Generalized covariation for Banach valued processes and Itô formula

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Abstract

This paper concerns the notion of quadratic variation and covariation for Banach valued processes and related Itô formula. If $\mathbb{X}$ and $\mathbb{Y}$ take respectively values in Banach spaces B_1 and B_2 (denoted by $(B_1\hat{\otimes}_\pi B_2)^*$) and χ is a suitable subspace of the dual of the projective tensor product of B_1 and B_2 we define the so-called χ-covariation of $\mathbb{X}$ and $\mathbb{Y}$. If $\mathbb{X} = \mathbb{Y}$ the χ-covariation is called χ-quadratic variation. The notion of χ-quadratic variation is a natural generalization of the one introduced by Métivier-Pellaumail and Dinculeanu which is too restrictive for many applications. In particular, if χ is the whole space $(B_1\hat{\otimes}_\pi B_1)^*$ then the χ-quadratic variation coincides with the quadratic variation of a B_1-valued semimartingale. We evaluate the χ-covariation of various processes for several examples of χ with a particular attention to the case $B_1 = B_2 = C([-\tau, 0])$ for some $\tau > 0$ and $\mathbb{X}$ and $\mathbb{Y}$ being *window processes*. If X is a real process, we call window process associated with X the $C([-\tau, 0])$-valued process $\mathbb{X} := X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$.

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1 Introduction

The present paper settles the basis for calculus via regularization for processes with values in an infinite dimensional separable Banach space B. The extension of Itô's stochastic integration theory for Hilbert valued processes dates only from the eighties, the results of which can be found in the monographies [18, 19, 5] and [31] with different techniques. However the discussion of this last approach is not the aim of this paper. Extension to nuclear valued spaces is simpler and was done in [15, 30]. One of the most natural but difficult situations arises when the processes are Banach valued.
As for the real case, a possible tool of infinite dimensional stochastic calculus is the concept of quadratic variation, or more generally of covariation. The notion of covariation is historically defined for two real valued $(\mathcal{F}_t)$-semimartingales X and Y and it is denoted by $[X,Y]$. This notion was extended to the case of general processes by mean of discretization techniques, for instance by [12], or via regularization, for instance in [26, 28]. In this paper we will follow the language of regularization as in [28]; for simplicity we suppose that either X or Y is continuous. In the whole paper T will be a fixed positive number.

Definition 1.1. Let X and Y be two real processes such that X is continuous and Y has almost surely locally integrable paths. For $\epsilon > 0$, we denote

$$[X,Y]^{\epsilon}_t = \int_0^t \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds , \quad t > 0 , \tag{1.1}$$

$$I^{-}(\epsilon, Y, dX)_t = \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds , \quad t > 0 . \tag{1.2}$$

1) We say that X and Y admit a **covariation** (denoted by $[X,Y]$) if $\lim_{\epsilon\to 0}[X,Y]^{\epsilon}_t$ exists in the ucp (**uniform convergence in probability** on each compact) sense with respect to $t > 0$. If $[X,X]$ exists, we say that X has a quadratic variation and it will also be denoted by $[X]$. If $[X] = 0$ we say that X is a zero quadratic variation process.

2) The **forward integral** $\int_0^t Y_s d^- X_s$ is a continuous process Z, such that whenever it exists, $\lim_{\epsilon\to 0} I^{-}(\epsilon, Y, dX)_t = Z_t$ in probability for every $t > 0$.

Let $(\mathcal{F}_t)$ be a usual filtration. If X is an $(\mathcal{F}_t)$-semimartingale and Y is $(\mathcal{F}_t)$-progressively measurable and cadlag (resp. an $(\mathcal{F}_t)$-semimartingale) $\int_0^{\cdot} Y_s d^- X_s$ (resp. $[X,Y]$) coincides with the classical Itô's integral $\int_0^{\cdot} Y dX$ (resp. the classical covariation). The class of real finite quadratic variation processes is much richer than the one of semimartingales. Typical examples of finite quadratic variation processes are $(\mathcal{F}_t)$-Dirichlet processes. D is called $(\mathcal{F}_t)$-**Dirichlet process** if it admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process. In that case it holds $[D] = [M]$. This class of processes generalizes the semimartingales since a locally bounded variation process has zero quadratic variation. A slight generalization of that notion is the one of weak Dirichlet process, which was

introduced in [11]. X is called $(\mathcal{F}_t)$-**weak Dirichlet** if it admits a decomposition $X = M + A$ where M is an $(\mathcal{F}_t)$ local martingale and A is a process such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N. A process A with that property is called a $(\mathcal{F}_t)$**-martingale orthogonal process**. An $(\mathcal{F}_t)$-weak Dirichlet process is not necessarily a finite quadratic variation process. On the other hand if A has finite quadratic variation then it holds $[X] = [M] + [A]$. Another interesting example is the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0,1)$ and $K \in (0,1]$ which has finite quadratic variation if and only if $HK \geq 1/2$, see [24]. Notice that if $K = 1$, then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. If $HK = 1/2$ it holds $[B^{H,K}] = 2^{1-K}t$; if $K \neq 1$ this process is not even Dirichlet with respect to its own filtration. Other significant examples are the so-called weak k-order Brownian motions, for fixed $k \geq 1$, constructed by [13], which are in general not Gaussian. X is a weak k-order Brownian motion if for every $0 \leq t_1 \leq \cdots \leq t_k < +\infty$, $(X_{t_1}, \cdots, X_{t_k})$ is distributed as $(W_{t_1}, \cdots, W_{t_k})$. If $k \geq 4$ then $[X]_t = t$. Among Gaussian processes, the processes admitting a *covariance measure structure* have also finite (deterministic) quadratic variation, see [16].
One object of this paper consists in investigating a possible useful generalization of the notions of covariation and quadratic variation of Banach valued processes. Applications are given for instance in [6] and in a paper in preparation [8]. Particular emphasis will be devoted to *window processes* with values in the Banach space of real continuous functions defined on $[-\tau, 0]$. Given $0 < \tau \leq T$ and a real continuous process $X = (X_t)_{t\in[0,T]}$ one can link to it a natural infinite dimensional valued process defined as follows.

Definition 1.2. We call **window process** associated with X, denoted by $X(\cdot)$, the $C([-\tau, 0])$-valued process

$$X(\cdot) = \big(X_t(\cdot)\big)_{t\in[0,T]} = \{X_t(u) := X_{t+u}; u \in [-\tau, 0], t \in [0,T]\} .$$

In the present paper, W will always denote a real standard Brownian motion. The window process $W(\cdot)$ associated with $X = W$ will be called **window Brownian motion**.

Those processes naturally appear in functional dependent stochastic differential equations as delay equations. We emphasize that $C([-\tau, 0])$ is typical a non-reflexive Banach space. So we will introduce a notion of covariation for processes with values in general Banach spaces but which will be performing also for window processes.

Let B_1, B_2 be two general Banach spaces. In this paper $\mathbb{X}$ (resp. $\mathbb{Y}$) will be a B_1 (resp. B_2) valued stochastic process. It is not obvious to define an exploitable notion of covariation (resp. quadratic variation) of $\mathbb{X}$ and $\mathbb{Y}$ (resp. of $\mathbb{X}$). When $\mathbb{X}$ is an H-valued martingale and $B_1 = B_2 = H$ is a separable Hilbert space, [5] introduces an operational notion of quadratic variation for martingales with values in H. This is implemented in the theory of stochastic partial differential equations. [9] introduces in Definitions A.1 in Chapter 2.15 and B.9 in Chapter 6.23 the appealing notions of *semilocally summable* and *locally summable processes with respect to a given bilinear mapping on* $B \times B$. See also Definition C.8 in Chapter 2.9 for the definition of *summable* process. Similar notions appears in [20]. Those processes are very close to Banach valued semimartingales. If B is a Hilbert space, a semimartingale is semilocally summable when the bilinear form is the inner product. For previous processes [9] defines two natural notions of quadratic variation: the real quadratic variation and the tensor quadratic variation. Even though [20, 9] make use of discretizations, we define here, for commodity, two very similar objects but in our regularization language. Moreover, the notion below extends to the covariation of two processes $\mathbb{X}$ and $\mathbb{Y}$ for which we remove the assumption of *semilocally summable* or *locally summable*.

Definition 1.3. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1 (resp. B_2) valued stochastic process.

1. $\mathbb{X}$ and $\mathbb{Y}$ are said to admit a **real covariation** if the limit for $\epsilon \downarrow 0$ of the sequence

$$[\mathbb{X},\mathbb{Y}]^{\mathbb{R},\epsilon}_{\cdot} = \int_0^{\cdot} \frac{\|\mathbb{X}_{s+\epsilon}-\mathbb{X}_s\|_{B_1}\|\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s\|_{B_2}}{\epsilon} ds \tag{1.3}$$

exists ucp. That limit will be called indeed real covariation of $\mathbb{X}$ and $\mathbb{Y}$ and it will be simply denoted by $[\mathbb{X},\mathbb{Y}]^{\mathbb{R}}$. The real covariation $[\mathbb{X},\mathbb{X}]^{\mathbb{R}}$ will be called **real quadratic variation** of $\mathbb{X}$ and simply denoted by $[\mathbb{X}]^{\mathbb{R}}$.

2. $\mathbb{X}$ and $\mathbb{Y}$ admit a **tensor covariation** if there exists a $(B_1\hat{\otimes}_\pi B_2)$-valued process denoted by $[\mathbb{X},\mathbb{Y}]^{\otimes}$ such that the sequence of Bochner $(B_1\hat{\otimes}_\pi B_2)$-valued integrals

$$[\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_{\cdot} = \int_0^{\cdot} \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)}{\epsilon} ds \tag{1.4}$$

converges (according to the strong topology) to $[\mathbb{X},\mathbb{Y}]^{\otimes}$ ucp for $\epsilon \downarrow 0$.
$[\mathbb{X},\mathbb{Y}]^{\otimes}$ will be indeed called tensor covariation of $\mathbb{X}$ and $\mathbb{Y}$. The tensor covariation $[\mathbb{X},\mathbb{X}]^{\otimes}$ will be called **tensor quadratic variation** and simply denoted by $[\mathbb{X}]^{\otimes}$.

Remark 1.4.

1. According to Lemma 2.1, if $[\mathbb{X},\mathbb{Y}]^{\mathbb{R},\epsilon}_{\cdot}$ converges for any $t \geq 0$ to Z_t, where Z is a continuous process, then the real covariation exists and $[\mathbb{X},\mathbb{Y}]^{\mathbb{R}} = Z$.

2. If $\mathbb{X}$ and $\mathbb{Y}$ admit both real and tensor covariation, the tensor covariation process has bounded variation and its total variation is bounded by the real covariation which is clearly an increasing process.

3. If $\mathbb{X}$ and $\mathbb{Y}$ admit a tensor covariation we have in particular

$$\frac{1}{\epsilon}\int_0^t \langle \phi, (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle\, ds \xrightarrow[\epsilon\longrightarrow 0]{ucp} \langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes}\rangle \tag{1.5}$$

for every $\phi \in (B_1\hat{\otimes}_\pi B_2)^*$.

4. If $\mathbb{X}$ and $\mathbb{Y}$ are such that $[\mathbb{X},\mathbb{Y}]^{\mathbb{R}} = 0$, then $\mathbb{X}$ and $\mathbb{Y}$ admits a tensor covariation which also vanishes.

A sketch of the proof of the two propositions below are given in the appendix.

Proposition 1.5. Let $\mathbb{X}$ be an $(\mathcal{F}_t)$-adapted semilocally summable process with respect to the bilinear forms $B \times B \longrightarrow B\hat{\otimes}_\pi B$, $(a, b) \mapsto a \otimes b$ and $(a, b) \mapsto b \otimes a$. Then $\mathbb{X}$ admits a tensor quadratic variation.

Proposition 1.6. Let $\mathbb{X}$ be an Hilbert valued continuous $(\mathcal{F}_t)$-semimartingale in the sense of [20], section 10.8. Then $\mathbb{X}$ admits a real quadratic variation.

Corollary 1.7. If $\mathbb{X}$ is a Banach valued process admitting a real quadratic variation being semilocally summable with respect to the tensor products, then $\mathbb{X}$ admits a tensor quadratic variation process which has bounded variation.

Remark 1.8. The tensor quadratic variation can be linked to the one of [5]; see chapter 6 in [7] for details. Let H be a separable Hilbert space. If $\mathbb{V}$ is an H-valued Q-Brownian motion with $Tr(Q) < +\infty$ (see [5] section 4), then $\mathbb{V}$ admits a real quadratic variation $[\mathbb{V}]^{\mathbb{R}}_t = t\,Tr(Q)$ and a tensor quadratic variation $[\mathbb{V}]^{\otimes}_t = tq$ where q is the tensor associated to the nuclear operator tQ.

Unfortunately, even the window process $W(\cdot)$ associated with a real Brownian motion W, does not admit a real quadratic variation. In fact the limit of

$$\int_0^t \frac{\|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|^2_{C([-\tau,0])}}{\epsilon}\, ds\ , \quad t \geq 0\ , \tag{1.6}$$

for ϵ going to zero does not converge, as we will see in Proposition 4.7. This suggests that when $\mathbb{X}$ is a window process, the tensor quadratic variation is not the suitable object in order to perform stochastic calculus. On the other hand in Proposition 4.5, we remark that $W(\cdot)$ is not a $C([-\tau, 0])$-valued semimartingale.
Let $\mathbb{X}$ (resp. $\mathbb{Y}$) a B_1 (resp. B_2)-valued process. In Definition 3.9 we will introduce a notion of covariation of $\mathbb{X}$ and $\mathbb{Y}$ (resp. quadratic variation when $\mathbb{X} = \mathbb{Y}$) which generalizes the tensor covariation (resp. tensor quadratic variation). This will be called χ-covariation (resp. χ-quadratic variation) in reference to a topological subspace χ of the dual of $B_1\hat{\otimes}_\pi B_2$ (resp. when $B_1 = B_2$). According to our strategy, we will suppose that

$$\frac{1}{\epsilon}\int_0^t \langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\rangle\, ds \tag{1.7}$$

converges for every $\phi \in \chi$. When χ is separable and it coincides with the whole space $(B_1\hat{\otimes}_\pi B_2)^*$, this convergence is strongly related to the weak star topology in $(B_1\hat{\otimes}_\pi B_2)^{**}$.
Our χ-covariation generalizes the concept of tensor covariation at two levels.

- First we replace the (strong) convergence of (1.4) with a weak type topology convergence of (1.7).

- Secondly the choice of a suitable subspace χ of $(B_1 \hat{\otimes}_\pi B_2)^*$ gives a degree of freedom. For instance, compatibly with (1.6), a window Brownian motion $\mathbb{X} = W(\cdot)$ admits a χ- quadratic variation only for strict subspaces χ.

When χ equals the whole space $(B_1 \hat{\otimes}_\pi B_2)^*$ (resp. $(B \hat{\otimes}_\pi B)^*$) this will be called *global covariation* (resp. *global quadratic variation*). This situation corresponds for us to the *elementary situation.*
When B_1 and B_2 are the finite dimensional space $\mathbb{R}^n$ and $\mathbb{R}^m$, Corollary 3.29 says that $(\mathbb{X}, \mathbb{Y})$ admits all its mutual brackets if and only if $\mathbb{X}$ and $\mathbb{Y}$ have a global covariation. It is well known that, in that case, $(B_1 \hat{\otimes}_\pi B_2)^*$ can be identified with the space of matrix $\mathbb{M}_{n\times m}(\mathbb{R})$. If χ is finite dimensional, then Proposition 3.28 gives a simple characterization for $\mathbb{X}$ to have a χ-quadratic variation.
Propositions 1.5, 1.6, 3.16 and Remark 1.8 will essentially imply that whenever $\mathbb{X}$ admits one of the classical quadratic variations (in the sense of [5, 20, 9]), it admits a global quadratic variation and they are essentially equal. In this paper we calculate the χ-covariation of Banach valued processes in various situations with a particular attention for window processes associated to real finite quadratic variation processes (for instance semimartingales, Dirichlet processes, bifractional Brownian motion).

The notion of covariation intervenes in Banach valued stochastic calculus for semimartingales, especially via Itô's type formula, see for [9] and [20]. An important result of this paper is an Itô's formula for Banach valued processes admitting a χ-quadratic variation, see Theorem 5.2. This generalizes the corresponding formula for real valued processes which is stated below, see [26]. Let X be a real finite quadratic variation process and $f \in C^{1,2}([0,T] \times \mathbb{R})$ than the forward integral $\int_0^t \partial_x f(s, X_s) d^- X_s$ exists and

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) d^- X_s + \frac{1}{2} \int_0^t \partial^2_{xx} f(s, X_x) d[X]_s \; . \tag{1.8}$$

[12] gives a similar formula in the discretization approach to pathwise stochastic integration.
For that purpose, given $\mathbb{Y}$ (resp. $\mathbb{X}$) a B^*-valued strongly measurable with locally bounded paths (resp. B-valued continuous) process, we define a real valued forward-type integral $\int_0^t {}_{B^*}\langle Y, d^- X \rangle_B$, see Definition 5.1. We remark that Theorem 5.2 constitutes a generalization of the Itô formula in [20], section 3.7, (see also [9]) for two reasons: first of all the integrator process $\mathbb{X}$ is not necessarily in the class considered by those authors and moreover the space χ corresponding to their case would be the full space $\chi = (B \hat{\otimes}_\pi B)^*$.

The paper is organised as follows. After this introduction, Section 2 contains general notations and some preliminary results. Section 3 will be devoted to the definition of χ-covariation and χ-quadratic variation and some related results. In Section 4, we will give examples of evaluation of χ-covariation variation for different classes of processes. Section 5 is devoted to the definition of a forward integral for Banach valued processes and related Itô's formula. The final section gives some illustrating examples of Itô's formula.

2 Preliminaries

In this section we recall some definitions and notations concerning the whole paper. Let A and C be two general sets such that $A \subset C$; $\mathbb{1}_A : C \to \{0,1\}$ will denote the indicator function of the set A, so $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. We also write $\mathbb{1}_A(x) = \mathbb{1}_{\{x \in A\}}$. Throughout this paper we will denote by $(\Omega, \mathbb{F}, \mathbb{P})$ a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ fulfilling the usual conditions. Let K be a compact space; $C(K)$ denotes the linear space of real continuous functions defined on K, equipped with the uniform norm denoted by $\|\cdot\|_\infty$. $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the set of finite signed Borel measures on K. In particular, if $a < b$ be two real numbers, $C([a,b])$ will denote the Banach linear space of real continuous functions. If E is a topological space, $\mathcal{B}or(E)$ will denote its Borel σ-algebra. The letters B, B_1, B_2 (respectively H) will denote a separable Banach (respectively Hilbert) space over the scalar field $\mathbb{R}$. Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E, we say that $\|\cdot\|_1 \leq \|\cdot\|_2$ if for every $x \in E$ there is a positive constant c such that $\|x\|_1 \leq c\,\|x\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist positive real numbers c and C such that $c\,\|x\|_2 \leq \|x\|_1 \leq C\,\|x\|_2$ for all $x \in E$. In particular they define the same topology.

The topological dual space of B will be denoted by B^*. If ϕ is a linear functional on B, we shall denote the value of ϕ at an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b \rangle$ or even ${}_{B^*}\langle \phi, b \rangle_B$. Throughout the paper the symbols $\langle \cdot, \cdot \rangle$ will denote always some type of duality that will change depending on the context. Let B be a normed space; a sequence of B^*-valued elements $(\phi^n)_{n \in \mathbb{N}}$ converges **weak star** to $\phi \in B^*$, i.e. in symbols $\phi^n \xrightarrow[n \longrightarrow +\infty]{w^*} \phi$, if ${}_{B^*}\langle \phi^n, b \rangle_B \xrightarrow[n \longrightarrow +\infty]{} {}_{B^*}\langle \phi, b \rangle_B$ for every $b \in B$. Given a Banach space B and its topological bidual space B^{**} the application $J : B \to B^{**}$ will denote the natural injection between a Banach space and its bidual. J is an injective linear mapping, though it is not surjective unless B is reflexive. J is an isometry with respect to the topology defined by the norm in B, the so-called *strong topology*, and $J(B)$ which is weak star dense in B^{**}. The weak star topology on B^* is the weak topology induced by the image $J(B) \subset B^{**}$ of J. By definition, the weak star topology is weaker than the weak topology on B^*. Let E, F, G be Banach spaces. $L(E;F)$ stands for the Banach space of linear bounded maps from E to F. We shall denote the space of $\mathbb{R}$-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E \times F)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{|\phi(e,f)| : \|e\|_E \leq 1; \|f\|_F \leq 1\}$. Our principal references about functional analysis and about Banach spaces topologies are [10, 1].

