SOME RESULTS CONCERNING THE p-ROYDEN AND p-HARMONIC BOUNDARIES OF A GRAPH OF BOUNDED DEGREE

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Abstract. Let p be a real number greater than one and let Γ be a connected graph of bounded degree. We show that the p-Royden boundary of Γ with the p-harmonic boundary removed is a F_σ-set. We also characterize the p-harmonic boundary of Γ in terms of the intersection of the extreme points of a certain subset of one-sided infinite paths in Γ.

1. Introduction

Let Γ be a graph with vertex set V_Γ and edge set E_Γ. We will write V for V_Γ and E for E_Γ if it is clear what graph Γ we are working with. For $x \in V$, $\deg(x)$ will denote the number of neighbors of x and N_x will be the set of neighbors of x. A graph Γ is said to be of *bounded degree* if there exists a positive integer k such that $\deg(x) \leq k$ for every $x \in V$. A path γ in Γ is a sequence of vertices $x_1, x_2, \ldots, x_n$ where $x_{i+1} \in N_{x_i}$ for $1 \leq i \leq n-1$ and $x_i \neq x_j$ if $i \neq j$. A graph is connected if any two given vertices of the graph are joined by a path. All graphs considered in this paper will be connected, of bounded degree with no self-loops and have countably infinite number of vertices. We shall say that a subset S of V is connected if the subgraph of Γ induced by S is connected. The Cayley graph of a finitely generated group is an example of the type of graph the we study in this paper. By assigning length one to each edge of Γ, V becomes a metric space with respect to the shortest path metric. We will denote this metric by $d(x, y)$, where x and y are vertices of Γ. Thus $d(x, y)$ gives the length of the shortest path joining the vertices x and y. Finally, if $x \in V$ and $n \in \mathbb{N}$, then $B_n(x)$ will denote the metric ball that contains all elements of V that have distance less than n from x.

Let p be a real number greater than one. In Section 2 we will define the p-Royden boundary of Γ, which we will indicate by $R_p(\Gamma)$. We will also define the p-harmonic boundary of Γ, which is a subset of $R_p(\Gamma)$. We will use $\partial_p(\Gamma)$ to denote the p-harmonic boundary. Our motivation for investigating the p-harmonic boundary of a graph is its connection to the vanishing of the first reduced ℓ_p-cohomology space of a finitely generated group. More specifically, this space vanishes if and only if its p-harmonic boundary is empty or contains exactly one element, see [5, Section 7] for the details of this fact. Gromov conjectured in [1, page 150] that the first reduced ℓ_p-cohomology space of a finitely generated amenable group vanishes. Thus, a better

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understanding of the p-harmonic boundary could be helpful in resolving Gromov's conjecture.

Recall that in a topological space a set is said to be F_σ if it is a countable union of closed sets. In this paper we will prove that $R_p(\Gamma) \setminus \partial_p(\Gamma)$ is F_σ. For each infinite path in Γ we can associate a set of extreme points, which is roughly the "points at infinity" of the path with respect to the p-Royden boundary. Our other main result in this paper is that the p-harmonic boundary is precisely the intersection of the extreme points of a certain subset of one-sided infinite paths in Γ.

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2. The p-Royden and p-harmonic boundaries

Let $1 < p \in \mathbb{R}$. In this section we construct the p-Royden and p-harmonic boundaries of Γ. For a more detailed discussion about this construction see Section 2.1 of [5]. Before we can give these definitions we need to define the space of p-Dirichlet finite functions on V. For any $S \subset V$, the outer boundary ∂S of S is the set of vertices in $V \setminus S$ with at least one neighbor in S. For a real-valued function f on $S \cup \partial S$ we define the p-th power of the *gradient*, the *p-Dirichlet sum*, and the *p-Laplacian* of $x \in S$ by

$$|Df(x)|^p = \sum_{y \in N_x} |f(y) - f(x)|^p,$$

$$I_p(f, S) = \sum_{x \in S} |Df(x)|^p,$$

$$\Delta_p f(x) = \sum_{y \in N_x} |f(y) - f(x)|^{p-2}(f(y) - f(x)).$$

In the case $1 < p < 2$, we make the convention that $|f(y)-f(x)|^{p-2}(f(y)-f(x)) = 0$ if $f(y) = f(x)$. Let $S \subseteq V$. A function f is said to be p-harmonic on S if $\Delta_p f(x) = 0$ for all $x \in S$. We shall say that f is *p-Dirichlet finite* if $I_p(f, V) < \infty$. The set of all p-Dirichlet finite functions on G will be denoted by $D_p(G)$. With respect to the following norm $D_p(G)$ is a reflexive Banach space,

$$\| f \|_{D_p} = (I_p(f, V) + |f(o)|^p)^{1/p},$$

where o is a fixed vertex of Γ and $f \in D_p(\Gamma)$. We use $HD_p(\Gamma)$ to represent the set of p-harmonic functions on V that are contained in $D_p(\Gamma)$. Let $\ell^\infty(\Gamma)$ denote the set of bounded functions on V and let $\| f \|_\infty = \sup_V |f|$ for $f \in \ell^\infty(\Gamma)$. Set $BD_p(\Gamma) = D_p(\Gamma) \cap \ell^\infty(\Gamma)$. The set $BD_p(\Gamma)$ is a Banach space under the norm

