# Arakelov motivic cohomology I

Andreas Holmstrom* and Jakob Scholbach†

December 14, 2010

## Contents



**Abstract**

This paper introduces the notion of Arakelov motivic cohomology. We construct a ring spectrum $\mathrm{H}_{\mathcal{D}}$ representing Deligne cohomology which is shown to enjoy a unique algebra structure over the spectrum $\mathrm{H}_{\mathcal{B}}$ representing motivic cohomology. We define Arakelov motivic cohomology to be represented by the homotopy fiber of the regulator map $\mathrm{H}_{\mathcal{B}} \to \mathrm{H}_{\mathcal{D}}$. Taking advantage of the framework of the stable homotopy category of schemes over an arithmetic ring, we establish a number of properties such as pushforwards, localization sequences, and $h$-descent.

*IMB, Université Bordeaux 1, 351 Cours de la Libération, F-33405 Talence cedex, France, andreas.holmstrom@gmail.com

†Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany, jakob.scholbach@uni-muenster.de

# 1 Introduction

The aim of this paper is to construct a new cohomology theory for schemes of finite type over an arithmetic ring. The main motivation for this Arakelov-theoretic version of motivic cohomology is the conjecture on special values of L-functions and zeta functions formulated by the second author [Sch10]. Additional motivation comes from the hope that these cohomology groups together with some version of higher arithmetic K-theory will allow for the formulation of a higher arithmetic Riemann-Roch theorem.

Compared to earlier definitions of higher arithmetic Chow groups by Goncharov and Burgos-Feliu, the main advantages of our cohomology theory is that it is defined for schemes over arithmetic rings like $\operatorname{Spec}\mathbb{Z}$ and not just over fields, and that it has pushforward functoriality for projective morphisms between regular schemes.

The following theorem summarizes the results of this paper:

**Theorem 1.1.** *Let $S$ be a regular scheme over a number field $F$ or a number ring $\mathcal{O}_F$, or $\mathbb{R}$ or $\mathbb{C}$. In the stable homotopy category $\mathbf{SH}(S)$ (cf. Section 2.1) there is a spectrum $\mathrm{H}_\mathrm{D}$ representing Deligne cohomology of smooth schemes $X/S$ (Theorem 3.6). This spectrum $\mathrm{H}_\mathrm{D}$ enjoys a unique $\mathrm{H}_{Б,S}$-algebra structure, where $\mathrm{H}_{Б,S}$ is Riou's spectrum representing motivic cohomology (Theorem 3.7).*

*We define the Arakelov motivic cohomology spectrum $\widehat{\mathrm{H}_{Б,S}}$ as the homotopy fiber of $\rho_\mathrm{D} : \mathrm{H}_Б \to \mathrm{H}_\mathrm{D}$, the unique $\mathrm{H}_Б$-module map. We define* Arakelov motivic cohomology *to be the theory represented by this spectrum, that is to say*

$$\widehat{\mathrm{H}}^n(M, p) := \mathrm{Hom}_{\mathbf{SH}(S)}(M, \widehat{\mathrm{H}_{Б,S}}(p)[n])$$

*for any $M \in \mathbf{SH}(S)$.*

*Arakelov motivic cohomology shares the structural properties known for motivic cohomology, for example a projective bundle formula, a localization sequence, and h-descent (Theorem 4.4). It also has the expected functoriality: pullback for arbitrary morphisms of schemes (or motives) and pushforward along projective maps between regular schemes (Theorem 4.8).*

In a second paper. we will show that the map obtained from the $\mathrm{H}_{Б,S}$-algebra structure of $\mathrm{H}_\mathrm{D}$,

$$\mathrm{ch}_\mathrm{D} : \mathrm{BGL} \to \oplus_{p\in\mathbb{Z}} \mathrm{H}_{Б,S}(p)[2p] \to \oplus \mathrm{H}_\mathrm{D}(p)[2p],$$

is such that the induced map of motivic cohomology to Deligne cohomology

$$K_n(X) = \mathrm{Hom}_{\mathbf{SH}(X)}(S^n, \mathrm{BGL}) \to \oplus_p \mathrm{H}_\mathrm{D}^{2p-n}(X, p) = \mathrm{Hom}_{\mathbf{SH}(X)}(S^n, \mathrm{H}_\mathrm{D}(p)[2p])$$

agrees with the Beilinson regulator. Moreover, we compare the Arakelov motivic cohomology groups with Takeda's higher arithmetic $K$-theory and with Gillet-Soulé arithmetic Chow groups. Moreover, the height pairing is shown to arise as a special case of a natural pairing between between (usual) motivic homology and Arakelov motivic cohomology.

### 1.1 Acknowledgements

We thank Denis-Charles Cisinski and Frédéric Déglise for many helpful discussions.

# 2 Preliminaries

## 2.1 The stable homotopy category

This section sets the notation and recalls some results pertaining to the homotopy theory of schemes due to Morel and Voevodsky and Riou [MV99, Rio07].

Let $S$ be a Noetherian scheme. We only use schemes which are of finite type over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. Unless explicitly mentioned otherwise, all morphisms of schemes are understood to be separated and of finite type. Let $\mathbf{Sm}/S$ be the category of smooth schemes over $S$. The category of presheaves of pointed sets on this category is denoted $\mathbf{PSh}_\bullet := \mathbf{PSh}_\bullet(\mathbf{Sm}/S)$. We often regard a scheme $X \in \mathbf{Sm}/S$ as the presheaf (of sets) represented by $X$, and we write $X_+ := X \sqcup \{*\}$ for a pointed version. The projective line $\mathbb{P}^1_S$ is pointed by $\infty$. The prefix $\Delta^{\mathrm{op}}-$ indicates simplicial objects in a category. The simplicial $n$-sphere is denoted $S^n$, this should not cause confusion with the base scheme $S$. Recall the definition of the *stable homotopy category* $\mathbf{SH}(S)$. In a nutshell, it is built by the following Quillen adjunctions. (Underneath we give the notation for the corresponding homotopy category.)

$$\begin{array}{ccccccccc} \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\leftrightarrows}} & \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\leftrightarrows}} & \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \overset{\Omega^\infty}{\underset{\Sigma^\infty_{\mathbb{P}^1}}{\leftrightarrows}} & \mathbf{Spt}^{\mathbb{P}^1}_\bullet & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\leftrightarrows}} & \mathbf{Spt}^{\mathbb{P}^1}_\bullet \\ \mathbf{Ho}_{\mathrm{sect},\bullet} & & \mathbf{Ho}_{\mathrm{Nis},\bullet} & & \mathbf{Ho}_\bullet & & \mathbf{SH}_p & & \mathbf{SH} \end{array} \quad (1)$$

From left to right, the involved model structures are the following (recall the 2-out-of-3-principle, by which any two classes among fibrations, cofibrations and weak equivalences determine the third): the *sectionwise model structure* on the category of pointed simplicial presheaves on smooth schemes over $S$ is defined by sectionwise weak equivalences and monomorphisms as cofibrations. Second, the *Nisnevich local model structure* is determined by weak equivalences on Nisnevich stalks and monomorphisms as cofibrations. The $\mathbb{A}^1$-*local model structure* on presheaves is given by $\mathbb{A}^1$-equivalences and monomorphisms. The former are maps $f : X \to Y$ such that $f^* : \mathrm{Hom}_{\mathbf{Ho}_{\mathrm{Nis},\bullet}}(Y, Z) \to \mathrm{Hom}_{\mathbf{Ho}_{\mathrm{Nis},\bullet}}(X, Z)$ is an isomorphism for all $\mathbb{A}^1$-local objects $Z$, that is, objects satisfying the condition that

$$\mathrm{Hom}_{\mathbf{Ho}_{\mathrm{Nis},\bullet}}(T \times \mathbb{A}^1_S, Z) \to \mathrm{Hom}_{\mathbf{Ho}_{\mathrm{Nis},\bullet}}(T, Z)$$

be a weak equivalence (of simplicial sets) for all "test" objects $T \in \mathbf{Ho}_{\mathrm{Nis},\bullet}$.

The next category, $\mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$, consists of symmetric $\mathbb{P}^1_S$-spectra, that is, sequences $E = (E_n)_{n \geq 0}$ of simplicial presheaves which are equipped with an action of the symmetric group $S_n$ and $S_n$-equivariant bonding maps $\mathbb{P}^1 \wedge E_n \to E_{n+1}$ (and the obvious morphisms). Here, the presheaf (represented by the scheme) $\mathbb{P}^1$ is pointed by $\infty$. It is endowed with the *projective model structure* [Jar00, 2.1]: weak equivalences and fibrations are maps $E \to F$ such that each $E_n \to F_n$ is an $\mathbb{A}^1$-equivalence, and a fibration in the $\mathbb{A}^1$-local model structure, respectively. Moreover, cofibrations are maps such that $E_0 \to F_0$ is a cofibration and

$$\mathbb{P}^1 \wedge F_n \wedge_{\mathbb{P}^1 \wedge E_n} E_{n+1} \to F_{n+1}$$

is a cofibration. The functor $\Sigma^\infty_{\mathbb{P}^1} : \Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet) \ni F \mapsto \left((\mathbb{P}^1)^{\wedge n} \wedge F\right)_{n \geq 0}$ (bonding maps are identity maps, $S_n$ acts by permuting the factors $\mathbb{P}^1$) is left adjoint to $\Omega^\infty : (E_n) \mapsto E_0$. Often, we will not distinguish between a simplicial presheaf $F$ and $\Sigma^\infty_{\mathbb{P}^1}(F)$. Finally, the *stable model structure* is defined by projective cofibrations (the same as in the previous step) and stable $\mathbb{A}^1$-equivalences. The latter are defined as follows: an $\Omega$-spectrum is an object $(E_n) \in \mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ such that the maps $E_n \to \mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, E_{n+1})$ is an $\mathbb{A}^1$-local weak equivalence for all $n$. Here $\mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, -)$ is the derived functor of the right adjoint to $\mathbb{P}^1 \wedge -$ and the above map is the adjoint to the bonding map of $E$. A stable weak equivalence is a map $E \to F$ of spectra such that for any $\Omega$-spectrum $V$ the induced map

$$\mathrm{Hom}_{\mathbf{SH}_p(S)}(F, V) \to \mathrm{Hom}_{\mathbf{SH}_p(S)}(E, V)$$

is a bijection. Note that the objects in the image of $\Sigma^\infty_{\mathbb{P}^1}$ consists are cofibrant.

