Setting  $F_n(z) = \frac{nz-i}{nz+i}$ , we rewrite equation (22) as

$$(F_1(z))^n = (-1)^{n+1} F_n(z).$$

Setting  $F_1(z) = \zeta$ ,  $z = F_1^{-1}(\zeta)$ , and using that

$$\begin{bmatrix} n & -i \\ n & i \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} ni+i & ni-i \\ ni-i & ni+i \end{bmatrix} \sim \begin{bmatrix} n+1 & n-1 \\ n-1 & n+1 \end{bmatrix},$$

we obtain equation (21).

We have proved that the transformation  $\zeta = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \circ z$  induces a multiplicity preserving isomorphism between the roots  $z$  of  $R_n$  such that  $F_1(z) \neq \infty$  and the solutions  $\zeta \neq F_1(\infty)$  of equation (21). Using that  $F_1(0) = -1$ ,  $F_1(\infty) = 1$ , and the information about the roots of  $R_n$  contained in Lemma 4, we obtain the claim.  $\blacksquare$

**Remark 4.** Proposition 5 singles out the roots  $\zeta = \pm 1$  of equation (21). Observe that  $-1$  is always a root of multiplicity three for this equation, while  $1$  is a (simple) root iff  $n$  is odd. To explain this, we note that  $-1 = F_1(0)$ , while  $1 = F_1(\infty)$ . Observe that  $0$  is a multiplicity three root of  $R_n$ ; the appearance of  $\infty$  as a “root” of  $R_n$  is due to the circumstance that in the beginning of the proof of Proposition 5 we have put the equation  $R_n(z) = 0$  in the form

$$(23) \quad (nz+i)(z-i)^n = (-1)^{n+1}(nz-i)(z+i)^n.$$

If  $n$  is odd, the leading terms in both sides of equation (23) have the same sign when  $z \rightarrow \infty$ ; if  $n$  is even, the signs are opposite.

Equation (21) involves a rational function whose denominator is  $(n-1)\zeta + (n+1)$ . Getting rid of the denominator and using the variable  $x = -\zeta$ , we obtain an equivalent polynomial equation:

$$-(n-1)x^{n+1} + (n+1)x^n - (n+1)x + (n-1) = 0.$$

The two corollaries below follow immediately from Proposition 5 and the preceding discussion.

**Corollary 6.** *Let  $n > 1$ . Set*

$$(24) \quad S_n(x) = (n-1)[x^{n+1} - 1] - (n+1)[x^n - x].$$

*Then all roots of the polynomials  $S_n$  belong to the unit circle  $\{|x| = 1\}$ . The number  $1$  is a root of multiplicity three. The number  $-1$  is a simple root of  $S_n$  if  $n$  is odd, and  $S_n(-1) \neq 0$  if  $n$  is even. The remaining roots of  $R_n$  are simple.*

In what follows we will refer to the roots  $x \neq \pm 1$  of  $S_n$  as the *complex roots*.

**Corollary 7.** *Let  $n > 1$ . The transformation  $z \mapsto x$  given by*

$$x = -\frac{z - i}{z + i}$$

*induces a 1-to-1 correspondence between the nonzero roots of the polynomial  $R_n$  and the complex roots of the polynomial  $S_n$ . Moreover, this transformation sends the positive (resp. negative) roots of  $R_n$  to the roots of  $S_n$  such that  $\Im x > 0$  (resp.  $\Im x < 0$ ).*

### 8.2. Polynomials $S_n$ : Conjectures and supporting evidence.

Let  $0 < \delta < \pi$ ,  $\delta \neq \pi/2$ , be an element in  $A$ . We want to describe the set of billiard tables having caustics  $\Gamma_\delta$ . The conjectures below aim at answering Question 1.

**Conjecture 1.** *Let  $m, n > 1$  be distinct integers; let  $S_m, S_n$  be the corresponding polynomials in equation (24). Then their sets of complex roots are disjoint.*

The material of section 7 and section 8.1 yields that Conjecture 1 is equivalent to the following claim.