$$\| f \|_{BD_p} = (I_p(f, V))^{1/p} + \| f \|_\infty,$$

where $f \in BD_p(\Gamma)$. Let $BHD_p(\Gamma)$ be the set of bounded p-harmonic functions contained in $D_p(\Gamma)$. The space $BD_p(\Gamma)$ is also closed under the usual operations of scalar multiplication, addition and pointwise multiplication. Furthermore, $\| fg \|_{BD_p} \leq \| f \|_{BD_p} \| g \|_{BD_p}$ for $f, g \in BD_p(\Gamma)$. Thus $BD_p(\Gamma)$ is a commutative Banach algebra. Let $C_c(\Gamma)$ be the set of functions on V with finite support. Indicate the closure of $C_c(\Gamma)$ in $D_p(\Gamma)$ by $\overline{C_c(\Gamma)}_{D_p}$. Set $B(\overline{C_c(\Gamma)}_{D_p}) = \overline{C_c(\Gamma)}_{D_p} \cap \ell^\infty(\Gamma)$. Using the fact that the inequality $(a + b)^{1/p} \leq a^{1/p} + b^{1/p}$ is true when $a, b \geq 0$

and $1 < p \in \mathbb{R}$, we see immediately that $\| f \|_{D_p} \leq \| f \|_{BD_p}$. It now follows that $B(\overline{C_c(\Gamma)}_{D_p})$ is closed in $BD_p(\Gamma)$.

Let $Sp(BD_p(\Gamma))$ denote the set of complex-valued characters on $BD_p(\Gamma)$, that is the nonzero ring homomorphisms from $BD_p(\Gamma)$ to $\mathbb{C}$. Then with respect to the weak $*$-topology, $Sp(BD_p(\Gamma))$ is a compact Hausdorff space. Given a topological space X, let $C(X)$ denote the ring of continuous functions on X endowed with the sup-norm. The Gelfand transform defined by $\hat{f}(\chi) = \chi(f)$ yields a monomorphism of Banach algebras from $BD_p(\Gamma)$ into $C(Sp(BD_p(\Gamma)))$ with dense image. Furthermore the map $i \colon V \to Sp(BD_p(\Gamma))$ given by $(i(x))(f) = f(x)$ is an injection, and $i(V)$ is an open dense subset of $Sp(BD_p(\Gamma))$. For the rest of this paper, we shall write f for $\hat{f}$, where $f \in BD_p(\Gamma)$. The *p-Royden boundary* of Γ, which we shall denote by $R_p(\Gamma)$, is the compact set $Sp(BD_p(\Gamma)) \setminus i(V)$. The *p-harmonic boundary* of Γ is the following subset of $R_p(\Gamma)$:

$$\partial_p(\Gamma) \colon = \{\chi \in R_p(\Gamma) \mid \hat{f}(\chi) = 0 \text{ for all } f \in B(\overline{C_c(\Gamma)}_{D_p})\}.$$

Let S be an infinite subset of V and let A and B be disjoint nonempty subsets of $S \cup \partial S$. The *p-capacity* of the condenser (A, B, S) is defined by

$$cap_p(A, B, S) = \inf_u I_p(u),$$

where the infimum is taken over all functions $u \in D_p(\Gamma)$ with $u = 0$ on A and $u = 1$ on B. Such a function is called *admissible*. Set $cap_p(A, B, S) = \infty$ if the set of admissible functions is empty.

Let A be a finite subset of $S \cup \partial S$ and let (U_n) be an exhaustion of V by finite connected subsets such that $A \subset U_1$. We now define

$$cap_p(A, \infty, S) = \lim_{n\to\infty} cap_p(A, (\partial S \cup S) \setminus U_n, S).$$

Since $cap_p(A, (\partial S \cup S) \setminus U_n, S) \geq cap_p(A, (\partial S \cup S) \setminus U_{n+1}, S)$, the above limit exists. We shall say that S is *p-hyperbolic* if there exists a finite subset A of $S \cup \partial S$ that satisfies $cap_p(A, \infty, S) > 0$. If S is not p-hyperbolic, then it is said to be *p-parabolic*. An equivalent definition of p-hyperbolic is that S is p-hyperbolic if and only if $1_S \in \overline{C_c(\Gamma_S)}_{D_p}$, where 1_S is the constant function 1 on S and Γ_S the subgraph of Γ induced by S, [8, Theorem 3.1]. We will define a graph Γ to be p-hyperbolic (p-parabolic) if its vertex set V is p-hyperbolic (p-parabolic). It was shown in [5, Proposition 4.2] that Γ is p-parabolic if and only if $\partial_p(\Gamma) = \emptyset$. A useful property of p-hyperbolic graphs that we will use throughout this paper is the following p-Royden decomposition, see [5, Theorem 4.6] for a proof.