The identity functors and $(\Sigma^\infty_{\mathbb{P}^1}, \Omega^\infty)$ are Quillen adjunctions with respect to these model structures (i.e., the left adjoint preserves cofibrations and weak equivalences). This is mostly just by definition. For the first adjunction, for any point $x$ in the Nisnevich topology (i.e., Henselian local ring) of some $X \in \mathbf{Sm}/S$, and any simplicial presheaf $A$, $\pi_n(A_x) = \varinjlim \pi_n(A(U))$; the (filtered) limit is over all Nisnevich neighborhoods $U$ of $x$.

The stable homotopy categories are triangulated categories. We write $\mathbf{SH}(S)_{\mathbb{Q}}$ for the category obtained by tensoring all Hom-groups with $\mathbb{Q}$.

For any map $f : T \to S$, not necessarily of finite type, the stable homotopy categories are connected by adjunctions [CD09, 1.1.11, 1.1.13; 2.4.4., 2.4.10]

$$f^* : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}(T) : f_* \tag{2}$$

and, if $f$ is separated (of finite type)

$$f_! : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}(T) : f^!. \tag{3}$$

The right adjoint to $S^1 \wedge - : \mathbf{Ho}_\bullet(S) \to \mathbf{Ho}_\bullet(S)$ (or on $\mathbf{SH}(S)$) is denoted $\mathrm{R}\Omega$.

In $\mathbf{Ho}(S)$, there is an isomorphism $\mathbb{P}^1_S \cong \mathbb{G}_{m,S} \wedge S^1$ (smash product of $\mathbb{G}_{m,S}$, pointed by 1, with the simplicial one-sphere). In particular, smashing with $S^1$ is invertible on $\mathbf{SH}(S)$ and $M[p] := M \wedge (S^1)^{\wedge p} = M \wedge S^p$ for $p \geq 0$ and the unique object $M'$ such that $M'[-p] = M$ for $p < 0$. Twists are defined by $M(p) := M \wedge \mathbb{G}_{m,S}^{\wedge p}[-p]$ with a similar convention for negative twists. For any $p \in \mathbb{Z}$, we will sometimes write

$$M\{p\} := M(p)[2p].$$

## 2.2 Motives

Let $S$ be a Noetherian scheme of finite dimension. Recall the following facts and definitions due to Riou [Rio07, IV.46, IV.72]. There is an object $\mathrm{BGL}_S \in \mathbf{SH}(S)$ representing algebraic $K$-theory in the sense that

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, \mathrm{BGL}_S) = K_n(X) \tag{4}$$

for any regular scheme $S$ and any smooth scheme $X/S$, functorially (with respect to pullback) in $X$. Also recall $\mathrm{BGL}_S \otimes \mathbb{Q}$ decomposes as

$$\mathrm{BGL}_S \otimes \mathbb{Q} = \oplus_{n \in \mathbb{Z}} \mathrm{BGL}_S^{(n)}$$

such that the pieces $\mathrm{BGL}_S^{(n)}$ represent the graded pieces of the $\gamma$-filtration on $K$-theory:

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, \mathrm{BGL}_S^{(i)}) \cong \mathrm{gr}^i_\gamma \, K_n(X)_{\mathbb{Q}}. \tag{5}$$

The *Beilinson motivic cohomology spectrum* $\mathrm{H}_{\mathrm{B}}$ is defined by

$$\mathrm{H}_{\mathrm{B},S} := \mathrm{BGL}_S^{(0)}. \tag{6}$$

The parts of the $K$-theory spectrum are related by natural isomorphisms

$$\mathrm{BGL}_S^{(p)} = \mathrm{H}_{\mathrm{B},S}\{p\}. \tag{7}$$

For any map $f : T \to S$, not necessarily of finite type, there are natural isomorphisms

$$f^*\mathrm{BGL}_S = \mathrm{BGL}_T, \;\; f^*\mathrm{H}_{\mathrm{B},S} = \mathrm{H}_{\mathrm{B},T}. \tag{8}$$

**Definition 2.1.** [CD09, Section 12.3] By a result of Röndigs, Spitzweck and Ostvaer, $\mathrm{BGL}_S \in \mathbf{SH}(S)$ is weakly equivalent to a strict ring spectrum $\mathrm{BGL}'_S$, that is to say a ring object in the underlying model category $\mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$. Thus it makes sense to look at the subcategory of $\mathrm{BGL}'_S$-modules. Its model category structure is defined such that the adjunction

$$- \wedge \mathrm{BGL}'_S : \mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet) \leftrightarrows \mathrm{BGL}'_S - \text{modules in } \mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet) : \text{forget} \tag{9}$$

is a Quillen adjunction. The category $\mathbf{DM}_{\mathrm{BGL}}(S)$ is defined to be the homotopy category of the right hand category. It is a full subcategory of $\mathbf{SH}(S)$. Similarly, the category $\mathbf{DM}_{\mathrm{B}}(S) \subset \mathbf{SH}(S)_{\mathbb{Q}}$ is defined to be the homotopy category of the subcategory of $\mathrm{H}_{\mathrm{B},S}$-modules. Objects in this category will be referred to as *Beilinson motives* (or just motives) over $S$.

*Motivic cohomology* of any object $M$ in $\mathbf{SH}(S)_{\mathbb{Q}}$ is defined as

$$\mathrm{H}^n(M,p) := \mathrm{Hom}_{\mathbf{SH}(S)}(M, \mathrm{H}_{\mathrm{B}}(p)[n]) \overset{(9)}{=} \mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(S)}(M \otimes \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_{\mathrm{B},S}(p)[n]).$$

**Theorem 2.2.** *[CD09, 2.4.21, 13.4.1] The functors $f_*$, $f^*$, $f_!$ and $f^!$ of (2), (3) preserve the subcategories $\mathbf{DM}_{\mathrm{BGL}}(-)$ and $\mathbf{DM}_{Ƃ}(-)$ of $\mathbf{SH}(-)$. Moreover,* relative purity *holds: for any smooth quasi-projective morphism $f : X \to Y$ of constant relative dimension $n$ and any $M \in \mathbf{DM}_{Ƃ}(Y)$, we have a natural isomorphism*

$$f^! M \cong f^* M\{n\}. \tag{10}$$

*In particular, $f^! \mathrm{H}_{Ƃ,Y} = \mathrm{H}_{Ƃ,X}\{n\}$. Secondly,* absolute purity *holds: for any closed immersion $i : X \to Y$ between two regular schemes $X$ and $Y$ with constant relative codimension $n$, there is an isomorphism*

$$i^! \mathrm{H}_{Ƃ,Y} \cong \mathrm{H}_{Ƃ,X}\{-n\}. \tag{11}$$

**Definition 2.3.** Let $f : X \to S$ be any map of finite type. We define the *motive* of $X$ over $S$ to be

$$\mathrm{M}(X) := \mathrm{M}_S(X) := f_! f^! \mathrm{H}_{Ƃ,S} \in \mathbf{DM}_{Ƃ}(S).$$

**Remark 2.4.** In [CD09, 1.1.33] the motive of a smooth scheme $f : X \to S$ is defined as $f_\sharp f^* \mathrm{H}_{Ƃ,S}$. Here, $f_\sharp$ is the left adjoint to $f^*$. These two definitions agree up to isomorphism: we can assume that is of constant relative dimension $d$. By relative purity, the functors $f^!$ and $f^*\{d\}$ are isomorphic. Thus their left adjoints, namely $f_!$ and $f_\sharp\{-d\}$ agree, too. Therefore, $f_! f^! \mathrm{H}_{Ƃ,S} = f_! f^* \mathrm{H}_{Ƃ,S}\{d\} = f_\sharp f^* \mathrm{H}_{Ƃ,S}$.

## 2.3 Deligne cohomology

We recall the properties of Deligne cohomology that we need in the sequel. For the construction of a spectrum representing Deligne cohomology in Section 3 it is necessary to have an explicit, functorial down-to-earth complex whose cohomology groups identify with Deligne cohomology. This is due to Burgos [Bur97].

**Definition 2.5.** [GS90, 3.1.1.] An *arithmetic ring* is a datum $(S, \Sigma, \mathrm{Fr}_\infty)$, where $S$ is a ring, $\Sigma = \{\sigma_1, \ldots, \sigma_n : S \to \mathbb{C}\}$ is a set of embeddings of $S$ into $\mathbb{C}$ and $\mathrm{Fr}_\infty : \mathbb{C}^\Sigma (:= \oplus_1^n \mathbb{C}) \to \mathbb{C}^\Sigma$ is a $\mathbb{C}$-antilinear involution (called *infinite Frobenius*) such that $\mathrm{Fr}_\infty \circ \sigma = \sigma$, where $\sigma = (\sigma_i)_i : S \to \mathbb{C}^\Sigma$ consists of the $\sigma_i$. If $S$ happens to be a field, this datum is called *arithmetic field*. For example, the generic fiber $\eta : S_\eta := S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q} \to S$ of an arithmetic ring is an arithmetic field. For any scheme $X$ over an arithmetic ring $S$, we write

$$X_\mathbb{C} := \sqcup_{\sigma \in \Sigma} X \times_{S,\sigma} \mathbb{C}$$

and $X(\mathbb{C})$ for the associated complex-analytic space (with its classical topology). We also write $\mathrm{Fr}_\infty : X_\mathbb{C} \to X_\mathbb{C}$ for the pullback of infinite Frobenius on the base.