**Conjecture 2.** *Let  $m, n > 1$  be distinct integers. Then equations  $\tan mx = m \tan x$ ,  $\tan nx = n \tan x$  have no common solutions in  $(0, \pi/2)$ .*

For reader's convenience, we outline a proof that the two conjectures are equivalent. Recall that  $B_k$  denotes the set of roots of the equation  $\tan kx = k \tan x$  in  $(0, \pi/2)$ . By Lemma 3, Proposition 4, and Corollary 5, the set  $\{\tan x : x \in B_k\}$  is the set of positive roots of the polynomial  $R_k$ . See equation (20). Corollary 7 provides a fractional linear transformation that sends the positive roots of  $R_k$ ,  $k > 1$ , to the roots of  $S_k$  in the semi-circle  $\{|z| = 1, \Im z > 0\}$ . Now the information about the roots of  $S_k$  contained in Corollary 6 implies the claim.

The following proposition lends support to Conjecture 2.

**Proposition 6.** *Let  $n > 1$ . We will say that a solution  $x$  is nontrivial if  $\tan x \neq 0$ .*

1. *The system*

$$(25) \quad \tan nx = n \tan x, \quad \tan(n+k)x = (n+k) \tan x$$

*has no nontrivial solutions for  $k = 1, 2$ .*

2. *The system*

$$(26) \quad \tan nx = n \tan x, \quad \tan knx = kn \tan x$$



has no nontrivial solutions for  $k = 2, 3$ .

3. The systems

$$(27) \quad \tan nx = n \tan x, \quad \tan(2n \pm 1)x = (2n \pm 1) \tan x$$

have no nontrivial solutions.

*Proof.* We have

$$\tan(n+k)x = \frac{\tan nx + \tan kx}{1 - \tan nx \tan kx}.$$

Substituting this into equation (25) and using that  $\tan x \neq 0$ , we obtain  $1 + n(n+1)\tan^2 x = 0$  in the case  $k = 1$ , and  $(n+1)^2 \tan^2 x = 0$  in the case  $k = 2$ . This proves claim 1.

We have

$$\tan 2nx = \frac{2 \tan nx}{1 - \tan^2 nx}, \quad \tan 3nx = \frac{3 \tan nx - \tan^3 nx}{1 - 3 \tan^2 nx}.$$

Substituting these identities into equation (26), and assuming  $\tan x \neq 0$ , we obtain  $1 - n^2 \tan x = 1$ ,  $8n^2 \tan x = 0$  if  $k = 2$ ,  $k = 3$  respectively. This proves claim 2.

Equation (27) and the identity

$$\tan(2n \pm 1)x = \frac{\tan 2nx \pm \tan x}{1 \mp \tan 2nx \tan x}$$

yield the relationship  $n^3 \pm 2n^2 + n = 0$  which has no solutions  $n > 1$ . ■

**Remark 5.** A refinement of the above approach yields that the systems

$$\tan nx = n \tan x, \quad \tan(3n \pm 1)x = (3n \pm 1) \tan x$$

do not have nontrivial solutions as well. The proof is rather long, and we do not reproduce it here.

Proposition 6 and Remark 5 yield particular families of pairs of integers  $m \neq n$  such that the system  $\tan mx = m \tan x$ ,  $\tan nx = n \tan x$  has no nontrivial solutions. This provides direct evidence supporting Conjecture 2. The work [7] provides additional support for the equivalent Conjecture 1. We will now elaborate on this.

Let  $n \geq 4$ . Set  $\tilde{S}_n(x) = S_n(x)/(x-1)^3(x+1)$  if  $n$  is odd and  $\tilde{S}_n(x) = S_n(x)/(x-1)^3$  if  $n$  is even. By Corollary 6,  $\tilde{S}_n$  are polynomials with integer coefficients; their roots are simple and belong to the unit circle. Let  $X$  be a property that holds for some natural numbers. Denote by  $\mathbb{N}(X) \subset \mathbb{N}$  be the set of natural numbers having property  $X$ . We say that property  $X$  holds for almost all positive integers if  $\mathbb{N}(X) \subset \mathbb{N}$  is a subset of density one. A property that holds for almost all pairs of positive integers is defined analogously.