Theorem 2.1. *(p-Royden decomposition) Let $1 < p \in \mathbb{R}$ and suppose $f \in BD_p(\Gamma)$. Then there exists a unique $u \in B(\overline{C_c(\Gamma)}_{D_p})$ and a unique $h \in BHD_p(\Gamma)$ such that $f = u + h$.*

3. Statement of main results

In this section we will state our main results. In section 4 we will prove

Theorem 3.1. *Let $1 < p \in \mathbb{R}$ and let Γ be a graph of bounded degree. The set $R_p(\Gamma) \setminus \partial_p(\Gamma)$ is F_σ.*

Before we state our other main result we need to define the set of extreme points of a path in Γ. Let P be the set of all one-sided infinite paths in Γ. For a real-valued function f on V and a path $\gamma \in P$, the limit of f as we follow γ to infinity is given by $\lim_{n\to\infty} f(x_n)$, where $x_0, x_1, \ldots, x_n, \ldots$ is the vertex representation of the path γ. Sometimes we write $f(\gamma) = \lim_{n\to\infty} f(x_n)$ to indicate this limit. Let $\gamma \in P$ and denote by $V(\gamma)$ the set of vertices on γ. The closure of $i(V(\gamma))$ in $Sp(BD_p(\Gamma))$ will be indicated by $\overline{V}(\gamma)$. Recall that $Sp(BD_p(\Gamma))$ is endowed with the weak$*$-topology. Thus $\chi \in \overline{V}(\gamma)$ if and only if there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} f(x_{n_k}) = \chi(f)$ for all $f \in BD_p(\Gamma)$. The *extreme points* of a path γ is defined to be

$$E(\gamma) = \overline{V}(\gamma) \cap R_p(\Gamma).$$

Let $f \in B(\overline{C_c(\Gamma)}_{D_p})$ and set $A_f = \{\gamma \in P \mid f(\gamma) \neq 0\}$. Set

$$E_f = \overline{\{\cup_\gamma E(\gamma) \mid \gamma \in P \setminus A_f\}}.$$

In Section 5 we shall prove

Theorem 3.2. *Let $1 < p \in \mathbb{R}$ and let Γ be a graph of bounded degree. Then*

$$\partial_p(\Gamma) = \bigcap_{f \in B(\overline{C_c(\Gamma)}_{D_p})} E_f.$$

Let $1 < p \in \mathbb{R}$. If Γ is p-parabolic, then $\partial_p(\Gamma) = \emptyset$ and Theorem 3.1 is true. Also for the p-parabolic case, $1_V \in B(\overline{C_c(\Gamma)}_{D_p})$ by [8, Theorem 3.2], where 1_V is the constant function one on V. Then $E_{1_V} = \emptyset$ and Theorem 3.2 follows. Thus for the rest of the paper we will assume Γ is p-hyperbolic.

4. Proof of Theorem 3.1

In this section we will prove Theorem 3.1. We will start by giving some needed definitions and proving a comparison principle. A comparison principle for finite subsets of V was proved in [2, Theorem 3.14]. Our proof follows theirs in spirit.

Let f and h be elements of $BD_p(\Gamma)$ and let $1 < p \in \mathbb{R}$. Define

$$\langle \Delta_p h, f \rangle :\, = \sum_{x\in V} \sum_{y \in N_x} |h(y) - h(x)|^{p-2}(h(y) - h(x))(f(y) - f(x)).$$

The sum exists since

$$\sum_{x\in V} \sum_{y \in N_x} \left| |h(y) - h(x)|^{p-2}(h(y) - h(x)) \right|^q = I_p(h, V) < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For notational convenience let

$$T(h, f, x, y) = |h(y) - h(x)|^{p-2}(h(y) - h(x))(f(y) - f(x)).$$

In order to prove Theorem 3.1 we will need the following:

Lemma 4.1. *(Comparison principle) Let h_1, h_2 be elements of $BHD_p(\Gamma)$ and suppose $h_1(x) \leq h_2(x)$ for all $x \in \partial_p(\Gamma)$. Then $h_1 \leq h_2$ on V.*

Proof. Define a function f on V by $f = \min\{h_2 - h_1, 0\}$. Theorem 4.8 of [5] says $f \in B(\overline{C_c(\Gamma)}_{D_p})$ since $f = 0$ on $\partial_p(\Gamma)$. By Lemma 4.6 of [5] we have $\langle \Delta_p h_1, f\rangle = 0$ and $\langle \Delta_p h_2, f\rangle = 0$, which implies $\langle \Delta_p h_1 - \Delta_p h_2, f\rangle = 0$. Now set

$$\begin{aligned} A &= \{x \in V \mid h_1(x) \le h_2(x)\}, \\ B &= \{x \in V \mid h_2(x) < h_1(x)\}, \end{aligned}$$

and for $a \in V$ let

$$\begin{aligned} C_a &= \{y \in V \mid y \in N_a \text{ and } h_1(y) \le h_2(y)\}, \\ D_a &= \{y \in V \mid y \in N_a \text{ and } h_2(y) < h_1(y)\}. \end{aligned}$$