**Example 2.6.** We have the following examples: $S = F$ (or $S = \mathcal{O}_F$), a number field (or its ring of integers), with its set $\Sigma$ of archimedean places, and $\mathrm{Fr}_\infty$ given by $(z_v)_{v\in\Sigma} \mapsto (\overline{z}_{\overline{v}})_v$. Moreover, we have $(\mathbb{R}, \text{standard inclusion}, \mathrm{id})$, as well as $(\mathbb{C}, \{\mathrm{id}, \overline{?}\}, \mathrm{Fr}_\infty : \mathbb{C}^2 \ni (a,b) \mapsto (\overline{b}, \overline{a}))$.

In the remainder of this subsection, $X/S$ is a smooth scheme (of finite type) over an arithmetic field.

**Definition 2.7.** [Bur97, Def. 1.2, Thm. 2.6] Let $E^*(X(\mathbb{C}))$ be the following complex:

$$E^*(X(\mathbb{C})) := \varinjlim E^*_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C})),$$

where the colimit is over all smooth compactifications $\overline{X}$ of $X$ such that $D := \overline{X} \backslash X$ is a divisor with normal crossings. The complex $E^*_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C}))$ is the complex of $C^\infty$-differential forms that have at most logarithmic poles along the divisor (see *loc. cit.* for details). The complex is a commutative differential graded algebra. We write

$$E^*(X) := (E^*(X(\mathbb{C})))^{\overline{\mathrm{Fr}^*_\infty}}$$

for the subcomplex of elements fixed under the $\overline{\mathrm{Fr}^*_\infty}$-action. Forms in $E^*(X)$ that are fixed under complex conjugation are referred to as real forms and denoted $E_\mathbb{R}(X)$. As usual, a twist is written as $E_\mathbb{R}(X)(p) := (2\pi i)^p E_\mathbb{R}(X) \subset E^*(X)$. The complex $E^*(X)$ is filtered by

$$F^p E^*(X) := \oplus_{a \geq p, a+b=*} E^{a,b}(X).$$

Let $\mathrm{D}^*(X,p)$ be the complex defined by

$$\mathrm{D}^n(X,p) := \begin{cases} E^{2p+n-1}_\mathbb{R}(X)(p-1) \cap \oplus_{a+b=2p+n-1, a,b<p} E^{a,b}(X) & n<0 \\ E^{2p+n}_\mathbb{R}(X)(p) \cap \oplus_{a+b=2p+n, a,b \geq p} E^{a,b}(X) & n \geq 0 \end{cases}$$

The differential $d_\mathrm{D}(x)$, $x \in \mathrm{D}^n(X,p)$, is defined as $-\mathrm{proj}(dx)$ $(n<-1)$, $-2\partial\overline{\partial}x$ $(n=-1)$, and $dx$ $(n \geq 0)$. Here $d$ is the standard exterior derivative, and proj denotes the projection onto the space of forms of the appropriate bidegrees. The usual pullback of differential forms turns D into complexes of presheaves on $\mathbf{Sm}/S$. We also set

$$\mathrm{D} := \bigoplus_{p \in \mathbb{Z}} \mathrm{D}(p).$$

*Deligne cohomology* of $X$ is defined as

$$\mathrm{H}^n_\mathrm{D}(X,p) := \mathrm{H}^{n-2p}(E^*(X)(p)).$$

For a scheme $X$ over an arithmetic ring, such that $X \times_S S_\eta$ is smooth (possibly empty), we set $\mathrm{H}^n_\mathrm{D}(X,p) := \mathrm{H}^n_\mathrm{D}(X \times_S S_\eta)$.

Recall that a complex of presheaves $X \mapsto F_*(X)$ on $\mathbf{Sm}/S$ is said to have *étale descent* if for any $X \in \mathbf{Sm}/S$ and any étale map $f: Y \to X$ the canonical map

$$F_*(X) \to \mathrm{Tot}(F_*(\ldots \to Y \times_X Y \to Y))$$

is a quasi-isomorphism. The right hand side is the total complex of $F_*$ applied to the Čech nerve. At least if $F$ is a complex of presheaves of $\mathbb{Q}$-vector spaces, this is equivalent to the requirement that

$$F_*(X) \to \mathrm{Tot}(F_*(\mathcal{Y}))$$

is a quasi-isomorphism for any étale hypercover $\mathcal{Y} \to X$. Indeed the latter is equivalent to Galois descent, cf. (19) and Nisnevich descent in the sense of hypercovers, which is equivalent to the one in the sense of Cech nerves by the Morel-Voevodsky criterion (see e.g. [CD09, 3.3.2, Theorem 3.3.22]).

**Theorem 2.8.** *(i) The previous definition of Deligne cohomology agrees with the classical one (for which see e.g. [EV88a]). In particular, there is a long exact sequence*

$$\mathrm{H}^n_{\mathrm{D}}(X,p) \to \mathrm{H}^n(X(\mathbb{C}),\mathbb{R}(p))^{(-1)^p} \to (\mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}})/F^p\mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}}))^{\mathrm{Fr}_\infty} \to \mathrm{H}^{n+1}_{\mathrm{D}}(X,p) \quad (12)$$

*involving Deligne cohomology, the $(-1)^p$-eigenspace of the $\overline{\mathrm{Fr}^*_\infty}$ action on Betti cohomology, and the invariant subspace of de Rham cohomology modulo the Hodge filtration.*

*(ii) The complex* $\mathrm{D}(p)$ *is* homotopy invariant *in the sense that the projection map $X\times\mathbb{A}^1 \to X$ induces a quasi-isomorphism* $\mathrm{D}(\mathbb{A}^1\times X) \to \mathrm{D}(X)$ *for any $X \in \mathbf{Sm}/S$.*

*(iii) There is a functorial* first Chern class *map*

$$c_1 : \mathrm{Pic}(X) \to \mathrm{H}^2_{\mathrm{D}}(X,1). \quad (13)$$

*(iv) Let $E$ be a vector bundle of rank $r$ over $X$. Let $P := \mathbf{P}(E)$ be the projectivization of $E$ with tautological bundle $\mathcal{O}_P(-1)$.*[1] *Then there is an isomorphism*

$$\oplus_{i=0}^{r-1} \mathrm{H}^{n-2i}_{\mathrm{D}}(X,p-i) \overset{-\cup(c_1(\mathcal{O}_P(1)))^i}{\longrightarrow} \mathrm{H}^n_{\mathrm{D}}(P,p). \quad (14)$$

*In particular the following Künneth-type formula holds:*

$$\mathrm{H}^n_{\mathrm{D}}(\mathbb{P}^1\times X,p) \cong \mathrm{H}^{n-2}_{\mathrm{D}}(X,p-1) \oplus \mathrm{H}^n_{\mathrm{D}}(X,p). \quad (15)$$

*(v) The complex of presheaves* $\mathrm{D}(p)$ *satisfy étale descent.*

*(vi) The complex* $\mathrm{D}$ *is a unital differential bigraded $\mathbb{Q}$-algebra which is associative and commutative up to homotopy. (The product of two sections will be denoted by $a \cdot_{\mathrm{D}} b$.) The induced product on Deligne cohomology agrees with the classical product on these groups [EV88b, Section 3].*

*Moreover, for a section $x \in \mathrm{D}_0(X)$ satisfying $d_{\mathrm{D}}(x)(= dx) = 0$ and any two sections $y, z \in \mathrm{D}_*(X)$, we have*

$$x \cdot_{\mathrm{D}} (y \cdot_{\mathrm{D}} z) = (x \cdot_{\mathrm{D}} y) \cdot_{\mathrm{D}} z \quad (16)$$

*and*

$$x \cdot_{\mathrm{D}} y = y \cdot_{\mathrm{D}} x. \quad (17)$$

[1] We use the notation of [Har77]: $\mathcal{O}(-1)$ is the generator of $\mathrm{Pic}(\mathbb{P}^1)$ of degree $-1$, i.e., whose only global section is 0.

***Proof:*** (i): This explicit presentation of Deligne cohomology is due to Burgos [Bur97, Prop. 1.3.]. The sequence (12) is a consequence of this and the degeneration of the Hodge to de Rham spectral sequence. See e.g. [EV88b, Cor. 2.10].

(ii): This follows from (12) and the homotopy invariance of Betti cohomology, de Rham cohomology, and, by functoriality of the Hodge filtration, homotopy invariance of $F^p \mathrm{H}^n_{\mathrm{dR}}(-)$.

(iii): See [BGKK07, Section 5.1.] (or [EV88b, Section 7] for the case of a proper variety). In fact, there is a unique such morphism such that its composition with the map to Betti cohomology coincides with the usual first Chern class [EV88b, Prop. 8.2.].

(iv): See e.g. [EV88b, Prop. 8.5.].

