ABOUT SOME FAMILY OF ELLIPTIC CURVES

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ABSTRACT. We examine the moduli space $\mathcal{E} \cong \mathbf{T}^*$ of complex tori $\mathbf{T}(\tau) \cong \mathbb{C}/L(\tau)$ where $L_(\tau) = const \cdot \eta^2(\tau)L_\tau$. We find that the Dedekind eta function furnishes a bridge between the euclidean and hyperbolic structures on $\mathbf{T}^* \cong \mathbb{C} - L_0/L_0$ as well as between the doubly periodic Weierstrass function $\wp$ on $\mathbf{T}^*$ and the theta function for the lattice E_8. The former one allows us to rewrite the Lame equation for the Bers embedding of $\mathcal{T}_{1,1}$ in a new form. We show that L_0 has natural decomposition into 8 sublattices (each equivalent to L_0), together with appropriate half-points and that this leads to local functions of the form $\vartheta_l^8(0, \tau_\alpha)$ for a local map (U_α, τ_α) and to a relation with E_8.

1. INTRODUCTION

We have shown in [1] that the natural algebraic structures associated to the punctured torus $\mathbf{T}^* \cong H/\Gamma'$, (here Γ' is the commutator group of the modular group $\Gamma = SL_2\mathbb{Z}$, $\Gamma' = [\Gamma, \Gamma]$) viewed as the Veech modular curve of complex tori, produce exactly the generating matrix for the binary error correcting Golay code G_{24}. This is a reason why in this paper we investigate the (on the other hand well known) punctured torus $\mathbf{T}^*$ more carefully. We will find that the Dedekind eta function η plays very important role. It furnishes not only a bridge between the hyperbolic and euclidean geometries on $\mathbf{T}^*$ but it also connects (see the formula (5.5)) the doubly-periodic Weierstrass function $\wp(p(z_\alpha), L_0)$ on $\mathbf{T}^*$ with the theta function for the lattice E_8, that is with $\Theta_{E_8}(\tau_\alpha) = \sum_{m=0}^{\infty} r_{E_8}(m) q_\alpha^m$ (here $\tau_\alpha = \tau_\alpha(x')$, $z_\alpha = p(\tau_\alpha)$, $x' \in U_\alpha \subset \mathbf{T}^*$ and $r_{E_8}(m)$ is the number of elements $\underline{v} \in E_8$ such that $\underline{v} \cdot \underline{v} = 2m$).

Since the Veech modular curve $\mathbf{T}^*$ naturally carries the modular J-invariant we may view each of the objects G_{24} and E_8 as a sort of a hidden structure associated to the Klein J-function that is encoded in the projection $J\colon \mathbf{T}^* \to Y(1) = H/\Gamma$.

In [2] we have shown that, similarly to strong consequences coming from relations between Γ' and the subgroups $\Gamma(2)$, $\Gamma(3)$, Γ_c and $\Gamma_{ns}^+(3)$ of the modular group Γ (and investigated in this note) the relations between $\Gamma = SL_2\mathbb{Z}$ and $\Gamma_0(p)$ (for the supersingular primes) introduce a hidden structure asociated to the J-function whose the full symmetry group $\mathcal{K}$ must have the order that is devided by each of these primes p. Since the full automorphism group of G_{24} (given by the Matieu group $\mathcal{M}_{24}$) must be a subgroup of $\mathcal{K}$, the conditions that $p||\mathcal{K}|$ together with the requirement that $\mathcal{K}$ is a simple group implies that $\mathcal{K}$ has to be the monster group $\mathbf{M}$.

We will start with the family of lattices $L(\tau) = const \cdot \eta^2(\tau)L_\tau$ on H, where $L_\tau = [1, \tau]$ and we will show that the moduli space for complex tori $\mathbf{T}(\tau) \cong \mathbb{C}/L(\tau)$