Now

$$0 = \sum_{x\in V}\sum_{y\in N_x} (T(h_1, f, x, y) - T(h_2, f, x, y)) = T_1 + T_2 + T_3 \tag{4.1}$$

where

$$\begin{aligned} T_1 &= \sum_{x\in A}\sum_{y\in C_x} (T(h_1, f, x, y) - T(h_2, f, x, y)), \\ T_2 &= \left(\sum_{x\in A}\sum_{y\in D_x} + \sum_{x\in B}\sum_{y\in C_x}\right)(T(h_1, f, x, y) - T(h_2, f, x, y)), \end{aligned}$$

and

$$T_3 = \sum_{x\in B}\sum_{y\in D_x} (T(h_1, f, x, y) - T(h_2, f, x, y)).$$

Since $f(x) = f(y) = 0$ for $x \in A$ and $y \in C_x$ it follows that $T_1 = 0$. We now claim that $T_3 \le 0$. To see the claim let a and b be real numbers such that $a \ne b$. It follows from the inequality

$$|a|^{p-2}a(a-b) > |b|^{p-2}b(a-b)$$

that

$$T(h_1, h_1 - h_2, x, y) \ge T(h_2, h_1 - h_2, x, y). \tag{4.2}$$

Equality occurs if and only if $(h_1 - h_2)(x) = (h_1 - h_2)(y)$. Now if $x \in B$ and $y \in D_x$, then $f(y) - f(x) = (h_2 - h_1)(y) - (h_2 - h_1)(x)$. Combining (4.2) with the fact $T(h_k, h_1 - h_2, x, y) = -T(h_k, h_2 - h_1, x, y)$, where $k = 1$ or $k = 2$, we obtain $T_3 \le 0$, which is our claim.

We now proceed to show that if there is a pair of vertices x and y that satisfy $x \in A, y \in D_x$ or $x \in B, y \in C_x$, then $T_2 < 0$. Suppose $x \in A$ and $y \in D_x$. Then $f(y) - f(x) = h_2(y) - h_1(y) < 0$ and

$$T(h_1, f, x, y) - T(h_2, f, x, y) = (h_2(y) - h_1(y)) \times \\ \left(|h_1(y) - h_1(x)|^{p-2}(h_1(y) - h_1(x)) - |h_2(y) - h_2(x)|^{p-2}(h_2(y) - h_2(x))\right).$$

Also $h_1(y) - h_1(x) > h_2(y) - h_2(x)$ because $h_1(y) - h_2(y) > 0 \ge h_1(x) - h_2(x)$. So if $h_2(y) \ge h_2(x)$ we see that $T(h_1, f, x, y) - T(h_2, f, x, y) < 0$ since $h_1(y) - h_1(x) > h_2(y) - h_2(x)$. On the other hand if $h_2(y) < h_2(x)$ and $h_1(y) > h_1(x)$ we obtain

$$\begin{aligned} &T(h_1, f, x, y) - T(h_2, f, x, y) \\ &= (h_2(y) - h_1(y))\left(|h_1(y) - h_1(x)|^{p-1} + |h_2(y) - h_2(x)|^{p-1}\right) < 0 \end{aligned}$$

since $|h_2(y)-h_2(x)| = -(h_2(y)-h_2(x))$. The only other possibility is $h_2(y) < h_2(x)$ and $h_1(y) \leq h_1(x)$. If this is the case then $h_2(y) < h_1(y) \leq h_1(x) \leq h_2(x)$ due to $x \in A$ and $y \in D_x$. Consequently, $h_2(y) - h_2(x) < h_1(y) - h_1(x)$ and $h_1(x) - h_1(y) < h_2(x) - h_2(y)$; hence, $|h_1(y) - h_1(x)| < |h_2(y) - h_2(x)|$. It now follows that

$$T(h_1, f, x, y) - T(h_2, f, x, y)$$
$$= (h_2(y) - h_1(y)) \left(|h_2(y) - h_2(x)|^{p-1} - |h_1(y) - h_1(x)|^{p-1}\right) < 0.$$

A similar argument can be used to show that $T(h_1, f, x, y) - T(h_2, f, x, y) < 0$ for each $x \in B$ and $y \in C_x$. Hence, if $x \in A, y \in D_x$ or $x \in B, y \in C_x$, then $T_2 < 0$. Since $T_1 = 0$ and $T_3 \leq 0$, it follows from (4.1) that it must be the case $T_2 = 0$. Thus it is impossible to have a pair of vertices x and y with $x \in A, y \in D_x$ or $x \in B, y \in C_x$.