(v): This statement is part of the existence statement of the absolute Hodge realization functor [Hub00, Corollary 2.3.5] (and also seems to be folklore). Since it is crucial for us in Theorem 3.7, we give a proof here. Let

$$\tilde{\mathrm{D}}^*(X,p) := \mathrm{cone}(E^*(X)_{\mathbb{R}}(p) \oplus F^p E^*(X) \stackrel{(+1,-1)}{\longrightarrow} E^*(X))[-1+2p].$$

By [Bur97, Theorem 2.6.], there is a natural (fairly concrete) homotopy equivalence between the complexes of presheaves $\tilde{\mathrm{D}}(p)$ and $\mathrm{D}(p)$. The descent statement is stable under quasi-isomorphisms of complexes of presheaves and cones of maps of such complexes. Therefore it is sufficient to show descent for the three constituent parts of $\tilde{\mathrm{D}}^*(X,p)$, namely $X \mapsto E^*_{\mathbb{R}}(X)(p)$, $X \mapsto F^p E^*(X)$, $X \mapsto E^*(X)$. Taking invariants of these complex under the $\overline{\mathrm{Fr}^*_\infty}$-action is an exact functor, so we can (and will) assume $S = \mathbb{C}$ in this proof. Let $j : X \to \overline{X}$ be an open immersion into a smooth compactification such that $D := \overline{X} \backslash X$ is a divisor with normal crossings. The inclusion

$$\Omega^*_{\overline{X}}(\log D) \subset E^*_X(\log D)$$

of holomorphic forms into $C^\infty$-forms (both with logarithmic poles) yields quasi-isomorphisms of complexes of vector spaces

$$\mathrm{R}\Gamma \mathrm{R} j_* \mathbb{C} \to \mathrm{R}\Gamma \mathrm{R} j_* \Omega^*_{X(\mathbb{C})} \leftarrow \mathrm{R}\Gamma \Omega^*_{\overline{X}}(\log D) \to \Gamma E^*_{X(\mathbb{C})}(\log D)$$

that are compatible with both the real structure and the Hodge filtration [Bur94, Theorem 2.1.], [Del71, 3.1.7, 3.1.8]. Here (R)$\Gamma$ denotes the (total derived functor of the) global section functor on $\overline{X}(\mathbb{C})$, i.e., with respect to the analytic topology. The complex $E^*(X)$, whose cohomology is $\mathrm{H}^*(X(\mathbb{C}), \mathbb{C})$, is known to satisfy étale descent [Hub00, Prop. 2.1.7]. This also applies to $E^*_{\mathbb{R}}(X)(p)$ instead of $E^*(X)$. (Alternatively for the former, see also [CD07, 3.1.3] for the étale descent of the algebraic de Rham complex $\Omega^*_X$.)

It remains to show the descent for $X \mapsto F^p E^*(X)$. Consider a distinguished square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y, \end{array}$$

i.e., cartesian such that $Y' \to Y$ is an open immersion, $X/Y$ is étale and induces an isomorphism $(X\backslash X')_{\mathrm{red}} \to (Y\backslash Y')_{\mathrm{red}}$. Then the sequence

$$\mathrm{H}^n(F^pE^*(Y)) \to \mathrm{H}^n(F^pE^*(Y')) \oplus \mathrm{H}^n(F^pE^*(X)) \to \mathrm{H}^n(F^pE^*(X')) \to \mathrm{H}^{n+1}(F^pE^*(Y)) \tag{18}$$

is exact: indeed, $\mathrm{H}^n(F^pE_{\overline{X}}(\log D))$ maps injectively into $\mathrm{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D)$, and the image is precisely the $p$-th filtration step of the Hodge filtration on $\mathrm{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D)) = \mathrm{H}^n(X, \mathbb{C})$. Similarly for $X'$ etc., so that the exactness of (18) results from the one of the sequence featuring the Betti cohomology groups of $Y$, $Y' \sqcup X$ and $X'$, together with the strictness of the Hodge filtration [Del71, Th. 1.2.10]. This shows Nisnevich descent for the Hodge filtration. Secondly, for any scheme $X$ and a Galois cover $Y \to X$ with group $G$, the pullback map into the $G$-invariant subspace

$$\mathrm{H}^n(F^pE^*(X)) \to \mathrm{H}^n(F^pE^*(Y)^G) \tag{19}$$

is an isomorphism. Indeed, the similar statement holds for $E^*(-)$ instead of $F^pE^*(-)$. We work with $\mathbb{Q}$-coefficients, so taking $G$-invariants is an exact functor, hence $\mathrm{H}^n(F^pE^*(Y)^G) = (\mathrm{H}^n(F^pE^*(Y)))^G = (F^p\mathrm{H}^n_{\mathrm{dR}}(Y))^G = F^p(\mathrm{H}^n_{\mathrm{dR}}(Y)^G)$, the last equality by functoriality of the Hodge filtration. Then, again using the strictness of the Hodge filtration, the claim follows. Hence (this uses $\mathbb{Q}$-coefficients) the presheaf $X \mapsto F^pE^*(X)$ has étale descent. The statement (v) is shown.

(vi): [Bur97, Theorem 3.3.].[2] □

# 3 The Deligne cohomology spectrum

Let $S$ be smooth scheme (of finite type) over an arithmetic field (Definition 2.5). The aim of this section is to construct a ring spectrum in $\mathbf{SH}(S)$ which represents Deligne cohomology for smooth schemes $X$ over $S$. The method is a slight variation of the method of Cisinski and Deglise used in [CD07] to construct a spectrum for any mixed Weil cohomology, such as algebraic or analytic de Rham cohomology, Betti cohomology, and (geometric) étale cohomology. The difference compared to their setting is that the Tate twist on Deligne cohomology groups is not an isomorphism of vector spaces.

In this section, all complexes of (presheaves of) abelian groups are considered with homological indexing: the degree of the differential is $-1$ and $C[1]$ is the complex whose $n$-th group is $C_{n+1}$. As usual, any cohomological complex is understood as a homological one by relabeling the indices. In particular, we apply this to $\mathrm{D}(p)$, $\mathrm{D}$ (Definition 2.7) and let

$$\mathrm{D}_n := \mathrm{D}^{-n} = \oplus_{p\in\mathbb{Z}} \mathrm{D}^{-n}(p). \tag{20}$$

[2] Actually, the product on $\mathrm{D}(X)$ is commutative on the nose. We shall only use the commutativity in the case stated in (17) and the associativity as in (16), cf. Definition and Lemma 3.3.

In order to have a complex of simplicial presheaves (as opposed to a complex of abelian groups), we use the Dold-Kan-equivalence

$$\mathcal{K} : \mathbf{Com}_{\geq 0}(\mathbf{Ab}) \rightleftarrows \Delta^{\mathrm{op}}(\mathbf{Ab}) : \mathcal{N}$$

between homological complexes concentrated in degrees $\geq 0$ and simplicial abelian groups.

As usual, we write $\tau_{\geq n}$ for the good truncation of a complex.

**Definition 3.1.** We write

$$\mathrm{D}_s := \mathcal{K}(\tau_{\geq 0}\mathrm{D}),$$
$$\mathrm{D}_s(p) := \mathcal{K}(\tau_{\geq 0}\mathrm{D}(p)).$$

**Lemma 3.2.** *For $X$ smooth over $S$ and any $k \geq 0$, $p \in \mathbb{Z}$ we have:*

$$\mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^k \wedge X_+, \mathrm{D}_s(p)) = \mathrm{H}_{\mathrm{D}}^{2p-k}(X, p) \tag{21}$$

*and similarly for* $\mathrm{D}_s$*.*

***Proof:*** The statement does hold if we take the Hom-group in $\mathbf{Ho}_{\mathrm{sect},\bullet}$ (cf. the discussion following (1) for the notation) instead of $\mathbf{Ho}_\bullet$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Ho}_{\mathrm{sect},\bullet}}(S^k \wedge X_+, \mathcal{K}(\tau_{\geq 0}(\mathrm{D}))) &= \pi_k \mathcal{K}(\tau_{\geq 0}(\mathrm{D}(X))) \\
&= \mathrm{H}_k(\tau_{\geq 0}(\mathrm{D}(X))) \\
&= \oplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}^{2p-k}(X, p).
\end{aligned}$$

(We have used the identification $\pi_n(A, 0) = \mathrm{H}_n(\mathcal{N}(A))$ for any simplicial abelian group and the fact that $\mathcal{K}(-)$ is as a simplicial abelian group a fibrant simplicial set.)

The presheaf $\mathrm{D}_s$ is fibrant with respect to the $\mathbb{A}^1$- local model structure, since Deligne cohomology satisfies Nisnevich descent and is $\mathbb{A}^1$-invariant by Theorem 2.8 (v) and (ii). Thus the Hom-groups agree when taken in $\mathbf{Ho}_{\mathrm{sect},\bullet}$ and $\mathbf{Ho}$, respectively. □

Via the Alexander-Whitney map, the product on D transfers to a product

$$\mathrm{D}_s(i) \wedge \mathrm{D}_s(j) \stackrel{\mu_{i,j}}{\longrightarrow} \mathrm{D}_s(i+j).$$

**Definition and Lemma 3.3.** *The* Deligne cohomology spectrum $\mathrm{H}_{\mathrm{D}}$ *is the symmetric $\mathbb{P}^1$-spectrum consisting of the $\mathrm{D}_s(p)$ ($p \geq 0$), equipped with the trivial action of the symmetric group $\Sigma_p$. We define the bonding maps to be the composition*

$$\sigma_p : \mathbb{P}^1_S \wedge \mathrm{D}_s(p) \stackrel{c^* \wedge \mathrm{id}}{\to} \mathrm{D}_s(1) \wedge \mathrm{D}_s(p) \stackrel{\mu_{1,p}}{\to} \mathrm{D}_s(p+1).$$

*Here $c^*$ is the map induced by $c := c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathrm{D}_0(1)(\mathbb{P}^1)$, the first Chern form of the bundle $\mathcal{O}(1)$ which is equipped with the Fubini-Study metric. Moreover, $\mu_{1,p}$ is the product map mentioned above.*

*We equip* $\mathrm{H_D}$ *with the following monoid structure: the product* $\mu : \mathrm{H_D} \wedge \mathrm{H_D} \to \mathrm{H_D}$ *is induced by the products* $\mu_{p,p'} : \mathrm{D}_s(p) \wedge \mathrm{D}_s(p') \to \mathrm{D}_s(p+p')$*. The unit map* $\eta : \Sigma^\infty_{\mathbb{P}^1} S_+ \to \mathrm{H_D}$ *is defined in degree zero by the unit of the DGA* $\mathrm{D}(0)$*. In higher degrees, we put*

$$\eta_p : (\mathbb{P}^1)^{\wedge p} \xrightarrow{(c^*)^{\wedge p}} \mathrm{D}_s(1)^{\wedge p} \xrightarrow{\mu} \mathrm{D}_s(p).$$