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is an elliptic open curve $\mathcal{E}: t^2 = 4u^3 - 1$. Using the ramification scheme for appropriate natural projections we obtain that $u(\tau)$ and $t(\tau)$ coicide with the absolute invariants for $\Gamma^+_{ns}(3)$ and for Γ_c respectively as well as that the curve $\mathcal{E}$ is analytically isomorphic to $\mathbf{T}^* \cong H/\Gamma' \cong \mathbb{C} - L_0/L_0$. In section 3 we find relations $dz_\alpha = s\eta^4(\tau_\alpha)d\tau_\alpha$, (s is a global constant) between local coordinates on $\mathbf{T}^*$ and we investigate their consequences. We introduce some Hecke operators and we find their images on some, important for us , automorphic functions and forms. The expression of the standard holomorphic quadratic differential $(dz^2_\alpha)_\alpha$ on $\mathbf{T}^*$ in terms of the Dedekind eta function allows us to find the Bers embedding of $\mathcal{T}_{1,1}$ using the equation(3.22) instead of working with the Lame equation (3.23). In section 4 we investigate different realizations (4.3) of the quotient $\widetilde{\Gamma}/\Gamma'$, ($\widetilde{\Gamma} = PSL_2\mathbb{Z}$) that are naturally associated to the standard quadrilateral $\mathfrak{F}'_4$ and hexagonal $\mathfrak{F}'_6$ fundamental domains for Γ' respectively. Only the latter one determines very important (although a non-unitary) representation of Γ in the 2-dimentional vector space spanned by the Weierstrass functions $\wp$ and $\wp'$ on $\mathbf{T}^*$. In section 5 we construct the decomposition of the lattice $L_0 \cong p(\Gamma'(\infty))$, (p is the natural projection $p: H \to \mathbb{C} - L_0$) into eight disjoint subsets $\widetilde{\mathcal{L}}_k$. The symmetries of the lattice L_0 allow us to realize each $\widetilde{\mathcal{L}}_k$, $k = 1, \dots, 8$, as a sublattice of L_0 (which is the 4-dilate of L_0) together with its appropriate half-points in three distinct ways. In section 6 we investigate conclusions of these decompositions and we find some sort of hidden E_8-symmetry on $\mathbf{T}^*$.

2. Preliminaries

2.1. **Curve $\mathcal{E}$.** Each element τ of the upper half-plane H determines a lattice $L_\tau = [1, \tau]$ and a complex torus $\mathbf{T}_\tau = \mathbb{C}/L_\tau$. However, we will consider, instead of the standard family $\{\mathbf{T}_\tau\}_{\tau\in H}$ of compact complex tori, a family $\{\mathbf{T}(\tau) = \mathbb{C}/L(\tau)\}_{\tau\in H}$ where $L(\tau) = \mu(\tau)L_\tau$, $\mu(\tau) = 2\pi 3^{-\frac{1}{4}}\eta^2(\tau)$ and $\eta(\tau)$ is the standard Dedekind eta function. Now, each torus $\mathbf{T}(\tau)$ is analytically isomorphic to the curve

$$E_{L(\tau)} : Y^2 = 4X^3 - g_2(L(\tau))X - g_3(L(\tau)) \tag{2.1}$$

and we will define a function $u(\tau)$ as given by $u(\tau) := \frac{1}{3\sqrt[3]{4}} g_2(L(\tau))$ and the function $t(\tau) := g_3(L(\tau))$. We have

$$g_2(L(\tau)) = \mu(\tau)^{-4} g_2(\tau) = \frac{3}{(2\pi)^4}\frac{g_2(\tau)}{\eta(\tau)^8} \tag{2.2}$$

and

$$g_3(L(\tau)) = \mu(\tau)^{-6} g_3(\tau) = \frac{3^{\frac{3}{2}}}{(2\pi)^6}\frac{g_3(\tau)}{\eta(\tau)^{12}} \tag{2.3}$$

where $g_k(\tau) = g_k(L_\tau)$ for $k = 2, 3$ are the standard Eisenstein series. We see that the functions $u(\tau)$ and $t(\tau)$ satisfy the equation $4u^3 - t^2 - 1 = 0$ and hence determine an elliptic open curve

$$\mathcal{E} : t^2 = 4u^3 - 1 \tag{2.4}$$

Each point $(u(\tau), t(\tau)) \in \mathcal{E}$ corresponds to a curve

$$E_{u,t} : Y^2 = 4X^3 - 3\sqrt[3]{4}uX - t \tag{2.5}$$

When point $P = (u, t)$ of $\mathcal{E}$ has both coordinates different from zero then there exist exactly six distict points $(\rho^k u, \pm t)$ $(k = 0, 1, 2,\ \rho = e^{\frac{2\pi i}{3}})$ on $\mathcal{E}$ which correspond to six isomorphic elliptic curves representing the same equivalent class of complex tori. When $P = (u, 0)$ then we must have $u = 4^{-\frac{1}{3}}\rho^k$ and points $(\rho^k 4^{-\frac{1}{3}}, 0)$ with $k = 0, 1, 2$ correspond to three isomorphic curves representing the equivalence class $[\mathbb{C}/\mathbb{Z}[i]]$. When $P = (0, t)$ then we must have $t = \pm i$ and both curves $E_{0,\pm i}$ represent the equivalence class $[\mathbb{C}/\mathbb{Z}[\rho]]$ of tori. (Here the square bracket denotes the equivalence class of complex tori i.e. a point of the modular space H/Γ, $\Gamma = SL_2\mathbb{Z}$.)