Now assume that $h_1(z) > h_2(z)$ for some $z \in V$. We claim that there exists vertices x_0, y_0 in V for which $y_0 \in N_{x_0}, h_1(x_0) > h_2(x_0)$ and $h_1(y_0) \leq h_2(y_0)$. To see the claim suppose $h_1 = h_2$ on $\partial_p(\Gamma)$, then $h_1 = h_2$ on V by [5, Corollary 4.9]. So there exists an $x \in \partial_p(\Gamma)$ that satisfies $h_1(x) < h_2(x)$. Let $(x_n) \to x$ where (x_n) is a sequence in V. Now there exists a term x_m in this sequence such that $h_1(x_m) < h_2(x_m)$. Since Γ is connected there is a path from z to x_m. Thus there are vertices x_0 and y_0 on this path with $y_0 \in N_{x_0}, h_1(x_0) > h_2(x_0)$, and $h_1(y_0) < h_2(y_0)$ because $h_1(z) > h_2(z)$ and $h_1(x_m) < h_2(x_m)$. Thus $x_0 \in B$ and $y_0 \in C_{x_0}$, a contradiction. Therefore, $h_1(z) \leq h_2(z)$ for all $z \in V$. □

Proof of Theorem 3.1. Let $1 < p \in \mathbb{R}$. Since $Sp(BD_p(\Gamma))$ is a normal space, there exists for each $x \in R_p(\Gamma)$ a sequence $(U_j(x))$ of open sets containing x such that $\overline{U}_{j+1}(x) \subseteq U_j(x)$. For each $j \in \mathbb{N}$ there exists a finite number of points $x_{j,k}, 1 \leq k \leq N_j$ such that $U_j(x_{j,k})$ cover $R_p(\Gamma)$. For notational simplicity we will denote $U_j(x_{j,k})$ by $U_{j,k}$. Using Urysohn's lemma we can construct a continuous function $\phi_{j,k}$ with $\phi_{j,k} = 2$ on $U_{j,k}$ and $\phi_{j,k} = -1$ on $Sp(BD_p(\Gamma)) \setminus U_{j-1,k}$. By the density of $BD_p(\Gamma)$ in $C(Sp(BD_p(\Gamma)))$ there exists a $g \in BD_p(\Gamma)$ such that $|\phi_{j,k} - g| < \frac{1}{2}$. Set $f_{j,k} = \max(\min(1, g), 0)$, so $f_{j,k} \in BD_p(\Gamma), 0 \leq f_{j,k} \leq 1, f_{j,k} = 1$ on $U_{j,k}$ and $f_{j,k} = 0$ on $Sp(BD_p(\Gamma)) \setminus U_{j-1,k}$. The p-Royden decomposition of $BD_p(\Gamma)$ yields a unique p-harmonic function $h_{j,k} \in BHD(\Gamma)$ and a unique $u_{j,k} \in B(\overline{C_c(\Gamma)}_{D_p})$ such that $f_{j,k} = u_{j,k} + h_{j,k}$. Because $u_{j,k} = 0$ on $\partial_p(\Gamma)$ by [5, Theorem 4.8], we see that $f_{j,k} = h_{j,k}$ on $\partial_p(\Gamma)$. New define

$$R_{j,k} = \{x \in R_p(\Gamma) \cap \overline{U}_{j,k} \mid \lim_{x_n \to x} h_{j,k}(x_n) < f_{j,k}(x) = 1\},$$

where (x_n) is a sequence in V. Observe that if $R_{j,k}$ is nonempty, then it only contains elements of $R_p(\Gamma) \setminus \partial_p(\Gamma)$.

Let $x \in R_p(\Gamma) \setminus \partial_p(\Gamma)$. We will now show that there exists $j, k \in \mathbb{N}$ such that $x \in R_{j,k}$. Since $x \notin \partial_p(\Gamma)$ there exists a $u \in B(\overline{C_c(\Gamma)}_{D_p})$ such that $u(x) \neq 0$. Since $B(\overline{C_c(\Gamma)}_{D_p})$ is an ideal we may assume that $u \geq 0$ on V and $u(x) > 0$. Replacing u by $u^{-1}(x)u$ if necessary we may assume that $u(x) = 1$. Let $h \in BHD_p(\Gamma)$ that satisfies $h \geq 1$ on V. Set $f = u + h$, so $f \in BD_p(\Gamma)$ and $f = h$ on $\partial_p(\Gamma)$. Let (x_n) be a sequence in V that converges to x. Now $\lim_{n\to\infty} h(x_n) < f(x)$. Because f is continuous we can find an open set $U_{j,k}$ that contains x and satisfies

$$m = \inf_{U_{j-1,k} \cap R_p(\Gamma)} f > \lim_{n\to\infty} h(x_n).$$

It now follows

$$f_{j,k} \leq \frac{f}{m} \text{ on } R_p(\Gamma),$$

which implies that $h_{j,k} \leq \frac{h}{m}$ on $\partial_p(\Gamma)$. An appeal to the comparison principle gives us

$$\lim_{n\to\infty} h_{j,k}(x_n) \leq \frac{1}{m} \lim_{n\to\infty} h(x_n) < 1 = f_{j,k}(x),$$

hence $x \in R_{j,k}$. Furthermore,

$$R_{j,k} = \bigcup_{i=1}^{\infty} \left(R_p(\Gamma) \cap \overline{U}_{j,k} \cap \overline{\{y \in V \mid h_{j,k}(y) < 1 - 1/i\}} \right).$$

Thus $R_{j,k}$ is a countable union of compact sets. Theorem 3.1 now follows because

$$R_p(\Gamma) \setminus \partial_p(\Gamma) = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{N_j} R_{j,k}.$$

5. Proof of Theorem 3.2

Before we prove Theorem 3.2 we need to state some definitions and prove several preliminary results.