*Equivalently,* $\eta_p := \sigma_{p-1} \circ (\mathrm{id}_{\mathbb{P}^1} \wedge \eta_{p-1})$*.*

*This defines a symmetric* $\mathbb{P}^1$*-spectrum in* $\Delta^{\mathrm{op}}\mathbf{PSh}(\mathbf{Sm}/S))$*. As an object in* $\mathbf{Ho}_{\mathrm{sect},\bullet}(S)$*,* $\mathrm{H_D}$ *is a ring spectrum.*

***Proof:*** First, recall that $c$ is a $(1,1)$-form which is invariant under $\mathrm{Fr}^*_\infty$ and under complex conjugation, so $c$ is indeed an element of $\mathrm{D}_0(1)(\mathbb{P}^1)$. Secondly, if we write $\infty$ for the immersion of the infinite point in $\mathbb{P}^1_S$, we have $\infty^* c = 0 \in \mathrm{D}_0(1)(S)$, since the pullback of $c$ is a 2-form, but $\dim S = 0$. That is, $c$ is a pointed map $(\mathbb{P}^1, \infty) \to (\mathrm{D}_0(1), 0)$. Thirdly, we have to show that the map

$$\begin{array}{rcl} {\mathbb{P}^1}^{\wedge m} \wedge \mathrm{D}_s(n) & \xrightarrow{\mathrm{id}^{\wedge m-1} \wedge c^* \wedge \mathrm{id}} & {\mathbb{P}^1}^{\wedge m-1} \wedge \mathrm{D}_s(1) \wedge \mathrm{D}_s(n) \\ & \xrightarrow{\mu_{1,n}} & {\mathbb{P}^1}^{\wedge m-1} \wedge \mathrm{D}_s(m+1) \\ & \to & \dots \\ & \to & \mathrm{D}_s(m+n) \end{array}$$

is a $\Sigma_{m+n}$ equivariant map of presheaves on $\mathbf{Sm}/S$, i.e., invariant under permuting the $m$ wedge factors $\mathbb{P}^1$. Given some map $f : U \to {\mathbb{P}^1}^{\times m}$ with $U \in \mathbf{Sm}/S$, let $f_i : U \to \mathbb{P}^1$ be the $i$-th projection of $f$ and $c_i := f_i^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Given some form $\omega \in \mathrm{D}(n)(U)$ (in some unspecified degree), the map is given by

$$(f, \omega) \mapsto c_1 \cdot_{\mathrm{D}} (c_2 \cdot_{\mathrm{D}} (\dots (c_m \cdot_{\mathrm{D}} \omega) \dots)).$$

Here $\cdot_{\mathrm{D}}$ denotes the product on $\mathrm{D}(*)$ (also denoted $\mu_{1,*}$). The forms $c_i \in \mathrm{D}_0(1)(U)$ are closed differential forms, so by Theorem 2.8(vi) the right hand expression is associative and commutative, i.e. invariant under the permutation action of $\Sigma_m$ on ${\mathbb{P}^1}^{\times m}$.

By *loc. cit.*, the product on D is (graded) commutative and associative up to homotopy, thus the diagrams checking, say, the commutativity of $\mathrm{H_D} \wedge \mathrm{H_D} \to \mathrm{H_D}$ do hold in the homotopy category $\mathbf{Ho}_{\mathrm{sect},\bullet}(S)$. The details of that verification are omitted. □

**Remark 3.4.**

1. The spectrum $\bigoplus_{p \in \mathbb{Z}} \mathrm{H_D}\{p\}$ is given by replacing the $p$-th level $\mathrm{D}_s(p)$ of $\mathrm{H_D}$ by D. Indeed, to see that the two agree, it is enough to check that $\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, -)$ induces an isomorphism when applied to the map in question. By the compactness of $S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+$ in $\mathbf{SH}(S)$, this Hom-group commutes with the direct sum. Then the claim is trivial.

2. Choosing another metric on $\mathcal{O}(1)$ in the above definition, the resulting spectrum would be weakly equivalent because the Chern class (as opposed to the Chern form) is independent of the choice of the metric.

**Lemma 3.5.** *The Deligne cohomology spectrum* $\mathrm{H}_\mathrm{D}$ *is an* $\Omega$*-spectrum (with respect to* $\mathbb{P}^1$*).*

***Proof:*** We have to check that the adjoint map to to $\sigma_p$ (Definition and Lemma 3.3),

$$b_p : \mathrm{D}_s(p) \to \mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p+1)),$$

is a $\mathbb{A}^1$-local weak equivalence. As $\mathbb{P}^1$ is cofibrant and $\mathrm{D}_s(p+1)$ is fibrant, the non-derived $\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p))$ is fibrant and agrees with $\mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p))$. The map is actually a sectionwise weak equivalence, i.e., an isomorphism in $\mathbf{Ho}_{\mathrm{sect},\bullet}(S)$. To see this, it is enough to check that the map

$$\mathrm{D}_s(p)(U) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p+1)(U))$$

is a weak equivalence of simplicial sets for all $U \in \mathbf{Sm}/S$ [MV99, 1.8., 1.10, p. 50]. The $m$-th homotopy group of the left hand side is $\mathrm{H}_\mathrm{D}^{2p-m}(U,p)$ (Lemma 3.2), while $\pi_m$ of the right hand simplicial set identifies with those elements of $\pi_m(\underline{\mathrm{Hom}}(\mathbb{P}^1 \times U, \mathrm{D}_s(p+1)) = \mathrm{H}_\mathrm{D}^{2(p+1)-m}(\mathbb{P}^1 \times U, p+1)$ which restrict to zero when applying the restriction to the point $\infty \to \mathbb{P}^1$. By the projective bundle formula (15), the two terms agree. $\square$

**Theorem 3.6.** *The ring spectrum* $\mathrm{H}_\mathrm{D}$ *represents Deligne cohomology in* $\mathbf{SH}(S)$*: for any smooth variety* $X$ *over* $S$*, and any* $n$*,* $m \in \mathbb{Z}$ *we have*

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1}({\mathbb{P}^1_S}^{\wedge m} \wedge X_+), \mathrm{H}_\mathrm{D}) = \mathrm{H}_\mathrm{D}^{-n-2m}(X, -m).$$

*(For* $n < 0$ *the left hand group is the same as* $\mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, S^{-n} \wedge \mathrm{H}_\mathrm{D})$*, taking into account that smashing with* $S^1$ *is invertible in* $\mathbf{SH}(S)$*. Likewise for* $m < 0$*, see p. 4.)*

***Proof:*** By Lemma 3.5, $\mathrm{H}_\mathrm{D}$ is an $\Omega$-spectrum. Thus the claim follows from Lemma 3.2. $\square$

**Theorem 3.7.** *The Deligne cohomology spectrum* $\mathrm{H}_\mathrm{D}$ *has a unique structure* $\mathrm{H}_{\text{Ɓ},S}$*-algebra and* $\oplus_{p\in\mathbb{Z}}\mathrm{H}_\mathrm{D}\{p\}$ *has a unique structure of an* $\mathrm{BGL}_S$*-algebra. In particular,* $\mathrm{H}_\mathrm{D}$ *is an object in* $\mathbf{DM}_{\text{Ɓ}}(S)$*.*

**Definition 3.8.** The maps induced by the unit of $\mathrm{H}_\mathrm{D}(\subset \oplus_p \mathrm{H}_\mathrm{D}\{p\})$ are denoted

$$\rho_\mathrm{D} : \mathrm{H}_{\text{Ɓ}} \to \mathrm{H}_\mathrm{D}, \quad \mathrm{ch}_\mathrm{D} : \mathrm{BGL} \to \oplus_p \mathrm{H}_\mathrm{D}\{p\}.$$

***Proof:*** By construction, $\mathrm{H}_\mathrm{D}$ is a commutative ring spectrum. Recall the definition of étale descent for spectra and that for this it is sufficient that the individual pieces of the spectrum have étale descent [CD09, Def. 3.2.5, Cor. 3.2.18]. Thus, $\mathrm{H}_\mathrm{D}$ satisfies étale descent by Theorem 2.8(v). Any commutative ring spectrum in $\mathbf{SH}(S)_\mathbb{Q}$ satisfying étale descent admits a unique structure of an $\mathrm{H}_{\text{Ɓ}}$-algebra [CD09, Corollary to Theorem 13, p. 7; 13.2.15]. This settles the claim for $\mathrm{H}_\mathrm{D}$. Secondly, the natural map (in $\mathbf{SH}(S)_\mathbb{Z}$)

$$\mathrm{ch}_\mathrm{D} : \mathrm{BGL} \to \mathrm{BGL}_\mathbb{Q} \overset{(7)}{\cong} \oplus_{p\in\mathbb{Z}} \mathrm{H}_{\text{Ɓ}}\{p\} \overset{\rho_\mathrm{D}\{p\}}{\longrightarrow} \oplus_p \mathrm{H}_\mathrm{D}\{p\}$$

and the ring structure of $\oplus \mathrm{H_D}\{p\}$ defines a BGL-algebra structure on $\oplus \mathrm{H_D}\{p\}$. This uses that the isomorphism (7) is an isomorphism of ring objects [CD09, 13.2.16]. The unicity of that structure follows from the unicity of the one on $\mathrm{H_D}$ and $\mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Z}}(\mathrm{BGL}_\mathbb{Q}, \oplus \mathrm{H_D}\{p\}) = \mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(\mathrm{BGL}_\mathbb{Q}, \oplus \mathrm{H_D}\{p\})$, since $\mathrm{H_D}$ is a spectrum of $\mathbb{R}$- (a fortiori: $\mathbb{Q}$-)vector spaces. □

# 4 Arakelov motivic cohomology

Let $S$ be a regular scheme of finite type over an arithmetic ring $B$ (Definition 2.5). Let $\eta : S_\eta := S \times_\mathbb{Z} \mathbb{Q} \to S$ be the generic fiber of $S$. The scheme $S_\eta$ is smooth over the arithmetic field belonging to $B$, so Section 3 yields a Deligne cohomology spectrum in $\mathbf{SH}(S_\eta)$. We glue this with $\mathrm{BGL}_S$ and $\mathrm{H}_{\mathrm{B},S}$ to define a Arakelov motivic cohomology spectrum. The framework of the stable homotopy category and motives readily implies a number of formal properties, most notably pushforwards for Arakelov motivic cohomology.