The capital letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ (resp. X, Y, Z) will generally denote Banach valued (resp. real valued) processes indexed by the time variable $t \in [0,T]$ with $T > 0$ (or $t \in \mathbb{R}_+$). A stochastic process $\mathbb{X}$ will also be denoted by $(\mathbb{X}_t)_{t \in [0,T]}$ or $(\mathbb{X}_t)_{t \geq 0}$. A B-valued (resp. $\mathbb{R}$-valued) stochastic process $\mathbb{X} : \Omega \times [0,T] \to B$ (resp. $\mathbb{X} : \Omega \times [0,T] \to \mathbb{R}$) is said to be measurable if $\mathbb{X} : \Omega \times [0,T] \longrightarrow B$ (resp. $\mathbb{X} : \Omega \times [0,T] \to \mathbb{R}$) is measurable with respect to the σ-algebras $\mathcal{F} \otimes \mathcal{B}or([0,T])$ and $\mathcal{B}or(B)$ (resp. $\mathcal{B}or(\mathbb{R})$). We recall that $\mathbb{X} : \Omega \times [0,T] \longrightarrow B$ (resp. $\mathbb{R}$) is said to be **strongly measurable** (or **measurable in the Bochner sense**) if it is the limit of measurable countable valued functions. If $\mathbb{X}$ is measurable and cadlag with B

separable then $\mathbb{X}$ is strongly measurable. If B is finite dimensional then a measurable process $\mathbb{X}$ is also strongly measurable. All the processes indexed by $[0,T]$ (respectively $\mathbb{R}^+$) will be naturally prolongated by continuity setting $\mathbb{X}_t = \mathbb{X}_0$ for $t \le 0$ and $\mathbb{X}_t = \mathbb{X}_T$ for $t \ge T$ (respectively $\mathbb{X}_t = \mathbb{X}_0$ for $t \le 0$). A sequence $(\mathbb{X}^n)_{n\in\mathbb{N}}$ of continuous B-valued processes indexed by $[0,T]$, will be said to converge **ucp** (**uniformly convergence in probability**) to a process $\mathbb{X}$ if $\sup_{0\le t\le T} \|\mathbb{X}^n - \mathbb{X}\|_B$ converges to zero in probability when $n \to \infty$. The space $\mathscr{C}([0,T])$ will denote the linear space of continuous real processes; it is a Fréchet space (or F-space shortly) if equipped with the metric $d(X,Y) = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right]$ which governs the ucp topology, see Definition II.1.10 in [10]. For more details about F-spaces and their properties see section II.1 in [10].
We recall Lemma 3.1 from [27] which constitutes a stochastic version of Dini's lemma.

Lemma 2.1. Let $(Z^\epsilon)_{\epsilon>0}$ be a family of continuous real processes. We suppose the following.

1) $\forall \epsilon > 0$, $t \longrightarrow Z_t^\epsilon$ is increasing.

2) There is a continuous process $(Z_t)_{t\in[0,T]}$ such that $Z_t^\epsilon \to Z_t$ in probability for any $t \in [0,T]$ when ϵ goes to zero.

Then Z^ε converges to Z ucp.

We go on with other notations.
The direct sum of two Banach spaces E_1 and E_2 will be denoted by $E := E_1 \oplus E_2$ and it is still a Banach space. If E_1 and E_2 are Hilbert spaces then E is a Hilbert space with scalar product given by $\langle e_1 + e_2, f_1 + f_2\rangle_E = \sum_{i=1}^2 \langle e_i, f_i\rangle_i$, where $\langle\cdot,\cdot\rangle_i$ is the scalar product in E_i. We observe that E_1 and E_2 are closed normed subspaces of E and it holds $\overline{Span\{E_1, E_2\}} = E_1 \oplus E_2$.
We recall now some basic concepts and results about tensor products of two Banach spaces E and F. For details and a more complete description of these arguments, the reader may refer to [29], the material when E and F are Hilbert spaces being particularly exhaustive in [21]. If E and F are Banach spaces, $E\hat{\otimes}_\pi F$ (resp. $E\hat{\otimes}_h F$) is a Banach space which denotes the **projective (resp. Hilbert) tensor product** of E and F. We recall that $E\hat{\otimes}_\pi F$ (resp. $E\hat{\otimes}_h F$) is obtained by a completion of the algebraic tensor product $E\otimes F$ equipped with the projective norm π (resp. Hilbert norm h). For a general element $u = \sum_{i=1}^n e_i \otimes f_i$ in $E \otimes F$, $e_i \in E$ and $f_i \in F$, it holds $\pi(u) = \inf\{\sum_{i=1}^n \|e_i\|_E \|f_i\|_F : u = \sum_{i=1}^n e_i \otimes f_i, e_i \in E, f_i \in F\}$. For the definition of the Hilbert tensor norm h the reader may refer [29], Chapter 7.4. We remind that if E and F are Hilbert spaces the Hilbert tensor product $E\hat{\otimes}_h F$ is also Hilbert and its inner product between $e_1 \otimes f_1$ and $e_2 \otimes f_2$ equals $\langle e_1, e_2\rangle_E \cdot \langle f_1, f_2\rangle_F$. Let $e \in E$ and $f \in F$, symbol $e \otimes f$ (resp. $e\otimes^2$) will denote an elementary element of the algebraic tensor product $E \otimes F$ (resp. $E \otimes E$). The Banach space $(E\hat{\otimes}_\pi F)^*$ denotes, as usual, the topological dual of the projective tensor product equipped with the operator norm. If $\tilde{T} : E\times F \to \mathbb{R}$ is a continuous bilinear form, there exists a unique bounded linear operator $T : E\hat{\otimes}F \to \mathbb{R}$ satisfying ${}_{(E\hat{\otimes}_\pi F)^*}\langle T, e\otimes f\rangle_{E\hat{\otimes}_\pi F} = T(e\otimes f) = \tilde{T}(e,f)$ for every $e \in E$, $f \in F$. We observe moreover that there exists a canonical identification between $\mathcal{B}(E\times F)$ and $L(E; F^*)$ which identifies $\tilde{T}$ with $\bar{T} : E \to F^*$

by $\tilde{T}(e,f) = \bar{T}(e)(f)$. Summarizing, there is an isometric isomorphism between the dual space of the projective tensor product and the space of bounded bilinear forms equipped with the usual norm, i.e.

$$(E\hat{\otimes}_\pi F)^* \cong \mathcal{B}(E\times F) \cong L(E;F^*)\ . \tag{2.1}$$

With this identification, the action of a bounded bilinear form T as a bounded linear functional on $E\hat{\otimes}_\pi F$ is given by

$$_{(E\hat{\otimes}_\pi F)^*}\langle T, \sum_{i=1}^n x_i\otimes y_i\rangle_{E\hat{\otimes}_\pi F} = T\left(\sum_{i=1}^n x_i\otimes y_i\right) = \sum_{i=1}^n \tilde{T}(x_i,y_i) = \sum_{i=1}^n \bar{T}(x_i)(y_i). \tag{2.2}$$

In the sequel that identification will often be used without explicit mention.
The importance of tensor product spaces and their duals is justified first of all by identification (2.1): indeed the second order Fréchet derivative of a real function defined on a Banach space E belongs to $\mathcal{B}(E\times E)$.

We state a useful result involving Hilbert tensor products and Hilbert direct sums.

Proposition 2.2. Let X and Y_1, Y_2 be Hilbert spaces such that $Y_1\cap Y_2=\{0\}$. We consider $Y=Y_1\oplus Y_2$ equipped with the Hilbert direct norm. Then $X\hat{\otimes}_h Y = (X\hat{\otimes}_h Y_1)\oplus(X\hat{\otimes}_h Y_2)$.

Proof. Since $X\otimes Y_i\subset X\otimes Y$, $i=1,2$ we can write $X\otimes_h Y_i\subset X\otimes_h Y$ and so

$$(X\hat{\otimes}_h Y_1)\oplus(X\hat{\otimes}_h Y_2)\subset X\hat{\otimes}_h Y \tag{2.3}$$

Since we handle with Hilbert norms, it is easy to show that the norm topology of $X\hat{\otimes}_h Y_1$ and $X\hat{\otimes}_h Y_2$ is the same that the one induced by $X\hat{\otimes}_h Y$.
It remains to show the converse inclusion of (2.3). This follows because $X\otimes Y\subset X\hat{\otimes}_h Y_1\oplus X\hat{\otimes}_h Y_1$. □

We recall another important property.

$$\mathcal{M}([-\tau,0]^2) = \big(C([-\tau,0]^2)\big)^* \subset \big(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0])\big)^* \cong \mathcal{B}(C([-\tau,0])\times C([-\tau,0]))\,. \tag{2.4}$$

With every $\mu\in\mathcal{M}([-\tau,0]^2)$ we can associate an operator $T^\mu\in\mathcal{B}(C([-\tau,0])\times C([-\tau,0]))$ defined by $T^\mu(f,g)=\int_{[-\tau,0]^2} f(x)g(y)\mu(dx,dy)$.
Let η_1, η_2 be two elements in $C([-\tau,0])$. The element $\eta_1\otimes\eta_2$ in the algebraic tensor product $C([-\tau,0])\otimes^2$ will be identified with the element η in $C([-\tau,0]^2)$ defined by $\eta(x,y)=\eta_1(x)\eta_2(y)$ for all x, y in $[-\tau,0]$. So if μ is a measure on $\mathcal{M}([-\tau,0]^2)$, the pairing duality $_{\mathcal{M}([-\tau,0]^2)}\langle\mu,\eta_1\otimes\eta_2\rangle_{C([-\tau,0]^2)}$ has to be understood as the following pairing duality:

$$_{\mathcal{M}([-\tau,0]^2)}\langle\mu,\eta\rangle_{C([-\tau,0]^2)} = \int_{[-\tau,0]^2}\eta(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2}\eta_1(x)\eta_2(y)\mu(dx,dy)\ . \tag{2.5}$$

Along the paper, the spaces $\mathcal{M}([-\tau,0])$ and $\mathcal{M}([-\tau,0]^2)$ and their subsets will play a central role. We will introduce some other notations that will be used in the sequel. Let $-\tau=a_N<a_{N-1}<\dots a_1<a_0=0$

be $N+1$ fixed points in $[-\tau, 0]$. Symbols a and A will refer respectively to the vector $(a_N, a_{N-1}, \dots, a_1, 0)$ and to the matrix $(A_{i,j})_{0 \le i,j \le N} = ((a_i, a_j))_{0 \le i,j \le N}$. Vector a identifies $N+1$ points on $[-\tau, 0]$ and matrix A identifies $(N+1)^2$ points on $[-\tau, 0]^2$.

- Symbol $\mathcal{D}_i([-\tau, 0])$ (shortly $\mathcal{D}_i$), will denote the one-dimensional space of multiples of Dirac's measure concentrated at $a_i \in [-\tau, 0]$, i.e.

$$\mathcal{D}_i([-\tau, 0]) := \{\mu \in \mathcal{M}([-\tau, 0]);\ s.t. \mu(dx) = \lambda\, \delta_{a_i}(dx) \text{ with } \lambda \in \mathbb{R}\}\ ; \tag{2.6}$$

  the space $\mathcal{D}_0$ (resp. $\mathcal{D}_{-\tau}$) will be the space of multiples of Dirac measure concentrated at 0 (resp. at $-\tau$).

- Symbol $\mathcal{D}_{i,j}([-\tau, 0]^2)$ (shortly $\mathcal{D}_{i,j}$), will denote the one-dimensional space of the multiples of Dirac measure concentrated at $(a_i, a_j) \in [-\tau, 0]^2$, i.e.

$$\mathcal{D}_{i,j}([-\tau, 0]^2) := \{\mu \in \mathcal{M}([-\tau, 0]^2);\ s.t. \mu(dx, dy) = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_i \hat{\otimes}_h \mathcal{D}_j\ . \tag{2.7}$$

  The space $\mathcal{D}_{0,0}$ (resp. $\mathcal{D}_{-\tau,-\tau}$) will be the space of Dirac's measures concentrated at $(0, 0)$ (resp. $(-\tau, -\tau)$).

- Symbol $\mathcal{D}_a([-\tau, 0])$ (shortly $\mathcal{D}_a$), will denote the $(N+1)$- dimensional space of linear combinations of Dirac measures concentrated at $(N+1)$ fixed points in $[-\tau, 0]$ identified by a.

$$\mathcal{D}_a([-\tau, 0]) := \{\mu \in \mathcal{M}([-\tau, 0])\, s.t.\ \mu(dx) = \sum_{i=0}^{N} \lambda_i \delta_{a_i}(dx);\ \lambda_i \in \mathbb{R},\ i = 0, \dots, N\} = \bigoplus_{i=0}^{N} \mathcal{D}_i\ . \tag{2.8}$$

- Symbol $\mathcal{D}_A([-\tau, 0]^2)$ (shortly $\mathcal{D}_A$), will denote the $(N+1)^2$-dimensional space of linear combination of Dirac measures concentrated at points $(a_i, a_j)_{0 \le i,j \le N}$ in $[-\tau, 0]^2$, i.e.

$$\mathcal{D}_A([-\tau, 0]^2) := \{\mu \in \mathcal{M}([-\tau, 0]^2);\ s.t. \mu(dx, dy) = \lambda_{i,j}\, \delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda_{i,j} \in \mathbb{R},\ i, j = 0, \dots, N\}\ . \tag{2.9}$$

Remark 2.3. There are natural identifications $\mathcal{D}_i \cong \mathcal{D}_{i,j} \cong \mathbb{R}$, $\mathcal{D}_a \cong \mathbb{R}^{N+1}$ and $\mathcal{D}_A \cong \mathbb{M}_{(N+1)\times(N+1)}(\mathbb{R}) \cong \mathbb{R}^{N+1} \otimes \mathbb{R}^{N+1}$. All those spaces are finite dimensional separable Hilbert spaces which are subspaces of the Banach space $\mathcal{M}([-\tau, 0])$ or $\mathcal{M}([-\tau, 0]^2)$.

We give some examples of infinite dimensional subspaces of measures intervening in the sequel.

- $L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$, as well as $L^2([-\tau,0]^2) \cong L^2([-\tau,0])\hat{\otimes}_h^2$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$, both equipped with the norm derived from the usual scalar product. The Hilbert tensor product $L^2([-\tau,0])\hat{\otimes}_h^2$ will be always identified with $L^2([-\tau,0]^2)$, conformally to a quite canonical procedure, see [21], chapter 6.

- $\mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$ for $i \in \{0,\ldots,N\}$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$. This is a direct sum in the space of measures $\mathcal{M}([-\tau,0])$. In fact a measure $\mu \in \mathcal{M}([-\tau,0])$ decomposes uniquely into $\mu^{ac} + \mu^s$ where μ^{ac} (respectively μ^s) is absolutely continuous (resp. singular) with respect to Lebesgue measure.
  If $\mu = \mu^1 + \mu^2$, $\mu^1 \in \mathcal{D}_i([-\tau,0])$ and $\mu^2 \in L^2([-\tau,0])$, obviously $\mu^1 = \mu^s$ and $\mu^2 = \mu^{ac}$.
  The particular case when $i = 0$, the space $\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0])$, shortly $\mathcal{D}_0 \oplus L^2$, will be often recalled in the paper. As generalization of previous spaces we provide an ulterior subspace of measures.

- $\mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0]) = \bigoplus_{i=0}^N \mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$, this is a Hilbert separable subspace of $\mathcal{M}([-\tau,0])$.

- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$.

- $Diag([-\tau,0]^2)$ (shortly $Diag$), will denote the subset of $\mathcal{M}([-\tau,0]^2)$ defined as follows:

$$Diag([-\tau,0]^2) := \left\{\mu^g \in \mathcal{M}([-\tau,0]^2)\, s.t.\, \mu^g(dx,dy) = g(x)\delta_y(dx)dy;\ g \in L^\infty([-\tau,0])\right\} \ . \quad (2.10)$$

  $Diag([-\tau,0]^2)$, equipped with the norm $\|\mu^g\|_{Diag([-\tau,0]^2)} = \|g\|_\infty$, is a Banach space. Let f be a function in $C([-\tau,0]^2)$; the pairing duality between f and $\mu(dx,dy) = g(x)\delta_y(dx)dy \in Diag$ gives

$${}_{C([-\tau,0]^2)}\langle f,\mu\rangle_{Diag([-\tau,0]^2)} = \int_{[-\tau,0]^2} f(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} f(x,y)g(x)\delta_y(dx)dy = \int_{-\tau}^0 f(x,x)g(x)dx \ . \quad (2.11)$$

  A closed subspace of $Diag([-\tau,0]^2)$ is the set denoted by $Diag_c([-\tau,0]^2)$ (resp. $Diag_d([-\tau,0]^2)$) equipped with the sup norm. By convention it will be constituted by the set of $\mu^g \in Diag([-\tau,0]^2)$ for which g belongs to $C([-\tau,0])$ (resp. in $D([-\tau,0])$). We recall that $D([-\tau,0])$ is the set of the (classes of) bounded functions $g : [-\tau,0] \longrightarrow \mathbb{R}$ admitting a cadlag version.

3 Chi-covariation and Chi-quadratic variation

3.1 Notion and examples of Chi-subspaces

Let B_1, B_2, B be three Banach spaces.

Definition 3.1. A Banach subspace $(\chi, \|\cdot\|_\chi)$ continuously injected into $(B_1 \hat{\otimes}_\pi B_2)^*$ will be called a **Chi-subspace**.

Proposition 3.2. Let χ be a Banach space. χ is a Chi-subspace if and only if

$$\|\cdot\|_{(B_1\hat{\otimes}_\pi B_2)^*} \leq \|\cdot\|_\chi \ , \tag{3.1}$$

where $\|\cdot\|_\chi$ is a norm related to the topology of χ.

Proof. Let $E := (B_1 \hat{\otimes}_\pi B_2)^*$. The injection $i : \chi \longrightarrow E$ is linear and continuous by definition. For every $e \in \chi$, we have

$$\|e\|_E \leq \|i\|_{L(\chi;E)} \cdot \|e\|_\chi \ .$$

In particular $\|e\|_E \leq \|e\|_\chi$. Conversely if for any $e \in \chi$, $\|e\|_E \leq \|e\|_\chi$ obviously χ is continuously injected into $(B_1 \hat{\otimes}_\pi B_2)^*$. $\square$

The result below follows immediately from the definition.

Proposition 3.3. Any closed subspace of a Chi-subspace is a Chi-subspace.

We are interested in expressing subsets of $(B_1 \hat{\otimes}_\pi B_2)^*$ as direct sums of Chi-subspaces. This, together with Proposition 3.18 will help us to evaluate the χ-covariations and the χ-quadratic variations of different processes.

Proposition 3.4. Let $\chi_1, \cdots, \chi_n$ be Chi-subspaces such that $\chi_i \bigcap \chi_j = \{0\}$ for any $1 \leq i \neq j \leq n$. Then the normed space $\chi = \chi_1 \oplus \cdots \chi_n$ is a Chi-subspace.

Proof. Reasoning by induction, it is enough to prove the result for the case $n = 2$. If $\mu \in \chi$, then it admits decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \chi_1, \quad \mu_2 \in \chi_2$. It holds $\|\mu\|_{(B_1\hat{\otimes}_\pi B_2)^*} \leq \|\mu_1\|_{(B_1\hat{\otimes}_\pi B_2)^*} + \|\mu_2\|_{(B_1\hat{\otimes}_\pi B_2)^*}$ and (3.1) for χ_1 and χ_2 implies that $\|\mu_i\|_{(B_1\hat{\otimes}_\pi B_2)^*} \leq \|\mu_i\|_{\chi_i}$ for $i = 1, 2$. It follows then $\|\mu\|_{(B_1\hat{\otimes}_\pi B_2)^*} \leq \|\mu_1\|_{\chi_1} + \|\mu_2\|_{\chi_2}$. Then (3.1) it yields for $\chi = \chi_1 \oplus \chi_2$. $\square$

Before providing the definition of the so-called χ-covariation (and of the χ-quadratic variation) between a B_1-valued and a B_2-valued stochastic processes, we will give some examples of Chi-subspaces that we will use frequently in the paper. For the notations we remind back to the preliminaries.

Example 3.5. Let B_1, B_2 be two Banach spaces.

- $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$. This corresponds to our elementary situation anticipated, see also Proposition 3.16.

Example 3.6. Let $B_1 = B_2 = B = C([-\tau,0])$.

This is the natural value space for all the window (continuous) processes. We list some examples of Chi-subspaces χ for which window processes have a χ-covariation or a χ-quadratic variation. Our basic reference Chi-subspace of $(C([-\tau,0]\hat{\otimes}_\pi C([-\tau,0]))^*$ will be $\mathcal{M}([-\tau,0]^2)$ equipped with the usual total variation norm, denoted by $\|\cdot\|_{Var}$. This is in fact a proper subspace as it will be illustrated in the following lines. All the other spaces considered in the sequel of the present example will be shown to be closed subspaces of $\mathcal{M}([-\tau,0]^2)$; by Proposition 3.3 they are Chi-subspaces. Here is the list.