Fix a real number $p > 1$. Recall that E denotes the edge set of a graph Γ. Denote by $\mathcal{F}(E)$ the set of all real-valued functions on E and let $\mathcal{F}^+(E)$ be the subset of $\mathcal{F}(E)$ that consists of all nonnegative functions. For $\omega \in \mathcal{F}(E)$ set

$$\mathcal{E}_p(\omega) = \sum_{e\in E} |\omega(e)|^p.$$

The edge set of a path γ in Γ will be denoted by $Ed(\gamma)$, remember $E(\gamma)$ represents the extreme points of γ. Let Q be a set of paths in Γ, denote by $\mathcal{A}(Q)$ the set of all $\omega \in \mathcal{F}^+(E)$ that satisfy $\mathcal{E}_p(\omega) < \infty$ and $\sum_{e\in Ed(\gamma)} \omega(e) \geq 1$ for each $\gamma \in Q$. The *extremal length* of order p for Q is defined by

$$\lambda_p(Q)^{-1} = \inf\{\mathcal{E}_p(\omega) \mid \omega \in \mathcal{A}(Q)\}.$$

A variation of the next lemma was proved for the case $p = 2$ in [6, Lemma 6.13]. In the $p = 2$ case the conclusion of the lemma is stronger in that g belongs to $\overline{C_c(\Gamma)}_{D_2}$ instead of the larger space $D_2(\Gamma)$.

Lemma 5.1. *Let K be a compact subset of $R_p(\Gamma)$ with $K \cap \partial_p(\Gamma) = \emptyset$. Then there exists a function $g \in D_p(\Gamma)$ that satisfies $g = \infty$ on K and $g = 0$ on $\partial_p(\Gamma)$.*

Proof. By Urysohn's lemma there exists an $f \in C(Sp(BD_p(\Gamma)))$ that satisfies the following: $0 \leq f \leq 1, f = 1$ on K and $f = 0$ on $\partial_p(\Gamma)$. Using the argument from the first paragraph of the proof of Theorem 3.1 we may and do assume $f \in BD_p(\Gamma)$.

Let (U_n) be an exhaustion of V by finite connected subsets. Applying Theorem 3.1 of [2] yields a function h_n that is p-harmonic on U_n and equals f on $V \setminus U_n$. It follows from the minimizer property of p-harmonic functions on U_n that $I_p(h_n, V) \leq I_p(f, V)$. Hence, $h_n \in BD_p(\Gamma)$ for each $n \in \mathbb{N}$. Also, $h_n = 0$ on $\partial_p(\Gamma), h_n = 1$ on K and $0 \leq h_n \leq 1$ for each n. By passing to a subsequence if necessary, we may assume that (h_n) converges pointwise to a function h because $\overline{\{h_n(x) \mid n \in \mathbb{N}\}}$ is compact for each $x \in V$. By Lemma 3.2 of [2], h is p-harmonic on V. Since the sequence $(I_p(h_n, V))$ is bounded, Theorem 1.6 on page 177 of [7] says that by passing to a subsequence if necessary, we may assume that (h_n) converges weakly to a function $\overline{h} \in D_p(\Gamma)$. Because evaluation by $x \in V$ is a continuous linear functional on $D_p(\Gamma)$, we have that $h_n(x) \to \overline{h}(x)$ for each $x \in V$. Thus $h = \overline{h}$ and $h \in BD_p(\Gamma)$. It follows from [5, Corollary 4.9] that $h = 0$ on V, due to that fact $h = 0$ on $\partial_p(\Gamma)$.

Since $h_n \to h$ pointwise on U_k for $k \in \mathbb{N}$, it follows $I_p(h_n, U_k) \to I_p(h, U_k) = 0$ for each k. Consequently, $I_p(h_n, V) \to 0$. By taking a subsequence if necessary, we may assume that $\| h_n \|_{D_p} \leq 2^{-n}$. Let $\epsilon > 0$ be given and for $m \in \mathbb{N}$, let $g_m = \sum_{k=1}^m h_k$. There exists $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$. For $m, n \in \mathbb{N}$ with $m > n \geq N$ we see that

$$\| g_m - g_n \|_{D_p} = \| \sum_{k=n+1}^{m} h_k \|_{D_p} \leq \sum_{k=n+1}^{m} 2^{-k} < 2^{-n} < \epsilon.$$

Hence, the Cauchy sequence (g_m) converges to $g = \sum_{k=1}^\infty h_k$ in the D_p-norm. Thus $g \in D_p(\Gamma)$. For $x \in K, g_m(x) = m$, so $g(x) = \infty$; also $g = g_m = 0$ on $\partial_p(\Gamma)$. The proof of the lemma is complete. □

The next result was shown to be true for the case $p = 2$ in [6, Theorem 6.16]. Our proof is essentially the same, and we include it for completeness.

Lemma 5.2. *Let P be a family of one-sided infinite paths in Γ and let*

$$K = \overline{\cup_{\gamma \in P} E(\gamma)}.$$

If K is disjoint from $\partial_p(\Gamma)$, then $\lambda_p(P) = \infty$.