The work [7] puts forward several conjectures about irreducibility of polynomials over  $\mathbb{Q}$ . It conjectures, in particular, that polynomials  $\tilde{S}_n$  are irreducible. See Conjecture 3 in [7]. Let  $m \neq n$  be natural numbers. We will say that *Conjecture 1 holds for the pair  $m, n$*  if the sets of complex roots of the polynomials  $S_m, S_n$  are disjoint.

**Proposition 7.** *Conjecture 1 holds for almost all pairs of positive integers.*

*Proof.* By the preceding discussion, it suffices to show that for almost all pairs  $m \neq n$  the root sets of  $\tilde{S}_m, \tilde{S}_n$  are disjoint. Let  $\mathbb{I} \subset \mathbb{N}$  be the set of integers  $k$  such that  $\tilde{S}_k$  is irreducible. Let  $\mathbb{J} \subset \mathbb{I} \times \mathbb{I}$  be the set of distinct pairs. By Theorem 4 in [7],  $\mathbb{I} \subset \mathbb{N}$  is a set of density one. Thus, the sets  $\mathbb{J} \subset \mathbb{I} \times \mathbb{I} \subset \mathbb{N} \times \mathbb{N}$  have density one. But for pairs  $(m, n) \in \mathbb{J}$  the polynomials  $\tilde{S}_m, \tilde{S}_n$  have disjoint root sets. ■

Our next conjecture addresses Question 2.

**Conjecture 3.** *Let  $x \neq 0$  satisfy  $\tan nx = n \tan x$  for some  $n > 1$ . Then  $x$  is  $x/\pi$  is irrational.*

The motivation for Conjecture 3 is as follows. If  $0 < x < \pi/2$  satisfies equation (10) then there is a continuous family<sup>4</sup> of billiard tables  $\Omega$  with the invariant curve  $\Gamma_x$ . The billiard map on  $\Omega$ , restricted to  $\Gamma_x$ , is the rotation by  $2x$ . See the proof of Proposition 1. If  $x/\pi$  is rational, then we acquire a lot of examples of billiard tables with invariant circles realizing *rational rotations*. The billiard literature indicates that such invariant circles are extremely rare [21, 19].

Let us now look at examples. The smallest  $n$  for which equation  $\tan nx = n \tan x$  has nontrivial solutions is 4. Below we analyze its solutions for  $n = 4, 5$ .

**Example 1.** Set  $z = \tan x$ . From the recurrence relations equation (17), we easily obtain  $P_4 = 4z - z^3, Q_4 = 1 - 6z^2 + z^4, P_5 = 5z - 10z^3 + z^5, Q_5 = 1 - 10z + 5z^4$ . Thus, the equation  $\tan 4x = 4 \tan x$  is equivalent to  $z^4 - 5z^2 = 0$ . Since  $z \neq 0$ , this yields  $z = \pm\sqrt{5}$ . Denote by  $x_4$  the solution of  $\tan 4x = 4 \tan x$  in  $(0, \pi/2)$ . Then  $x_4 = \arctan(\sqrt{5}) = \arcsin(\sqrt{5}/\sqrt{6})$ . We analyze the equation  $\tan 5x = 5 \tan x$  in the same way. Denote by  $x_5$  the unique solution of  $\tan 5x = 5 \tan x$  in  $(0, \pi/2)$ . Then  $x_5 = \arctan(\sqrt{5}/3) = \arcsin(\sqrt{5}/2\sqrt{2})$ .

Using the formula  $\tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$ , we obtain the bounds

$$\frac{\pi}{4} < x_5 < \frac{3\pi}{10} < \frac{\pi}{3} < x_4 < \frac{\pi}{2}.$$

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<sup>4</sup>A one-parameter family, if Conjecture 2 holds.