- $\mathcal{M}([-\tau,0]^2)$. This space, equipped with the total variation norm, is a Banach space and it is a subspace of $(B\hat{\otimes}_\pi B)^*$ according to (2.4). Relation (3.1) is verified since

  $$\|T^\mu\|_{(B\hat{\otimes}_\pi B)^*} = \sup_{\|f\|\le 1, \|g\|\le 1} |T^\mu(f,g)| \le \|\mu\|_{Var}$$

  for every $\mu \in \mathcal{M}([-\tau,0]^2)$.
- $\mathcal{D}_{ij}([-\tau,0]^2)$ for every $i,j = 0,\dots,N$. If $\mu = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy)$, $\|\mu\|_{Var} = |\lambda| = \|\mu\|_{\mathcal{D}_{i,j}}$.
- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$. For a general element in this space $\mu = \lambda\delta_{a_i}(dx)\phi(y)dy$, $\phi \in L^2([-\tau,0])$, we have $\|\mu\|_{Var} \le \|\mu\|_{L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i([-\tau,0])} = |\lambda| \cdot \|\phi\|_{L^2}$.
- $\chi^2([-\tau,0]^2) := (L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0]))\hat{\otimes}_h^2$. This space will be frequently shortly denoted by χ^2. This is a well defined Hilbert space with the scalar product which derives from the scalar products in every Hilbert space and it is a closed subspace of $\mathcal{M}([-\tau,0]^2)$ and consequently also of Chi-subspace $(B\hat{\otimes}_\pi B)^*$. Using Proposition 2.2, we obtain

  $$\chi^2([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_a([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2 . \tag{3.2}$$

  Using again Proposition 2.2 with (2.8) and (2.9), we can expand every addend in the right-hand side of (3.2), into a sum of elementary addends. For instance we have $L^2\hat{\otimes}_h \mathcal{D}_a = \bigoplus_{i=0}^N \left(L^2\hat{\otimes}_h \mathcal{D}_i\right)$ and $\mathcal{D}_a\hat{\otimes}_h^2 = \mathcal{D}_A = \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}$ so that (3.2) equals

  $$L^2([-\tau,0]^2) \oplus \bigoplus_{i=0}^N \left(L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i([-\tau,0])\right) \oplus \bigoplus_{i=0}^N \left(\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])\right) \oplus \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}([-\tau,0]^2) . \tag{3.3}$$

  Being χ^2 a finite direct sum of Chi-subspaces, Proposition 3.4 affirms that it is a Chi-subspace.
- As a particular case of $\chi^2([-\tau,0]^2)$ we will denote $\chi^0([-\tau,0]^2)$, χ^0 shortly, the subspace of measures defined as

  $$\chi^0([-\tau,0]^2) := (\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0]))\hat{\otimes}_h^2 .$$

Again using Proposition 2.2, we obtain

$$\chi^0([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_0([-\tau,0]) \oplus \mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_{0,0}([-\tau,0]^2)\ . \tag{3.4}$$

Remark 3.7. 1. For every μ in $\chi^2([-\tau,0]^2)$ there exist $\mu_1 \in L^2([-\tau,0]^2)$, $\mu_2 \in L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_a([-\tau,0])$, $\mu_3 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ and $\mu_4 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2$ such that

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{3.5}$$

with $\mu_1 = \phi_1$, $\mu_2 = \sum_{i=0,\ldots,N} \phi_2^i \otimes \delta_{a_i}$, $\mu_3 = \sum_{i=0,\ldots,N} \delta_{a_i} \otimes \phi_3^i$ and $\mu_4 = \sum_{i,j=0,\ldots,N} \lambda_{i,j}\delta_{a_i}\otimes\delta_{a_j}$, where $\phi_1 \in L^2([-\tau,0]^2)$, $\phi_2^i, \phi_3^i \in L^2([-\tau,0])$ and $\lambda_{i,j}$ are real numbers for every $i,j = 0,\ldots,N$. Components μ_1, μ_2 and μ_3 are singular with respect to the Dirac's measure on $\{(a_i,a_j)\}_{0\le i,j\le N}$, remarking that $\delta_{(a_i,a_j)} = \delta_{a_i} \otimes \delta_{a_j}$; in particular $\mu_k\{(a_i,a_j)\} = 0$ for $k = 1,2,3$. For a general μ it follows

$$\mu\{(a_i,a_j)\} = \mu_4\{(a_i,a_j)\} = \lambda_{i,j}\ . \tag{3.6}$$

2. Consequently, an element $\mu \in \chi^0([-\tau,0]^2)$ can be uniquely decomposed as

$$\mu = \phi_1 + \phi_2 \otimes \delta_0 + \delta_0 \otimes \phi_3 + \lambda\delta_0 \otimes \delta_0, \tag{3.7}$$

where $\phi_1 \in L^2([-\tau,0]^2)$, ϕ_2, ϕ_3 are functions in $L^2([-\tau,0])$ and λ, α, β are real numbers and

$$\mu\left(\{0,0\}\right) = \mu_4\left(\{0,0\}\right) = \lambda\ . \tag{3.8}$$

3.2 Definition of χ-covariation and some related results

In this subsection, we introduce the definition of χ-covariation between a B_1-valued stochastic process $\mathbb{X}$ and a B_2-valued stochastic process $\mathbb{Y}$. We remind that $\mathscr{C}([0,T])$ denotes the space of continuous processes equipped with the ucp topology.
Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be B_1 (resp. B_2) valued stochastic process. Let χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$ and $\epsilon > 0$. We denote by $[\mathbb{X},\mathbb{Y}]^\epsilon$, the following application

$$[\mathbb{X},\mathbb{Y}]^\epsilon : \chi \longrightarrow \mathscr{C}([0,T]) \quad \text{defined by} \quad \phi \mapsto \left(\int_0^t {}_{\chi}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\right)}{\epsilon} \rangle_{\chi^*}\, ds \right)_{t\in[0,T]} \tag{3.9}$$

where $J : B_1\hat{\otimes}_\pi B_2 \longrightarrow (B_1\hat{\otimes}_\pi B_2)^{**}$ is the canonical injection between a space and its bidual. With application $[\mathbb{X},\mathbb{Y}]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon$, defined by

$$\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon(\omega,\cdot) : [0,T] \longrightarrow \chi^* \quad \text{given by} \quad t \mapsto \left(\phi \mapsto \int_0^t {}_{\chi}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon}(\omega)-\mathbb{X}_s(\omega))\otimes(\mathbb{Y}_{s+\epsilon}(\omega)-\mathbb{Y}_s(\omega))\right)}{\epsilon} \rangle_{\chi^*}\, ds \right).$$

Remark 3.8.

1. We recall that $\chi \subset (B_1\hat{\otimes}_\pi B_2)^*$ implies $(B_1\hat{\otimes}_\pi B_2)^{**} \subset \chi^*$.

2. As indicated, ${}_\chi\langle \cdot, \cdot \rangle_{\chi^*}$ denotes the duality between the space χ and its dual χ^*. In fact by assumption, ϕ is an element of χ and element $J\left((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\right)$ naturally belongs to $(B_1\hat{\otimes}_\pi B_2)^{**} \subset \chi^*$.

3. With a slight abuse of notation, in the sequel the injection J from $B_1\hat{\otimes}_\pi B_2$ to its bidual will be omitted. The tensor product $(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)$ has to be considered as the element $J\left((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\right)$ which belongs to χ^*.

4. Suppose $B_1 = B_2 = B = C([-\tau, 0])$ and let χ be a Chi-subspace.
   An element of the type $\eta = \eta_1 \otimes \eta_2$, η_1, $\eta_2 \in B$, can be either considered as an element of the type $B\hat{\otimes}_\pi B \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$ or as an element of $C([-\tau, 0]^2)$ defined by $\eta(x, y) = \eta_1(x)\eta_2(y)$. When χ is indeed a closed subspace of $\mathcal{M}([\tau, 0]^2)$, then the pairing between χ and χ^* will be compatible with the pairing duality between $\mathcal{M}([\tau, 0]^2)$ and $C([-\tau, 0]^2)$ given by
$$ {}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta \rangle_{C([-\tau,0]^2)} = \int_{[-\tau,0]^2} \eta(x, y)\mu(dx, dy) = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\mu(dx, dy)\ . \tag{3.10}$$

Definition 3.9. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be B_1 (resp. B_2) valued stochastic process. We say that $\mathbb{X}$ **and** $\mathbb{Y}$ **admit a** χ**-covariation** if the following assumption hold.

H1 For all (ϵ_n) it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| \langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)}{\epsilon_{n_k}} \rangle \right| ds = \sup_k \int_0^T \frac{\left\| (\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s) \right\|_{\chi^*}}{\epsilon_{n_k}} ds \ < \infty \ a.s. \tag{3.11}$$

H2 (i) There exists an application $\chi \longrightarrow \mathscr{C}([0, T])$, denoted by $[\mathbb{X}, \mathbb{Y}]$, such that
$$[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi) \xrightarrow[\epsilon \longrightarrow 0_+]{ucp} [\mathbb{X}, \mathbb{Y}](\phi) \tag{3.12}$$
for every $\phi \in \chi \subset (B_1\hat{\otimes}_\pi B_2)^*$.

(ii) There is a measurable process $\widetilde{[\mathbb{X}, \mathbb{Y}]} : \Omega \times [0, T] \longrightarrow \chi^*$, such that

- for almost all $\omega \in \Omega$, $\widetilde{[\mathbb{X}, \mathbb{Y}]}(\omega, \cdot)$ is a (cadlag) bounded variation function,
- $\widetilde{[\mathbb{X}, \mathbb{Y}]}(\cdot, t)(\phi) = [\mathbb{X}, \mathbb{Y}](\phi)(\cdot, t)$ a.s. for all $\phi \in \chi$, $t \in [0, T]$.

If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation we will call χ-**covariation** of $\mathbb{X}$ and $\mathbb{Y}$ the χ^*-valued process $(\widetilde{[\mathbb{X},\mathbb{Y}]})_{0\leq t\leq T}$. By abuse of notation, $[\mathbb{X},\mathbb{Y}]$ will also be called χ-covariation and it will be sometimes confused with $\widetilde{[\mathbb{X},\mathbb{Y}]}$.

Definition 3.10. Let $\mathbb{X}=\mathbb{Y}$ be a B-valued stochastic process and χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. The χ-covariation $[\mathbb{X},\mathbb{X}]$ (or $\widetilde{[\mathbb{X},\mathbb{X}]}$) will also be denoted by $[\mathbb{X}]$ and $\widetilde{[\mathbb{X}]}$; it will be called χ**-quadratic variation of** $\mathbb{X}$ and we will say that $\mathbb{X}$ has a χ-quadratic variation.

Remark 3.11.

1. For every fixed $\phi\in\chi$, the processes $\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,t)(\phi)$ and $[\mathbb{X},\mathbb{Y}](\phi)(\cdot,t)$ are indistinguishable. In particular the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}$ is weakly star continuous, i.e. $\widetilde{[\mathbb{X},\mathbb{Y}]}(\phi)$ is continuous for every fixed ϕ.

2. In fact the existence of $\widetilde{[\mathbb{X},\mathbb{Y}]}$ guarantees that $[\mathbb{X},\mathbb{Y}]$ admits a proper bounded variation version which allows to consider it as pathwise integral.

3. The quadratic variation $\widetilde{[\mathbb{X}]}$ will be the object intervening in the second order term of the Itô formula expanding $F(\mathbb{X})$ for some C^2-Fréchet function F, see Theorem 5.2.

4. In Corollaries 3.26 and 3.27 we will show that, whenever χ is separable (most of the cases) Condition **H2** can be relaxed in a significant way. In fact Condition **H2**(i) reduces to the convergence in probability of (3.12) on a dense subspace and **H2**(ii) will be automatically granted.

Remark 3.12.

1. A practical criterion to verify Condition **H1** is

$$\frac{1}{\epsilon}\int_0^T \|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}\,ds\leq B(\epsilon) \tag{3.13}$$

where $B(\epsilon)$ converges in probability when ϵ goes to zero. In fact the convergence in probability implies the a.s. convergence of a subsequence.

2. A consequence of Condition **H1** is that for all $(\epsilon_n)\downarrow 0$ there exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \|\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon_{n_k}}\|_{Var([0,T])}<\infty \quad a.s. \tag{3.14}$$

In fact $\|\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}\|_{Var([0,T])}\leq\frac{1}{\epsilon}\int_0^T\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}ds$, which implies that $\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}$ is a χ^*-valued process of bounded variation on $[0,T]$. As a consequence, for a χ-valued continuous stochastic process $\mathbb{Z}$, $t\in[0,T]$, the integral $\int_0^t {}_{\chi}\langle\mathbb{Z}_s, d\widetilde{[\mathbb{X},\mathbb{Y}]}_s^{\epsilon_{n_k}}\rangle_{\chi^*}$ is a well-defined Lebesgue-Stieltjes type integral for almost all $\omega\in\Omega$.

Remark 3.13.

1. To a Borel function $G : \chi \longrightarrow C([0,T])$ we can associate $\tilde{G} : [0,T] \longrightarrow \chi^*$ setting $\tilde{G}(t)(\phi) = G(\phi)(t)$. By definition $\tilde{G} : [0,T] \longrightarrow \chi^*$ has bounded variation if

$$\|\tilde{G}\|_{Var([0,T])} := \sup_{\sigma \in \Sigma_{[0,T]}} \sum_{i|(t_i)_i=\sigma} \left\|\tilde{G}(t_{i+1}) - \tilde{G}(t_i)\right\|_{\chi^*} = \sup_{\sigma \in \Sigma_{[0,T]}} \sum_{i|(t_i)_i=\sigma} \sup_{\|\phi\|_\chi \leq 1} |G(\phi)(t_{i+1}) - G(\phi)(t_i)| < +\infty\,,$$

where $\Sigma_{[0,T]}$ is the set of all possible partitions $\sigma = (t_i)_i$ of the interval $[0,T]$. This quantity is the **total variation** of $\tilde{G}$.
For example if $G(\phi) = \int_0^t \dot{G}_s(\phi)\, ds$ with $\dot{G} : \chi \to C([0,T])$ Bochner integrable, then $\|G\|_{Var[0,T]} \leq \int_0^T \sup_{\|\phi\|_\chi \leq 1} |\dot{G}_s(\phi)|\, ds$.

2. If $G(\phi)$, $\phi \in \chi$ is a family of stochastic processes, it is not obvious to find a good version $\tilde{G} : [0,T] \longrightarrow \chi^*$ of G. This will be the object of Theorem 3.24.

Definition 3.14. If the χ-covariation exists with $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$, we say that $\mathbb{X}$ and $\mathbb{Y}$ admit a **global covariation**. Analogously if $\mathbb{X}$ is B-valued and the χ-quadratic variation exists with $\chi = (B \hat{\otimes}_\pi B)^*$, we say that $\mathbb{X}$ admits a **global quadratic variation**.

Remark 3.15. 1. $\widetilde{[\mathbb{X},\mathbb{Y}]}$ takes values "a priori" in $(B_1 \hat{\otimes}_\pi B_2)^{**}$.

2. If $[\mathbb{X},\mathbb{Y}]^{\mathbb{R}}$ exists then Condition **H1** follows by Remark 3.12.1.

Proposition 3.16. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) process such that $\mathbb{X}$ and $\mathbb{Y}$ admit real and tensor covariation. Then $\mathbb{X}$ and $\mathbb{Y}$ admit a global covariation. In particular the global covariation takes values in $B_1 \hat{\otimes}_\pi B_2$ and $\widetilde{[\mathbb{X},\mathbb{Y}]} = [\mathbb{X},\mathbb{Y}]^{\otimes}$ a.s.

Proof. We set $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$. Taking into account Remark 3.15.2, it will be enough to verify Condition **H2**. Recalling the definition of $[\mathbb{X},\mathbb{Y}]^\epsilon$ at (3.9) and the definition of injection J we observe that

$$\begin{aligned}
[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t) &= \int_0^t {}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, \frac{J\left((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\right)}{\epsilon} \rangle_{(B_1 \hat{\otimes}_\pi B_2)^{**}}\, ds \\
&= \int_0^t {}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)}{\epsilon} \rangle_{B_1 \hat{\otimes}_\pi B_2}\, ds\,.
\end{aligned} \tag{3.15}$$

Since Bochner inegrability implies Pettis integrability, for every $\phi \in (B_1 \hat{\otimes}_\pi B_2)^*$, we also have

$${}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t \rangle_{B_1 \hat{\otimes}_\pi B_2} = \int_0^t {}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)}{\epsilon} \rangle_{B_1 \hat{\otimes}_\pi B_2}\, ds\,. \tag{3.16}$$

(3.15) and (3.16) imply that

$$[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t) = {}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t \rangle_{B_1 \hat{\otimes}_\pi B_2} \qquad \text{a.s.} \tag{3.17}$$

In order to conclude the proof of Condition **H2** we will show that that

$$\sup_{t \leq T} \left| [\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t) - {}_{(B_1 \hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes}_t \rangle_{B_1 \hat{\otimes}_\pi B_2} \right| \xrightarrow[\epsilon \longrightarrow 0]{\mathbb{P}} 0\,. \tag{3.18}$$

Developing the left-hand side of (3.18) and using (3.17), we obtain

$$\sup_{t\leq T}\left|[\mathbb{X},\mathbb{Y}]^{\epsilon}(\phi)(\cdot,t) - {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes}_t\rangle_{B_1\hat{\otimes}_\pi B_2}\right| = \sup_{t\leq T}\left|{}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t - [\mathbb{X},\mathbb{Y}]^{\otimes}_t\rangle_{B_1\hat{\otimes}_\pi B_2}\right|$$
$$\leq \|\phi\|_{(B_1\hat{\otimes}_\pi B_2)^*}\sup_{t\leq T}\left\|[\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t - [\mathbb{X},\mathbb{Y}]^{\otimes}_t\right\|_{B_1\hat{\otimes}_\pi B_2},$$

where the last quantity converges to zero in probability by Definition 1.3.2 of tensor quadratic variation. This implies (3.18). The tensor quadratic variation has always bounded variation because of item 2. of Remark 1.4. In particular **H2**(ii) is also verified. □

Remark 3.17. We observe some interesting features related to the global covariation, i.e. when $\chi = (B_1\hat{\otimes}_\pi B_2)^*$.

1. When χ is separable, for any $t\in[0,T]$, there exists a null subset N of Ω and a sequence (ϵ_n) such that
$$\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon_n}(\omega,t) \xrightarrow[\epsilon\longrightarrow 0]{} \widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,t)$$
weak star for $\omega\notin N$, see Lemma A.1. This shows that the global covariation is related to a weak star convergence in the space $(B_1\hat{\otimes}_\pi B_2)^{**}$ for elements $\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}$.

2. We recall that $J(B_1\hat{\otimes}_\pi B_2)$ is isometrically embedded (and weak star dense) in $(B_1\hat{\otimes}_\pi B_2)^{**}$. In particular it is the case if B_1 or B_2 has infinite dimension. If the Banach space $B_1\hat{\otimes}_\pi B_2$ is not reflexive, then $(B_1\hat{\otimes}_\pi B_2)^{**}$ strictly contains $B_1\hat{\otimes}_\pi B_2$. The weak star convergence is weaker then the strong convergence in $J(B_1\hat{\otimes}_\pi B_2)$, required in the definition of tensor quadratic variation, see Definition 1.3.2. The global covariation is therefore truly more general than the tensor covariation.

3. In general $B_1\hat{\otimes}_\pi B_2$ is not reflexive even if B_1 and B_2 are Hilbert spaces, see for instance [29] at Section 4.2.

We go on with some related results about the χ-covariation and the χ-quadratic variation.

Proposition 3.18. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) process and χ_1, χ_2 be two Chi-subspaces of $(B_1\hat{\otimes}_\pi B_2)^*$ with $\chi_1\cap\chi_2=\{0\}$. Let $\chi=\chi_1\oplus\chi_2$. If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ_i-covariation $[\mathbb{X},\mathbb{Y}]_i$ for $i=1,2$ then they admit a χ-covariation $[\mathbb{X},\mathbb{Y}]$ and it holds $[\mathbb{X},\mathbb{Y}](\phi)=[\mathbb{X},\mathbb{Y}]_1(\phi_1)+[\mathbb{X},\mathbb{Y}]_2(\phi_2)$ for all $\phi\in\chi$ with unique decomposition $\phi=\phi_1+\phi_2$, $\phi_1\in\chi_1$ and $\phi_2\in\chi_2$.

