MOMENT ASYMPTOTICS FOR THE PARABOLIC ANDERSON PROBLEM WITH A PERTURBED LATTICE POTENTIAL

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ABSTRACT. The parabolic Anderson problem with a random potential obtained by attaching a long tailed potential around a randomly perturbed lattice is studied. The moment asymptotics of the total mass of the solution is derived. The results show that the total mass of the solution concentrates on a small set in the space of configuration.

1. INTRODUCTION

This paper is a continuation of [4]. We consider the initial value problem of the heat equation with a random potential

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,x) = \frac{1}{2}\Delta v(t,x) - V_\xi(x)v(t,x), \qquad (t,x)\in(0,\infty)\times\mathbb{R}^d,\\ &v(0,x) = \delta_{x_0}(x), \qquad x\in\mathbb{R}^d, \end{aligned} \tag{1.1}$$

where Δ is the Laplacian, $x_0 \in \mathbb{R}^d$, and

$$V_\xi(x) := \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q) \tag{1.2}$$

with $\xi = (\xi_q)_{q\in\mathbb{Z}^d}$ a collection of independent and identically distributed random vectors. Under appropriate assumptions, (1.1) has a solution $v_\xi(t,x;x_0)$ represented by the Feynman-Kac formula

$$v_\xi(t,x;x_0) = E_{x_0}\left[\exp\left\{-\int_0^t V_\xi(B_s)ds\right\}\middle| B_t = x\right]\frac{1}{(2\pi t)^{d/2}}\exp\Big(-\frac{|x-x_0|^2}{2t}\Big), \tag{1.3}$$

where $(B_s)_{s\geq 0}$ is the Brownian motion on $\mathbb{R}^d$ and E_{x_0} is the expectation of the Brownian motion starting at x_0.

In this paper, we investigate the long time asymptotics of the moment of the total mass

$$v_\xi(t;x_0) := \int_{\mathbb{R}^d} v_\xi(t,x;x_0)dx_0 = E_{x_0}\left[\exp\left\{-\int_0^t V_\xi(B_s)ds\right\}\right]. \tag{1.4}$$

Our main result is Theorem 1.2, which deals with the first moment. We also obtain results on the higher moments in Section 3 below.

The operator $H_\xi = -\Delta/2+V_\xi$ is the Hamiltonian of the so-called random displacement model in the theory of random Schrödinger operators and there has recently been an increase in research, see e.g. [1, 2, 3, 4, 7]. Also, the initial value problem (1.1) itself is called the "parabolic Anderson problem" in literature (see e.g. a

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survey article by Gärtner and König [5]). The solution of the parabolic Anderson problem is believed to concentrate on a relatively small region and there are many results support this concentration. We shall discuss this aspect in more detail in Subsection 3.2 below.

1.1. **Basic assumptions.** We are mainly interested in the case where the single site potential and the displacement variables satisfy the following: (i) u is a nonnegative function belonging to the Kato class K_d (cf. [8]) and

$$u(x) = C_0|x|^{-\alpha}(1 + o(1)) \tag{1.5}$$

as $|x| \to \infty$ for some $\alpha > d$ and $C_0 > 0$; (ii) each ξ_q has the explicit distribution

$$\mathbb{P}_\theta(\xi_q \in dx) = \frac{1}{Z(d,\theta)} \sum_{p \in \mathbb{Z}^d} \exp(-|p|^\theta)\delta_p(dx) \tag{1.6}$$

for some $\theta > 0$ and the normalizing constant $Z(d, \theta)$.

We also consider the case that u is a nonpositive function. For this case, we assume $\inf u = u(0) > -\infty$, (1.5) for some $C_0 < 0$, and that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $u(x) \le u(0) + \varepsilon$ for $|x| < R_\varepsilon$. Nevertheless, our main interest is the nonnegative case and we assume $u \ge 0$ unless otherwise specified.

1.2. **Motivation.** In Theorem 6.3 of the preceding paper [4], we have shown the following:

Theorem 1.1. *Let us define*

$$c(d, \alpha, \theta, C_0) := \int_{\mathbb{R}^d} dq \inf_{y \in \mathbb{R}^d} (\frac{C_0}{|q + y|^\alpha} + |y|^\theta). \tag{1.7}$$

(i) *Assume that $d = 1$ and that $\operatorname{ess\,inf}_{B(R)} u > 0$ for any $R \ge 1$ if $\alpha \le 3$. Then we have*

$$\log \mathbb{E}_\theta[v_\xi(t; x_0)] \begin{cases} \sim -t^{(1+\theta)/(\alpha+\theta)} c(1, \alpha, \theta, C_0) & (1 < \alpha < 3) \\ \asymp -t^{(1+\theta)/(3+\theta)} & (\alpha = 3), \\ \sim -t^{(1+\theta)/(3+\theta)} \dfrac{3+\theta}{1+\theta} (\dfrac{\pi^2}{8})^{(1+\theta)/(3+\theta)} & (\alpha > 3) \end{cases} \tag{1.8}$$

as $t \to \infty$, where $f(t) \sim g(t)$ means $\lim_{t\to\infty} f(t)/g(t) = 1$ and $f(t) \asymp g(t)$ means $0 < \underline{\lim}_{t\to\infty} f(t)/g(t) \le \overline{\lim}_{t\to\infty} f(t)/g(t) < \infty$.

(ii) *Assume that $d = 2$ and that $\operatorname{ess\,inf}_{B(R)} u > 0$ for any $R \ge 1$ if $\alpha \le 4$. Then we have*

$$\log \mathbb{E}_\theta[v_\xi(t; x_0)] \begin{cases} \sim -t^{(2+\theta)/(\alpha+\theta)} c(2, \alpha, \theta, C_0) & (2 < \alpha < 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} & (\alpha = 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} (\log t)^{-\theta/(4+\theta)} & (\alpha > 4) \end{cases} \tag{1.9}$$

as $t \to \infty$.

(iii) *Assume that* $d \geq 3$ *and that* $\operatorname{ess\,inf}_{B(R)} u > 0$ *for any* $R \geq 1$ *if* $\alpha \leq d+2$. *Then we have*

$$\log \mathbb{E}_\theta[v_\xi(t;x_0)] \begin{cases} \sim -t^{(d+\theta)/(\alpha+\theta)} c(d,\alpha,\theta,C_0) & (d<\alpha<d+2), \\ \asymp -t^{(d+\theta\mu)/(d+2+\theta\mu)} & (\alpha \geq d+2) \end{cases} \tag{1.10}$$

as $t \to \infty$, *where*

$$\mu = \frac{2(\alpha-2)}{d(\alpha-d)}. \tag{1.11}$$

(iv) *Assume* $u \leq 0$, $\sup u = u(0) > -\infty$, *and the existence of* $R_\varepsilon > 0$ *for any* $\varepsilon > 0$ *such that* $\operatorname{ess\,sup}_{B(R_\varepsilon)} u \leq u(0)+\varepsilon$. *Then we have*

$$\log \mathbb{E}_\theta[v_\xi(t;x_0)] \sim t^{1+d/\theta} c_-(d,\theta,u(0)) \tag{1.12}$$

as $t \to \infty$, *where*

$$c_-(d,\theta,K) := \frac{2\pi^{d/2}\theta|K|^{1+d/\theta}}{d(d+\theta)\Gamma(d/2)} \tag{1.13}$$

for $K \in \mathbb{R}$.