Proof. By Lemma 5.1 there exists a $g \in D_p(\Gamma)$ such that $g = \infty$ on K and $g = 0$ on $\partial_p(\Gamma)$. Let $\gamma \in P$ and let $x_1, x_2, x_3 \ldots$ be the vertex representation of γ. Since $E(\gamma) \subseteq K$ we have that $g(\gamma) = \lim_{k\to\infty} g(x_k) = \infty$. Thus

$$\sum_{k=1}^{\infty} |g(x_k) - g(x_{k+1})| \geq \lim_{k\to\infty} (g(x_k) - g(x_1)) = \infty.$$

By [3, Lemma 2.3] we obtain $\lambda_p(P) = \infty$. □

A connected infinite subset D of V with $\partial D \neq \emptyset$ is defined to be D_p-massive if there exists a p-harmonic function u on D that satisfies the following: $0 \leq u \leq 1, u = 0$ on $\partial D, \sup_D u = 1$ and $I_p(u, D) < \infty$. The function u is known as an inner potential of D.

Proposition 5.3. *Let D be a D_p-massive subset, with inner potential u, of V. Denote by P_D the set of all one-sided infinite paths contained in $D \cup \partial D$. Then $\lambda_p(P_D) < \infty$.*

Proof. Let $a \in D$ and let P_a be the set of all paths in P_D with initial point a. If $\lambda_p(P_a) < \infty$, then $\lambda_p(P_D) < \infty$ by [3, Lemma 2.1]. Let (B_n) be an exhaustion of V by finite connected subsets of V such that $B_1 \cap \partial D \neq \emptyset$. Pick an $a \in B_1 \cap \partial D$. By combining Theorem 2.1 and Theorem 2.4 of [4] we see that $\lambda_p(P_a) < \infty$ if and only if $cap_p(\{a\}, \infty, D) > 0$. Thus to finish the proof we need to show $cap_p(\{a\}, \infty, D) > 0$, which we now proceed to do.

Choose admissible functions $\omega_k, k \geq 2$, for condensers $(\{a\}, (D \cup \partial D) \setminus B_k, D)$ such that

$$I_p(\omega_k, D \cap B_k) \leq cap_p(\{a\}, (D \cup \partial D) \setminus B_k, D) + \frac{1}{k}. \tag{5.1}$$

Replacing all values of $\omega_k(x)$ on $D \cap B_k$ for which $\omega_k(x) < 0$ by 0 and replacing all values of $\omega_k(x)$ on $D \cap B_k$ for which $\omega_k(x) > 1$ by 1 decreases the value of $I_p(\omega_k, D \cap B_k)$. Thus we may and do assume $0 \leq \omega_k \leq 1$ on $D \cap B_k$. Theorem 3.11 of [2] tells us that there exists a unique p-harmonic function v_2 on $D \cap B_2$ such that $v_2 = \omega_2$ on $\partial(D \cap B_2)$. Extend v_2 to all of D by setting $v_2 = 1$ on $D \setminus B_2$. By the minimizing property of p-harmonic functions,

$$I_p(v_2, D \cap B_2) \leq I_p(\omega_2, D \cap B_2).$$

Since u is p-harmonic on D and $u(x) \leq v_2(x)$ for all $x \in \partial(D \cap B_2), u \leq v_2$ on $D \cap B_2$ by [2, Theorem 3.14]. Pick ω_3. The set $A = \{x \in D \mid \omega_3(x) > v_2(x)\}$ is a subset of $D \cap B_2$. If $A \neq \emptyset$, redefine ω_3 by setting $\omega_3 = v_2$ on A. The redefined ω_3 decreases $I_p(\omega_3, D \cap B_3)$, so (5.1) remains true. By continuing as above, we obtain a decreasing sequence of functions (v_k) such that v_k is p-harmonic on $B \cap B_k, v_k \geq u$, and

$$I_p(v_k, D \cap B_k) \leq I_p(\omega_k, D \cap B_k).$$

Now assume that $cap_p(\{a\}, (D \cup \partial D) \setminus B_k, D) \to 0$. Then $I_p(v_k, D \cap B_k) \to 0$. Since $v_k \geq u$ and $\sup_D u = 1$, it must be the case that $(v_k) \to 1_D$, the constant function 1 on D. This is a contradiction because (v_k) is a decreasing sequence of functions, $0 \leq v_2 \leq 1$ and $v_2 \neq 1$. Thus, $cap_p(\{a\}, \infty, D) > 0$ and the proof of the proposition is complete. $\square$

Our next result is [6, Theorem 6.18] for the case $p = 2$. We give a different proof of the result.