Proof. χ is a Chi-subspace because of Proposition 3.4. It will be enough to show the result for a fixed norm in the space χ. We set $\|\phi\|_\chi = \|\phi_1\|_{\chi_1}+\|\phi_2\|_{\chi_2}$ and we remark that $\|\phi\|_\chi\geq\|\phi_i\|_{\chi_i}$, $i=1,2$. Condition **H1** follows immediately by inequality

$$\int_0^T \sup_{\|\phi\|_\chi\leq 1}\left|{}_\chi\langle\phi,(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi^*}\right| ds \leq \int_0^T \sup_{\|\phi_1\|_{\chi_1}\leq 1}\left|{}_{\chi_1}\langle\phi_1,(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi_1^*}\right| ds +$$
$$+\int_0^T \sup_{\|\phi_2\|_{\chi_2}\leq 1}\left|{}_{\chi_2}\langle\phi_2,(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi_2^*}\right| ds\,.$$

Condition **H2**(i) follows by linearity; in fact

$$
\begin{aligned}
[\mathbb{X},\mathbb{Y}]^{\epsilon}(\phi) &= \int_0^t {}_{\chi}\langle \phi_1+\phi_2, (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi^*}\, ds = \\
&= \int_0^t {}_{\chi_1}\langle \phi_1, (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi_1^*}\, ds + \int_0^t {}_{\chi_2}\langle \phi_2, (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle_{\chi_2^*}\, ds \\
&\xrightarrow[\epsilon\to 0]{ucp} [\mathbb{X},\mathbb{Y}]_1(\phi_1)+[\mathbb{X},\mathbb{Y}]_2(\phi_2)\ .
\end{aligned}
$$

Concerning Condition **H2**(ii), for $\omega\in\Omega$, $t\in[0,T]$ we can obviously set $\widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,t)(\phi)=\widetilde{[\mathbb{X},\mathbb{Y}]}_1(\omega,t)(\phi_1)+\widetilde{[\mathbb{X},\mathbb{Y}]}_2(\omega,t)(\phi_2)$. $\square$

Proposition 3.19. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process.

1. Let χ_1 and χ_2 be two subspaces $\chi_1\subset\chi_2\subset(B_1\hat{\otimes}_\pi B_2)^*$, χ_1 being a Banach subspace continuously injected into χ_2 and χ_2 a Chi-subspace. If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ_2-covariation $[\mathbb{X},\mathbb{Y}]_2$, then they also admit a χ_1-covariation $[\mathbb{X},\mathbb{Y}]_1$ and it holds $[\mathbb{X},\mathbb{Y}]_1(\phi)=[\mathbb{X},\mathbb{Y}]_2(\phi)$ for all $\phi\in\chi_1$.
2. In particular if $\mathbb{X}$ and $\mathbb{Y}$ admit a tensor quadratic variation, then $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-quadratic variation for any Chi-subspace χ.

Proof. 1. If Condition **H1** is valid for χ_2 then it is also verified for χ_1. In fact we remark that $(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)$ is an element in $(B_1\hat{\otimes}_\pi B_2)\subset(B_1\hat{\otimes}_\pi B_2)^{**}\subset\chi_2^*\subset\chi_1^*$. If $A:=\{\phi\in\chi_1\,;\|\phi\|_{\chi_1\le 1}\}$ and $B:=\{\phi\in\chi_2\,;\|\phi\|_{\chi_2\le 1}\}$, then $A\subset B$ and clearly $\int_0^t\sup_A|\langle\phi,(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle|ds\le\int_0^t\sup_B|\langle\phi,(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\rangle|ds$. This implies the inequality $\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi_1^*}\le\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi_2^*}$ and Assumption **H1** follows immediately. Assumption **H2**(i) is trivially verified because, by restriction, we have $[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)\xrightarrow[\epsilon\to 0]{ucp}[\mathbb{X},\mathbb{Y}]_2(\phi)$ for all $\phi\in\chi_1$. We define $[\mathbb{X},\mathbb{Y}]_1(\phi)=[\mathbb{X},\mathbb{Y}]_2(\phi)$, $\forall\ \phi\in\chi_1$ and $\widetilde{[\mathbb{X},\mathbb{Y}]}_1(\omega,t)(\phi)=\widetilde{[\mathbb{X},\mathbb{Y}]}_2(\omega,t)(\phi)$, for all $\omega\in\Omega$, $t\in[0,T]$, $\phi\in\chi_1$. Condition **H2**(ii) follows because given $G:[0,T]\longrightarrow\chi_1$ we have $\|G(t)-G(s)\|_{\chi_1^*}\le\|G(t)-G(s)\|_{\chi_2^*}$, $\forall\,0\le s\le t\le T$.

2. It follows from 1. and Proposition 3.16.

$\square$

We continue with some general properties of the χ-covariation.

Lemma 3.20. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process and χ be a Chi-subspace. Suppose that $\frac{1}{\epsilon}\int_0^T\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}\,ds$ converges to 0 in probability when ϵ goes to zero.

1. Then $\mathbb{X}$ and $\mathbb{Y}$ admit a zero χ-covariation.
2. If $\chi=(B_1\hat{\otimes}_\pi B_2)^*$ then $\mathbb{X}$ and $\mathbb{Y}$ admit a zero real and tensor covariation.

Proof. Condition **H1** is verified because of Remark 3.12.1. We verify **H2**(i) directly. For every fixed $\phi \in \chi$ we have

$$
\begin{aligned}
|[\mathbb{X},\mathbb{Y}]^{\epsilon}(\phi)(t)| &= \left| \int_0^t {}_{\chi}\langle \phi, \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)}{\epsilon} \rangle_{\chi^*}\, ds \right| \leq \\
&\leq \int_0^T \left| {}_{\chi}\langle \phi, \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)}{\epsilon} \rangle_{\chi^*} \right| ds\ .
\end{aligned}
$$

So we obtain

$$
\sup_{t\in[0,T]} |[\mathbb{X},\mathbb{Y}]^{\epsilon}(\phi)(t)| \leq \|\phi\|_{\chi} \frac{1}{\epsilon} \int_0^T \|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}\, ds \xrightarrow[\epsilon\longrightarrow 0]{} 0
$$

in probability by the hypothesis. Since condition **H2**(ii) holds trivially, we can conclude. By definition, the real covariation is zero which also forces the tensor covariation to be zero, see Remark 1.4, item 4. ☐

3.3 Technical issues

3.3.1 Convergence of infinite dimensional Stieltjes integrals

We state now an important technical lemma which will be used in the proof of Itô Theorem 5.2.

Proposition 3.21. Let χ be a separable Banach space, a sequence $F^n : \chi \longrightarrow \mathscr{C}([0,T])$ of linear continuous maps and measurable random fields $\widetilde{F}^n : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\widetilde{F}^n(\cdot,t)(\phi) = F^n(\phi)(\cdot,t)$ a.s. $\forall\, t \in [0,T]$, $\phi \in \chi$. We suppose the following.

i) For all (n_k) it exists (n_{k_j}) such that $\sup_j \|\widetilde{F}^{n_{k_j}}\|_{Var[0,T]} < \infty$.

ii) There is a linear continuous map $F : \chi \longrightarrow \mathscr{C}([0,T])$ such that for all $t \in [0,T]$ and for every $\phi \in \chi$ $F^n(\phi)(\cdot,t) \longrightarrow F(\phi)(\cdot,t)$ in probability.

iii) There is measurable random field $\widetilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ of such that for ω a.s. $\widetilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ has bounded variation and $\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$a.s. $\forall\, t \in [0,T]$ and $\phi \in \chi$.

iv) $F^n(\phi)(0) = 0$ for every $\phi \in \chi$.

Then for every $t \in [0,T]$ and every continuous process $H : \Omega \times [0,T] \longrightarrow \chi$

$$
\int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{F}^n(\cdot,s)\rangle_{\chi^*} \longrightarrow \int_0^t {}_{\chi}\langle H(\cdot,s), d\widetilde{F}(\cdot,s)\rangle_{\chi^*} \qquad \text{in probability.} \tag{3.19}
$$

Proof. See Appendix A. ☐

Corollary 3.22. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_{\pi}B_2)^*$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two stochastic processes with values in B_1 and B_2 admitting a χ-covariation and $\mathbb{H}$ a continuous measurable process $\mathbb{H} : \Omega \times [0,T] \longrightarrow \mathcal{V}$ where $\mathcal{V}$ is a closed separable subspace of χ. Then, for every $t \in [0,T]$,

$$\int_0^t {}_{\chi}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}(\cdot,s)\rangle_{\chi^*} \xrightarrow[\epsilon\longrightarrow 0]{} \int_0^t \langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,s)\rangle_{\chi^*} \tag{3.20}$$

in probability.

Proof. By Proposition 3.3, $\mathcal{V}$ is a Chi-subspace. By Proposition 3.19, $\mathbb{X}$ and $\mathbb{Y}$ admit a $\mathcal{V}$-covariation $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}$ and $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}(\phi) = [\mathbb{X},\mathbb{Y}](\phi)$ for all $\phi \in \mathcal{V}$; in the sequel of the proof, $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}$ will be still denoted by $[\mathbb{X},\mathbb{Y}]$. Since the ucp convergence implies the convergence in probability for every $t \in [0,T]$, by Proposition 3.21 and definition of $\mathcal{V}$-covariation, it follows

$$\int_0^t {}_{\mathcal{V}}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}(\cdot,s)\rangle_{\mathcal{V}^*} \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} \int_0^t {}_{\mathcal{V}}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,s)\rangle_{\mathcal{V}^*} \,. \tag{3.21}$$

Since the pairing duality between χ and χ^* is compatible with the one between $\mathcal{V}$ and $\mathcal{V}^*$, the result (3.20) is now established. $\square$

3.3.2 Weaker conditions for the existence of the χ-covariation

An important and useful theorem which helps to find sufficient conditions for the existence of the χ-quadratic variation of a Banach valued process is given below. It will be a consequence of a Banach-Steinhaus type result for Fréchet spaces, see Theorem II.1.18, pag. 55 in [10]. We start with a remark.

Remark 3.23.

1. The following notion plays a role in Banach-Steinhaus theorem in [10]. Let E be a Fréchet spaces, F-space shortly. A subset C of E is called **bounded** if for all $\epsilon > 0$ it exists δ_ϵ such that for all $0 < \alpha \leq \delta_\epsilon$, αC is included in the open ball $\mathcal{B}(0,\epsilon) := \{e \in E; d(0,e) < \epsilon\}$.

2. Let $(\mathbb{Y}^n)$ be a sequence of random elements with values in a Banach space $(B, \|\cdot\|_B)$ such that $\sup_n \|\mathbb{Y}^n\|_B \leq Z$ a.s. for some real positive random variable Z. Then $(\mathbb{Y}^n)$ is bounded in the F-space of random elements equipped with the convergence in probability which is governed by the metric

   $$d(\mathbb{X},\mathbb{Y}) = \mathbb{E}\left[\|\mathbb{X}-\mathbb{Y}\|_B \wedge 1\right].$$

   In fact by Lebesgue dominated convergence theorem it follows $\lim_{\gamma\to 0}\mathbb{E}[\gamma Z \wedge 1] = 0$.

3. In particular taking $B = C([0,T])$ a sequence of continuous processes $(\mathbb{Y}^n)$ such that $\sup_n \|\mathbb{Y}^n\|_\infty \leq Z$ a.s. is bounded for the usual metric in $\mathscr{C}([0,T])$ equipped with the topology related to the ucp convergence.

Theorem 3.24. Let $F^n : \chi \longrightarrow \mathscr{C}([0,T])$ be a sequence of linear continuous maps such that $F^n(\phi)(0) = 0$ a.s. and there is $\tilde{F}^n : \Omega \times [0,T] \longrightarrow \chi^*$ a.s. for which we have the following.

i) $F^n(\phi)(\cdot,t) = \tilde{F}^n(\cdot,t)(\phi)$ a.s. $\forall\, t \in [0,T]$, $\phi \in \chi$.

ii) $\forall\, \phi \in \chi$, $t \mapsto \tilde{F}^n(\cdot,t)(\phi)$ is cadlag.

iii) $\sup_n \|\tilde{F}^n\|_{Var([0,T])} < \infty$ a.s.

iv) There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathscr{C}([0,T])$ such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$.
Condition **iv)** can be replaced with the one below **iv')**.

iv') There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathscr{C}([0,T])$ such that for every $\phi \in \mathcal{S}$.

- $F^n(\phi)(t) \longrightarrow F(\phi)(t)$ for every $t \in [0,T]$ in probability.
- $F^n(\phi)$ is an increasing process.

1) Suppose that χ is separable.
Then there is a linear and continuous extension $F : \chi \longrightarrow \mathscr{C}([0,T])$ and there is a measurable random field $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$ a.s. for every $t \in [0,T]$. Moreover the following properties hold.

a) For every $\phi \in \chi$, $F^n(\phi) \xrightarrow{ucp} F(\phi)$.
In particular for every $t \in [0,T]$, $\phi \in \chi$, $F^n(\phi)(\cdot,t) \xrightarrow{\mathbb{P}} F(\phi)(\omega,t)$.

b) $\tilde{F}$ has bounded variation a.s. and $t \mapsto \tilde{F}(\omega,t)$ is ω-a.s. weakly star continuous.

2) Suppose the existence of a measurable $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $t \mapsto \tilde{F}(\omega,t)$ has bounded variation and weakly star cadlag such that

$$\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) \qquad \text{a.s.} \qquad \forall\, t \in [0,T],\ \forall\, \phi \in \mathcal{S}\ .$$

Then point **a)** still follows.

Remark 3.25. In point **2)** we do not necessarily suppose χ to be separable.

Proof. See Appendix A. □

Important implications of Theorem 3.24 are Corollaries 3.26 and 3.27, which give us easier conditions for the existence of the χ-covariation as anticipated in Remark 3.11.4.

Corollary 3.26. Let B_1 and B_2 be Banach spaces, $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process and χ be a separable Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. We suppose the following.

H0' There is $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty .$$

H2' There is $\mathcal{T} : \chi \longrightarrow \mathscr{C}([0,T])$ such that $[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ ucp for all $\phi \in \mathcal{S}$.

Then $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation and application $[\mathbb{X}, \mathbb{Y}]$ is equal to $\mathcal{T}$.

Proof. Condition **H1** is verified by assumption. Conditions **H2**(i) and (ii) follow by Theorem 3.24 setting $F^n(\phi)(\cdot, t) = [\mathbb{X}, \mathbb{Y}]^{\epsilon_n}(\phi)(t)$ and $\tilde{F}^n = \widetilde{[\mathbb{X}, \mathbb{Y}]}^{\epsilon_n}$ for a suitable sequence (ϵ_n). □

In the case $\mathbb{X} = \mathbb{Y}$ and $B = B_1 = B_2$ we can further relax the hypotheses.

Corollary 3.27. Let B be a Banach space, $\mathbb{X}$ a be B-valued stochastic processes and χ be a separable Chi-subspace. We suppose the following.

H0" There are subsets $\mathcal{S}$, $\mathcal{S}^p$ of χ such that $\overline{Span(\mathcal{S})} = \chi$, $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ and $\mathcal{S}^p$ is constituted by **positive definite** elements ϕ in the sense that $\langle \phi, b \otimes b \rangle \ge 0$ for all $b \in B$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s)^{\otimes 2}}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty .$$

H2" There is $\mathcal{T} : \chi \longrightarrow \mathscr{C}([0,T])$ such that $[\mathbb{X}]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ in probability for every $\phi \in \mathcal{S}$ and for every $t \in [0,T]$.

Then $\mathbb{X}$ admits a χ-quadratic variation and application $[\mathbb{X}]$ is equal to $\mathcal{T}$.

Proof. We verify the conditions of Corollary 3.26. Conditions **H0'** and **H1** are verified by assumption. We observe that, for every $\phi \in \mathcal{S}^p$, $[\mathbb{X}]^\epsilon(\phi)$ is an increasing process. By linearity, it follows that for any $\phi \in \mathcal{S}^p$, $[\mathbb{X}]^\epsilon(\phi)(t)$ converges in probability to $\mathcal{T}(\phi)(t)$ for any $t \in [0,T]$. Lemma 2.1 implies that $[\mathbb{X}]^\epsilon(\phi)$ converges ucp for every $\phi \in \mathcal{S}^p$ and therefore in $\mathcal{S}$. Conditions **H2'** of Corollary 3.26 is now verified. □

When χ has finite dimension the notion of χ-quadratic variation becomes very natural.

Proposition 3.28. Let $\chi = Span\{\phi_1, \ldots, \phi_n\}$, $\phi_1, \ldots, \phi_n \in (B \hat{\otimes}_\pi B)^*$ of positive type and linearly independent. $\mathbb{X}$ has a χ-quadratic variation if and only if there are continuous processes Z^i such that $[\mathbb{X}]^\epsilon_t(\phi_i)$ converges in probability to Z^i_t for ϵ going to zero for all $t \in [0,T]$ and $i = 1, \ldots, n$.

Proof. We only need to show that the condition is sufficient, the converse implication resulting immediately. We verify the hypotheses of Corollary 3.27 taking $\mathcal{S} = \{\phi_1, \dots, \phi_n\}$. Without restriction to generality we can suppose $\|\phi_i\|_{(B\hat{\otimes}_\pi B)^*} = 1$, for $1 \leq i \leq n$. Conditions **H0"** and **H2"** are straightforward. It remains to verify **H1**. Since χ is finite dimensional it can be equipped with the norm $\|\phi\|_\chi = \sum_{i=1}^n |a_i|$ if $\phi = \sum_{i=1}^n a_i\,\phi_i$ with $a_i \in \mathbb{R}$. For ϕ such that $\|\phi\|_\chi = \sum_{i=1}^n |a_i| \leq 1$ we have

$$
\begin{aligned}
\frac{1}{\epsilon}\int_0^T \left|\langle \phi\,,\, \mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle\right| ds &\leq \sum_{i=1}^n \frac{1}{\epsilon}\int_0^T \left|\langle a_i\,\phi_i\,,\, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle\right| ds \\
&= \sum_{i=1}^n \frac{|a_i|}{\epsilon}\int_0^T \langle \phi_i\,,\, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle ds
\end{aligned}
\tag{3.22}
$$

because ϕ_i are of positive type. Previous expression is smaller or equal than

$$
\sum_{i=1}^n \frac{1}{\epsilon}\int_0^T \langle \phi_i\,,\, (X_{s+\epsilon} - X_s)\otimes^2\rangle = \sum_{i=1}^n [\mathbb{X}]_T^\epsilon(\phi_i)
$$

because $|a_i| \leq 1$ for $1 \leq i \leq n$. Taking the supremum over $\|\phi\|_\chi \leq 1$ and using the hypothesis of convergence in probability of the quantity $[\mathbb{X}]_T^\epsilon(\phi_i)$ for $1 \leq i \leq n$, the result follows. $\square$

Corollary 3.29. Let $B_1 = B_2 = \mathbb{R}^n$. $\mathbb{X}$ admits all its mutual brackets if and only if $\mathbb{X}$ admits a global quadratic variation.

Our $\chi-$covariation methodology provides a simple property related to the covariation of real processes which was not formally stated in the literature.

Proposition 3.30. Let X and Y be two real continuous processes such that

i) $[X, Y]$ exists and

ii) for every sequence $(\epsilon_n) \downarrow 0$, it exists a subsequence (ϵ_{n_k}) such that

$$
\sup_k \frac{1}{\epsilon_{n_k}}\int_0^T \left|X_{s+\epsilon_{n_k}} - X_s\right| \cdot \left|Y_{s+\epsilon_{n_k}} - Y_s\right| ds \quad < +\infty\,. \tag{3.23}
$$

Then the real covariation process $[X, Y]$ has bounded variation.

Proof. The processes X and Y take values in $B = \mathbb{R}$ and the (separable) space $\chi = (B\hat{\otimes}_\pi B)^*$ coincides with $\mathbb{R}$. Taking into account Corollary 3.26, the processes X and Y admit therefore a global covariation which coincides with the classical covariation $[X, Y]$ defined in Definition 1.1 and in particular $[X, Y]$ has bounded variation. $\square$

Remark 3.31. 1. A sufficient condition to ensure that $[X, Y]$ has bounded variation is that X, Y and $X + Y$ are finite quadratic variation processes.

In this case, the bilinearity of the real covariation implies that $[X, Y]$ is difference of increasing processes and has therefore bounded variation. However the mentioned condition is too strong. Consider for instance the following example. Let X be any continuous process and V be a bounded variation process; then $[X, V] = 0$, see [28], Proposition 1, 6). On the other hand, it is easy to show that (3.23) is verified even if X is not a finite quadratic variation process, so that Proposition 3.30 provides a new argument for $[X, V] = 0$.

2. If X, Y are two continuous processes such that (X, Y) has all its mutual covariations then conditions i) and ii) of Proposition 3.30 are fulfilled. In fact by Cauchy-Schwarz inequality we have

$$\frac{1}{\epsilon}\int_0^T |X_{s+\epsilon} - X_s| \cdot |Y_{s+\epsilon} - Y_s| \, ds \leq \sqrt{\int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds}\sqrt{\int_0^T \frac{(Y_{s+\epsilon} - Y_s)^2}{\epsilon} ds} =: A(\epsilon)$$

where the sequence $A(\epsilon)$ converges in probability to $\sqrt{[X]_T [Y]_T}$. This implies of course (3.23).