We have precise forms of the leading terms for the one-dimensional case with $\alpha \neq 3$, the general dimensional case with $d < \alpha < d+2$, and the case of $u \leq 0$. Furthermore, if one goes into the proof of these results, it will be observed that only a very small set in ξ-space contributes the leading terms of the asymptotics. More precisely, when $u \geq 0$ and $d < \alpha < d+2$ for instance, the y-variable in the definition of $c(d,\alpha,\theta,C_0)$ corresponds to the displacement ξ_q from q. Therefore taking the infimum in the definition of $c(d,\alpha,\theta,C_0)$ with respect to y means minimizing the sum of the contribution of $u(-q-\xi_q)$ to $V_\xi(0)$ and the cost for displacement for each q. With these interpretation, the above theorem says that only the *optimal* configuration contributes the leading term. This kind of concentration in ξ-space is sometimes regarded as a collateral evidence of the aforementioned spatial irregularity of $v_\xi(t,x;x_0)$, see Sect. 1.3 of [5]. The aim of this paper is to find a variational expression for the leading part in the remaining cases to see a concentration phenomenon similar to above.

1.3. **Main result.** We need to introduce some notations to state the results. We write Λ_r for $[-r/2,r/2]^d$ and introduce scaling factors

$$r = \begin{cases} t^{1/(3+\theta)} & (d=1 \text{ and } \alpha=3), \\ t^{1/(4+\theta)}(\log t)^{\theta/(8+2\theta)} & (d=2 \text{ and } \alpha>4), \\ t^{1/(d+2+\mu\theta)} & (d\geq 3 \text{ and } \alpha \geq d+2 \text{ or } (d,\alpha)=(2,4)). \end{cases} \tag{1.14}$$

For any open set U and $\xi = (\xi_q)_{q\in\mathbb{Z}^d} \in (\mathbb{Z}^d)^{\mathbb{Z}^d}$, we denote by $\lambda_\xi^r(U)$ the bottom of the spectrum of

$$-\frac{1}{2}\Delta + V_\xi^r$$

in U with the Dirichlet boundary condition, where

$$V_\xi^r(x) := \sum_{q\in\mathbb{Z}^d} r^2 u(rx - q - \xi_q).$$

Finally, let $\Omega_t = (\mathbb{Z}^d)^{\Lambda_t\cap\mathbb{Z}^d}$, which is the set of possible configurations of $(\xi_q)_{q\in\Lambda_t\cap\mathbb{Z}^d}$, and we write $\lambda_\xi^r(U)$ for the same object as above also for $\xi\in\Omega_t$ with the potential replaced by

$$V_\xi^r(x) := \sum_{q\in\mathbb{Z}^d\cap\Lambda_t} r^2 u(rx - q - \xi_q).$$

Theorem 1.2. *Assume that $\alpha = 3$ for $d = 1$ and $\alpha \geq d+2$ for $d \geq 2$. Under the above setting, we have*

$$\log \mathbb{E}_\theta[v_\xi(t;x_0)] = -tr^{-2}\inf_{\zeta\in\Omega_t}\left\{\lambda_\zeta^r(\Lambda_{t/r}) + \gamma(r)^\theta \sum_{q\in\Lambda_t\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta\right\}(1+o(1)) \tag{1.15}$$

as t goes to ∞, where

$$\gamma(r) = \begin{cases} 1 & (d=1 \text{ and } \alpha = 3),\\ \sqrt{(4+\theta)\log r} & (d=2 \text{ and } \alpha > 4),\\ r^{1-\mu} & (d\geq 3 \text{ or } (d,\alpha) = (2,4)),\end{cases} \tag{1.16}$$

and μ is the number defined in (1.11).

The interpretation of this result is as follows. For a given configuration $\xi = \zeta$, the eigenfunction expansion indicates that

$$v_\zeta(t,x) = \exp\left\{-\lambda_\zeta^1(\Lambda_t)t(1+o(1))\right\} \tag{1.17}$$

since the contribution from outside Λ_t is negligible. On the other hand, the probability to have such a configuration is formally given by

$$\mathbb{P}_\theta(\xi=\zeta) = \exp\left\{-\sum_{q\in\mathbb{Z}^d}|\zeta_q|^\theta(1+o(1))\right\}. \tag{1.18}$$

Therefore, the variational problem to minimize the sum of the decay rate for fixed configuration and the cost to realize it has the form

$$\inf_\zeta\left\{\lambda_\zeta^1(\Lambda_t)t + \sum_{q\in\mathbb{Z}^d}|\zeta_q|^\theta\right\}, \tag{1.19}$$

which becomes almost the same as the right hand side of (1.15) after the scaling. Hence, the above theorem says that only the *optimal* configuration contributes the leading part of the asymptotics, just as in the heavy tailed case.

2. Proof of Theorem 1.2

In Theorem 2.9 of [3], the leading term for $\log\mathbb{E}_\theta[v_\xi(t;x_0)]$ with compactly supported u was investigated by using Sznitman's "method of enlargement of obstacles". We shall apply the same method here.

2.1. **Method of enlargement of obstacles for the multidimensional case.** Let us first recall the elements of the methods developed in [3]. It is basically a coarse graining method to establish a certain variational principle by reducing the number of configurations contributing the asymptotics. In this subsection, we define a set of reduced configurations and show that its cardinality is indeed negligible compared with the decay of $\mathbb{E}_\theta[v_\xi(t;x_0)]$ (see (2.7) and (2.8) below).

We take $\chi \in ((\mu - 2/d)\theta, \mu\theta)$ and $\eta \in (0,1)$ so small that

$$\chi > \left(\mu - \frac{2}{d}\right)\theta + 2\eta^2 + \left(d - 2 + \frac{2\theta}{d}\right)\eta$$

and define

$$\gamma := \frac{d-2}{d} + \frac{2\eta}{d} < 1.$$

We further introduce a notation concerning a diadic decomposition of $\mathbb{R}^d$. For each $k \in \mathbb{Z}_+$, let $\mathcal{I}_k$ be the collection of indices $\imath = (i_0, i_1, \dots, i_k)$ with $i_0 \in \mathbb{Z}^d$ and $i_1, \dots, i_k \in \{0,1\}^d$. For each $\imath \in \mathcal{I}_k$, we associate the box

$$C_\imath = q_\imath + 2^{-k}[0,1]^d,$$

where

$$q_\imath = i_0 + 2^{-1}i_1 + \cdots + 2^{-k}i_k.$$

For $\imath \in \mathcal{I}_k$ and $\imath' \in \mathcal{I}_{k'}$ with $k' \le k$, $\imath \preceq \imath'$ means that the first k' coordinates coincide. Finally, we introduce

$$n_\beta = \left[\beta \frac{\log r}{\log 2}\right]$$

for $\beta > 0$ so that $2^{-n_\beta - 1} < r^{-\beta} \le 2^{-n_\beta}$.

We can now define the <u>density</u> set, which we can discard from the consideration.

Definition 2.1. We call a unit cube C_q with $q \in \mathbb{Z}^d$ a <u>density</u> box if all $q \preceq \imath \in \mathcal{I}_{n_{\eta\gamma}}$ satisfy the following: for at least half of $\imath \preceq \imath' \in \mathcal{I}_{n_\gamma}$,

$$(q_{\imath'} + 2^{-n_\gamma - 1}[0,1]^d) \cap \{(q + \xi_q)/r : q \in \mathbb{Z}^d\} \neq \emptyset. \tag{2.1}$$

The union of all <u>density</u> boxes is denoted by $\underline{\mathcal{D}}_r(\xi)$.

The following theorem tells us that we can replace $\underline{\mathcal{D}}_r(\xi)$ by a hard trap without causing a substantial increase in the principal eigenvalue.

Spectral control. There exists $\rho > 0$ such that for all $M > 0$ and sufficiently large r,

$$\sup_{\xi \in (\mathbb{R}^d)^{\mathbb{Z}^d}} \left(\lambda_\xi^r\left(\underline{\mathcal{R}}_r(\xi)\right) \wedge M - \lambda_\xi^r\left(\Lambda_{t/r}\right) \wedge M\right) \le r^{-\rho}, \tag{2.2}$$

where $\underline{\mathcal{R}}_r(\xi) = \Lambda_{t/r} \setminus \underline{\mathcal{D}}_r(\xi)$.