These inequalities do not imply that the numbers  $x_4/\pi, x_5/\pi$  are irrational; however, they show that the denominators cannot be small.

Our next proposition provides more evidence for Conjecture 3.

**Proposition 8.** *Let  $\Omega$  be a regular billiard table. Suppose that  $\Gamma_{\pi/4}$  or  $\Gamma_{\pi/3}$  is a caustic for  $\Omega$ . Then  $\Omega$  is circular.*

*Proof.* We will show that  $\pi/3, \pi/4$  do not satisfy equation (6) for any  $n > 1$ . Set  $\delta = \pi/4$  and examine both sides of equation (6) for  $n = 2, 3, \dots$ . By periodicity of the sine function, everything is determined by the residue  $n \pmod 4$ . Let  $n \equiv 1 \pmod 4$ . Then in the left hand side of equation (6) the numerator is  $\sin(k\pi) = 0$ ; in the right hand side of equation (6) the numerator is  $\sin(k\pi + \pi/2) = \pm 1$ . Analogous considerations show that for  $n \equiv 3 \pmod 4$  equation (6) is not satisfied.

Let now  $n \equiv 0 \pmod 4$ . Thus  $n - 1 \equiv 3 \pmod 4, n + 1 \equiv 1 \pmod 4$ . Then one of the two numerators in equation (6) is  $\pm 1$  while the other is  $\mp 1$ , hence equation (6) does not hold. An analogous argument disposes of the case  $n \equiv 2 \pmod 4$ . This proves the claim for  $\pi/4$ . The argument for  $\delta = \pi/3$  follows the same pattern, and we leave it to the reader. ■

## 9. CONDITIONAL IMPLICATIONS FOR THE BILLIARD AND THE FLOATING

We will now deduce some implications of the above conjectures to billiard dynamics and capillary floating. We begin with the billiard.

### 9.1. Billiard tables with constant angle caustics.

**Proposition 9.** *Let  $\delta \in A \setminus \{\pi/2\}$ , and let  $\Omega \subset \mathbb{R}^2$  be a noncircular, regular billiard table with the caustic  $\Gamma_\delta$ .*

*Suppose that Conjecture 2 holds. Then there is a unique integer  $n \geq 4$  and a unique parameter  $0 < \tau < 1$  such that  $\Omega$  is conformally equivalent to the table  $\Omega_{n,\tau}$  given by equations (11), (12).*

*Proof.* Let  $\rho(\alpha)$  be the radius of curvature function for  $\partial\Omega$ . By Theorem 1, there are unique constants  $a, b, c$  satisfying  $0 < a^2 + b^2 < c$  such that  $\rho(\alpha) = c + a \cos n\alpha + b \sin n\alpha$ . The claim now follows from Corollary 2. ■

We say that a set  $\Omega \subset \mathbb{R}^2$  is *rotationally symmetric of order  $n > 1$*  if there is  $P_0 \in \mathbb{R}^2$  such that  $\Omega$  is invariant under the group  $\mathbb{Z}/n\mathbb{Z}$  of rotations of  $\mathbb{R}^2$  about  $P_0$ .

**Theorem 2.** *Let  $\Omega \subset \mathbb{R}^2$  be a noncircular, regular billiard table. The following statements are equivalent.*

1. *The table  $\Omega$  has a caustic  $\Gamma_\delta$ ,  $\delta \neq \pi/2$ .*
2. *There is  $n > 3$  such that the Fourier coefficients of the radius of curvature  $\rho(\cdot)$  of  $\partial\Omega$  satisfy i)  $c_n \neq 0$ ; ii)  $c_k = 0$  for all positive  $k \neq n$ .*
3. *There is  $n > 3$  and  $0 < \tau < 1$  such that  $\Omega$  is conformally equivalent to the table  $\Omega_{n,\tau}$  given by equations (11), (12).*

*Proof.* Proposition 9 proves the implication  $1 \Rightarrow 3$ , while  $2 \Rightarrow 1$  is a byproduct of Corollary 2. The implication  $3 \Rightarrow 2$  is obvious. ■

The preceding propositions rely on Conjecture 2.<sup>5</sup> The proof of the following claim relies on Conjecture 3.