4 Evaluations of χ-covariations for window processes

In this section we consider X and Y as real continuous processes as usual prolongated by continuity and $X(\cdot)$ and $Y(\cdot)$ their associated window processes. We set $B = C([-\tau, 0])$. We will proceed to the evaluation of some χ-covariations (resp. χ-quadratic variations) for window processes $X(\cdot)$ and $Y(\cdot)$ (resp. for process $X(\cdot)$) with values in $B = C([-\tau, 0])$. We start with some examples of χ-covariation calculated directly through the definition.

Proposition 4.1. Let X and Y be two real valued processes with Hölder continuous paths of parameters γ and δ such that $\gamma + \delta > 1$. Then $X(\cdot)$ and $Y(\cdot)$ admit a zero real and tensor covariation. In particular $X(\cdot)$ and $Y(\cdot)$ admit a zero global covariation.

Proof. By Proposition 3.16 we only need to show that $X(\cdot)$ and $Y(\cdot)$ admit a zero real and tensor covariation. By Lemma 3.20, point 2, we only need to show the convergence to zero in probability of following quantity.

$$\frac{1}{\epsilon}\int_0^T \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_B \|Y_{s+\epsilon}(\cdot) - Y_s(\cdot)\|_B \, ds = \frac{1}{\epsilon}\int_0^T \sup_{u\in[-\tau,0]} |X_{s+u+\epsilon} - X_{s+u}| \sup_{v\in[-\tau,0]} |Y_{s+v+\epsilon} - Y_{s+v}| \, ds \, . \tag{4.1}$$

Since X (resp. Y) is a.s. γ-Hölder continuous (resp. δ-Hölder continuous), there is a non-negative finite random variable Z such that the right-hand side of (4.1) is bounded by a sequence of random variables $Z(\epsilon)$ defined by $Z(\epsilon) := \epsilon^{\gamma+\delta-1} Z T$. This implies that (4.1) converges to zero a.s. for $\gamma + \delta > 1$. □

Remark 4.2. As a consequence of previous proposition every window process $X(\cdot)$ associated with a continuous process with Hölder continuous paths of parameter $\gamma > 1/2$ admits zero real, tensor and global quadratic variation.

Remark 4.3. Let B^H (resp. $B^{H,K}$) be a real fractional Brownian motion with parameters $H \in]0,1[$ (resp. real bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$). See [24] and [14] for elementary facts about the bifractional Brownian motion. As immediate applications of Proposition 4.1 we obtain the following results.

1. The fractional window Brownian motion $B^H(\cdot)$ with $H > 1/2$ admits a zero real, tensor and global quadratic variation.

2. The bifractional window Brownian motion $B^{H,K}(\cdot)$ with $KH > 1/2$ admits a zero real, tensor and global quadratic variation.

3. We recall that the paths of a Brownian motion W are a priori only a.s. Hölder continuous of parameter $\gamma < 1/2$ so that we can not use Proposition 4.1.

Propositions 4.5 and 4.7 show that the stochastic calculus developed by [5], [9] and [20] cannot be applied for $\mathbb{X}$ being a window Brownian motion $W(\cdot)$.

Definition 4.4. Let B be a Banach space and $\mathbb{X}$ be a B-valued stochastic process. We say that $\mathbb{X}$ is a **Pettis semimartingale** if, for every $\phi \in B^*$, $\langle \phi, \mathbb{X}_t \rangle$ is a real semimartingale.

We remark that if $\mathbb{X}$ is a B-valued semimartingale in the sense of Section 1.17, [20], then it is also a Pettis semimartingale.

Proposition 4.5. The $C([-\tau,0])$-valued window Brownian $W(\cdot)$ motion is not a Pettis semimartingale.

Proof. It is enough to show that there exists an element μ in $B^* = \mathcal{M}([-\tau,0])$ such that $\langle \mu, W_t(\cdot) \rangle = \int_{[-\tau,0]} W_t(x)\mu(dx)$ is not a semimartingale with respect to any filtration. We will proceed by contradiction: we suppose that $W(\cdot)$ is a Pettis semimartingale, then in particular if we take $\mu = \delta_0 + \delta_{-\tau}$, the process $\langle \delta_0 + \delta_{-\tau}, W_t(\cdot) \rangle = W_t + W_{t-\tau} := X_t$ has to be a semimartingale with respect to some filtration $(\mathcal{G}_t)$. Let $(\mathcal{F}_t)$ be the natural filtration generated by the real Brownian motion W. Now $W_t + W_{t-\tau}$ is $(\mathcal{F}_t)$-adapted, so by Stricker's theorem (see Theorem 4, pag. 53 in [23]), X is a semimartingale with respect to filtration $(\mathcal{F}_t)$. On the other hand $(W_{t-\tau})_{t \geq \tau}$ is a strongly predictable process with respect to $(\mathcal{F}_t)$, see Definition 3.5 in [4]. By Proposition 4.11 in [3], it follows that $(W_{t-\tau})_{t\geq\tau}$ is an $(\mathcal{F}_t)$-martingale orthogonal process. Since W is an $(\mathcal{F}_t)$-martingale, the process $X_t = W_t + W_{t-\tau}$ is an $(\mathcal{F}_t)$-weak Dirichlet process. By uniqueness of the decomposition for $(\mathcal{F}_t)$-weak Dirichlet processes, $(W_{t-\tau})_{t\geq\tau}$ has to be a bounded variation process. This generates a contradiction because $(W_{t-\tau})_{t\geq\tau}$ is not a zero quadratic variation process. In conclusion $\langle \mu, W_t(\cdot) \rangle$ is not a semimartingale. □

Remark 4.6. Process X defined by $X_t = W_t + W_{t-\tau}$ is an example of $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation which is not an $(\mathcal{F}_t)$-Dirichlet process.

Proposition 4.7. If W is a classical Brownian motion, then $W(\cdot)$ does not admit a real quadratic variation. In particular $W(\cdot)$ does not admit a global quadratic variation.

Proof. We can prove that

$$\int_0^T \frac{1}{\epsilon}\,\|W_{u+\epsilon}(\cdot) - W_u(\cdot)\|_B^2\,du \geq T\,A^2(\tilde{\epsilon})\ln(1/\tilde{\epsilon}) \quad \text{where} \quad \tilde{\epsilon} = \frac{2\epsilon}{T} \tag{4.2}$$

and $(A(\epsilon))$ is a family of non negative r.v. such that $\lim_{\epsilon\to 0} A(\epsilon) = 1$ a.s. In fact the left-hand side of (4.2) gives

$$\begin{aligned}
\int_0^T \frac{1}{\epsilon}\sup_{x\in[0,u]}|W_{x+\epsilon} - W_x|^2 du &\geq \int_{T/2}^T \frac{1}{\epsilon}\sup_{x\in[0,u]}|W_{x+\epsilon} - W_x|^2 du \geq \\
&\geq \int_{T/2}^T \frac{1}{\epsilon}\sup_{x\in[0,T/2-\epsilon]}|W_{x+\epsilon} - W_x|^2 du \\
&= \frac{T}{2\epsilon}\sup_{x\in[0,T/2-\epsilon]}|W_{x+\epsilon} - W_x|^2 .
\end{aligned}$$

Clearly we have $W_t = \sqrt{\frac{T}{2}}B_{\frac{2t}{T}}$ where B is another standard Brownian motion. Previous expression gives

$$\frac{T^2}{4\epsilon}\sup_{x\in[0,T/2-\epsilon]}|B_{(x+\epsilon)\frac{2}{T}} - B_{\frac{2x}{T}}|^2 = \frac{T^2}{4\epsilon}\sup_{y\in[0,1-\frac{2\epsilon}{T}]}|B_{y+\frac{2\epsilon}{T}} - B_y|^2$$

We choose $\tilde{\epsilon} = \frac{2\epsilon}{T}$. Previous expression gives

$$T\ln(1/\tilde{\epsilon})A^2(\tilde{\epsilon})$$

where

$$A(\epsilon) = \left(\frac{\sup_{x\in[0,1-\epsilon]}|B_{x+\epsilon} - B_x|}{\sqrt{2\,\epsilon\,\ln(1/\epsilon)}}\right) .$$

According to Theorem 1.1 in [2], $\lim_{\epsilon\to 0} A(\epsilon) = 1$ a.s. and the result is established. □

Remark 4.8. The window Brownian motion $W(\cdot)$ is not a $C([-\tau,0])$-valued semimartingale. In fact there is $\mu \in \mathcal{M}([-\tau,0])$ (take for instance $\mu = \delta_{-\tau/2} + \delta_0$) for which ${}_{\mathcal{M}([-\tau,0])}\langle \mu\,,\,W(\cdot)\rangle_{C([-\tau,0])}$ is not a real semimartingale; see Introduction of [6].

Below we will see that $W(\cdot)$, even if it does not admit a global quadratic variation, admits a χ-quadratic variation for several Chi-subspaces χ. More generally we can state a significant existence result of a χ-covariation for finite quadratic variation processes with the help of Corollaries 3.26 and 3.27. We remind that $\mathcal{D}_i([-\tau,0])$ and $\mathcal{D}_{i,j}([-\tau,0]^2)$ were defined at (2.6) and (2.7).

Proposition 4.9. Let X and Y be two real continuous processes with finite quadratic variation and $0 < \tau \leq T$. The following properties hold true.

1) $X(\cdot)$ and $Y(\cdot)$ admit a zero χ-covariation, where $\chi = L^2([-\tau,0]^2)$.

2) $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation for every $i \in \{0,\ldots,N\}$, where $\chi = L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i([-\tau,0])$.

If moreover the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\ldots,N\}$, the following statement is valid.

3) $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation, where $\chi = \mathcal{D}_{i,j}([-\tau,0]^2)$ and it equals

$$[X(\cdot),Y(\cdot)](\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}], \quad \forall \mu \in \chi. \tag{4.3}$$

Proof. The proof will be analogous in all the three cases. Example 3.6 says that the three involved sets χ are separable Chi-subspaces.
Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for $L^2([-\tau,0])$; $\{f_i = \delta_{a_i}\}$ is clearly a basis for $\mathcal{D}_i([-\tau,0])$. Then $\{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0]^2)$, $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])$ and $\{f_i \otimes f_j\}$ is a basis of $\mathcal{D}_{i,j}([-\tau,0]^2)$. The results will follow using Corollary 3.27. To verify Condition **H1** we consider

$$A(\epsilon) := \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (Y_{s+\epsilon}(\cdot) - Y_s(\cdot))\rangle_{\chi^*} \right| ds$$

for the three Chi-subspaces mentioned above. In all the three situations we will show the existence of a family of random variables $\{B(\epsilon)\}$ converging in probability to some random variable B, such that $A(\epsilon) \le B(\epsilon)$ a.s. By Remark 3.12.1 this will imply Assumption **H1**.

1) Suppose $\chi = L^2([-\tau,0]^2)$. By Cauchy-Schwarz inequality we have

$$\begin{aligned} A(\epsilon) &\le \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-\tau,0]^2)} \le 1} \|\phi\|^2_{L^2([-\tau,0]^2)} \cdot \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_{L^2([-\tau,0])} \cdot \|Y_{s+\epsilon}(\cdot) - Y_s(\cdot)\|_{L^2([-\tau,0])}\, ds \le \\ &\le \frac{1}{\epsilon}\int_0^T \sqrt{\int_0^s (X_{u+\epsilon} - X_u)^2\, du}\sqrt{\int_0^s (Y_{v+\epsilon} - Y_v)^2\, dv}\, ds \le B(\epsilon) \end{aligned}$$

where

$$B(\epsilon) = T\sqrt{\int_0^T \frac{(X_{u+\epsilon} - X_u)^2}{\epsilon}\, du \int_0^T \frac{(Y_{v+\epsilon} - Y_v)^2}{\epsilon}\, dv}$$

which converges in probability to $T\sqrt{[X]_T[Y]_T}$.

2) We proceed similarly for $\chi = L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])$.

We consider ϕ of the form $\phi = \tilde{\phi} \otimes \delta_{\{a_i\}}$, where $\tilde{\phi}$ is an element of $L^2([-\tau,0])$. We first observe

$$\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i} = \left\|\tilde{\phi}\right\|_{L^2([-\tau,0])} \cdot \left\|\delta_{\{a_i\}}\right\|_{\mathcal{D}_i} = \sqrt{\int_{[-\tau,0]} \tilde{\phi}(s)^2\, ds}\,.$$

Then

$$A(\epsilon) = \frac{1}{\epsilon} \int_0^T \sup_{\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i} \leq 1} \left| (X_{s+\epsilon}(a_i) - X_s(a_i)) \int_{[-\tau,0]} (Y_{s+\epsilon}(x) - Y_s(x))\, \tilde{\phi}(x)\, dx \right| ds \leq$$

$$\leq \frac{1}{\epsilon} \int_0^T \sup_{\|\phi\| \leq 1} \left\{ \left(\sqrt{(X_{s+\epsilon}(a_i) - X_s(a_i))^2} \right) \cdot \right.$$

$$\left. \cdot \left(\left\| \tilde{\phi} \right\|_{L^2([-\tau,0])} \sqrt{\int_{[-\tau,0]} (Y_{s+\epsilon}(x) - Y_s(x))^2 \, dx} \right) \right\} ds \leq$$

$$\leq \int_0^T \sqrt{\frac{(X_{s+\epsilon}(a_i) - X_s(a_i))^2}{\epsilon}} \sqrt{\int_{[-T,0]} \frac{(Y_{s+\epsilon}(x) - Y_s(x))^2}{\epsilon} dx}\; ds \leq B(\epsilon)$$

where

$$B(\epsilon) = \sqrt{T \int_0^T \frac{(X_{x+\epsilon} - X_x)^2}{\epsilon} dx \int_0^T \frac{(Y_{y+\epsilon} - Y_y)^2}{\epsilon} dy}\ ,$$

converges in probability to $\sqrt{T\,[X]_T [Y]_T}$ for $\epsilon \to 0$.

3) The last case is $\chi = \mathcal{D}_{i,j}([-\tau, 0]^2)$. A general element ϕ which belongs to χ admits a representation $\phi = \lambda\, \delta_{\{(a_i,a_j)\}}$, with norm equals to $\|\phi\|_{\mathcal{D}_{i,j}} = |\lambda|$. We have

$$A(\epsilon) = \frac{1}{\epsilon} \int_0^T \sup_{\|\phi\|_{\mathcal{D}_{i,j}} \leq 1} \left| \lambda \left(X_{s+a_i+\epsilon} - X_{s+a_i} \right) \left(Y_{s+a_j+\epsilon} - Y_{s+a_j} \right) \right| ds \leq$$

$$\leq \frac{1}{\epsilon} \int_0^T \left| (X_{s+a_i+\epsilon} - X_{s+a_i}) \left(Y_{s+a_j+\epsilon} - Y_{s+a_j} \right) \right| ds\ ; \tag{4.4}$$

using again Cauchy-Schwarz inequality, previous quantity is bounded by

$$\sqrt{\int_0^T \frac{\left(X_{s+a_i+\epsilon} - X_{s+a_i}\right)^2}{\epsilon} ds} \sqrt{\int_0^T \frac{\left(Y_{v+a_j+\epsilon} - Y_{v+a_j}\right)^2}{\epsilon} dv} \leq B(\epsilon) \tag{4.5}$$

where

$$B(\epsilon) = \sqrt{\int_0^T \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon} ds \int_0^T \frac{\left(Y_{v+\epsilon} - Y_v\right)^2}{\epsilon} dv}$$

which converges in probability to $\sqrt{[X]_T [Y]_T}$.

We verify now Conditions **H0"** and **H2"**.

1) A general element in $\{e_i \otimes e_j\}_{i,j \in \mathbb{N}}$ is difference of two positive definite elements in the set $\mathcal{S}^p = \{e_i \otimes^2, (e_i + e_j) \otimes^2\}_{i,j \in \mathbb{N}}$. We also define $\mathcal{S} = \{e_i \otimes e_j\}_{i,j \in \mathbb{N}}$ and The fact that $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ implies **H0"**. To conclude we need to show the validity of Condition **H2"**. For this we have to verify

$$[X(\cdot), Y(\cdot)]^\epsilon (e_i \otimes e_j)(t) \xrightarrow[\epsilon \longrightarrow 0]{} 0 \tag{4.6}$$

in probability for any $i, j \in \mathbb{N}$. Clearly we can suppose $\{e_i\}_{i\in\mathbb{N}} \in C^1([-\tau, 0])$. We fix $\omega \in \Omega$, outside some null set, fixed but omitted. We have

$$[X(\cdot), Y(\cdot)]^\epsilon(e_i \otimes e_j)(t) = \int_0^t \frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon} ds \tag{4.7}$$

where

$$\gamma_j(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_j(y)\,(X_{s+y+\epsilon} - X_{s+y})\,dy$$

and

$$\gamma_i(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_i(x)\,(Y_{s+x+\epsilon} - Y_{s+x})\,dx\ .$$

Without restriction of generality, in the purpose not to overcharge notations, we can suppose from now on that $\tau = T$.

For every $s \in [0, T]$, we have

$$\begin{aligned} |\gamma_j(s,\epsilon)| &= \left| \int_{-s}^0 (e_j(y-\epsilon) - e_j(y))\,X_{s+y}dy + \int_0^\epsilon e_j(y-\epsilon)X_{s+y}dy - \int_{-s}^{-s+\epsilon} e_j(y-\epsilon)X_{s+y}dy \right| \leq \\ &\leq \epsilon \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \sup_{s\in[0,T]} |X_s|\ . \end{aligned} \tag{4.8}$$

For $t \in [0, T]$, this implies that

$$\begin{aligned} \int_0^t \left| \frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon} \right| ds &\leq \int_0^T \left| \frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon} \right| ds \\ &\leq T\ \epsilon\ \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \left(\int_{-T}^0 |\dot{e_i}(y)|dy + 2\|e_i\|_\infty \right) \left(\sup_{s\in[0,T]} |X_s| \right) \left(\sup_{u\in[0,T]} |Y_u| \right) \end{aligned}$$

which trivially converges a.s. to zero when ϵ goes to zero which yields (4.6).

2) A general element in $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is difference of two positive definite elements of type $\{e_j \otimes^2, f_i \otimes^2, (e_j + f_i)\otimes^2\}_{j\in\mathbb{N}}$. This shows **H0"**. It remains to show that

$$[X(\cdot), Y(\cdot)]^\epsilon\,(e_j \otimes f_i)\,(t) \longrightarrow 0 \tag{4.9}$$

in probability for every $j \in \mathbb{N}$. In fact the left-hand side equals

$$\int_0^t \frac{\gamma_j(s,\epsilon)}{\epsilon}\,(X_{s+a_i+\epsilon} - X_{s+a_i})\ ds\ .$$

Using estimate (4.8), we obtain

$$\int_0^t \left| \frac{\gamma_j(s,\epsilon)}{\epsilon}\,(Y_{s+a_i+\epsilon} - Y_{s+a_i}) \right| ds \leq T \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty \right) \left(\sup_{s\in[0,T]} |X_s| \right) \varpi_Y(\epsilon) \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0$$

where $\varpi_Y(\epsilon)$ is the usual (random in this case) continuity modulus, so the result follows.

3) A general element $f_i \otimes f_j$ is difference of two positive definite elements $(f_i + f_j)\otimes^2$ and $f_i \otimes^2 + f_j \otimes^2$. So that Condition **H0"** is fulfilled. Concerning Condition **H2"** we have, for $0 \leq i, j \leq N$,

$$[X(\cdot), Y(\cdot)]^{\epsilon} (f_i \otimes f_j)(t) = \frac{1}{\epsilon} \int_0^t \left(X_{s+a_i+\epsilon} - X_{s+a_i}\right)\left(Y_{s+a_j+\epsilon} - Y_{s+a_j}\right) ds \; .$$

This converges to $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ which exists by hypothesis.

This finally concludes the proof of Proposition 4.9. □

We recall that χ^2 and χ^0 were defined respectively at (3.2) and (3.4).

Corollary 4.10. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X, Y]$ exist. Then for every $i \in \{0, \ldots, N\}$, it yields

4) $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation, where $\chi = \mathcal{D}_i([-\tau, 0])\hat{\otimes}_h L^2([-\tau, 0])$.

5) $X(\cdot)$ and $Y(\cdot)$ admit $\chi^0([-\tau, 0]^2)$-covariation which equals

$$[X(\cdot), Y(\cdot)](\mu) = \mu(\{0, 0\})[X, Y], \ \forall \mu \in \chi^0. \tag{4.10}$$

If moreover $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for all $i, j = 0, \ldots, N$, then

6) $X(\cdot)$ and $Y(\cdot)$ admit a $\chi^2([-\tau, 0]^2)$-covariation which equals

$$[X(\cdot), Y(\cdot)]_t(\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]_t, \forall \mu \in \chi^2([-\tau, 0]^2), t \in [0, T]. \tag{4.11}$$

Proof. The considered χ^2 and χ^0 admit a finite direct sum decomposition given by (3.3). The results follow immediately applying Propositions 3.18 and 4.9. □

When $\chi = \mathcal{D}_{0,0}([-\tau, 0]^2)$ the existence of a χ-covariation between X and Y holds even under more relaxed hypotheses.