Lemma 5.4. *Let P be the family of all one-sided infinite paths in Γ and let $P_\infty \subseteq P$ be any subfamily with $\lambda_p(P_\infty) = \infty$. Then*

$$\partial_p(\Gamma) \subseteq \overline{\{\cup_\gamma E(\gamma) \mid \gamma \in P \setminus P_\infty\}}.$$

Proof. Set $K = \overline{\{\cup_\gamma E(\gamma) \mid \gamma \in P \setminus P_\infty\}}$. Since our standing assumption is that Γ is p-hyperbolic, it follows from [4, theorem 2.1] that $\lambda_p(P) < \infty$. By [3, Lemma 2.2], $\lambda_p(P \setminus P_\infty) < \infty$. Lemma 5.2 tells us $K \cap \partial_p(\Gamma) \neq \emptyset$. For purposes of contradiction, assume that there exists a $\chi \in \partial_p(\Gamma)$ for which $\chi \notin K$. By Urysohn's lemma there exists a continuous function f on $Sp(BD_p(\Gamma))$ that satisfies the following: $0 \leq f \leq 1, f(\chi) = 1$ and $f = 0$ on $K \cap \partial_p(\Gamma)$. By density of $BD_p(\Gamma)$ in $C(Sp(BD_p(\Gamma)))$ we assume $f \in BD_p(\Gamma)$. The p-Royden decomposition for $BD_p(\Gamma)$ yields a unique p-harmonic function h on V and a unique $g \in B(\overline{C_c(\Gamma)}_{D_p})$ such that $f = g + h$. Theorem 4.8 of [5] shows that $g = 0$ on $\partial_p(\Gamma)$. Combining this fact with the maximum principe ([5, Theorem 4.7]) it follows that $0 < h < 1$ on $V, h(\chi) = 1$ and $h = 0$ on $\partial_p(\Gamma) \cap K$. Let

$$A = \{x \in V \mid h(x) > 1 - \epsilon\},$$

where $0 < \epsilon < 1$. Let B be a component of A. The set B is D_p-massive, see the proof of [5, Proposition 4.12] for a proof of this fact. Let P_A be the family of all one-sided infinite paths in A, and let P_B consist of all one-sided infinite paths in B. Since B is a D_p-massive set, $\lambda_p(P_B) < \infty$ by Proposition 5.3. It now follows from [3, Lemma 2.1] that $\lambda_p(P_A) < \infty$. Set

$$K_1 = \overline{\{\cup_\gamma E(\gamma) \mid \gamma \in P_A \setminus P_\infty\}}.$$

Another appeal to Lemma 5.2 shows $K_1 \cap \partial_p(\Gamma) \neq \emptyset$, because $\lambda_p(P_A \setminus P_\infty) < \infty$. Furthermore, $h = 0$ on $K_1 \cap \partial_p(\Gamma)$ since $K_1 \cap \partial_p(\Gamma) \subseteq K \cap \partial_p(\Gamma)$. However, $h(\gamma) \geq$

$1-\epsilon$ for all $\gamma \in P_A$. Thus we obtain the contradiction $h(x) \geq 1-\epsilon$ for all $x \in K_1$. Therefore, $\partial_p(\Gamma) \subseteq K$, as desired. □

Proof of Theorem 3.2. Let $f \in B(\overline{C_c(\Gamma)}_{D_p})$ and let $a \in V$. Denote by P_a the set of all one-sided infinite paths in Γ with initial point a. Set

$$A_{a,f} = \{\gamma \in P_a \mid f(\gamma) \neq 0\}.$$

By [3, Theorem 3.3], $\lambda_p(A_{a,f}) = \infty$. Also, [3, Lemma 2.2] tells us $\lambda_p(A_f) = \lambda_p(\cup_{a\in V} A_{a,f}) = \infty$. The definition of A_f above and E_f below were given in Section 3. Now Proposition 5.4 says that

$$\partial_p(\Gamma) \subseteq E_f.$$

For notational convenience set $F = \cap_f E_f$, where f runs through $B(\overline{C_c(\Gamma)}_{D_p})$. Thus, $\partial_p(\Gamma) \subseteq F$. We now proceed to prove the reverse inclusion. Suppose there exists a $\chi \in F$ for which $\chi \notin \partial_p(\Gamma)$. By [5, Theorem 4.8] we obtain an $f \in B(\overline{C_c(\Gamma)}_{D_p})$ for which $\chi(f) \neq 0$. Let $\alpha \sim x_0, x_1, \ldots, x_n, \ldots$ be a one-sided path with $\chi \in \overline{V}(\alpha)$. Because $\chi(f) \neq 0$, there is a subsequence (x_{n_k}) of (x_n) that satisfies $\lim_{k\to\infty} f(x_{n_k}) \neq 0$. Thus $f(\alpha) \neq 0$ and has a result $\alpha \in A_f$. Hence $\chi \notin \{\cup_\gamma E(\gamma) \mid \gamma \in P \setminus A_f\}$. We are assuming $\chi \in E_f$, so it must be the case that there is a sequence (χ_n) in $\{\cup_\gamma E(\gamma) \mid \gamma \in P \setminus A_f\}$ with $(\chi_n) \to \chi$. Since $f(\gamma) = 0$ for each $\gamma \in P \setminus A_f$ it follows immediately that $\chi_n(f) = 0$ for each $n \in \mathbb{N}$. This implies $\chi(f) = 0$, contradicting our assumption $\chi(f) \neq 0$. Therefore, $F \subseteq \partial_p(\Gamma)$. The proof of Theorem 3.2 is now complete.

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