By Proposition 2.7 in [3], the proof of this theorem is reduced to the extension of Theorem 4.2.3 in [8] from the compactly supported single site potentials to the Kato class single site potentials, which is straightforward.

For $\underline{\mathcal{R}}_r(\xi)$, we can give the following quantitative estimate on its volume:

Lemma 2.2. (i) *There exists a positive constant* c_1 *independent of* r *such that* $|\underline{\mathcal{R}}_r(\xi)| \geq r^\chi$ *implies*

$$\sum_{q \in \Lambda_t \cap \mathbb{Z}^d} |\xi_q|^\theta \geq c_1 r^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi}. \tag{2.3}$$

(ii) *There exists a positive constant* c_2 *independent of* r *such that*

$$\mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \geq r^\chi) \leq \exp(-c_2 r^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi}). \tag{2.4}$$

In particular, $\mathbb{P}_\theta(|\underline{\mathcal{R}}_r(\xi)| \geq r^\chi) = o(\mathbb{E}_\theta[v_\xi(t; x_0)])$.

Proof. Throughout the proof, c_1 and c_2 are positive constants whose values may change line by line. We consider the following necessary condition of $C_q \not\subset \underline{\mathcal{D}}_r(\xi)$:

$$\begin{aligned}&\text{there exists an } \imath \succeq q \text{ in } \mathcal{I}_{n_{\eta\gamma}} \text{ such that for a half of } \imath' \succeq \imath \text{ in } \mathcal{I}_{n_\gamma},\\ &\{r^{-1}q' + r^{-1}\xi_{q'} : q' \in (rC_{\imath'}) \cap \mathbb{Z}^d\} \not\subset q_{\imath'} + 2^{-n_\gamma-1}[0,1]^d.\end{aligned} \tag{2.5}$$

Note first that

$$\sum_{q' \in (rC_{\imath'}) \cap \mathbb{Z}^d} |\xi_{q'}|^\theta \geq \sum_{q' \in (rC_{\imath'}) \cap \mathbb{Z}^d} |d(q', \partial(rC_{\imath'}))|^\theta \geq c_1 r^{(1-\gamma)(d+\theta)}$$

for any configurations satisfying the second line in (2.5). Thus $C_q \not\subset \underline{\mathcal{D}}_r(\xi)$ implies

$$\begin{aligned}\sum_{q' \in (rC_q) \cap \mathbb{Z}^d} |\xi_{q'}|^\theta &\geq c_1 r^{(1-\gamma)(d+\theta)} 2^{d(n_\gamma - n_{\eta\gamma})-1}\\ &\geq c_2 r^{(1-\gamma)(d+\theta)+d\gamma(1-\eta)}\end{aligned} \tag{2.6}$$

and the first assertion follows from this.

For the second assertion, we use (2.6) and take the sum over the possibilities of the indices $\imath$ and $\imath'$'s in (2.5) to obtain

$$\begin{aligned}&\mathbb{P}_\theta((2.5) \text{ is satisfied})\\ &\leq 2^{dn_{\eta\gamma}} \binom{2^{d(n_\gamma - n_{\eta\gamma})}}{2^{d(n_\gamma - n_{\eta\gamma})-1}} \exp(-c_1 r^{(1-\gamma)(d+\theta)+d\gamma(1-\eta)})\\ &\leq \exp(-c_2 r^{d(1-\eta\gamma)+(1-\gamma)\theta})\end{aligned}$$

for large r. In the second line, the first factor represents the choice of the index $\imath$ and the second factor the choice of the indices $\imath'$'s. Since the variables $\{\xi_{q'} : q' \in C_q \cap \mathbb{Z}^d\}$ are independent in $q \in \mathbb{Z}^d$, we have

$$\begin{aligned}\mathbb{P}_\theta(|\Lambda_{t/r} \setminus \underline{\mathcal{D}}_r(\xi)| \geq r^\chi) &\leq t^{dr^\chi} \left(\exp(-c_2 r^{d(1-\eta\gamma)+(1-\gamma)\theta})\right)^{r^\chi}\\ &\leq \exp(-c_2 r^{d(1-\eta\gamma)+(1-\gamma)\theta+\chi}),\end{aligned}$$

which is the desired estimate.

Finally the third assertion follows from Theorem 1.1 and our choice of χ. □

With the help of this lemma, we may restrict ourselves on some special configurations. To see this, we introduce some more notations. A domain R is called a lattice animal if it is represented as

$$R = \left(\overline{\bigcup_{q \in S(R)} \Lambda_1(q)}\right)^\circ,$$

where $S(R) \subset \mathbb{Z}^d$ consists of adjacent sites. This means that R is a combination of unit cubes connected via faces. We set

$$\begin{aligned}\mathcal{S}_r = \big\{ &(R_r, \zeta = (\zeta_q)_{q\in(r[R_r:l])\cap\mathbb{Z}^d}) : R_r \text{ is a lattice animal included in } \Lambda_{t/r}, \\ &|R_r| < r^\chi, q + \zeta_q \in [\mathcal{T} : t^{1/(\mu\theta)}] \cap \mathbb{Z}^d \text{ for all } q \in (r[R_r : l]) \cap \mathbb{Z}^d \big\},\end{aligned} \tag{2.7}$$

where l is a positive number specified later, and $[A : l] = \{x \in \mathbb{R}^d : d(x, A) < l\}$ for any $A \subset \mathbb{R}^d$. For any $(R_r, \zeta) \in \mathcal{S}_r$, we write

$$V^r_\zeta(x) = \sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} r^2 u(rx - q - \zeta_q)$$

with a slight abuse of the notation and define $\lambda^r_\zeta(R_r)$ accordingly.

We now see that the *relevant* configurations of $(\underline{\mathcal{R}}_r(\xi), \xi)$ are only the pairs in $\mathcal{S}_r$. In fact removing the points $\{q + \xi_q : q \in \mathbb{Z}^d \setminus (r[R_r : l])\}$, which should be cared in proving the lower bound, is permitted as we will show in Lemma 2.5 below. We also have

$$\lambda^r_\xi (\underline{\mathcal{R}}_r(\xi)) = \lambda^r_\xi (R_r)$$

for some lattice animal R_r included in $\underline{\mathcal{R}}_r(\xi)$ and

$$\mathbb{P}_\theta(q + \xi_q \notin [\mathcal{T} : t^{1/(\mu\theta)}] \text{ for some } q \in (r[R_r : l]) \cap \mathbb{Z}^d)$$

decays exponentially in t. The latter easily follows by observing that

$$d(r[R_r : l], [\mathcal{T} : t^{1/(\mu\theta)}]^c) > t^{1/\theta},$$

which is due to $lr + t^{1/\theta} < t^{1/(\mu\theta)}$, for large t.

The key point in our coarse graining method is that the number of relevant configurations is estimated as

$$\#\mathcal{S}_r \le t^{dr^\chi}(t + 2t^{1/(\mu\theta)})^{dr^{d+\chi}c(1+l)} = o(\mathbb{E}_\theta[v_\xi(t; x_0)]^{-1}) \tag{2.8}$$

by an elementary counting argument, where c is a finite constant depending only on d. The second relation comes from our choice of χ.

2.2. **Proof of a modified statement for the multidimensional case.** We state and prove slightly modified versions of Theorem 1.2 in this section. They are shown to be equivalent to Theorem 1.2 in Subsection 2.4 below. Let us start with the multidimensional case.