Proposition 4.11. Let X, Y be continuous processes such that $[X, Y]$ exists and hypothesis (3.23) is verified. Then $X(\cdot)$ and $Y(\cdot)$ admit a $\mathcal{D}_{0,0}([-\tau, 0]^2)$-covariation and

$$[X(\cdot), Y(\cdot)]_t(\mu) = \mu(\{0, 0\})[X, Y]_t \; .$$

Proof. The proof is again very similar to the one of Proposition 4.9. The only relevant difference consists in checking the validity of condition **H1**. This will be verified identically until (4.4); the next step will follow by (3.23). □

Example 4.12. We list some examples of processes X for which $X(\cdot)$ admits a χ-quadratic variation through Corollary 4.10.

1) All continuous real semimartingales S (for instance Brownian motion). In fact S has finite quadratic variation and it holds $[S_{\cdot+a_i}, S_{\cdot+a_j}] = 0$ for $i \neq j$, it follows easily by Corollary 3.11 in [4].

2) Consider a bifractional Brownian motion $B^{H,K}$ with parameters H and K.

   Proposition 4.13. Let $B^{H,K}$ be a Bifractional Brownian motion with $HK = 1/2$. Then $[B^{H,K}] = 2^{1-K}t$ and $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$.

   Remark 4.14.

   - If $K = 1$, then $H = 1/2$ and $B^{H,K}$ is a Brownian motion, case already treated.
   - In the case $K \neq 1$ we recall that the bifractional Brownian motion $B^{H,K}$ is not a semimartingale, see Proposition 6 from [24].

   Proof of Proposition 4.13. Proposition 1 in [24] says that $B^{H,K}$ has finite quadratic variation which is equal to $[B^{H,K}] = 2^{1-K}t$. By Proposition 1 and Theorem 2 in [17] there are two constants α and β depending on K, a centered Gaussian process $X^{H,K}$ with absolutely continuous trajectories on $[0, +\infty[$ and a standard Brownian motion W such that $\alpha X^{H,K} + B^{H,K} = \beta W$. Then

$$[\alpha X^{H,K}_{\cdot+a_i} + B^{H,K}_{\cdot+a_i}, \alpha X^{H,K}_{\cdot+a_j} + B^{H,K}_{\cdot+a_j}] \quad = \quad \beta^2 [W_{\cdot+a_i}, W_{\cdot+a_j}]. \tag{4.12}$$

   Using the bilinearity of the covariation, we expand the left-hand side in (4.12) into a sum of four terms

$$\alpha^2 [X^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha [B^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha [X^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] + [B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] \tag{4.13}$$

   Since $X^{H,K}$ has bounded variation then first three terms on (4.13) vanish because of point 6) of Proposition 1 in [28]. On the other hand term the right-hand side in (4.12) is equal to zero for $i \neq j$ since W is a semimartingale, see point 1). We conclude that $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$. $\square$

3) Let D be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $D = M + A$, M local martingale and A zero quadratic variation process. Then D satisfies the hypotheses of the Corollary 4.10. In fact $[D] = [M]$ and $[D_{\cdot+a_i}, D_{\cdot+a_j}] = 0$ for $i \neq j$. Consequently the associated window Dirichlet process admits a χ^2-quadratic variation.

4) Let D be a real $(\mathcal{F}_t)$-Dirichlet process with decomposition $M + A$, M being the $(\mathcal{F}_t)$-local martingale part and let N be a real $(\mathcal{F}_t)$-local martingale. Then $D(\cdot)$ and $N(\cdot)$ admit a χ^2-covariation given by $[D(\cdot), N(\cdot)](\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[D_{\cdot+a_i}, N_{\cdot+a_j}]. = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[M_{\cdot+a_i}, N_{\cdot+a_j}] = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[M_{\cdot+a_i}, N_{\cdot+a_i}]$.

5) Similar examples can be produced considering the window of a weak Dirichlet process with finite quadratic variation.

We go on with evaluations of χ-covariation.

Proposition 4.15. Let V and Z be two real absolutely continuous processes such that $V', Z' \in L^2([0,T])$ ω-a.s. Then the associated window processes $V(\cdot)$ and $Z(\cdot)$ have zero real and tensor covariation. In particular they have zero global covariation.

Proof. Similarly as in the proof of Proposition 4.1, using Lemma 3.20 point 2. and Proposition 3.16, we only need to show the convergence to zero in probability of the quantity

$$\int_0^T \frac{1}{\epsilon} \|V_{s+\epsilon}(\cdot) - V_s(\cdot)\|_B \|Z_{s+\epsilon}(\cdot) - Z_s(\cdot)\|_B \, ds \tag{4.14}$$

which equals

$$\int_0^T \frac{1}{\epsilon} \|(V_{s+\epsilon}(\cdot) - V_s(\cdot)) \otimes (Z_{s+\epsilon}(\cdot) - Z_s(\cdot))\|_{(B\hat{\otimes}_\pi B)^{**}} \, ds \, .$$

Using Cauchy-Schwarz (4.14) is bounded by

$$\sqrt{\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} |V_{s+\epsilon}(x) - V_s(x)|^2 \, ds} \cdot \sqrt{\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} |Z_{u+\epsilon}(x) - Z_u(x)|^2 \, du} \; ; \tag{4.15}$$

in fact we will even show the a.s. convergence of (4.15). The square of the first term in (4.15) equals

$$\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} \left| \int_{s+x}^{s+x+\epsilon} V'(y)dy \right|^2 ds \le \int_0^T \frac{1}{\epsilon} \max_{x\in[-\tau,0]} \int_{s+x}^{s+x+\epsilon} V'(y)^2 dyds \le T\, \varpi_{\int_0^{\cdot}(V'^2)(y)dy}(\epsilon) \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0 \, ,$$

since $\varpi_{\int_0^{\cdot}(V'^2)(y)dy}(\epsilon)$ denotes the modulus of continuity of the a.s. continuous function $t \mapsto \int_0^t (V'^2)(y)dy$. The square of the second term in (4.15) can be treated analogously and the result is finally established. $\square$

We will show now that, if X is a finite quadratic variation processes X then $\mathbb{X} = X(\cdot)$ admits a $Diag([-\tau,0]^2)$-quadratic variation, where $Diag([-\tau,0]^2)$ was defined in (2.10).

Proposition 4.16. Let $0 < \tau \le T$. Let X and Y be two real continuous processes such that $[X,Y]$ exists and (3.23) is verified. Then $X(\cdot)$ and $Y(\cdot)$ admit a $Diag([-\tau,0]^2)$-covariation. Moreover we have

$$\widetilde{[X(\cdot), Y(\cdot)]}_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X,Y]_{t-x} dx \, , \qquad t \in [0,T] \, , \tag{4.16}$$

where μ is a generic element in $Diag([-\tau,0]^2)$ of type $\mu(dx,dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$.

Remark 4.17. Taking into account the usual convention $[X,Y]_t = 0$ for $t < 0$, the process $\left(\int_0^{t\wedge\tau} g(-x)[X,Y]_{t-x}dx\right)_{t\ge 0}$ can also be written as $\left(\int_0^\tau g(-x)[X,Y]_{t-x}dx\right)_{t\ge0}$.

Proof. We recall that for a generic element μ we have $\|\mu\|_{Diag} = \|g\|_\infty$.
First we verify Condition **H1**. We can write

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{Diag}\leq 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (Y_{s+\epsilon}(\cdot) - Y_s(\cdot))\rangle\right| \, ds \\
&\leq \frac{1}{\epsilon}\int_0^T \sup_{\|g\|_\infty\leq 1} \left|\int_{-T}^0 g(x)\,(X_{s+\epsilon}(x) - X_s(x))\,(Y_{s+\epsilon}(x) - Y_s(x))\, dx\right| ds = \\
&= \int_0^T \sup_{\|g\|_\infty\leq 1} \left|\int_0^s \frac{(X_{x+\epsilon} - X_x)\,(Y_{x+\epsilon} - Y_x)}{\epsilon} g(x-s)\, dx\right| ds \,.
\end{aligned}$$

Condition **H1** is verified because of Hypothesis (3.23).
It remains to prove Condition **H2**. Using Fubini's theorem, we write

$$\begin{aligned}
[X(\cdot), Y(\cdot)]^\epsilon_t(\mu) &= \frac{1}{\epsilon}\int_0^t \langle \mu(dx,dy), (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (Y_{s+\epsilon}(\cdot) - Y_s(\cdot))\rangle \, ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-\tau,0]^2} (X_{s+\epsilon}(x) - X_s(x))\,(Y_{s+\epsilon}(y) - Y_s(y))\, g(x)\delta_x(dy) dx\, ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-\tau,0]} (X_{s+\epsilon}(x) - X_s(x))\,(Y_{s+\epsilon}(x) - Y_s(x))\, g(x) dx\, ds = \\
&= \int_{(-t)\vee(-\tau)}^0 g(x) \int_{-x}^t \frac{(X_{s+x+\epsilon} - X_{s+x})\,(Y_{s+x+\epsilon} - Y_{s+x})}{\epsilon}\, ds\, dx = \\
&= \int_{(-t)\vee(-\tau)}^0 g(x) \int_0^{t+x} \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds\; dx = \\
&= \int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds\; dx \,.
\end{aligned} \tag{4.17}$$

It remains to show the ucp convergence,

$$\left(\int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds\; dx\right)_{t\in[0,T]} \xrightarrow[\epsilon\longrightarrow 0]{ucp} \left(\int_0^{t\wedge\tau} g(-x) [X,Y]_{t-x}\, dx\right)_{t\in[0,T]}$$

i.e.

$$\sup_{t\leq T}\left|\int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds - [X,Y]_{t-x}\, dx\right| \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0 \tag{4.18}$$

The left-hand side of (4.18) is bounded by

$$\begin{aligned}
&\int_0^T |g(-x)| \sup_{t\in[0,T]}\left|\int_0^{t-x} \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds - [X,Y]_{t-x}\right| dx \\
&\leq T\,\|g\|_\infty \sup_{t\in[0,T]}\left|\int_0^t \frac{(X_{s+\epsilon} - X_s)\,(Y_{s+\epsilon} - Y_s)}{\epsilon}\, ds - [X,Y]_t\right| \,.
\end{aligned}$$

Since X and Y admit a covariation, previous expression converges to zero.
More explicitly we obtain

$$[X(\cdot),Y(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X,Y]_{t-x}\,dx = \begin{cases} \displaystyle\int_0^t g(-x)[X,Y]_{t-x}dx & 0\le t\le\tau \\ \displaystyle\int_0^\tau g(-x)[X,Y]_{t-x}dx & \tau<t\le T \end{cases} .$$

Previous expression has an obvious modification $\widetilde{[X(\cdot),Y(\cdot)]}$ which has finite variation with values in χ^*. The total variation is in fact easily dominated by $\int_0^T |[X,Y]_x|dx$. □

A useful proposition is the following. We recall notation given in (2.10) for $Diag_d([-\tau,0]^2)$ and for $D([-\tau,0])$, the space of cadlag functions equipped with the uniform topology.

Proposition 4.18. Let X be a finite quadratic variation process. Let $G:[0,T]\longrightarrow\chi := Diag_d([-\tau,0]^2)$, cadlag. We have

$$\int_0^T {}_{\chi}\langle G(s)\,,\,d\widetilde{[X(\cdot)]}_s\rangle_{\chi^*} = \int_0^\tau\left(\int_x^T g(s,-x)[X]_{ds-x}\right)dx = \int_0^\tau\left(\int_0^{T-x} g(s+x,-x)d[X]_s\right)dx \quad (4.19)$$

where $G(s)=g(s,x)\delta_y(dx)dy$ for some Borel function $g:[0,T]\times[-\tau,0]\longrightarrow\mathbb{R}$ and $[X]_{ds-x}$ the derivative of function $s\mapsto[X]_{s+x}$.

Remark 4.19. We recall that $t\mapsto g(t,\cdot)$ is continuous from $[0,T]$ to $D([-\tau,0])$ equipped with the $\|\cdot\|_\infty$ norm.

Proof. By Propositions 3.3 and 4.16 $X(\cdot)$ admits a χ-quadratic variation. The proof will be established fixing $\omega\in\Omega$. We first suppose that

$$G(s)=\sum_{i=1}^N A_i\mathbb{1}_{]t_i,t_{i+1}]}(s)+A_0\mathbb{1}_{\{0\}}(s) \quad (4.20)$$

where $0=t_0<\ldots<t_N=T$ is a subdivision of $[0,T]$ for some positive integer $N\in\mathbb{N}$, $A_0,\ldots,A_N\in\chi$; in particular there are $a_0,\ldots,a_N\in D([-\tau,0])$ with

$$A_i(dx,dy)=a_i(x)\delta_y(dx)dy \quad \text{for all } i\in\{0,\ldots,N\}\ . \quad (4.21)$$

Then (4.19) holds by use of Proposition 4.16.
To treat the general case we approach a general G by a sequence (G^n) of type (4.20), i.e.

$$G^n(s)=\sum_{i=1}^N A_i^n\mathbb{1}_{]t_i,t_{i+1}]}(s)+A_0^n\mathbb{1}_{\{0\}}(s) \quad (4.22)$$

where $A_i^n = G(t_{i+1})$, $0 \le i \le N-1$, $0 = t_0 < \ldots < t_N = T$ is a subdivision whose mesh goes to zero when N going to infinity and there exist $a_0^n, \ldots, a_N^n \in D([-\tau, 0])$ related to $A_0^n, \ldots, A_N^n$ through relation (4.21). Consequently we have

$$\int_0^T {}_{\chi}\langle G^n(s)\,,\, d\widetilde{[X(\cdot)]}_s\rangle_{\chi^*} = \int_0^\tau \left(\int_x^T g^n(s,-x)[X]_{ds-x}\right) dx \tag{4.23}$$

with $g^n(s,x) = \sum_{i=1}^N a_i^n(x) 1\!\!1_{]t_i, t_{i+1}]}(s) + a_0^n$. In particular $a_i^n = g(t_{i+1}, \cdot)$.
By assumption, for every $s \in [0,T]$ we have

$$\lim_{n\to+\infty} \sup_{x\in[-\tau,0]} |g^n(s,x) - g(s,x)| = 0 \; .$$

Consequently for every $x \in [0,\tau]$

$$\lim_{n\to+\infty} \int_x^T \left(g^n(s,-x) - g(s,-x)\right)[X]_{ds-x} = 0 \; .$$

Moreover

$$\left| \int_x^T \left(g^n(s,-x) - g(s,-x)\right)[X]_{ds-x} \right| \le \left(\sup_n \|g^n\|_\infty + \|g\|_\infty \right) [X]_T \; .$$

By Lebesgue dominated convergence theorem the right-hand side of (4.23) converges to the right-hand side of (4.19) and the result follows. □

Remark 4.20. If $[X]$ is absolutely continuous with respect to Lebesgue, (4.19) in the statement would be valid with $\chi = Diag([-\tau,0]^2)$.

5 Itô's formula

We need now to formulate the definition of the forward type integral for B-valued integrator and B^*-valued integrand, where B is a separable Banach space.

Definition 5.1. Let $(\mathbb{X}_t)_{t\in[0,T]}$ (respectively $(\mathbb{Y}_t)_{t\in[0,T]}$) be a B-valued (respectively a B^*-valued) stochastic process. We suppose $\mathbb{X}$ to be continuous and $\mathbb{Y}$ to be strongly measurable such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s.
For every fixed $t \in [0,T]$ we define the **definite forward integral of** $\mathbb{Y}$ **with respect to** $\mathbb{X}$ denoted by $\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B$ as the following limit in probability:

$$\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B := \lim_{\epsilon\to 0} \int_0^t {}_{B^*}\langle \mathbb{Y}_s, \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon}\rangle_B ds \; . \tag{5.1}$$

We say that the **forward stochastic integral of $\mathbb{Y}$ with respect to $\mathbb{X}$** exists if the process

$$\left(\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B\right)_{t\in[0,T]}$$

admits a continuous version. In the sequel indices B and B^* will often be omitted.

We are now able to state an Itô formula for stochastic processes with values in a general Banach space.

Theorem 5.2. Let χ be a Chi-subspace and $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation. Let $F : [0,T]\times B \longrightarrow \mathbb{R}$ of class $C^{1,2}$ Fréchet. such that

$$D^2F : [0,T]\times B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ continuously with respect to } \chi\ . \tag{5.2}$$

Then for every $t\in[0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B$$

exists and following formula holds:

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0)+\int_0^t \partial_t F(s,\mathbb{X}_s)ds+\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B+\frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}\ a.s. \tag{5.3}$$

Proof. We fix $t\in[0,T]$ and we observe that the quantity

$$I_0(\epsilon,t) = \int_0^t \frac{F(s+\epsilon,\mathbb{X}_{s+\epsilon})-F(s,\mathbb{X}_s)}{\epsilon}ds \tag{5.4}$$

converges ucp for $\epsilon\to 0$ to $F(t,\mathbb{X}_t)-F(0,\mathbb{X}_0)$ since $\big(F(s,\mathbb{X}_s)\big)_{s\geq 0}$ is continuous. At the same time, using Taylor's expansion, (5.4) can be written as the sum of the two terms:

$$I_1(\epsilon,t) = \int_0^t \frac{F(s+\epsilon,\mathbb{X}_{s+\epsilon})-F(s,\mathbb{X}_{s+\epsilon})}{\epsilon}ds \tag{5.5}$$

and

$$I_2(\epsilon,t) = \int_0^t \frac{F(s,\mathbb{X}_{s+\epsilon})-F(s,\mathbb{X}_s)}{\epsilon}ds\ , \quad \epsilon>0, \quad t\ \in[0,T]\ . \tag{5.6}$$

We fix $t\in[0,T]$ and we prove that

$$I_1(\epsilon,t)\longrightarrow \int_0^t \partial_t F(s,\mathbb{X}_s)ds \tag{5.7}$$

in probability. In fact

$$I_1(\epsilon,t) = \int_0^t \partial_t F(s,\mathbb{X}_{s+\epsilon})ds + R_1(\epsilon,t) \tag{5.8}$$

where

$$R_1(\epsilon,t) = \int_0^t \int_0^1 \big(\partial_t F\big(s+\alpha\epsilon, \mathbb{X}_{s+\epsilon}\big) - \partial_t F(s,\mathbb{X}_{s+\epsilon})\big)\, d\alpha ds \ .$$

For fixed $\omega \in \Omega$ we denote by $\mathcal{V}(\omega) := \{\mathbb{X}_t(\omega);\ t\in[0,T]\}$ and

$$\mathcal{U} = \mathcal{U}(\omega) = \overline{conv(\mathcal{V}(\omega))}, \tag{5.9}$$

i.e. the set $\mathcal{U}$ is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B. For $x \in \Omega$, we have

$$\sup_{t\in[0,T]} |R_1(\epsilon,t)| \le T\, \varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$$

where $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus in ϵ of the application $\partial_t F : [0,T]\times B \longrightarrow \mathbb{R}$ restricted to $[0,T]\times\mathcal{U}$. From the continuity of the $\partial_t F$ as function from $[0,T]\times B$ to $\mathbb{R}$, it follows that the restriction on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Therefore we have proved that $R_1(\epsilon,\cdot) \to 0$ ucp as $\epsilon \to 0$.
On the other hand the first term in (5.8) can be rewritten as

$$\int_0^t \partial_t F(s,\mathbb{X}_s) ds + R_2(\epsilon,t)$$

where $R_2(\epsilon,t)\to 0$ ucp arguing similarly as for $R_1(\epsilon,t)$ and so the convergence (5.7) is established.
The second addend $I_2(\epsilon,t)$ in (5.6), can be approximated by Taylor's expansion and it can be written as the sum of the following three terms:

$$\begin{aligned}
I_{21}(\epsilon,t) &= \int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon}-\mathbb{X}_s}{\epsilon}\rangle_B ds \ ,\\
I_{22}(\epsilon,t) &= \frac{1}{2}\int_0^t {}_\chi\langle D^2F(s,\mathbb{X}_s), \frac{(X_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle_{\chi^*} ds \ ,\\
I_{23}(\epsilon,t) &= \int_0^t \left[\int_0^1 \alpha\, {}_\chi\langle D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle_{\chi^*}\, d\alpha\right] ds \ .
\end{aligned}$$

Since $D^2F : [0,T]\times B \longrightarrow \chi$ is continuous and B separable, we observe that the process H defined by $H_s = D^2F(s,X_s)$ takes values in a separable closed subspace $\mathcal{V}$ of χ. Applying Corollary 3.22, it yields

$$I_{22}(\epsilon,t) \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{1}{2}\int_0^t {}_\chi\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}$$

for every $t\in[0,T]$.
We analyse now $I_{23}(\epsilon,t)$ and we show that $I_{23}(\epsilon,t) \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0$. In fact we have

$$\begin{aligned}
|I_{23}(\epsilon,t)| &\le \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left|{}_\chi\langle D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s), (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2\rangle_{\chi^*}\right| d\alpha\, ds \le\\
&\le \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left\|D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s)\right\|_\chi \left\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2\right\|_{\chi^*} d\alpha\, ds \le\\
&\le \varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)\int_0^t \sup_{\|\phi\|_\chi\le 1}\left|\langle\phi, \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle\right| ds \ ,
\end{aligned}$$

where $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus of the application $D^2F : [0,T] \times B \longrightarrow \chi$ restricted to $[0,T]\times\mathcal{U}$ where $\mathcal{U}$ is the same random compact set introduced in (5.9). So again D^2F on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Taking into account condition **H1** in the definition of χ-quadratic variation, $I_{23}(\epsilon,t)\to 0$ in probability when ϵ goes to zero.
Since $I_0(\epsilon,t)$, $I_1(\epsilon,t)$, $I_{22}(\epsilon,t)$ and $I_{23}(\epsilon,t)$ converge in probability for every fixed $t\in[0,T]$, it follows that $I_{21}(\epsilon,t)$ converges in probability when $\epsilon\to 0$. Therefore the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B$$

exists by definition. This in particular implies the Itô's formula (5.3). $\square$

We make now some operational comments. The Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ constitutes a degree of freedom in the statement of Itô's formula. In order to find the suitable expansion for $F(t,\mathbb{X}_t)$ we may proceed as follows.