**Theorem 3.** *Let  $\Omega \subset \mathbb{R}^2$  be a regular billiard table, where  $\partial\Omega$  is not a curve of constant width. Suppose that the table  $\Omega$  has a caustic  $\Gamma_\delta$  of constant type. Then the restriction of the billiard map on  $\Omega$  to  $\Gamma_\delta$  is an irrational rotation.*

*Proof.* By the proof of Proposition 1, the transformation in question is the rotation by  $2\delta/\pi$ . The claim now follows from Conjecture 3. ■

From now until the end of the section, we will assume the truth of all conjectures in section 8.2.

**Theorem 4.** *There is a dense countable set  $R \subset (0, 1)$  of irrational numbers such that the following claims hold.*

1. *For every  $\rho \in R$  there is a one-parameter family of regular billiard tables  $\Omega_\tau$  having a constant angle caustic with the rotation number  $\rho$ . The curves  $\partial\Omega_\tau$  are real analytic. Every regular billiard table having a constant angle caustic with the rotation number  $\rho$  is conformally equivalent to a unique table  $\Omega_\tau$ .*
2. *Let  $\rho \in (0, 1) \setminus R$ . Suppose that a regular billiard table  $\Omega$  has a constant angle caustic with the rotation number  $\rho$ . i) If  $\rho = 1/2$  then  $\Omega$  has constant width. ii) If  $\rho \neq 1/2$  then  $\Omega$  is a disc.*

*Proof.* Claim 1 is immediate from Theorem 2 and Theorem 3. Claim 2 follows by combining these statements with Theorem 1. ■

## 9.2. Two-dimensional capillary floating.

In what follows we assume that all of the conjectures in section 8.2 hold.

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<sup>5</sup>Or the equivalent Conjecture 1.

**Theorem 5.** *Let  $\Omega \subset \mathbb{R}^2$  be a regular, compact, convex domain. Then the following holds.*

1. *Suppose that  $\Omega$  is not a disc. Then  $\Omega$  floats in neutral equilibrium at any orientation with the contact angle  $\pi - \delta \neq \pi/2$  if and only if there is  $n > 3$  and  $0 < \tau < 1$  such that  $\Omega$  is conformally equivalent to the domain  $\Omega_{n,\tau}$  given by equations (11), (12).*

2. *Suppose that  $\Omega$  is not a domain of constant width. If  $\Omega$  floats in neutral equilibrium at any orientation with the contact angle  $\gamma$  then  $\gamma/\pi$  is irrational.*

3. *There is a countable dense set  $A \subset (0, \pi)$  containing  $\pi/2$  and symmetric about this point such that the following holds:*

*If  $\Omega$  floats in neutral equilibrium at any orientation with the contact angle  $\gamma \in (0, \pi) \setminus A$  then  $\Omega$  is a disc.*

*Proof.* The claims are the counterparts of statements in Theorems 2, 3, and 4. ■

## REFERENCES

- [1] V.I. Arnold, *Singularities of caustics and wave fronts*, Kluwer Academic Publishers, Dordrecht, 1990.
- [2] H. Auerbach, *Sur un problème de M. Ulam concernant l'équilibre des corps flottants*, *Studia Math.* **7** (1938), 121 - 142.
- [3] V. Bangert, *Mather sets for twist maps and geodesics on tori*, *Dynamics reported* **1**, 1 - 56, Wiley, Chichester, 1988.
- [4] D. Bennequin, *Caustique mystique (d'après Arnold et al.)*, *Séminaire Bourbaki*, **1984/85**. Astérisque 133-134 (1986), 19 - 56.
- [5] G.D. Birkhoff, *Dynamical systems with two degrees of freedom*, *Trans. Amer. Math. Soc.* **18** (1917), 199-300.
- [6] V.G. Boltyanski and I.M. Yaglom, *Convex figures*, Rinehart and Winston, New York, 1960.
- [7] A. Borisov, M. Filaseta, T.Y. Lam, O. Trifonov, *Classes of polynomials having only one non-cyclotomic irreducible factor*, *Acta Arith.* **90** (1999), 121 - 153.
- [8] J.W. Bruce and P.J. Giblin, *Curves and Singularities. A geometrical introduction to singularity theory*, Cambridge University Press, Cambridge (1984).
- [9] R. Douady, *Applications du théorème des tores invariants*, Thèse de troisième cycle, Paris VI (1982).
- [10] R. Finn, *Floating bodies subject to capillary attractions*, *J. Math. Fluid Mech.* **11** (2009), 443 - 458.
- [11] R. Finn, *Remarks on "Floating bodies in neutral equilibrium"*, *J. Math. Fluid Mech.* **11** (2009), 466 - 467.