Theorem 2.3. *Let $d \ge 2$ and assume the setting of Theorem 1.2. Then we have the following:*

(i) *For any $\varepsilon > 0$ and $l > 0$, there exists $t_{\varepsilon,l} > 0$ such that*

$$\begin{aligned}&t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t; x_0)] \\ &\quad\le -(1-\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\}\end{aligned} \tag{2.9}$$

for any $t \ge t_{\varepsilon,l}$, where $\gamma(r)$ is the function defined in (1.16).

(ii) *If* $\alpha > d+2$, *then for any* $\varepsilon > 0$ *and* $l > 0$, *there exists* $t_{\varepsilon,l} > 0$ *such that*

$$\begin{aligned}&t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t;x_0)]\\ &\ge -(1+\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda_\zeta^r(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\}\end{aligned} \tag{2.10}$$

for any $t \ge t_{\varepsilon,l}$.

(iii) *If* $\alpha = d+2$, *then for any* $\varepsilon > 0$, *there exist* $t_\varepsilon > 0$ *and* $l_\varepsilon > 0$ *such that*

$$\begin{aligned}&t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t;x_0)]\\ &\ge -(1+\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \left\{ \lambda_\zeta^r(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\}\end{aligned} \tag{2.11}$$

for any $t \ge t_\varepsilon$ *and* $l \ge l_\varepsilon$.

Proof. We first prove the upper bound in (i). By a standard Brownian estimate and scaling, we have

$$\begin{aligned}&\mathbb{E}_\theta[v_\xi(t;x_0)]\\ &\le \mathbb{E}_\theta \otimes E_{x_0}\left[\exp\left\{-\int_0^t V_\xi(B_s)ds\right\} : \sup_{0\le s\le t}|B_s|_\infty < \frac{t}{2}\right] + e^{-ct}\\ &\le \mathbb{E}_\theta \otimes E_{x_0/r}\left[\exp\left\{-\int_0^{tr^{-2}} V_\xi^r(B_s)ds\right\} : \sup_{0\le s\le tr^{-2}}|B_s|_\infty < \frac{t}{2r}\right] + e^{-ct}.\end{aligned} \tag{2.12}$$

For any $\varepsilon \in (0,1)$, there exists a finite constant c_ε depending only on d and ε such that the first term of the right hand side is less than

$$c_\varepsilon \mathbb{E}_\theta\left[\exp\left\{-(1-\varepsilon)\lambda_\xi^r(\Lambda_{t/r})tr^{-2}\right\}\right]$$

by (3.1.9) of [8]. By the spectral control (2.2), Lemma 2.2, and (2.8), this quantity is less than

$$\begin{aligned}&o(\mathbb{E}_\theta[v_\xi(t;x_0)]^{-1}) \sup_{(R_r,\zeta)\in\mathcal{S}_r} \mathbb{P}_\theta(\xi_q = \zeta_q \text{ for all } q \in (r[R_r:l])\cap\mathbb{Z}^d)\\ &\times \exp\left\{-(1-\varepsilon)(\lambda_\zeta^r(R_r)\wedge M - r^{-\rho})tr^{-2}\right\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t;x_0)] \le &-(1-2\varepsilon) \inf_{(R_r,\zeta)\in\mathcal{S}_r} \Bigg\{ \lambda_\zeta^r(R_r)\wedge M - r^{-\rho}\\ &+ t^{-1}r^2 \sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} \left(|\zeta_q|^\theta + \log Z(d,\theta)\right)\Bigg\}\end{aligned} \tag{2.13}$$

for sufficiently large t. We can drop M and $r^{-\rho}$ from the right hand side since Theorem 1.1 tells us that the left hand side is bounded from below. Moreover, we can also neglect $\log Z(d,\theta)$ since

$$\#((r[R_r:l])\cap\mathbb{Z}^d) \le cr^{d+\chi} = o(tr^{-2}). \tag{2.14}$$

After removing the above three terms, (2.13) gives us the upper bound.

We next proceed to the lower bound. We pick a pair (R_r^*, ζ^*) which attains the infimum in the right hand side of (2.10). Then we have the following estimate for the L^2-normalized nonnegative eigenfunction ϕ^* corresponding to $\lambda^r_{\zeta^*}(R_r^*)$.

Lemma 2.4. *There exist $p^* \in (rR_r^*) \cap \mathbb{Z}^d$ and $c_0 > 0$ such that*

$$\sup_{x \in \Lambda_{2/r}(p^*/r)} V^r_{\zeta^*}(x) \le c_0 r^{d+\chi+2}$$

and

$$\int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)\, dx \ge \frac{1}{2\|\phi^*\|_\infty} r^{-d-\chi}. \tag{2.15}$$

Proof. We fix $1 < r_0 < \infty$ so that

$$\frac{C_0}{2|x|^\alpha} \le u(x) \le \frac{2C_0}{|x|^\alpha}$$

for all $|x| > r_0$ and take $k \in \mathbb{N}$ satisfying $2^{-k-3} \le r_0/r < 2^{-k-2}$. We divide R_r^* into subboxes of sidelength 2^{-k} as

$$R_r^* = \bigcup_{\imath \in \mathcal{I}^*} C_\imath \quad \text{for some } \mathcal{I}^* \subset \mathcal{I}_k.$$

Let $\mathcal{C}$ be the union of all boxes $C_\imath$ in R_r^* whose enlarged boxes $q_\imath + 2^{-k}[-1,2]^d$ intersect with $\{r^{-1}(q + \zeta_q^*) : q \in (r[R_r^* : l]) \cap \mathbb{Z}^d\}$. Then it is easy to see that if $C_\imath \subset \mathcal{C}$, there exist $a \in C_\imath$ and $c_1 > 0$ for which $V^r_{\zeta^*} \ge c_1 r^2 1_{B(a,1/r)}$. Thus, by using Lemma 3.5 in [4], which states

$$\inf\{\lambda_1((-\Delta + 1_{B(b,1)})^N_R) : b \in \Lambda_R\} \ge cR^{-d}, \tag{2.16}$$

and the scaling with the factor r, we have

$$\inf_{\phi \in C^\infty(C_\imath)} \left\{ \frac{1}{\|\phi\|_2^2} \int_{C_\imath} \left(\frac{1}{2}|\nabla\phi(x)|^2 + V^r_{\zeta^*}(x)\phi(x)^2 \right) dx \right\} \ge c_2 r^2$$

for all $C_\imath \subset \mathcal{C}$ and consequently

$$c_2 r^2 \int_{\mathcal{C}} \phi^*(x)^2 dx \le \int_{\mathcal{C}} \left(\frac{1}{2}|\nabla\phi^*|^2(x) + V^r_{\zeta^*}(x)\phi^*(x)^2 \right) dx.$$

Since the right hand side is bounded from above by $\lambda^r_{\zeta^*}(R_r^*)$, it follows that

$$\int_{\mathcal{C}} \phi^*(x)^2 dx \le c_3 r^{-2}.$$

This implies

$$\int_{R_r^* \setminus \mathcal{C}} \phi^*(x)^2 dx \ge 1/2$$

for large r and hence we can find a $\Lambda_{1/r}(p^*/r)$ in $R_r^* \setminus \mathcal{C}$ such that

$$\|\phi^*\|_\infty \int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)\, dx \ge \int_{\Lambda_{1/r}(p^*/r)} \phi^*(x)^2\, dx \ge \frac{1}{2} r^{-d-\chi}.$$

Finally, we show the bound $\sup_{x \in \Lambda_{2/r}(p^*/r)} V^r_{\zeta^*}(x) \le c_0 r^{d+\chi+2}$. Note first that we have $\sup_{x \in \Lambda_{2/r}(p^*/r)} r^2 u(rx - q - \zeta_q^*) \le c_4 r^2$ for each q since $R_r^* \setminus \mathcal{C}$ keeps the distance larger than $(r_0+1)/r$ from $\{r^{-1}(q + \zeta_q^*) : q \in (r[R_r^* : l]) \cap \mathbb{Z}^d\}$. Multiplying the total number of points $\#\{r^{-1}(q+\zeta_q^* : q \in (r[R_r^* : l]) \cap \mathbb{Z}^d\} \leq (2l+1)^d r^{d+\chi}$, we obtain the result. □