- Let $F:[0,T]\times B\longrightarrow\mathbb{R}$ of class $C^{1,1}([0,T]\times B)$ we compute the second order derivative D^2F if it exists.

- We look for the existence of a Chi-subspace χ for which the range of $D^2F:[0,T]\times B\longrightarrow(B\hat{\otimes}_\pi B)^*$ is included in χ and it is continuous with respect to the topology of χ.

- We verify that $\mathbb{X}$ admits a χ-quadratic variation.

We observe that whenever $\mathbb{X}$ admits a global quadratic variation, i.e. $\chi=(B\hat{\otimes}_\pi B)^*$, previous points reduce to check that $F\in C^{1,2}([0,T]\times B)$. When $\mathbb{X}$ is a semimartingale (or more generally a semilocally summable B-valued process with respect to the tensor product) then it admits a tensor quadratic variation and in particular previous result generalizes the classical Itô formula in [20], Section 3.7.

6 Applications of Itô formula for window process

Let $0\le\tau\le T$. The scope of this section is to illustrate some elementary applications of our Banach valued Itô formula. One more involved application appear in [6] whose Theorem 7.1 treats a Clark-Ocone formula of pathwise type related to a process with the same quadratic variation as Brownian motion. The basic tool of the proof is precisely Theorem 5.2.

The results in [6] will be expanded with several extensions in [8].

6.1 An elementary case

We set $B = C([-T,0])$ and we define $u : [0,T] \times B \to \mathbb{R}$ as

$$u(t,\eta) = \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right)^2 + \frac{(T-t)^3}{3}, t \in [0,T], \eta \in C([0,T]) \; . \tag{6.1}$$

If W is a Brownian motion with related canonical filtration $(\mathcal{F}_t)$, u naturally appears in the evaluation of the conditional expectation $\mathbb{E}\left[\left(\int_0^T W(s)ds \right)^2 | \mathcal{F}_t \right]$ which in particular gives $u(t, W_t(\cdot))$.

We observe that $u \in C^{1,2}([0,T] \times C([-T,0]))$ and we evaluate the corresponding derivatives obtaining

$$\partial_t u(t,\eta) = -2\eta(0) \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right) - (T-t)^2$$

$$D_{dx} u(t,\eta) = D^{ac}_x u(t,\eta)dx + D^{\delta_0} u(t,\eta)\delta_0(dx)$$

where

$$D^{ac}_x u(t,\eta) = 2 \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right) \mathbb{1}_{[-T,0]}(x)$$

$$D^{\delta_0} u(t,\eta) = 2 \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right) (T-t)$$

and

$$\begin{aligned} D^2_{dx\,dy} u(t,\eta) = {} & 2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy + \\ & + 2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy) + \\ & + 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy + \\ & + 2(T-t)^2\delta_0(dx)\,\delta_0(dy) \; . \end{aligned} \tag{6.2}$$

We observe that for any (t,η) in $[0,T] \times C([-T,0])$ the first Fréchet derivative $Du(t,\eta)$ is the sum of a measure which is absolute continuous with respect to Lesbegue, denoted by $D^{ac}u(t,\eta)$, and a multiple of a Dirac measure at 0, denoted by $D^{\delta_0}u(t,\eta)$. In particular $Du(t,\eta)$ belongs to $\mathcal{D}_0([-T,0]) \oplus L^2([-T,0])$. Let now consider a continuous process X such that $X_0 = 0$ and $[X]_t = t$. A rich class of examples of such processes are given in Example 4.12 and in [6].

Representing the martingale $M_t = u(t, W_t(\cdot))$ as a stochastic integral allows to obtain a representation of $u(T, W_T(\cdot)) = \left(\int_0^T W_s ds \right)^2$. The illustrating proposition below shows that a similar representation holds replacing W with X. This will be done applying Theorem 5.2 to $u(T, X_T(\cdot))$.

Proposition 6.1. Let X be a continuous process such that $X_0 = 0$ and $[X]_t = t$. The random variable $h := \left(\int_0^T X_t dt \right)^2 = u(T, X_T(\cdot))$ equals

$$h = H_0 + \int_0^T \xi_t \, d^- X_t \tag{6.3}$$

with $H_0 = u(0, X_0(\cdot)) = T^3/3$ and $\xi_t = D^{\delta_0} u\,(t, X_t(\cdot)) = 2(T-t)\int_0^t X_s\,ds + 2(T-t)^2 X_t$.

Proof. In order to apply Theorem 5.2 to $u(T, X_T(\cdot))$, we observe that for any (t,η), $D^2u(t,\eta)$ belongs to $\chi^0([-T,0]^2)$ and $D^2u : [0,T] \times C([-T,0]) \to \chi^0([-T,0]^2)$ is continuous. Corollary 4.10 point 5) says whenever $\mathbb{X} = \mathbb{Y}$ that any finite quadratic variation process admits a $\chi^0([-T,0]^2)$-quadratic variation. Therefore Itô formula (5.3) for $u(T, X_T(\cdot))$ gives

$$u(T, X_T(\cdot)) = I_0 + I_1 + I_2 + I_3 \tag{6.4}$$

where

$$\begin{aligned}
I_0 &= u(0, X_0(\cdot)) = \frac{T^3}{3}\\
I_1 &= \int_0^T \partial_t u(t, X_t(\cdot))dt\\
I_2 &= \int_0^T \langle Du(t, X_t(\cdot)), d^- X_t(\cdot)\rangle\\
I_3 &= \frac{1}{2}\int_0^T \langle D^2u(t, X_t(\cdot)), d\widetilde{[X(\cdot)]}_t\rangle\ .
\end{aligned}$$

We get

$$\begin{aligned}
I_1 &= -2\int_0^T X_t \int_{-T}^0 X_t(s)ds\,dt - 2\int_0^T X_t^2(T-t)dt - \int_0^T (T-t)^2 dt\\
&= -2\int_0^T X_t\left(\int_0^t X_u du\right)dt - 2\int_0^T X_t^2(T-t)dt - \int_0^T (T-t)^2 dt\ .
\end{aligned}$$

Concerning I_2 it holds $I_2 = I_{21} + I_{22}$ with

$$\begin{aligned}
I_{21} &= \int_0^T \langle D^{ac}u(t, X_t(\cdot)), d^- X_t(\cdot)\rangle = \lim_{\epsilon\to 0}\int_0^T \langle D^{ac}u(t, X_t(\cdot)), \frac{X_{t+\epsilon}(\cdot) - X_t(\cdot)}{\epsilon}\rangle dt = \lim_{\epsilon\to 0} I_{21}(\epsilon),\\
I_{21}(\epsilon) &= 2\int_0^T \left(\int_0^t X_s\,ds\right)\left(\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon}ds\right)dt + 2\int_0^T (T-t)X_t\left(\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon}ds\right)dt,\\
I_{22} &= \int_0^T D^{\delta_0}u(t, X_t(\cdot))d^- X_t,
\end{aligned} \tag{6.5}$$

provided that I_{21} and I_{22} exist.
Since

$$\int_0^t \frac{X_{s+\epsilon} - X_s}{\epsilon}ds \xrightarrow[\epsilon\longrightarrow 0]{a.s.} X_t - X_0 = X_t\ ,$$

by Lebesgue dominated convergence theorem we get the convergence in probability of $I_{21}(\epsilon)$ to

$$I_{21} := 2\int_0^T X_t\left(\int_0^t X_u du\right)dt + 2\int_0^T X_t^2(T-t)dt\ .$$

Since I_2 and I_{21} exist, so does I_{22}. We recall the $\chi^0([-T,0]^2)$-quadratic variation of $X(\cdot)$ was given by (4.10). For the last term we obtain that

$$I_3 = \frac{1}{2}\int_0^T 2(T-t)^2 dt = \int_0^T (T-t)^2 dt \, .$$

We observe that $I_1 = -I_2 - I_3$ so that (6.4) gives the desired representation for $h = H(X_T(\cdot)) = u(T, X_T(\cdot))$ in the form □

6.2 A toy model with anticipative integration

We recall that $0 < \tau \leq T$ and we set $B = C([-\tau, 0])$. We consider (X_t) be a real finite quadratic variation process such that $X_0 = 0$ a.s. and prolongated as usual by continuity to the real line. One motivation is to express, for $s \in [0,T]$,

$$\int_0^s \int_{(-T)\vee(-\tau)}^0 g\left(X_{t+x}, X_{t-\tau}\right) dx\, d^- X_t \qquad \left(= \int_0^s \int_{(-t)\vee(-\tau)}^0 g\left(X_{t+x}, X_{t-\tau}\right) dx\, d^- X_t \right)$$

for some smooth enough $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. We remark that previous forward integral is not an Itô integral since the integrand is anticipating (not adapted). In this perspective we consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of class $C^2(\mathbb{R}^2)$ such that $f(x,y) = \int_0^y g(x,z)dz$. In particular $g = \partial_2 f$. More generally we aim in finding some identities involving path-dependent Itô's or Skorohod integrals with forward integrals. For this purpose, we start expanding

$$\int_{-\tau}^0 f\left(X_{x+t}, X_{t-\tau}\right) dx$$

through our Itô's formula. We will apply Itô formula to $F\left(X_t(\cdot)\right)$ where F is the functional

$$F : C([-\tau,0]) \longrightarrow \mathbb{R} \, , \qquad F(\eta) = \int_{-\tau}^0 f\left(\eta(x), \eta(-\tau)\right) dx \, . \tag{6.6}$$

In fact $F \in C^2(B)$ and below we express the derivatives:

$$D_{dx}F\left(\eta\right) = \partial_1 f\left(\eta(x), \eta(-\tau)\right) 1\!\!1_{[-\tau,0]}(x)dx + \int_{-\tau}^0 \partial_2 f\left(\eta(z), \eta(-\tau)\right) dz\; \delta_{-\tau}(dx)$$

and the second derivatives are

$$\begin{aligned} D^2_{dx,\,dy}F(\eta) &= \partial^2_{1\,1} f\left(\eta(x), \eta(-\tau)\right) 1\!\!1_{[-\tau,0]}(x)\delta_y(dx)\, dy \\ &+ \partial^2_{2\,1} f\left(\eta(x), \eta(-\tau)\right) \delta_{-\tau}(dx)\, 1\!\!1_{[-\tau,0]}(y)dy \\ &+ \partial^2_{1\,2} f\left(\eta(x), \eta(-\tau)\right) 1\!\!1_{[-\tau,0]}(x)dx\, \delta_{-\tau}(dy) \\ &+ \int_{-T}^0 \partial^2_{2\,2} f\left(\eta(z), \eta(-\tau)\right) dz\; \delta_{-\tau}(dx)\, \delta_{-\tau}(dy) \, . \end{aligned} \tag{6.7}$$

The second order Fréchet derivative $D^2F(\eta)$ belongs to χ with $\chi := Diag \oplus \mathcal{D}_{-\tau} \otimes_h L^2 \oplus L^2 \otimes_h \mathcal{D}_{-\tau} \oplus \mathcal{D}_{-\tau,-\tau}$. Since X is a finite quadratic variation process, Propositions 4.9, 4.16 and 3.18 imply that $X(\cdot)$ admits a χ-quadratic variation. We apply now Theorem 5.2 to $F\left(X_T(\cdot)\right)$. The forward integral appearing in Itô formula

$$I_1 := \int_0^T \langle DF(X_t(\cdot))\,,\, d^- X_t(\cdot)\rangle$$

exists and it is given by $I_{11} + I_{12}$ where

$$I_{11} = \lim_{\epsilon \to 0} \int_0^T \int_{-\tau}^0 \partial_1 f\left(X_{t+x}, X_{t-\tau}\right) \frac{X_{t+x+\epsilon} - X_{t+x}}{\epsilon} dx\, dt$$

and

$$I_{12} = \lim_{\epsilon \to 0} \int_0^T \left(\int_{-\tau}^0 \partial_2 f\left(X_{t+x}, X_{t-\tau}\right) dx \right) \frac{X_{t-\tau+\epsilon} - X_{t-\tau}}{\epsilon} dt\ ,$$

provided that previous limits in probability exist.
We have

$$\begin{aligned} I_{11} &= \lim_{\epsilon \to 0} \int_0^T \int_{(-\tau)\vee(-t)}^0 \partial_1 f\left(X_{t+x}, X_{t-\tau}\right) \frac{X_{t+x+\epsilon} - X_{t+x}}{\epsilon} dx\, dt \\ &= \lim_{\epsilon \to 0} \int_0^T \int_{(t-\tau)\vee(0)}^t \partial_1 f\left(X_y, X_{t-\tau}\right) \frac{X_{y+\epsilon} - X_y}{\epsilon} dy\, dt\ . \end{aligned}$$

Fubini's theorem and previous limit give

$$I_{11} = \int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\left(X_y, X_{t-\tau}\right) dt \right) d^- X_y \tag{6.8}$$

provided that previous forward limit exists.

Remark 6.2. If X is an $(\mathcal{F}_t)$-semimartingale (6.8) is the Itô integral

$$\int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\left(X_y, X_{t-\tau}\right) dt \right) dX_y\ .$$

We go on specifying I_{12}

$$\begin{aligned} I_{12} &= \lim_{\epsilon \to 0} \int_\tau^T \left(\int_{-\tau}^0 \partial_2 f\left(X_{t+x}, X_{t-\tau}\right) dx \right) \frac{X_{t-\tau+\epsilon} - X_{t-\tau}}{\epsilon} dt \\ &= \lim_{\epsilon \to 0} \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(X_{y+x+\tau}, X_y\right) dx \right) \frac{X_{y+\epsilon} - X_y}{\epsilon} dy \\ &= \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(X_{y+x+\tau}, X_y\right) dx \right) d^- X_y \end{aligned}$$

provided that previous forward integral exists.
We evaluate now the integrals involving the second order derivative of F, i.e.

$$\frac{1}{2}\int_0^T {}_{\chi}\langle D^2F(X_t(\cdot))\,,\,d\widetilde{[X(\cdot)]}_t\rangle_{\chi^*}\ . \tag{6.9}$$

We remind that $D^2F(\eta)$ takes values in $\chi := Diag \oplus \mathcal{D}_{-\tau}\otimes_h L^2 \oplus L^2\otimes_h \mathcal{D}_{-\tau}\oplus \mathcal{D}_{-\tau,-\tau}$. The term (6.9) splits into a sum of four terms. Since by Proposition 4.9 item 2), $X(\cdot)$ has zero $\mathcal{D}_{-\tau}\otimes_h L^2$ and $L^2\otimes_h \mathcal{D}_{-\tau}$-quadratic variation, the only non vanishing integrals are the two terms I_{21} and I_{22} given respectively by the $\mathcal{D}_{-\tau,-\tau}$ and the $Diag$-quadratic variation. Again by Proposition 4.9 item 3), expression (6.9) becomes $I_{21}+I_{22}$ where

$$I_{21}=\frac{1}{2}\int_0^{T-\tau}\int_{-\tau}^0 \partial^2_{2\,2}f\,(X_{y+z+\tau},X_y)\,dz\,d[X]_y \qquad \text{and}$$

$$I_{22}=\frac{1}{2}\int_0^T {}_{Diag}\langle G(t)\,,\,d\widetilde{[X(\cdot)]}_t\rangle_{Diag^*}$$

and $G(t)=g(t,x)\delta_y(dx)dy$, with $g(t,x)=\partial^2_{1\,1}f\,(X_{t+x},X_{t-\tau})$. Since $\partial^2_{1\,1}f$ is a continuous function, Proposition 4.18 can be applied and we get

$$I_{22}=\frac{1}{2}\int_{-\tau}^0\left(\int_{-x}^T \partial^2_{1\,1}f\,(X_{t+x},X_{t-\tau})\ [X]_{dt+x}\right)dx\ .$$

In conclusion we obtain

$$\begin{aligned}\int_{-\tau}^0 f\,(X_{x+t},X_{t-\tau})\,dx &= \tau\, f(0,0)+\int_0^T\left(\int_y^{(y+\tau)\wedge T}\partial_1 f\,(X_y,X_{t-\tau})\,dt\right)d^-X_y\\ &+\int_0^{T-\tau}\left(\int_{-\tau}^0 \partial_2 f\,(X_{y+x+\tau},X_y)\,dx\right)d^-X_y+\frac{1}{2}\int_0^{T-\tau}\left(\int_{-\tau}^0\partial^2_{2\,2}f\,(X_{y+z+\tau},X_y)\,dz\right)d[X]_y\\ &+\frac{1}{2}\int_{-\tau}^0\left(\int_{-x}^T\partial^2_{1\,1}f\,(X_{t+x},X_{t-\tau})\ [X]_{dt+x}\right)dx\ .\end{aligned}$$

This leads to the following result.

Proposition 6.3. Let X be a finite quadratic variation process such that $X_0=0$. Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function of class C^2.

$$\begin{aligned}\int_{-\tau}^0 f\,(X_{x+t},X_{t-\tau})\,dx &= \tau\, f(0,0)+\int_0^T\left(\int_y^{(y+\tau)\wedge T}\partial_1 f\,(X_y,X_{t-\tau})\,dt\right)d^-X_y\\ &+\int_0^{T-\tau}\left(\int_{-\tau}^0 \partial_2 f\,(X_{y+x+\tau},X_y)\,dx\right)d^-X_y+\frac{1}{2}\int_0^{T-\tau}\left(\int_{-\tau}^0\partial^2_{2\,2}f\,(X_{y+z+\tau},X_y)\,dz\right)d[X]_y\\ &+\frac{1}{2}\int_{-\tau}^0\left(\int_{-x}^T\partial^2_{1\,1}f\,(X_{t+x},X_{t-\tau})\ [X]_{dt+x}\right)dx\end{aligned}$$

provided that at least one of the two forward integrals above exists.

Corollary 6.4. Let X be an $(\mathcal{F}_t)$-semimartingale and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of class $C^{2,1}(\mathbb{R}\times\mathbb{R})$. Then the forward integral $\int_0^{T-\tau} \left(\int_{-\tau}^0 g\left(X_{y+\tau+x}, X_y\right) dx \right) d^- X_y$ exists and it can be explicitly given.

Proof. We set $f(x, y) = \int_0^y g(x, z)dz$. The first forward integral in Proposition 6.3 exists and it is an Itô integral. We apply finally Proposition 6.3. □

Corollary 6.5. Let $X = W$ be a classical Wiener process, $f \in C^2(\mathbb{R}^2)$. We have the following identity.

$$\begin{aligned}
\int_{-\tau}^0 f\left(W_{x+t}, W_{t-\tau}\right) dx &= \tau f(0,0) + \int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\left(W_y, W_{t-\tau}\right) dt \right) dW_y \\
&\quad + \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(W_{y+x+\tau}, W_y\right) dx \right) \delta W_y + \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial^2_{2\,1} f\left(W_{t+\tau+z}, W_t\right) dz \right) dt \\
&\quad + \frac{1}{2} \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial^2_{2\,2} f\left(W_{y+z+\tau}, W_y\right) dz \right) dy + \frac{1}{2} \int_{-\tau}^0 \left(\int_{-x}^T \partial^2_{1\,1} f\left(W_{t+x}, W_{t-\tau}\right) dt \right) dx \,.
\end{aligned}$$

Remark 6.6. If $Y \in \mathbb{D}^{1,2}\left(L^2([0,T])\right)$, DY represents the Malliavin derivative and $\int_0^t Y_s \delta Y_s$, $t \in [0,T]$, is the Skorohod integral. The reader may consult for instance [22] for more details about Malliavin calculus. We recall that, by [25] and [28]

$$\int_0^t Y_s d^- W_s = \int_0^t Y_s \delta W_s + \left(Tr^- DY\right)(t) \tag{6.10}$$

where

$$\left(Tr^- DY\right)(t) = \lim_{\epsilon\to 0} \int_0^t \left(\int_s^{s+\epsilon} \frac{D_r Y_s}{\epsilon} dr \right) ds$$

in $L^2(\Omega)$.