## 4.1 Definition

**Definition 4.1.** Recall the spectra BGL and $\mathrm{H}_{\mathrm{B},S}$ from (4) and (6), respectively, as well as the regulator maps $\mathrm{ch_D} : \mathrm{BGL}_{S_\eta} \to \oplus \mathrm{H_D}\{p\}$, $\rho_\mathrm{D} : \mathrm{H}_{\mathrm{B},S_\eta} \to \mathrm{H_D}$ (Definition 3.8). Using the adjunction $\eta^* \rightleftarrows \eta_*$ we put:

$$\widehat{\mathrm{BGL}_S} := \mathrm{hofib}\left(\mathrm{BGL}_S \to \eta_* \mathrm{BGL}_{S_\eta} \xrightarrow{\mathrm{ch_D}} \eta_* \oplus_p \mathrm{H_D}\{p\}\right),$$

$$\widehat{\mathrm{H}_{\mathrm{B},S}} := \mathrm{hofib}\left(\mathrm{H}_{\mathrm{B},S} \to \eta_* \mathrm{H}_{\mathrm{B},S_\eta} \xrightarrow{\rho_\mathrm{D}} \eta_* \mathrm{H_D}\right)$$

The latter is called *Arakelov motivic cohomology spectrum*.

**Remark 4.2.**
- The map $\mathrm{ch_D}$ is a map in $\mathbf{DM}_{\mathrm{BGL}}(S) \subset \mathbf{SH}(S)$ (cf. Definition 2.1). In the underlying model category $\mathrm{BGL}' - \mathbf{Mod} \subset \mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$, $\mathrm{BGL}'$ is cofibrant and $\mathrm{H_D}$ is fibrant (Lemma 3.5). Thus, $\mathrm{ch_D}$ can be lifted to a map $\mathrm{ch}'_\mathrm{D}$ of $\mathrm{BGL}'$-modules. As an object of $\mathbf{DM}_{\mathrm{BGL}}(S)$, the homotopy fiber of $\mathrm{ch_D}$ is independent of the choice of the lift. (We refer to, say, [Hir03, Section 13.4] for generalities on homotopy fibers.) The adjunction (9) is a Quillen adjunction, in particular the forgetful functor preserves homotopy limits and in particular homotopy fibers. Thus, we could also have lifted $\mathrm{ch_D}$ to a map $\mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ and taken its homotopy fiber, without changing the resulting definition of $\widehat{\mathrm{BGL}_S}$. Similarly, $\widehat{\mathrm{H}_{\mathrm{B},S}}$ can be constructed indifferently in $\mathbf{SH}(S)_\mathbb{Q}$ or $\mathbf{DM}_\mathrm{B}(S)$.
- We are mainly interested in gluing motivic cohomology with Deligne cohomology. However, nothing is special about Deligne cohomology. In fact, given some scheme $f : T \to S$ (not necessarily of finite type), and complexes of presheaves of $\mathbb{Q}$-vector spaces $D(p)$ on $\mathbf{Sm}/T$ satisfying the

conclusion of Theorem 2.8(ii), (iii), (iv) (actually (15) suffices), (v), and (vi), everything could be done with $f_* D(p)$ instead of $\eta_* \mathrm{D}(p)$. This might be an approach to an adelic variant of Arakelov theory, as envisioned in [Sou92, O.1.5].

- By construction, the spectrum $\widehat{\mathrm{H}_{\mathrm{Б},S}}$ is a spectrum of $\mathbb{Q}$-vector spaces. A seemingly plausible modification of this definition to $\mathbb{Z}$-coefficients would be

$$\widehat{\mathrm{H}\mathbb{Z}}_S := \mathrm{hofib}\left(\mathrm{H}\mathbb{Z}_S \to \mathrm{H}\mathbb{Q}_S \xleftarrow{\sim} \mathrm{H}_{\mathrm{Б},S} \to \eta_* \mathrm{H}_{\mathrm{Б},S_\eta} \to \eta_* \mathrm{H}_{\mathrm{D}}\right).$$

Here, $\mathrm{HR}_S$ denotes the Eilenberg-MacLane spectrum which represents motivic cohomology with R-coefficients over the base scheme $S$ [Voe98, section 6.1]. The time being, questions such as $f^* \mathrm{H}\mathbb{Z}_S \cong \mathrm{H}\mathbb{Z}_T$ for a map $f : T \to S$ are only known when passing to rational coefficients, so at this point we do not study this variant any further.

**Definition 4.3.** For any $M \in \mathbf{SH}(S)$, we define

$$\widehat{\mathrm{H}}^n(M) := \mathrm{Hom}_{\mathbf{SH}(S)}(M, \widehat{\mathrm{BGL}_S}[n]),$$

$$\widehat{\mathrm{H}}^n(M,p) := \mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(M, \widehat{\mathrm{H}_{\mathrm{Б}}}(p)[n]).$$

The latter is called *Arakelov motivic cohomology* of $M$. For any finite type scheme $f : X \to S$, we define *Arakelov motivic cohomology* of $X$ as

$$\widehat{\mathrm{H}}^n(X/S,p) := \widehat{\mathrm{H}}^n(X,p) := \widehat{\mathrm{H}}^n(\mathrm{M}_S(X),p) = \mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(f_! f^! \mathrm{H}_{\mathrm{Б},S}, \widehat{\mathrm{H}_{\mathrm{Б},S}}(p)[n]).$$

For any compact object $M$ (such as $M = \mathrm{M}_S(X)$), we have a natural isomorphism

$$\widehat{\mathrm{H}}^n(M) \otimes_\mathbb{Z} \mathbb{Q} = \oplus_{p\in\mathbb{Z}} \widehat{\mathrm{H}}^{n+2p}(M,p). \tag{22}$$

Moreover $\widehat{\mathrm{H}}^n(M,p) = \widehat{\mathrm{H}}^n(M \otimes \mathrm{H}_{\mathrm{Б},S}, p)$ by (9).

## 4.2 First properties

**Theorem 4.4.** *Arakelov motivic cohomology satisfies h-descent (thus, a fortiori, Nisnevich, étale, cdh, qfh and proper descent), homotopy invariance:*

$$\widehat{\mathrm{H}}^n(X \times \mathbb{A}^1, p) \cong \widehat{\mathrm{H}}^n(X,p),$$

*and the projective bundle formula*

$$\widehat{\mathrm{H}}^n(\mathbf{P}(E),p) \cong \oplus_{i=0}^d \widehat{\mathrm{H}}^{n-2i}(X, p-i).$$

*Here $X/S$ is arbitrary (of finite type), $E \to X$ is a vector bundle of rank $d+1$, $\mathbf{P}(E)$ is its projectivization. Any distinguished triangle of motives induces long* *exact sequences of Arakelov motivic cohomology. For example, let $X/S$ be a scheme satisfying*

$$X \text{ is regular and is a closed subscheme of a smooth scheme } Y/S. \qquad (23)$$

*Let $Z \subset X$ be a closed regular subscheme of constant codimension $c$. Let $U$ be its complement. Then there is an exact sequence*

$$\widehat{\mathrm{H}}^{n-2c}(Z, p-c) \to \widehat{\mathrm{H}}^{n}(X, p) \to \widehat{\mathrm{H}}^{n}(U, p) \to \widehat{\mathrm{H}}^{n+1-2c}(Z, p-c).$$

***Proof:*** The h-descent is a general property of modules over $\mathrm{H}_{\text{Б},S}$ [CD09, Thm 15.1.3]. The $\mathbb{A}^1$-invariance and the bundle formula are immediate from $\mathrm{M}(X) \cong \mathrm{M}(X \times \mathbb{A}^1)$ and $\mathrm{M}(\mathbf{P}(E)) \cong \oplus_{i=0}^{d} \mathrm{M}(X)\{i\}$. For the last statement, we use the localization exact triangle [CD09, 2.3.5] to $U \xrightarrow{j} X \xleftarrow{i} Z$, as well as the absolute purity for the closed immersion $Z \subset X$ and the purity isomorphism $f^* \mathrm{H}_{\text{Б},S}\{d\} = f^! \mathrm{H}_{\text{Б},S}$ for the structural map $f : X \to S$ which is obtained by using relative purity (10) for $Y/S$ and absolute purity (11) for the closed immersion $X \subset Y$ (note that $Y$ is regular since $S$ is so by assumption).

$$f_! j_! j^! f^! \mathrm{H}_{\text{Б},S} \to f_! f^! \mathrm{H}_{\text{Б},S} \to f_! i_* i^* f^! \mathrm{H}_{\text{Б},S}.$$

Hom-ming this triangle into $\widehat{\mathrm{H}_{\text{Б},S}}(p)[n]$ gives the desired long exact sequence. □

**Proposition 4.5.** *For any $M \in \mathbf{SH}(S)$ there are long exact sequences relating Arakelov motivic cohomology to (usual) motivic cohomology (Definition 2.1) and, for appropriate motives, Deligne cohomology (Definition 2.7):*

$$\ldots \to \widehat{\mathrm{H}}^{n}(M, p) \to \mathrm{H}^{n}(M, p) \to \mathrm{Hom}_{\mathbf{SH}(S)}(M, \eta_* \mathrm{H}_{\mathrm{D}}(p)[n]) \to \widehat{\mathrm{H}}^{n+1}(M, p) \ldots \qquad (24)$$