We bound $\mathbb{E}_\theta[v_\xi(t;x_0)]$ from below by

$$
\begin{aligned}
&\mathbb{P}_\theta\left(\xi_q = \zeta^*_{p^*+q} \text{ for } q \in (r[R_r^* : l]) \cap \mathbb{Z}^d - p^*\right)\\
&\times \mathbb{P}_\theta\left(\sup_{x\in(rR_r^*-p^*)\cup\Lambda_2} \sum_{q\in\mathbb{Z}^d\setminus\{(r[R_r^*:l])\cap\mathbb{Z}^d-p^*\}} u(x-q-\xi_q) < \frac{c_1}{(rl)^{\alpha-d}}\right)\\
&\times E_{x_0}\Bigg[\exp\left\{-\int_0^t \sum_{q\in(r[R_r^*:l])\cap\mathbb{Z}^d-p^*} u(B_s-q-\zeta^*_{p^*+q})ds\right\} :\\
&\qquad B_s\in\Lambda_2 \text{ for } 0\le s\le 1, B_1\in\Lambda_1, B_s\in rR_r^*-p^* \text{ for } 1\le s\le t\Bigg]\\
&\times \exp\left(-\frac{c_1 t}{(rl)^{\alpha-d}}\right).
\end{aligned} \tag{2.17}
$$

The first factor is greater than or equal to

$$\exp\left(-\sum_{q\in(r[R_r:l])\cap\mathbb{Z}^d} |\zeta_q|^\theta - cr^{d+\chi}\right)$$

by the same argument using (2.14) as for the upper bound. The last factor is greater than $\exp(-\varepsilon t r^{-2})$ for sufficiently large r if $\alpha > d+2$, and for sufficiently large r and l if $\alpha = d+2$. To bound the second factor we use the following:

Lemma 2.5. *Let $\{R_r : r \geq 1\}$ be a family of lattice animals satisfying $R_r \subset \Lambda_{t/r}$ and $|R_r| < r^\chi$. Let $k, l > 0$. Then there exist $c_1, c_2, c_3 > 0$ independent of R_r such that*

$$\mathbb{P}_\theta\left(\sup_{x\in[rR_r:k]} \sum_{q\in\mathbb{Z}^d\setminus(r[R_r:l])} u(x-q-\xi_q) < c_1(rl)^{-\alpha+d}\right) \geq c_2 \tag{2.18}$$

for any $r \geq c_3$.

Proof. We consider the event

$$\left\{d(q+\xi_q,[rR_r:k]) \geq \frac{1}{2}d(q,[rR_r:k]) \text{ for all } q\in\mathbb{Z}^d\setminus(r[R_r:l])\right\}. \tag{2.19}$$

On this event, we have

$$
\begin{aligned}
&\sum_{q\in\mathbb{Z}^d\setminus(r[R_r:l])} |x-q-\xi_q|^{-\alpha} \leq \sum_{q\in\mathbb{Z}^d\setminus(r[R_r:l])} \left(\frac{2}{d(q,[rR_r:k])}\right)^\alpha\\
\leq& c_4 \sum_{q\in\mathbb{Z}^d:\, d(q,rR_r)\geq rl} d(q,[rR_r:k])^{-\alpha} \leq c_5(rl)^{-\alpha+d}
\end{aligned}
$$

for any $x \in [rR_r : k]$ and large r. By this estimate and the assumption $u(x) = C_0|x|^{-\alpha}(1+o(1))$, we see that the event in (2.19) implies the event in (2.18). Since the inequality in (2.19) is satisfied if

$$|\xi_q| \le d(q, [rR_r : k])/2 \text{ for all } q \in \mathbb{Z}^d \setminus (r[R_r : l]),$$

the probability of the event (2.19) is greater than or equal to

$$\prod_{q\in\mathbb{Z}^d\setminus(r[R_r:l])} \left(1 - \frac{1}{Z(d,\theta)} \sum_{y\in\mathbb{Z}^d:|y|\ge d(q,[rR_r:k])/2} \exp(-|y|^\theta)\right). \tag{2.20}$$

It is easy to see that

$$\frac{1}{Z(d,\theta)} \sum_{y\in\mathbb{Z}^d:|y|\ge d(q,[rR_r:k])/2} \exp(-|y|^\theta) \le \exp(-c_6 d(q,[rR_r:k])^\theta)$$

and

$$\#\{q \in \mathbb{Z}^d : n \le d(q, [rR_r : k]) < n+1\} \le c_7 r^{\chi+d} n^{d-1}.$$

By using also an elementary inequality $(1-x)^p \ge 1-px$ for any $p \ge 1$ and $0 < x < 1$, the quantity in (2.20) is greater than or equal to

$$\prod_{rl-k\le n\in\mathbb{N}} \left(1 - \exp(-c_6 n^\theta)\right)^{c_7 r^{\chi+d} n^{d-1}} \ge \prod_{rl-k\le n\in\mathbb{N}} \left(1 - c_8 r^{\chi+d}\exp(-c_9 n^\theta)\right).$$

Since the right hand side is a convergent infinite product, we conclude (2.18). □

It remains to bound the third factor in (2.17). We use the bound

$$\sup_{x\in\Lambda_{2/r}(p^*/r)} V^r_{\zeta^*}(x) \le c_0 r^{d+\chi+2}$$

in Lemma 2.4 for $0 \le s \le 1$ and the positivity of

$$\inf_{x,y\in\Lambda_1} \exp(\Delta_2^D/2)(x,y),$$

where $\exp(t\Delta_2^D/2)(x,y)$, $(t,x,y) \in (0,\infty)\times\Lambda_2\times\Lambda_2)$ is the integral kernel of the heat semigroup generated by the Dirichlet Laplacian on Λ_2 multiplied by $-1/2$. Then, we can show that the third factor is greater than

$$r^d \exp(-c_0 r^{d+\chi}) \int_{\Lambda_{1/r}} dy \int_{R_r^*-p^*/r} dz \exp(-(t-1)r^{-2}H^*)(y,z) \tag{2.21}$$

for large r by using a scaling, where $\exp(-tH^*)(x,y)$, $(t,x,y) \in (0,\infty)\times(R_r^* - p^*/r)\times(R_r^* - p^*/r))$ is the integral kernel of the heat semigroup generated by the Schrödinger operator

$$H^* = -\Delta/2 + \sum_{q\in(r[R_r^*:l])\cap\mathbb{Z}^d-p^*} r^2 u(rx - q - \zeta^*_{p^*+q})$$

in $R_r^* - p^*/r$ with the Dirichlet boundary condition. By (2.15), the integral in (2.21) is greater than or equal to

$$\begin{aligned}&\int_{\Lambda_{1/r}} dy \int_{R_r^*-p^*/r} dz \exp(-(t-1)r^{-2}H^*)(y,z)\frac{\phi^*(z+p^*/r)}{\|\phi^*\|_\infty}\\ &\ge \exp(-(t-1)r^{-2}\lambda^r_{\zeta^*}(R_r^*))/(2\|\phi^*\|^2_\infty r^{d+\chi}).\end{aligned}$$

Finally $\|\phi^*\|_\infty$ is bounded since

$$\phi^*(y) = \exp(\lambda^r_{\zeta^*}(R^*_r)) \int \exp(-H^0)(y,z)\phi^*(z)\,dz,$$