Proof of Corollary 6.5. We need to prove that

$$\int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(W_{y+x+\tau}, W_y\right) dx \right) d^- W_y$$

equals

$$\int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(W_{y+x+\tau}, W_y\right) dx \right) \delta W_y + \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial^2_{2\,1} f\left(W_{t+\tau+z}, W_t\right) dz \right) dt \,.$$

It follows from Proposition 6.3 and previous Remark 6.6 with

$$Y_s = \int_{-\tau}^0 \partial_2 f\left(W_{s+\tau+z}, W_s\right) dz \,.$$

In fact, for $r > s$, $D_r Y_s = \int_{r-s-\tau}^{0} \partial_{2\,1}^2 f\left(W_{s+\tau+z}, W_s\right) dz$ and so

$$\left(Tr^{-} DY\right)(t) = \lim_{r \downarrow s} \int_0^t D_r Y_s \, ds = \int_0^t \left(\int_{-\tau}^{0} \partial_{2\,1}^2 f\left(W_{s+\tau+z}, W_s\right) dz \right) ds \, . \tag{6.11}$$

Combining (6.11) with (6.10) for $t = T - \tau$ the result is now established. $\square$

Remark 6.7. Let X be a Gaussian centered process with covariance $R(s,t) = \mathbb{E}\left[X_s X_t\right]$ such that $\frac{\partial^2 R}{\partial_s \partial_t}$ is a signed finite measure μ. X has therefore a covariance measure structure according to [16]. We recall that in this case X is a finite quadratic variation process and $[X]_t = \mu(D_t)$ with $D_t = \{(s,s) | s \in [0,t]\}$. With some slight technical assumptions, the following relation holds:

$$\int_0^t Y_s d^- X_s = \int_0^t Y_s \delta X_s + \int_{[0,t]^2} D_{r^+} Y_s d\mu(r,s) \, . \tag{6.12}$$

This allows to show the existence of both the forward integrals in the statement of Proposition 6.3 using (6.12).

A Appendix: Proofs of some technical results

Sketch of the proof of the Proposition 1.5. Let $\mathbb{V}$ (resp. $\mathbb{Y}$) be an H-valued bounded variation (resp. continuous) process. Proceeding as for real valued processes, see for instance [28], Proposition 1.7)b), one can show that $\mathbb{V}$ and $\mathbb{Y}$ has a zero real covariation. A semilocally summable process is the sum of a locally summable process and a bounded variation process. Therefore, without restriction of generality, we can suppose that $\mathbb{X}$ is locally summable with respect to the tensor products. By localization we can suppose that $\mathbb{X}$ is summable with respect to the tensor products and bounded. Let $s \geq 0$ and consider the following identity

$$\mathbb{X}_{s+\epsilon}^{\otimes^2} - \mathbb{X}_s^{\otimes^2} = \mathbb{X}_s \otimes \left(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\right) + \left(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\right) \otimes \mathbb{X}_s + \left(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\right) \otimes^2 \, . \tag{A.1}$$

Dividing (A.1) by ϵ and integrating from 0 to t in the Bochner sense we obtain

$$I_0(t,\epsilon) = I_1(t,\epsilon) + I_2(t,\epsilon) + \int_0^t \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} ds \tag{A.2}$$

where

$$I_0(t,\epsilon) = \int_0^t \frac{\mathbb{X}_{s+\epsilon}^{\otimes^2} - \mathbb{X}_s^{\otimes^2}}{\epsilon} ds \, ,$$
$$I_1(t,\epsilon) = \int_0^t \frac{\mathbb{X}_s \otimes (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)}{\epsilon} ds \, ,$$
$$I_2(t,\epsilon) = \int_0^t \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes \mathbb{X}_s}{\epsilon} ds \, .$$

Let $t \in [0,T]$. Obviously we get

$$\lim_{\epsilon \to 0} I_0(t,\epsilon) = \mathbb{X}_t^{\otimes^2} - \mathbb{X}_0^{\otimes^2} .$$

By an elementary Fubini argument we can show that

$$I_1(t,\epsilon) = \int_0^t \left(\frac{1}{\epsilon} \int_{u-\epsilon}^u \mathbb{X}_s ds \right) \otimes d\mathbb{X}_u .$$

Since $\frac{1}{\epsilon}\int_{u-\epsilon}^u \mathbb{X}_s ds \longrightarrow \mathbb{X}_u$ for every $u \in [0,T]$ and $\omega \in \Omega$ and $\mathbb{X}$ being bounded, Theorem 1 in section 12. A of [9] allows to show that $I_1(t,\epsilon) \longrightarrow \int_0^t \mathbb{X}_s \otimes d\mathbb{X}_s$ in probability. Similarly one shows that $I_2(t,\epsilon) \longrightarrow \int_0^t d\mathbb{X}_s \otimes \mathbb{X}_s$. In conclusion $\mathbb{X}$ admits a tensor quadratic variation which equals

$$\mathbb{X}_t^{\otimes^2} - \int_0^t \mathbb{X}_s \otimes d\mathbb{X}_s - \int_0^t d\mathbb{X}_s \otimes \mathbb{X}_s .$$

$\square$

Sketch of the proof of Proposition 1.6. Let H be the Hilbert space values of $\mathbb{X}$. Let $\mathbb{V}$ (resp. $\mathbb{Y}$) be an H-valued bounded variation (resp. continuous) process. Without restriction of generality we can suppose that $\mathbb{X}$ is an $(\mathcal{F}_t)$-local martingale. After localization one can suppose that $\mathbb{X}$ is an $(\mathcal{F}_t)$-square integrable martingale. Proceeding similarly as for the proof of Proposition 1.5, using Remark 14.b) of Chapter 6.23 of [9], it is possible to show that

$$\frac{1}{\epsilon} \int_0^t \|\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\|_H^2 ds \xrightarrow[\epsilon \longrightarrow 0]{} \|\mathbb{X}_t\|_H^2 - 2 \int_0^t \langle \mathbb{X}_s, d\mathbb{X}_s \rangle_H .$$

The analogous of the bilinear forms considered in Proposition 1.5 proof will be the H inner product. $\square$

Before writing the proof of Proposition 3.21 we need a technical lemma. In the sequel the indices χ and χ^* in the duality, will often be omitted.

Lemma A.1. Let $t \in [0,T]$. There is a subsequence of (n_k) still denoted by the same symbol and a null subset N of Ω such that

$$\widetilde{F}^{n_k}(\omega,t)(\phi) \longrightarrow_{k \to \infty} \widetilde{F}(\omega,t)(\phi) \tag{A.3}$$

for every $\phi \in \chi$ and $\omega \notin N$.

Proof of Lemma A.1 . Let $\mathcal{S}$ be a dense countable subset of χ. By a diagonalization principle for extracting subsequences, there is a subsequence (n_k), a null subset N of Ω such that for all $\omega \notin \Omega$,

$$\widetilde{F}_\infty(\omega,t)(\phi) := \lim_{k \to +\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \tag{A.4}$$

exists for any $\phi \in \mathcal{S}$, $\omega \notin N$ and $\forall\, t \in [0,T]$.
By construction, for every $t \in [0,T]$, $\phi \in \mathcal{S}$

$$\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) = \widetilde{F}_\infty(\cdot,t)(\phi) \qquad a.s.$$

Let $t \in [0,T]$ be fixed. Since $\phi \in \mathcal{S}$ countable, a slight modification of the null set N, yields that for every $\omega \notin N$,

$$\widetilde{F}(\omega,t)(\phi) = \widetilde{F}_\infty(\omega,t)(\phi) \qquad \forall\, \phi \in \mathcal{S}\ .$$

At this point (A.4) becomes

$$\widetilde{F}(\omega,t)(\phi) = \lim_{k\to+\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \tag{A.5}$$

for every $\omega \notin N$, $\phi \in \mathcal{S}$.
It remains to show that (A.5) still holds for $\phi \in \chi$. Therefore we fix $\phi \in \chi$, $\omega \notin N$. Let $\epsilon > 0$ and $\phi_\epsilon \in \mathcal{S}$ such that $\|\phi - \phi_\epsilon\|_\chi \le \epsilon$. We can write

$$\begin{aligned}\left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| &\le \left|\widetilde{F}(\omega,t)(\phi-\phi_\epsilon)\right| + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right| + \left|\widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon - \phi)\right| \le \\ &\le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*} \|\phi-\phi_\epsilon\|_\chi + \sup_k \left\|\widetilde{F}^{n_k}(\omega,t)\right\|_{\chi^*} \|\phi-\phi_\epsilon\|_\chi + \\ &\quad + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right|\ .\end{aligned}$$

Taking the $\limsup_{k\to+\infty}$ in previous expression and using (A.5) yields

$$\limsup_{k\to+\infty} \left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| \le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*} \epsilon + \sup_k \left\|\widetilde{F}^{n_k}(\omega,\cdot)\right\|_{Var[0,T]} \epsilon\ .$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

Proof of Proposition 3.21 . Let $t \in [0,T]$ be fixed. We denote

$$I(n)(\omega) := \int_0^t \langle H(\omega,s), d\widetilde{F}^n(\omega,s)\rangle - \int_0^t \langle H(\omega,s), d\widetilde{F}(\omega,s)\rangle\ .$$

Let $\delta > 0$ and a subdivision of $[0,t]$ given by $0 = t_0 < t_1 < \cdots < t_m = t$ whose mesh is smaller than δ. Let (n_k) be a sequence diverging to infinity. We need to exhibit a subsequence (n_{k_j}) such that

$$I(n_{k_j})(\omega) \longrightarrow 0 \qquad \text{a.s.} \tag{A.6}$$

Lemma A.1 implies the existence of a null set N, a subsequence (n_{k_j}) such that

$$\left|\widetilde{F}^{n_{k_j}}(\omega,t_l)(\phi) - \widetilde{F}(\omega,t_l)(\phi)\right| \xrightarrow[j\longrightarrow+\infty]{} 0 \qquad \forall\, \phi \in \chi \quad \text{and for every} \quad l \in \{0,\dots,m\}\ . \tag{A.7}$$

Let $\omega \notin N$. We have

$$
\begin{aligned}
\big|I(n_{k_j})(\omega)\big| &= \left|\sum_{i=1}^m \left(\int_{t_{i-1}}^{t_i} \langle H(\omega,s), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle - \langle H(\omega,s), d\widetilde{F}(\omega,s)\rangle\right)\right| \leq \\
&\leq \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega,s) - H(\omega,t_{i-1}) + H(\omega,t_{i-1}), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle + \right. \\
&\qquad\qquad \left. - \int_{t_{i-1}}^{t_i} \langle H(\omega,s) - H(\omega,t_{i-1}) + H(\omega,t_{i-1}), d\widetilde{F}(\omega,s)\rangle\right| \leq \\
&\leq I_1(n_{k_j})(\omega) + I_2(n_{k_j})(\omega) + I_3(n_{k_j})(\omega)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega,s) - H(\omega,t_{i-1}), d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle\right| \leq \varpi_{H(\omega,\cdot)}(\delta)\ \sup_j \|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]} \\
I_2(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega,s) - H(\omega,t_{i-1}), d\widetilde{F}(\omega,s)\rangle\right| \leq \varpi_{H(\omega,\cdot)}(\delta)\, \|\widetilde{F}(\omega)\|_{Var[0,T]} \\
I_3(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega,t_{i-1}), d(\widetilde{F}^{n_{k_j}}(\omega,s) - \widetilde{F}(\omega,s))\rangle\right| = \\
&= \sum_{i=1}^m \left|\langle H(\omega,t_{i-1}), \widetilde{F}^{n_{k_j}}(\omega,t_i) - \widetilde{F}(\omega,t_i) - \widetilde{F}^{n_{k_j}}(\omega,t_{i-1}) + \widetilde{F}(\omega,t_{i-1})\rangle\right| \leq \\
&\leq \sum_{i=1}^m |F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_i) - F(H(\omega,t_{i-1}))(\omega,t_i)| + \\
&\qquad\qquad \sum_{i=1}^m |F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_{i-1}) - F(H(\omega,t_{i-1}))(\omega,t_{i-1})|\ .
\end{aligned}
$$

The notation $\varpi_{H(\omega,\cdot)}$ indicates the modulus of continuity for H and it is a random variable; in fact it depends on ω in the sense that

$$
\varpi_{H(\omega,\cdot)}(\delta) = \sup_{|s-t|\leq\delta} \|H(\omega,s) - H(\omega,t)\|_{\chi}\ .
$$

By (A.7) applied to $\phi = H(\omega,t_{i-1})$ we obtain

$$
\limsup_{j\to\infty} |I(n_{k_j})(\omega)| \leq \left(\sup_j \|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]} + \|\widetilde{F}(\omega)\|_{Var[0,T]}\right) \varpi_{H(\omega,\cdot)}(\delta)\,. \tag{A.8}
$$

Since $\delta > 0$ is arbitrary and H is uniformly continuous on $[0,t]$ so that $\varpi_{H(\omega,\cdot)}(\delta) \to 0$ a.s. for $\delta \to 0$, then $\limsup_{j\to\infty} |I(n_{k_j})(\cdot)| = 0$ a.s..
This concludes (A.6) and the proof of Proposition 3.21. □

Proof of Theorem 3.24. Supposing **iv')**, Lemma 2.1 implies that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$, since for every $\phi \in \mathcal{S}$, $F(\phi)$ is an increasing process, so **iv)** is established. We only show the result considering **iv)**.

a) We recall that $\mathscr{C}([0,T])$ is an F-space. Let $\phi \in \chi$. Clearly $(F^n(\phi)(\cdot,t))_t$ and $\left(\tilde{F}^n(\cdot,t)(\phi)\right)_t$ are indistinguishable processes and so $\left(\tilde{F}^n(\phi)(\cdot,t)\right)_t$ is a continuous process. So it follows

$$\begin{aligned}\|F^n(\phi)\|_\infty &= \sup_{t\in[0,T]} |F^n(\phi)(t)| = \sup_{t\in[0,T]} |\tilde{F}^n(\cdot,t)(\phi)| \leq \\ &\leq \sup_{t\in[0,T]} \left\|\tilde{F}^n(\cdot,t)\right\|_{\chi^*} \|\phi\|_\chi \leq \sup_n \|\tilde{F}^n\|_{Var([0,T])} \|\phi\|_\chi < +\infty\end{aligned}$$

a.s. by the hypothesis. By Remark 3.23.2. and 3. it follows that the set $\{F^n(\phi)\}$ is a bounded subset of the F-space $\mathscr{C}([0,T])$ for every fixed $\phi \in \chi$.

We can apply the Banach-Steinhaus Theorem II.1.18, pag. 55 in [10] and point **iv)**, which imply the existence of $F : \chi \longrightarrow \mathscr{C}([0,T])$ linear and continuous such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \chi$. So **a)** is established in both situations **1)** and **2)**.

b) It remains to show the rest in situation **1)**, i.e. when χ is separable.

b.1) We first prove the existence of a suitable version $\tilde{F}$ of F such that $\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ is weakly star continuous ω a.s.

Since χ is separable, we consider a dense countable subset $\mathcal{D} \subset \chi$. Point **a)** implies that for a fixed $\phi \in \mathcal{D}$ there is a subsequence (n_k) such that $F^{n_k}(\phi)(\omega,\cdot) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot)$ a.s. Since $\mathcal{D}$ is countable there is a null set N and a further subsequence still denoted by (n_k) such that

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot) \qquad \forall\, \phi \in \mathcal{D},\ \forall \omega \notin N\ . \tag{A.9}$$

For $\omega \notin N$, we set $\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t)\ \forall\ \phi \in \mathcal{S},\ t \in [0,T]$. By a slight abuse of notation the sequence $\tilde{F}^{n_k}$ can be seen as applications

$$\tilde{F}^{n_k}(\omega,\cdot) : \chi \longrightarrow C([0,T])$$

which are linear continuous maps verifying the following.

- $\tilde{F}^{n_k}(\omega,\cdot)(\phi) \longrightarrow \tilde{F}(\omega,\cdot)(\phi)$ in $C([0,T])$ for all $\phi \in \mathcal{D}$, because of (A.9).
- For every $\phi \in \chi$, we have

$$\begin{aligned}\sup_k \sup_{t\leq T} |\tilde{F}^{n_k}(\omega,t)(\phi)| &\leq \sup_k \sup_{t\leq T} \sup_{\|\phi\|_\chi\leq 1} |\tilde{F}^{n_k}(\omega,t)(\phi)|\ \|\phi\|_\chi \leq \sup_k \sup_{t\leq T} \|\tilde{F}^{n_k}(\omega,t)\|\ \|\phi\|_\chi \\ &\leq \sup_k \|\tilde{F}^{n_k}(\omega,\cdot)\|_{Var([0,T])} \|\phi\|_\chi < +\infty.\end{aligned}$$

Banach-Steinhaus thereom implies the existence of a linear random continuous map

$$\tilde{F}(\omega,\cdot):\chi\longrightarrow C([0,T])$$

extending previous map $\tilde{F}(\omega,\cdot)$ from $\mathcal{D}$ to χ with values on $C([0,T])$. Moreover

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi)\xrightarrow{C([0,T])}\tilde{F}(\omega,\cdot)(\phi)\qquad\forall\phi\in\chi,\,\forall\omega\notin N$$

and for every $\omega\notin N$ the application

$$\tilde{F}(\omega,\cdot):[0,T]\longrightarrow\chi^*\qquad t\mapsto\tilde{F}(\omega,t)$$

is weakly star continuous. $\tilde{F}$ is measurable from $\Omega\times[0,T]$ to χ^* being limit of measurable processes.

b.2) We prove now that the χ^*-valued process $\tilde{F}$ has bounded variation.
Let $\omega\notin N$ fixed again. Let $(t_i)_{i=0}^M$ be a subdivision of $[0,T]$ and let $\phi\in\chi$. Since the functions

$$F^{t_i,t_{i+1}}:\phi\longrightarrow\left(\tilde{F}(t_{i+1})-\tilde{F}(t_i)\right)(\phi)\qquad F^{n_k,t_i,t_{i+1}}:\phi\longrightarrow\left(\tilde{F}^{n_k}(t_{i+1})-\tilde{F}^{n_k}(t_i)\right)(\phi)$$

belong to χ^*, Banach-Steinhaus theorem says

$$\sup_{\|\phi\|\le1}\left|\left(\tilde{F}(t_{i+1})-\tilde{F}(t_i)\right)(\phi)\right|=\|F^{t_i,t_{i+1}}\|_{\chi^*}\le\liminf_{k\to\infty}\|F^{n_k,t_i,t_{i+1}}\|_{\chi^*}=$$
$$=\liminf_{k\to\infty}\sup_{\|\phi\|\le1}\left|\left(\tilde{F}^{n_k}(t_{i+1})-\tilde{F}^{n_k}(t_i)\right)(\phi)\right|\ .$$

Taking the sum over $i=0,\ldots,(M-1)$ we get

$$\sum_{i=0}^{M-1}\sup_{\|\phi\|\le1}\left|\left(\tilde{F}(t_{i+1})-\tilde{F}(t_i)\right)(\phi)\right|\le\sum_{i=0}^{M-1}\liminf_{k\to\infty}\sup_{\|\phi\|\le1}\left|\left(\tilde{F}^{n_k}(t_{i+1})-\tilde{F}^{n_k}(t_i)\right)(\phi)\right|\le$$
$$\le\sup_k\sum_{i=0}^{M-1}\sup_{\|\phi\|\le1}\left|\left(\tilde{F}^{n_k}(t_{i+1})-\tilde{F}^{n_k}(t_i)\right)(\phi)\right|\le\sup_k\|\tilde{F}^{n_k}\|_{Var([0,T])}\ ,$$

where the second inequality is justified by the relation $\liminf a_i^n+\liminf b_i^n\le\sup(a_i^n+b_i^n)$.
Taking the sup over all subdivision $(t_i)_{i=0}^M$ we obtain

$$\|\tilde{F}\|_{Var([0,T])}\le\sup_k\|\tilde{F}^{n_k}\|_{Var([0,T])}<+\infty\ .$$

This shows finally the fact that $\tilde{F}(\omega,\cdot):[0,T]\longrightarrow\chi^*$ has bounded variation.

□

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