From the form of the equation (2.4) the elliptic curve $\mathcal{E}$ is itself analytically isomorphic to a complex torus that belongs to the class $[\mathbb{C}/\mathbb{Z}[\rho]]$. Since both functions $u(\tau)$ and $t(\tau)$ have the hyperbolic nature to find their realizations in terms of the Weierstrass functions $\wp$ and $\wp'$ (which belong to the flat geometry) we must consider the relationships between $\mathcal{E}$ and some modular curves of level 2 and of level 3 structures respectively.

2.2. **Γ', Γ_c and $\Gamma(2)$.** . Let r_N denote the modulo N homomorphism r_N: $SL_2\mathbb{Z} \to SL_2(N)$. The image $r_2(\Gamma) = SL_2(2) \cong S_3$ whereas the image of $\Gamma' = [\Gamma, \Gamma]$ is the normal subgroup of S_3 given by $\mathcal{C}_3 \cong \mathbb{Z}_3$. Let Γ_c denote the subgroup ${r_2}^{-1}(\mathcal{C}_3)$ of Γ. It has genus zero, it has only one cusp of width 2 and it has index 2 in Γ. Moreover we may take $\{I, T\}$ as a set of its coset representatives in Γ, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\mathbf{T}^*$ denote the punctured torus H/Γ' which is analyticaly isomorphic to $\mathbb{C} - L_0/L_0$ for some lattice $L_0 = const \cdot L_\rho$ and let $\mathbf{X}'$ be $\mathbf{T}^* \cup \{\infty\} \cong H^*/\Gamma'$ where H^* denotes the extended half-plane $H \cup \mathbb{Q} \cup \{\infty\}$. We have the following natural projections: $\mathbf{X}' \overset{\pi'_c}{\to} \mathbf{X}_c \overset{\pi_c}{\to} \mathbf{X}(1)$ with $\mathbf{X}_c \cong H^*/\Gamma_c$, $\mathbf{X}(1) \cong H^*/\Gamma$ with projections π'_c of degree 3 and π_c of degree 2. The absolute invariant for Γ_c is given by $J_c(\tau) = (J(\tau)-1)^{\frac{1}{2}}$, [3], and it is also Γ' invariant (using (2.3) it may be identified with the function $t(\tau)$). The comparison of the ramification scheme for π'_c: $\mathbf{X}' \to \mathbf{X}_c$ and for $\wp'$: $\mathbf{X}' \to \mathbb{C}P_1$ implies that (after the identification of $\mathbb{C}P_1$ with J_c-plane $\mathbf{X}_c$) the Γ'-automorphic function $t(\tau)$ coincides with the lifting to H of the function $\wp'$ on $\mathbf{T}^*$. In other words we have shown that the following is true:

Lemma 1. *Let p: $H \to H/N$ be the natural projection coresponding to the group $N = [\Gamma', \Gamma']$ with $H/N \cong \mathbb{C} - L_0$. The lifting of $\wp'(z, L_0)$ on $\mathbb{C} - L_0$ to H determined by p produces exactly the Γ'-automorphic function $t(\tau)$.*

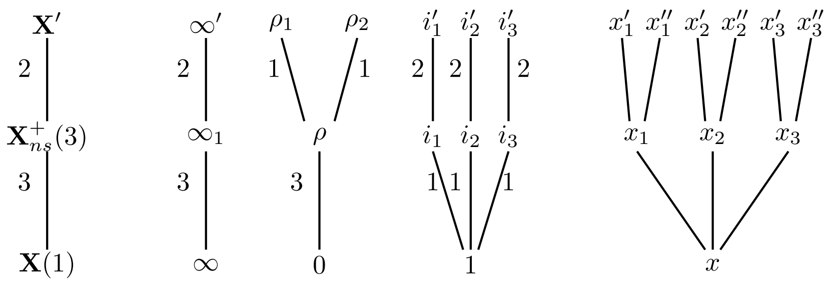
At this point it is worth to notice that (since the modulo 2 homomorphism maps both groups $\langle g \rangle$ and $\langle a \rangle$ onto $\mathcal{C}_3$ and since a is $\Gamma(2)$-equivalent to g^2 and a^2 is $\Gamma(2)$-equivalent to g) we may view the modular curve $\mathbf{X}_c$ (of $\mathcal{C}_3$-equivalent level two structures) as the quotient $\mathbf{X}(2)/\mathcal{C}_3$ (here $g = ST$, $a = TS$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$).