*The first map will sometimes be called the* forgetful map.*) For example, the motive $M = \mathrm{M}_S(X)$ (Definition 2.3 of a scheme $X/S$ satisfying (23) we get*

$$\widehat{\mathrm{H}}^{n}(X, p) \to K_{2p-n}(X)^{(p)}_{\mathbb{Q}} \to \mathrm{H}^{n}_{\mathrm{D}}(X, p) \to \widehat{\mathrm{H}}^{n+1}(X, p).$$

*Secondly, we have a sequence*

$$\cdots \to \widehat{\mathrm{H}}^{n}(X) \to K_{-n}(X) \to \oplus_p \mathrm{H}^{2p-n}_{\mathrm{D}}(X, p) \to \widehat{\mathrm{H}}^{n+1}(X) \to \cdots . \qquad (25)$$

*If $S' \xrightarrow{f} S$ is a scheme of characteristic $p > 0$ over $S$, the forgetful map*

$$\widehat{\mathrm{H}}^{n}(i_* M, p) \to \mathrm{H}^{n}(i_* M, p)$$

*is an isomorphism for any $M \in \mathbf{SH}(S')$.*

***Proof:*** The first statement is a general property of the homotopy fiber in any stable model category. For the second statement, let $f : X \to S$ be the structural map. In order to identify the motivic cohomology with the claimed Adams eigenspace in $K$-theory, we use the adjunction (3) and the purity isomorphism for $f$ (cf. the proof of Theorem 4.4). Consider the following cartesian diagram where $j$ is such that $f_U : X_U := X \times_S U \to U$ is smooth. Let $d := \operatorname{reldim}(f_U) = \operatorname{reldim}(f_\eta)$.

$$\begin{array}{ccccc} X_\eta & \xrightarrow{\eta'_U} & X_U & \longrightarrow & X \\ \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle f_U} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} F & \xrightarrow{\eta_U} & U & \xrightarrow{j} & S \end{array}$$

The group $\operatorname{Hom}_{\mathbf{SH}(S)}(f_! f^! \mathrm{H}_{\mathrm{B},S}, \eta_* \mathrm{H}_{\mathrm{D}}(p)[n])$ is easily seen to be isomorphic to

$$\operatorname{Hom}_{\mathbf{SH}(X_\eta)}(\mathrm{H}_{\mathrm{B},X_\eta}, f_\eta^* j^* \mathrm{H}_{\mathrm{D}}(p)[n]) \overset{3.6}{=} \mathrm{H}_{\mathrm{D}}^{2p-n}(X, p).$$

One uses $j_* j^* \eta_* \mathrm{H}_{\mathrm{D}} = \mathrm{H}_{\mathrm{D}}$ (by base-change applied to $F = F \times_{\mathbb{Z}} U$), relative purity (10), applied to $f_U$ and $f_\eta$, together with base-change ($\eta_U^* f_{U!} f_U^* = f_{\eta!} {\eta'_U}^* f_U^* = f_{\eta!} f_\eta^* \eta'^*$).

The last claim follows from $\eta^* i_* = 0$. □

**Example 4.6.** Let us list what the groups $\widehat{\mathrm{H}}^{-n} := \widehat{\mathrm{H}}^{-n}(\operatorname{Spec} \mathcal{O}_F)$ look like. This is well-known, cf. [Sou92, III.4], [Tak05, p. 623]. We refer to [Wei05] to facts pertaining to $K$-theory of $\mathcal{O}_F$ and the regulator in this case. Let $r_1$ and $r_2$ be the number of real and pairs of complex embeddings of $F$. The sequence (25) shows that for $n \leq -2$

$$\hat{H}^{-n} \overset{\cong}{\leftarrow} \oplus_p \mathrm{H}_{\mathrm{D}}^{2p-n-1}(\mathcal{O}_F, p) = \begin{cases} 0 & n = 2m \\ \mathbb{R}^{r_2} & n = 4m - 1 \\ \mathbb{R}^{r_1 + r_2} & n = 4m - 3 \end{cases}$$

Moreover, $\hat{H}^1 \cong \mathrm{H}_{\mathrm{D}}^0(0) / \operatorname{ch}_{\mathrm{D}}(K_0) = \mathbb{R}^{r_1+r_2}/\mathbb{Z}$. Here $\operatorname{ch}_{\mathrm{D}} : \mathbb{Z} = K_0/\operatorname{Pic}(\mathcal{O}_F) \to \mathrm{H}_{\mathrm{D}}^0(0)$ is the diagonal embedding. For $n = 2m \geq 0$, the group $\hat{H}^{-n}$ is an extension of $K_{n,\mathrm{tor}}$, the torsion part of the $K$-group (which is the class group in case $n = 0$) by $\mathrm{H}_{\mathrm{D}}^1(m+1) / \operatorname{ch}_{\mathrm{D}}(K_{n+1})$. For $n = 2m + 1 \geq 1$, $\hat{H}^{-n}$ maps isomorphically to $K_{n,\mathrm{tor}}$.

## 4.3 Functoriality

This section gathers a few functoriality properties of Arakelov motivic cohomology. They are consequences of Ayoub's six functors formalism. First of all, we unsurprisingly have pullbacks:

**Lemma 4.7.** *For any map $f : M \to M'$ in $\mathbf{SH}(S)$ there is a functorial pullback*

$$f^* : \widehat{\mathrm{H}}^n(M', p) \to \widehat{\mathrm{H}}^n(M, p), \quad f^* : \widehat{\mathrm{H}}^n(M') \to \widehat{\mathrm{H}}^n(M).$$

*This pullback is compatible with the long exact sequence (24) and, for compact motives $M$ and $M'$, with the decomposition (22). For example, for any map $f : X \to Y$ of schemes over $S$, there is a functorial pullback*

$$f^* : \widehat{\mathrm{H}}^n(Y, p) \to \widehat{\mathrm{H}}^n(X, p).$$

***Proof:*** The first statement is a trivial consequence of the definitions. The second follows by considering the map induced by the adjunction $f_! \leftrightarrows f^!$:

$$p_{X!}p_X{}^! = p_{Y!}f_!f^!p_Y{}^! \to p_{Y!}p_Y{}^!.$$

Here $p_X$ and $p_Y$ are the structural maps of $X$ and $Y$, respectively. □

We say that a map $f : X \to Y$ of schemes over $S$ is *regular projective* if both $X$ and $Y$ are regular and if $f = p \circ i : X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{p} Y$, where $i$ is a closed immersion and $p$ is the projection map. For any map $f : X \to Y$ we write $d_f := \dim X - \dim Y$.

**Theorem 4.8.** *Let $f_1 : X_1 \to X_2$ be a regular projective map. We also assume the structural map $p_2 : X_2 \to S$ is regular projective. For simplicity of notation, we also assume all schemes are connected. Then there is a natural pushforward*

$$f_{1!} : \widehat{\mathrm{H}}^n(X_1, p) \to \widehat{\mathrm{H}}^{n-2d_{f_1}}(X_2, p - d_{f_1})$$

$$f_{1!} : \widehat{\mathrm{H}}^n(X_1)_{\mathbb{Q}} \to \widehat{\mathrm{H}}^n(X_2)_{\mathbb{Q}}.$$

*It is functorial: given another regular projective map $f_2 : X_2 \to X_3$ then $f_2 \circ f_1$ is also regular projective and*

$$f_{2!} \circ f_{1!} = (f_2 \circ f_1)_!.$$

***Proof:*** In the following arguments, we refer to the following commutative diagram:

$$\begin{array}{ccccccccccccc}
 & & X'_2 & & & & X'_3 & & & & X'_4 & & \\
 & \overset{i_1}{\nearrow} & & \overset{p_2}{\searrow} & & \overset{i_2}{\nearrow} & & \overset{p_3}{\searrow} & & \overset{i_3}{\nearrow} & & \overset{p_4}{\searrow} & \\
X_1 & & \xrightarrow{f_1} & & X_2 & & \xrightarrow{f_2} & & X_3 & & \xrightarrow{f_3} & & X_4 := S
\end{array} \qquad (26)$$

In this proof $A \twoheadrightarrow B$ denotes a projection map $A = \mathbb{P}^n_B \to B$ and $\rightarrowtail$ denotes closed immersions. We let $p_n = f_3 \circ \cdots \circ f_n : X_n \to S$ be the structural map (for $n \leq 3$). By assumption, all schemes in this proof are regular.

Recall the map $(n \leq 2)$

$$f_{n*}\mathrm{BGL}_{X_n} = p_{n+1*}i_{n*}i_n^*\mathrm{BGL}X'_{n+1} \to p_{n+1*}\mathrm{BGL}X'_{n+1} \to \mathrm{BGL}_{X_{n+1}}$$

constructed in [CD09, 12.7.3]. It represents the $K$-theoretic pushforward. At least after tensoring with $\mathbb{Q}$, the map is therefore independent of the choice of the factorization $f_n = p_{n+1} \circ i_n$ by [Rio07, 6.1.3.2]. Consider the diagrams:

$$\begin{array}{ccc}
p_{n+1*}p^*_{n+1}\mathrm{H}_{\text{Б},X_{n+1}}\{d_{p_n}\} & \longrightarrow & \mathrm{H}_{\text{Б},X_{n+1}} \\
\big\downarrow & & \big\uparrow \\
p_{n+1*}p^*_{n+1}\mathrm{BGL}_{X_{n+1}} & \longrightarrow & \mathrm{BGL}_{X_{n+1}},
\end{array}$$

and

$$\begin{array}{ccc} i_{n*}i_n^*\mathrm{H}_{Б,X'_{n+1}}\{d_{i_n}\} & \longrightarrow & \mathrm{H}_{Б,X'_{n+1}} \\ \big\downarrow & & \big\uparrow \\ i_{n*}i_n^*\mathrm{BGL}_{X'_{n+1}} & \longrightarrow & \mathrm{BGL}_{X'_{n+1}}. \end{array}$$

The top map in the left diagram is the adjoint to the purity isomorphism (10), which induces the usual pushforward on motivic cohomology. Thus the commutativity of the diagram is essentially a restatement of the fact that the $K$-theoretic pushforward of $\mathcal{O}_{\mathbb{P}^n_A}(n) \in K_0(\mathbb{P}^n_A)$ along $\pi : \mathbb{P}^n_A \to A$ is $\mathcal{O}_A \in K_0(A)$, which is the same as the cycle-theoretic pushforward of the codimension-$n$-cycle on $\mathbb{P}^n_A$ represented by $\mathcal{O}_{\mathbb{P}^n_A}(n)$ (i.e., an $A$-point).