$\|\exp(-H^0)(y,\cdot)\|_2 \le 1$, and $\lambda^r_{\zeta^*}(R^*_r)$ is bounded by Theorem 1.1 and the upper bound in (i), where $\exp(-tH^0)(x,y)$, $(t,x,y) \in (0,\infty) \times R^*_r \times R^*_r$, is the integral kernel of the heat semigroup generated by the Schrödinger operator $H^0 = -\Delta/2 + V^r_{\zeta^*}$ in R^*_r with the Dirichlet boundary condition. By all these the lower bounds (ii) and (iii) are proven. □

2.3. **Proof of a modified statement for the one-dimensional case.** We first fix a constant $M > 0$ such that

$$\begin{aligned}\mathbb{P}_\theta\left(\{q+\xi_q : q \in \mathbb{Z}\} \cap (0, Mt^{1/(3+\theta)}) = \emptyset\right) &\le \exp\left\{-cM^{1+\theta}t^{(1+\theta)/(3+\theta)}\right\}\\ &= o(\mathbb{E}_\theta[v_\xi(t;x_0)]),\end{aligned}$$

which is possible in view of Theorem 1.1. We define the set $\mathcal{S}_r$ of relevant configurations by

$$\begin{aligned}\mathcal{S}_r = \big\{&((m,n), \zeta = (\zeta_q)_{q\in(m-lr,n+lr)\cap\mathbb{Z}})\\ &: m,n \in \mathbb{Z}, -t \le m < n \le t, n-m \le Mr, |\zeta_q| \le t^{1/\theta},\\ &\{q+\zeta_q : q \in (m-lr,n+lr)\cap\mathbb{Z}\} \cap (m,n) = \emptyset\big\}\end{aligned}$$

in this case. Now we can state the result.

Theorem 2.6. *Let $d = 1$ and assume the setting of Theorem 1.2. Then, for any $\varepsilon > 0$, there exist $t_\varepsilon > 0$ and $l_\varepsilon > 0$ such that*

$$\begin{aligned}&-(1+\varepsilon)\inf_{((m,n),\zeta)\in\mathcal{S}_r}\left\{\lambda^r_\zeta((m/r,n/r)) + \sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\left|\frac{\zeta_q}{r}\right|^\theta\right\}\\ &\le t^{-(1+\theta)/(3+\theta)}\log\mathbb{E}_\theta[v_\xi(t;x_0)]\\ &\le -(1-\varepsilon)\inf_{((m,n),\zeta)\in\mathcal{S}_r}\left\{\lambda^r_\zeta((m/r,n/r)) + \sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\left|\frac{\zeta_q}{r}\right|^\theta\right\},\end{aligned}$$

for all $t > t_\varepsilon$ and $l > l_\varepsilon$.

Proof. We only prove the upper bound. After having it, the lower bound follows exactly in the same way as for Theorem 2.3.

We use a simple version of the method of enlargement of obstacles where $\gamma = 1$ and any 2^{-n_1}-box containing a point of $\{r^{-1}(q+\xi_q) : q \in \mathbb{Z}\}$ is a density box. Such a box indeed satisfies the quantitative Wiener criterion (2.12) in page 152 of [8] since even a point has positive capacity when $d = 1$ (cf. page 153 of [8]). Then, the spectral control (2.2) implies that we can impose the Dirichlet boundary condition on each point in $\{r^{-1}(q+\xi_q)\}_{q\in\mathbb{Z}}$.

Combining this observation with a standard Brownian estimate and (3.1.9) in [8], we find

$$\begin{aligned}\mathbb{E}_\theta[v_\xi(t;x_0)] &\le \mathbb{E}_\theta\left[c\left(1 + \left(\lambda^1_\xi((-t,t))t\right)^{1/2}\right)\exp\left\{-\lambda^1_\xi((-t,t))t\right\}\right] + e^{-ct}\\ &\le c_\varepsilon\mathbb{E}_\theta\left[\sup_k \exp\left\{-(1-\varepsilon)\lambda^r_\xi\left(r^{-1}I_k\right)tr^{-2}\right\}\right] + e^{-ct},\end{aligned}$$

where ε is an arbitrary positive constant and $\{I_k\}_k$ are the random open intervals such that $\sum_k I_k = (-t,t) \setminus \{q+\xi_q : q \in \mathbb{Z}\}$. By considering all possibilities of I_k, we can bound the $\mathbb{E}_\theta$-expectation in the right hand side by

$$\sum_{m,n\in\mathbb{Z}: -t\le m<n\le t} \mathbb{E}_\theta\Big[\exp\left\{-(1-\varepsilon)\lambda_\xi^r\left((m/r,n/r)\right)tr^{-2}\right\} \\ : \{q+\xi_q : q\in\mathbb{Z}\}\cap(m,n)=\emptyset\Big].$$

Note that we can discard (m,n) whose interval $n-m>Mr$ thanks to our choice of M. Hence, we can restrict our consideration on $\mathcal{S}_r$ and we can also show $\#\mathcal{S}_r = \exp\{o(t^{(1+\theta)/(3+\theta)})\}$ by an elementary counting argument. Now, we have

$$\begin{aligned} &\mathbb{E}_\theta[v_\xi(t;x_0)] \\ &\le \sum_{((m,n),\zeta)\in\mathcal{S}_r} \exp\left\{-(1-\varepsilon)\lambda_\zeta^r\left((m/r,n/r)\right)tr^{-2}\right\}\mathbb{P}_\theta\left(\xi_q=\zeta_q \text{ for all } q\right) \\ &\quad + o(\mathbb{E}_\theta[v_\xi(t;x_0)]) \\ &\le \exp\Bigg\{-(1-2\varepsilon)t^{(1+\theta)/(3+\theta)} \\ &\quad \times \inf_{((m,n),\zeta)\in\mathcal{S}_r}\Bigg\{\lambda_\zeta^r((m/r,n/r)) + \sum_{q\in(m-lr,n+lr)\cap\mathbb{Z}} r^{-1}\Big|\frac{\zeta_q}{r}\Big|^\theta\Bigg\}\Bigg\}, \end{aligned}$$

which is the desired estimate. □

2.4. **Proof of Theorem 1.2.** In this section, we complete the proof of Theorem 1.2 by simplifying the variational expression in Theorem 2.3. We treat only the multidimensional case since the modification for the one-dimensional case is straightforward. Recall that $\Omega_t = (\mathbb{Z}^d)^{\Lambda_t\cap\mathbb{Z}^d}$ is the set of possible configurations of $\{\xi_q\}_{q\in\Lambda_t\cap\mathbb{Z}^d}$. We first show

$$\begin{aligned} &\inf_{(R_r,\,\zeta)\in\mathcal{S}_r}\Bigg\{\lambda_\zeta^r(R_r)+\gamma(r)^\theta\sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\Big|\frac{\zeta_q}{r}\Big|^\theta\Bigg\} \\ &\quad \ge (1-\varepsilon)\inf_{\zeta\in\Omega_t}\Bigg\{\lambda_\zeta^r(\Lambda_{t/r})+\gamma(r)^\theta\sum_{q\in\Lambda_t\cap\mathbb{Z}^d} r^{-d}\Big|\frac{\zeta_q}{r}\Big|^\theta\Bigg\}. \end{aligned} \tag{2.22}$$

for sufficiently large t (and l) if $\alpha\in(d,d+2)$ (resp. $\alpha=d+2$). Let $(R_r^*,\,\zeta^*)$ be a minimizer of the variational problem in the first line. We extend ζ^* to $\zeta^{**}\in\Omega_t$ by setting $\zeta_q^{**}=0$ for $q\in(\Lambda_t\setminus r[R_r^*:\,l])\cap\mathbb{Z}^d$. Then, it is obvious that

$$\sum_{q\in(r[R_r^*:\,l])\cap\mathbb{Z}^d} r^{-d}\Big|\frac{\zeta_q^*}{r}\Big|^\theta \ge \sum_{q\in\Lambda_t\cap\mathbb{Z}^d} r^{-d}\Big|\frac{\zeta_q^{**}}{r}\Big|^\theta. \tag{2.23}$$

Moreover, we can prove

$$\sup_{x\in rR_r^*}\sum_{q\in\mathbb{Z}^d\setminus(r[R_r^*:\,l])}|x-q-\zeta_q^{**}|^{-\alpha}\le c_1(rl)^{-\alpha+d} \tag{2.24}$$

for this ζ^{**}. Therefore, we have

$$\lambda_{\zeta^*}^r(R_r^*)+c_2r^{-\alpha+d+2}l^{-\alpha+d}\ge\lambda_{\zeta^{**}}^r(\Lambda_{t/r}) \tag{2.25}$$

and this yields (2.22).