2.3. **Γ', $\Gamma(3)$ and $\Gamma^+_{ns}(3)$.** A similar situation occurs when we pass to the modulo 3 homomorphism. The image $r_3(\Gamma')$ is the normal subgroup of $\Gamma/\Gamma(3) \cong SL_2(3)$ but now this subgroup is not an abelian one. It is isomorphic to the quaternion group Q_8 and we have

$$1 \longrightarrow Q_8 \longrightarrow SL_2(3) \longrightarrow \mathbb{Z}_3 \longrightarrow 1 \tag{2.6}$$

The subgroup ${r_3}^{-1}(Q_8)$ of Γ is associated to the non-split Cartan subgroup of $GL_2(3)$ and is usually denoted by $\Gamma^+_{ns}(3)$, [4]. It has index 3 in Γ and we may take the set $\{I, T, T^2\}$ as a set of its coset representatives. The modular curve $\mathbf{X}^+_{ns}(3)$ of Q_8-equivalent level 3 structures (in fact, since the normal subgroup $\mathcal{N}$ of Q_8 acts trivially, these structures are $Q_8/\mathcal{N}$ equivalent) has genus zero and only one cusp of width 3.

The absolute invariant for $\Gamma^+_{ns}(3)$ can be taken as $J_n(\tau) = J(\tau)^{\frac{1}{3}}$, [5], and hence this uniformizer of $\mathbf{X}^+_{ns}(3)$ coincides with the Γ'-automorphic function $\sqrt[3]{4}u(\tau)$ introduced earlier. Taking into account the ramification scheme given by Pict.1



Pict.1

we obtain immediately:

Lemma 2. *Let p be the natural projection $H \to \mathbb{C} - L_0$ introduced earlier. The lifting of the Weierstrass function $\wp(z, L_0)$ on $\mathbb{C} - L_0$ to H determined by p produces exactly the Γ'-automorphic function $u(\tau) = J_n(\tau)$.*

Since functions $u(\tau)$ and $t(\tau)$ are liftings to H of the Weierstrass functions $\wp$ and $\wp'$ respectively we have the following

Corollary 1. *An elliptic curve $\mathcal{E}$: $t^2 = 4u^3 - 1$ that forms the moduli space of elliptic curves associated to the family of lattices $\{L(\tau) = \mu(\tau)L_\tau\}$ with $\tau \in H$ is analytically isomorphic to the punctured torus $\boldsymbol{T}^* = H/\Gamma' \cong \mathbb{C} - L_0/L_0$ with isomorphism given by $z \to (u(\tau), t(\tau), 1)$ for any τ with the property that $p(\tau) \in z + L_0$.*

Thus, the Γ'-automorphic functions $u(\tau) = \frac{1}{3\sqrt[3]{4}} g_2(L(\tau))$ and $t(\tau) = g_3(L(\tau))$ are objects of both: of the euclidean geometry (since $u(\tau) = \wp(p(\tau), L_0)$ and $t(\tau) = \wp'(p(\tau), L_0)$) and of the hyperbolic geometry (as $u(\tau)$ is the lifting to Γ' of a Hauptmodule $J_n(\tau)$ for $\Gamma^+_{ns}(3)$ and $t(\tau)$ is the lifting of a Hauptmodule J_c for Γ_c). In other words we have the following commutative diagrams:

$$\begin{tikzcd} \mathbf{X}' \arrow[r, "\wp'"] \arrow[d, "\pi'_c"'] & \mathbb{C}P_1 \\ \mathbf{X}_c \arrow[ur, "J_c"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X}' \arrow[r, "\sqrt[3]{4}\wp"] \arrow[d, "\pi'_n"'] & \mathbb{C}P_1 \\ \mathbf{X}^+_{ns}(3) \arrow[ur, "J_n"'] & \end{tikzcd}$$