- [12] R. Finn and M. Sloss, *Floating Bodies in Neutral Equilibrium*, J. Math. Fluid Mech. **11** (2009) 459 – 463.
- [13] R. Finn and T. Vogel, *Floating criteria in three dimensions*, Analysis **29** (2009), 387 - 402.
- [14] E. Gutkin, *Billiard tables of constant width and dynamical characterizations of the circle*, pp. 21 – 24, Workshop on Dynamics and Related Questions, PennState University, 1993.
- [15] E. Gutkin, *A few remarks on the billiard ball problem*, pp. 157 – 165, Contemp. Math. **173**, A. M. S., Providence, RI, 1994.
- [16] E. Gutkin, *Two applications of calculus to triangular billiards*, Amer. Math. Monthly **104** (1997), 618 – 622.
- [17] E. Gutkin, *Billiard dynamics: A survey with the emphasis on open problems*, Reg. & Chaot. Dyn. **8** (2003), 1 – 13.
- [18] E. Gutkin and A. Katok, *Caustics for inner and outer billiards*, Comm. Math. Phys. **173** (1995), 101 – 133.
- [19] E. Gutkin and O. Knill, *Billiards that share a triangular caustic*, pp. 199 – 213, World Sci. Publ., River Edge, NJ, 1996.
- [20] E. Gutkin and M. Rams, *Growth rates for geometric complexities and counting functions in polygonal billiards*, Erg. Theory & Dyn. Sys. **29** (2009), 1163 - 1183.
- [21] N. Innami, *Convex curves whose points are vertices of billiard triangles*, Kodai Math. J. **11** (1988), 17 – 24.
- [22] A. Katok, *Billiard table as a playground for a mathematician*, pp. 216 – 242, London Math. Soc. Lecture Notes **321**, Cambridge University Press, Cambridge, 2005.
- [23] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge 1995.
- [24] P.S. de Laplace, *Traité de Mécanique Céleste*, Vol. 4, Suppléments au Livre X, Gauthier-Villars, Paris, 1806.
- [25] V.F. Lazutkin, *Existence of caustics for the billiard problem in a convex domain*, Izv. Akad. Nauk SSSR **37** (1973), 186 - 216.
- [26] J.N. Mather, G. Forni, *Action minimizing orbits in Hamiltonian systems. Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991)*, 92 - 186, Lecture Notes in Math. **1589**, Springer, Berlin, 1994.
- [27] E. Raphaël, J.-M. di Meglio, M. Berger, E. Calabi, *Convex particles at interfaces*, J. Phys. I France **2** (1992), 571 – 579.
- [28] *The Scottish Book. Mathematics from the Scottish Café. Selected papers presented at the Scottish Book Conference held at North Texas State University*, editor R.D. Mauldin, Birkhäuser, Boston, 1981.
- [29] S. Tabachnikov, *Billiards*, Soc. Math. de France, Paris, 1995.
- [30] S. Tabachnikov, *Geometry and billiards*, AMS, Providence, 2005.
- [31] P.L. Varkonyi, *Floating body problems in two dimensions*, Stud. Appl. Math. **122** (2009), 195 - 218.

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