We next show

$$\begin{aligned}&\inf_{(R_r,\,\zeta)\in\mathcal{S}_r}\left\{\lambda^r_\zeta(R_r)+\gamma(r)^\theta\sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta\right\}\\ &\quad\le(1+\varepsilon)\inf_{\zeta\in\Omega_t}\left\{\lambda^r_\zeta(\Lambda_{t/r})+\gamma(r)^\theta\sum_{q\in\Lambda_t\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta\right\}\end{aligned}\tag{2.26}$$

for sufficiently large t. It follows from Lemma 2.2 that if a sequence $\{\zeta^t\}_t$ of configurations satisfies $\zeta^t\in\Omega_t$ and $|\underline{\mathcal{R}}_r(\zeta^t)|\ge r^\chi$ for any t, then we have

$$\gamma(r)^\theta\sum_{q\in\Lambda_t\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta^t_q}{r}\right|^\theta\longrightarrow\infty.\tag{2.27}$$

as $t\to\infty$. Thus if each ζ^t is a minimizer of the right-hand side of (2.26), then we have $|\underline{\mathcal{R}}_r(\zeta^t)|<r^\chi$ for large t. We may also assume that $q+\zeta^t_q\in[\mathcal{T}:t^{1/(\mu\theta)}]$ for all $q\in(r[\underline{\mathcal{R}}_r(\zeta^t):l])\cap\mathbb{Z}^d$ since otherwise (2.27) holds. We here extend ζ^t to $(r[\underline{\mathcal{R}}_r(\zeta^t):l])\cap\mathbb{Z}^d$ by $\zeta^t_q=0$ for $q\in(r[\underline{\mathcal{R}}_r(\zeta^t):l])\cap\mathbb{Z}^d\setminus\Lambda_t$. There exists a lattice animal R^t_r in $\underline{\mathcal{R}}_r(\zeta^t)$ such that $\lambda^r_{\zeta^t}(\underline{\mathcal{R}}_r(\zeta^t))=\lambda^r_{\zeta^t}(R^t_r)$. Then it follows that $(R^t_r,(\zeta^t_q)_{q\in(r[R^t_r:\,l])\cap\mathbb{Z}^d})\in\mathcal{S}_r$ for sufficiently large t. Combining with Spectral control (2.2), we obtain (2.26).

3. Asymptotics of higher moments

In [3], a result on the asymptotics for higher moments of the survival probability is shown as an application of the precise form of the leading term. We shall extend the result to our cases in this section. Our objects are the p-th moments $\mathbb{E}_\theta[v_\xi(t;x_0)^p]$ for $p\ge1$. We consider their asymptotics in Subsection 3.1. In Subsection 3.2, we discuss a related quantitative estimate on intermittency for the parabolic Anderson problem.

3.1. **Asymptotics for each case.**

Proposition 3.1. *Under the settings in Theorem 1.2, there exist $c_1,c_2\in(0,\infty)$ depending on d,θ and u such that for any $p\ge1$,*

$$-c_1p^{(d+\mu\theta)/(d+2+\mu\theta)}\le t^{-1}r^2\log\mathbb{E}_\theta[v_\xi(t;x_0)^p]\le -c_2p^{(d+\mu\theta)/(d+2+\mu\theta)}$$

holds for sufficiently large t, uniformly in $x_0\in\Lambda_1$, where we take $\mu=1$ in the case $d=1$.

Proof. We first assume $d\ge3$ and $\alpha>d+2$. The same argument as in Section 1.3, using the scaling with factor $s=(pt)^{1/(d+2+\mu\theta)}$ instead of $r=t^{1/(d+2+\mu\theta)}$ in (2.12) and (2.21), yields

$$\begin{aligned}&\log\mathbb{E}_\theta[v_\xi(t;x_0)^p]\sim-(pt)^{(d+\mu\theta)/(d+2+\mu\theta)}\\ &\qquad\times\inf_{(R_s,\,\zeta)\in\mathcal{S}_s}\left\{\lambda^s_\zeta(R_s)+s^{(1-\mu)\theta}\sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d}\left|\frac{\zeta_q}{s}\right|^\theta\right\}\end{aligned}\tag{3.1}$$

as $t \to \infty$ for any l. Since we know

$$\begin{aligned} 0 &< \varliminf_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda^s_\zeta(R_s) + s^{(1-\mu)\theta} \sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d}\left|\frac{\zeta_q}{s}\right|^\theta \right\} \\ &\le \varlimsup_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda^s_\zeta(R_s) + s^{(1-\mu)\theta} \sum_{q\in(s[R_s:\,l])\cap\mathbb{Z}^d} s^{-d}\left|\frac{\zeta_q}{s}\right|^\theta \right\} < \infty \end{aligned} \tag{3.2}$$

from Theorems 1.1 and 2.3, the proof is completed. The other cases can be treated exactly in the same way. □

Remark 3.2. If $-\lim_{t\to\infty} t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t;x_0)]$ exists under the setting of the last proposition, denoting it by L, we have

$$t^{-1}r^2 \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] \sim -Lp^{(d+\mu\theta)/(d+2+\mu\theta)}. \tag{3.3}$$

Indeed, when $d \ge 2$ and $\alpha > d+2$, the existence of the above limit implies

$$\lim_{t\to\infty} \inf_{(R_r,\,\zeta)\in\mathcal{S}_r} \left\{ \lambda^r_\zeta(R_r) + \gamma(r)^\theta \sum_{q\in(r[R_r:\,l])\cap\mathbb{Z}^d} r^{-d}\left|\frac{\zeta_q}{r}\right|^\theta \right\} = L$$

by Theorem 2.3 and then (3.3) is obvious from the proof of the last proposition. When $\alpha = d+2$, we know only that the superior limit and the inferior limit in (3.2) tend to L as $l \to \infty$. This is still enough to show (3.3).

The above remark actually applies for the case $d = 1$ and $\alpha > 3$:

Proposition 3.3. *Under the conditions of Theorem 1.1-(i) with $\alpha > 3$, we have*

$$\lim_{t\uparrow\infty} t^{-(1+\theta)/(3+\theta)} \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] = -\frac{3+\theta}{1+\theta}\Big(\frac{p\pi^2}{8}\Big)^{(1+\theta)/(3+\theta)} \tag{3.4}$$

for any $p \ge 1$, uniformly in $x_0 \in \Lambda_1$.