The commutativity of the second diagram is by construction of the absolute purity isomorphism for $\mathrm{H}_Б$ [CD09, Th. 13.4.1.]. Therefore, the isomorphism

$$r_n^{\mathrm{H}_Б} : \mathrm{H}_{Б,X_n}\{d_{f_n}\} = i_n^*\mathrm{H}_{Б,X'_{n+1}} \to i_n^! p_{n+1}^* \mathrm{H}_{Б,X_{n+1}}\{d_{p_{n+1}}\} \to f_n^! \mathrm{H}_{Б,X_{n+1}}$$

also only depends on $f_n$, not on the factorization.

To define the pushforward map $f_{1!}$, look at the above diagram, where $X_3 = X'_4 = X_4 := S$. We define

$$f_{1!} : \widehat{\mathrm{H}}^n(X_1, p) \to \widehat{\mathrm{H}}^{n-2d_{f_1}}(X_2, p - d_{f_1})$$

to be $\mathrm{Hom}_{\mathbf{SH}(S)}(\phi_{12}, \widehat{\mathrm{H}_{Б,S}}(p)[n])$, where $\phi_{12}$ is the the composition

$$\begin{array}{rcl} f_{2!}f_2{}^!\mathrm{H}_Б\{d_{f_1}\} & \stackrel{(r_2^{\mathrm{H}_Б})^{-1}}{\longrightarrow} & f_{2!}f_2{}^*\mathrm{H}_Б\{d_{f_1}+d_{f_2}\} \\ & \stackrel{(2)}{\to} & f_{2!}f_{1*}f_1{}^*f_2{}^*\mathrm{H}_Б\{d_{f_1}+d_{f_2}\} \\ & \stackrel{r_1^{\mathrm{H}_Б}}{\to} & f_{2!}f_{1*}f_1{}^!f_2{}^*\mathrm{H}_Б\{d_{f_2}\} \\ & \stackrel{r_2^{\mathrm{H}_Б}}{\to} & f_{2!}f_{1!}f_1{}^!f_2{}^!\mathrm{H}_Б. \end{array}$$

Similarly, define $\widehat{\mathrm{H}}^n(X_1)_{\mathbb{Q}} \to \widehat{\mathrm{H}}^n(X_2)_{\mathbb{Q}}$ using $r_*^{\mathrm{BGL}}$ instead of $r_*^{\mathrm{H}_Б}$.

The compatibility of $f_1 \mapsto f_{1!}$ with compositions follows from completing (26) by schemes $X'_{n-1} \rightarrowtail X''_n \twoheadrightarrow X'_n$ ($n = 2, 3, 4$) making the above commutative. Indeed, given a commutative diagram (we could even assume that it is cartesian)

$$\begin{array}{ccc} X'_2 & \rightarrowtail & X''_3 \\ \big\downarrow & \searrow^{\phi} & \big\downarrow \\ X_2 & \rightarrowtail & X'_3 \end{array}$$

the two possible (iso)morphisms $\phi^*\mathrm{BGL}_{X'_3} \to \phi^!\mathrm{BGL}_{X'_3}$ (and similarly with $\mathrm{H}_Б$ instead of BGL) resulting from the two factorizations of $\phi$ agree, by the same argument as above. $\square$

**Remark 4.9.** (i) Unlike with pullbacks, the pushforward is *not* compatible with the decomposition (22). To elucidate this, let $f_1 : X_1 \to X_2 = S$ be a projective regular map such that $S$ is an arithmetic ring. In the notation of the proof, applying $\mathrm{Hom}_{\mathbf{SH}(S)}(\phi_{12}, -)$ to the triangle $\widehat{\mathrm{BGL}_S} \to \mathrm{BGL}_S \to \oplus \mathrm{H}_{\mathrm{D}}\{p\}$ yields a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \widehat{\mathrm{H}}^n(X_1) & \longrightarrow & K_{-n}(X_1) & \longrightarrow & \oplus_p \mathrm{H}_{\mathrm{D}}^{2p+n}(X_1, p) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{1!}} & & \downarrow{\scriptstyle f_{1!}} & & \downarrow{\scriptstyle f_{1!}\circ(-\cup \mathrm{Td}(X_1))} & & \\
\cdots & \longrightarrow & \widehat{\mathrm{H}}^n(S) & \longrightarrow & K_{-n}(S) & \longrightarrow & \oplus_p \mathrm{H}_{\mathrm{D}}^{2p+n}(S, p) & \longrightarrow & \cdots
\end{array}$$

Here $f_{1!}$ in the middle and right column is the usual pushforward in $K$-theory and Deligne cohomology (the latter being given by integration over the fiber) and $\mathrm{Td}(X_1)$ denotes the Todd class of the tangent bundle of $X_1 \times_S S_\eta$, which is a smooth scheme over $S_\eta$. Indeed, the statement about the induced map on $K$-theory is by definition. To identify the induced map on Deligne cohomology with the stated one, one can replace $S$ by an open subscheme thereof and assume $f_1$ smooth. The Grothendieck-Riemann-Roch theorem for smooth maps [Rio09, 6.3.1] yields the claim, since $\mathrm{ch}_{\mathrm{D}} : \mathrm{BGL} \to \oplus \mathrm{H}_{\mathrm{D}}\{p\}$ factors over the Chern character $\mathrm{BGL} \to \oplus_p \mathrm{H}_{\text{Б}}\{p\}$, (22), by definition.

(ii) For comparison, the pushforward of arithmetic $K$-theory groups $\hat{K}_n(-)$ defined by Takeda applies to smooth projective maps $f : X \to Y$ between arithmetic varieties (flat over $\mathbb{Z}$ and regular). The definition needs an auxiliary choice of a metric on the relative tangent bundle [Tak05, Section 7.3.]. Neither the independence of this choice nor the functoriality seem to be known. The situation is similar for the pushforward defined by Roessler on $\hat{K}_0$ [Roe99, 3.1]. The pushforward on arithmetic Chow groups [GS90, Theorem 3.6.1] is defined for all proper maps between arithmetic varieties. For the time being, no pushforward has been established for the higher arithmetic Chow groups of Feliu and Burgos.

# References


[BGKK07] J. I. Burgos Gil, J. Kramer, and U. Kühn. Cohomological arithmetic Chow rings. *J. Inst. Math. Jussieu*, 6(1):1–172, 2007.

[Bur94] José Ignacio Burgos. A $C^\infty$ logarithmic Dolbeault complex. *Compositio Math.*, 92(1):61–86, 1994.

[Bur97] Jose Ignacio Burgos. Arithmetic Chow rings and Deligne-Beilinson cohomology. *J. Algebraic Geom.*, 6(2):335–377, 1997.

[CD07] Denis-Charles Cisinski and Frédéric Déglise. Mixed Weil cohomologies. 2007.

[CD09] Denis-Charles Cisinski and Frédéric Déglise. Triangulated categories of mixed motives. 2009.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–57, 1971.

[EV88a] Hélène Esnault and Eckart Viehweg. Deligne-Beĭlinson cohomology. In *Beĭlinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 43–91. Academic Press, Boston, MA, 1988.

[EV88b] Hélène Esnault and Eckart Viehweg. Deligne-Beĭlinson cohomology. In *Beĭlinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 43–91. Academic Press, Boston, MA, 1988.

[GS90] Henri Gillet and Christophe Soulé. Arithmetic intersection theory. *Inst. Hautes Études Sci. Publ. Math.*, (72):93–174 (1991), 1990.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[Hub00] Annette Huber. Realization of Voevodsky's motives. *J. Algebraic Geom.*, 9(4):755–799, 2000.

[Jar00] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Rio07] Joël Riou. Opérations sur la $K$-théorie algébrique et régulateurs via la théorie homotopique des schémas. *C. R. Math. Acad. Sci. Paris*, 344(1):27–32, 2007.

[Rio09] Joel Riou. Algebraic $K$-theory, $\mathbf{A}^1$-homotopy and Riemann-Roch theorems. Preprint, Jul. 2009, http://arxiv.org/abs/0907.2710, 2009.

[Roe99] Damian Roessler. An Adams-Riemann-Roch theorem in Arakelov geometry. *Duke Math. J.*, 96(1):61–126, 1999.

[Sch10] Jakob Scholbach. Special $L$-values of geometric motives. Preprint, Feb. 2010, http://arxiv.org/abs/1003.1215, 2010.

[Sou92] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[Tak05] Yuichiro Takeda. Higher arithmetic $K$-theory. *Publ. Res. Inst. Math. Sci.*, 41(3):599–681, 2005.

[Voe98] Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604 (electronic), 1998.

[Wei05] Charles Weibel. Algebraic $K$-theory of rings of integers in local and global fields. In *Handbook of $K$-theory. Vol. 1, 2*, pages 139–190. Springer, Berlin, 2005.