Proof. As in the proof of the last proposition we have

$$\begin{aligned} \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] &\sim -(pt)^{(1+\theta)/(3+\theta)} \\ &\quad\times \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda^s_\zeta((m/s,n/s)) + \sum_{q\in(m-ls,n+ls)\cap\mathbb{Z}} s^{-1}\left|\frac{\zeta_q}{s}\right|^\theta \right\} \end{aligned} \tag{3.5}$$

as $t \to \infty$ for any l in the notations of Subsection 2.3, where $s = (pt)^{1/(3+\theta)}$. When $\alpha > 3$, we know the limit

$$\begin{aligned} &\lim_{s\to\infty} \inf_{(R_s,\,\zeta)\in\mathcal{S}_s} \left\{ \lambda^s_\zeta((m/s,n/s)) + \sum_{q\in(m-ls,n+ls)\cap\mathbb{Z}} s^{-1}\left|\frac{\zeta_q}{s}\right|^\theta \right\} \\ &= \frac{3+\theta}{1+\theta}\Big(\frac{\pi^2}{8}\Big)^{(1+\theta)/(3+\theta)}. \end{aligned}$$

□

Proposition 3.4. *Under the conditions of Theorem 1.1 with $\alpha < d+2$, we have*

$$\lim_{t\uparrow\infty} t^{-(d+\theta)/(\alpha+\theta)} \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] = -p^{(d+\theta)/(\alpha+\theta)} c(d,\alpha,\theta,C_0) \tag{3.6}$$

for any $p \ge 1$, uniformly in $x_0 \in \Lambda_1$.

Proof. We have only to show

$$\lim_{t\uparrow\infty} t^{-(d+\theta)/(\alpha+\theta)} \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] = -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d}\Big(\frac{pC_0}{|q+y|^\alpha} + |y|^\theta\Big).$$

The upper estimate is easy since we have

$$\begin{aligned}\mathbb{E}_\theta[v_\xi(t;x_0)^p] \le& \mathbb{E}_\theta\bigg[E_{x_0}\bigg[\exp\bigg\{-\int_0^t V_\xi(B_s)ds\bigg\}\bigg]^p\bigg]\\ \le& \mathbb{E}_\theta\otimes E_{x_0}\bigg[\exp\bigg\{-p\int_0^t V_\xi(B_s)ds\bigg\}\bigg]\end{aligned}$$

by removing the Dirichlet condition and using the Hölder inequality. For the lower estimate, we take R, R_1 and β as in the proof of Proposition 2.2 in [4] and restrict the integral as

$$\begin{aligned}\mathbb{E}_\theta[v_\xi(t;x_0)^p] \ge \mathbb{E}_\theta\bigg[E_{x_0}\bigg[&\exp\bigg\{-\int_0^t V_\xi(B_s)ds\bigg\}\\ &: B_s\in\Lambda_R \text{ for } 0\le s\le t\bigg]^p : \Xi_t\bigg]\end{aligned}$$

for $t^\beta \ge 2(R_1 + R\sqrt{d})$, where Ξ_t is the set of configurations defined by

$$\{|\xi_q| \le |q|/2 \text{ for } |q|\ge t^\beta, \text{ and } |q+\xi_q| \ge R_1 + R\sqrt{d} \text{ for } |q| < t^\beta\}.$$

The right hand side is bounded from below by

$$\mathbb{E}_\theta\bigg[\exp\bigg\{-pt\sup_{y\in\Lambda_R} V_\xi(y)\bigg\} : \Xi_t\bigg]\exp(-cptR^{-2}).$$

This is estimated by the same method as in our proof of Proposition 2.2 in [4]. □

Proposition 3.5. *Under the conditions of Theorem 1.1 with $u\le 0$, we have*

$$\lim_{t\uparrow\infty} t^{-(1+d/\theta)} \log \mathbb{E}_\theta[v_\xi(t;x_0)^p] = c_-(d,\theta,pu(0))$$

for any $p\ge 1$, uniformly in $x_0\in\Lambda_1$.

Proof. The upper and lower estimates are obtained by similar ways to the proof of Proposition 3.4 and that of (1.12) respectively. □

3.2. **Intermittency.** The initial value problem of the form (1.1) is called the "parabolic Anderson problem" in literature, see e.g. a survey article by Gärtner and König [5]. For a wide class of random potentials, it is believed that the solution of parabolic Anderson problem consists of high peaks which are far from each other. A manifestation of this phenomenon formulated by Gärtner and Molchanov [6] is so-called "intermittency" defined by

$$\frac{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_2}]^{1/p_2}}{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_1}]^{1/p_1}} \overset{t\to\infty}{\longrightarrow} \infty \quad \text{for } p_1 < p_2. \tag{3.7}$$

Although (3.7) implies the concentration of $v_\xi(t;x_0)$ in the ξ-space, there is a way to relate this to the spatial concentration of the solution through the ergodic theorem. See Sect. 1.3 of [5] for this point. Gärtner and Molchanov also proved in [6] that the intermittency holds for a quite general class of potentials. In particular, if we consider a slightly different moment

$$\mathbb{E}_\theta\left[\int_{\Lambda_1} v_\xi(t;x_0)^p dx_0\right] \tag{3.8}$$

in our model, then the intermittency follows by the same argument as for Theorem 3.2 of [6].

Our main result Theorem 1.2 gives a more detailed description of the concentration in the configuration space. Indeed, it says that the main contribution to $\mathbb{E}_\theta[v_\xi(t;x_0)]$ comes only from minimizers of the right hand side of (1.15). Furthermore, we can derive the rates of the divergence in (3.7) from the results in the previous subsection as follows:

(i) Under the settings in Theorem 1.2, we have

$$\frac{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_2}]^{1/p_2}}{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_1}]^{1/p_1}}\begin{cases}\geq \exp\left\{tr^{-2}\left(c_2p_1^{-2/(d+2+\mu\theta)}-c_1p_2^{-2/(d+2+\mu\theta)}\right)\right\}\\ \leq \exp\left\{tr^{-2}\left(c_1p_1^{-2/(d+2+\mu\theta)}-c_2p_2^{-2/(d+2+\mu\theta)}\right)\right\},\end{cases}$$

for sufficiently large t, where $\infty > c_1 \geq c_2 > 0$ are the constants in Proposition 3.1.

(ii) Under the settings in Theorem 1.1 with $d=1$ and $\alpha > 3$, it holds that

$$\begin{aligned}&\frac{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_2}]^{1/p_2}}{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_1}]^{1/p_1}}\\ &=\exp\left\{\frac{3+\theta}{1+\theta}\left(\frac{\pi^2 t}{8}\right)^{(1+\theta)/(3+\theta)}\left(p_1^{-2/(3+\theta)}-p_2^{-2/(3+\theta)}+o(1)\right)\right\}\end{aligned}$$

as t goes to ∞.

(iii) Under the settings in Theorem 1.1 with $\alpha < d+2$, it holds that

$$\begin{aligned}&\frac{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_2}]^{1/p_2}}{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_1}]^{1/p_1}}\\ &=\exp\left\{c(d,\alpha,\theta,C_0)t^{(d+\theta)/(\alpha+\theta)}\left(p_1^{(d-\alpha)/(\alpha+\theta)}-p_2^{(d-\alpha)/(\alpha+\theta)}+o(1)\right)\right\}\end{aligned}$$

as t goes to ∞.

(iv) Under the settings in Theorem 1.1 with $u \leq 0$, it holds that

$$\frac{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_2}]^{1/p_2}}{\mathbb{E}_\theta[v_\xi(t;x_0)^{p_1}]^{1/p_1}}=\exp\left\{c_-(d,\theta,u(0))t^{1+d/\theta}\left(p_2^{d/\theta}-p_1^{d/\theta}+o(1)\right)\right\}$$

as t goes to ∞.

Note that in the first case, the left hand side goes to infinity only when p_2/p_1 is sufficiently large. On the other hand, the left hand sides go to infinity for any $p_2/p_1 > 1$ in other cases. This is slightly better than Theorem 3.2 of [6] where $p_2 \geq 2$ is required. Note also that all these estimates hold uniformly in $x_0 \in \Lambda_1$ and therefore, the same estimates hold for (3.8) as well.

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