3. A Matter of the Dedekind Eta Function

3.1. **Hyperbolic and Euclidean.** We have already introduced a universal covering p which projects H onto the infinite punctured plane $\mathbb{C} - L_0$ with the deck group corresponding to a homomorphism of $\Pi_1(\mathbb{C} - L_0) \to N$, $N = [\Gamma', \Gamma']$. So, $N\tau \Leftrightarrow z \in \mathbb{C} - L_0$, $\Gamma'\tau \Leftrightarrow z + L_0$ and $L_0 = c[1, \rho]$ for some constant c. Let r be the local inverse of p, that is, $\{r,z\} = \frac{1}{2}\wp(z,L_0)$ (here $\{\}$ denotes the Schwarzian derivative). Now the Γ'-automorphic functions u and t can be locally viewed as

$$u(r(z)) = \wp(z,L_0) \qquad t(r(z)) = \wp'(z,L_0) \tag{3.1}$$

Let $\{(U_\alpha,\tau_\alpha\}_\alpha$ be an atlas on $\mathbf{T}^* \cong H/\Gamma'$ coming from the universal covering π': $H \to \mathbf{T}^*$ i.e. for $(u,t) = x' \in U_\alpha \cap U_\beta$, we have $\tau_\beta(x') = \gamma\tau_\alpha(x')$ for some $\gamma \in \Gamma'$. Since the multiplier system of $\eta^2(\tau)$ restricted to the subgroup Γ' of Γ is a trivial one, on any intersection $U_\alpha \cap U_\beta$ we obtain

$$L(\tau_\beta) = \mu(\tau_\beta)L_{\tau_\beta} = \mu(\tau_\alpha)L_{\tau_\alpha} = L(\tau_\alpha) \tag{3.2}$$

This means that at each point $x' \in \mathbf{T}^*$, $x' = (u,t)$, we have well define lattice $L(x') = L(\tau_\alpha(x')) = L(\tau_\beta(x'))$ and hence we have an analytic isomorphism between $\mathbb{C}/L(x')$ and $E_{u,t}$: $Y^2 = 4X^3 - 3\sqrt[3]{4}uX - t$.

Let us introduce another atlas $\{(U_\alpha,z_\alpha)\}_\alpha$ on $\mathbf{T}^*$ with holomorphic bijections z_α:$U_\alpha \to V_\alpha \subset \mathbb{C} - L_0$ coming from the projection $p' : \mathbb{C} - L_0 \to \mathbf{T}^*$ and with the property that

$$\tau_\alpha(p'(z_\alpha)) = r(z_\alpha) \qquad z_\alpha(\pi'(\tau_\alpha)) = p(\tau_\alpha) \tag{3.3}$$

(If necessary we may pass to some refinement of an open covering $\{U_\alpha\}$ of $\mathbf{T}^*$.) Now, for each $x' \in U_\alpha \cap U_\beta$ we have $\tau_\beta(x') = \gamma\tau_\alpha(x')$ for some $\gamma \in \Gamma'$ and $z_\beta(x') = z_\alpha(x') + w$ for some $w \in L_0$. Since

$$u(\tau_\alpha) = \wp(p(\tau_\alpha),L_0) = \wp(p(\tau_\beta),L_0) = u(\tau_\beta)$$

and analogously

$$t(\tau_\alpha) = \wp'(z_\alpha,L_0) = \wp'(z_\beta,L_0) = t(\tau_\beta)$$

the relation

$$t(\tau_\alpha) = \wp'(z_\alpha,L_0) = \frac{d\wp(p(\tau_\alpha),L_0)}{dp(\tau_\alpha)} = \frac{d\wp(p(\tau_\alpha))}{d\tau_\alpha}\frac{d\tau_\alpha}{dz_\alpha} = \frac{du(\tau_\alpha)}{d\tau_\alpha}\frac{d\tau_\alpha}{dz_\alpha}$$

implies that

$$\frac{du(\tau_\alpha)}{d\tau_\alpha} = t(\tau_\alpha)\frac{dz_\alpha}{d\tau_\alpha} \tag{3.4}$$

.

Since we already know that

$$u(\tau_\alpha) = (\frac{1}{4}J(\tau_\alpha))^{\frac{1}{3}}, \qquad t(\tau_\alpha) = (J(\tau_\alpha) - 1)^{\frac{1}{2}}$$

we may use the well known formula [6]

$$\eta^{24}(\tau) = \frac{1}{(48\pi^2)^3}\frac{J'(\tau)^6}{(J(\tau))^4(1-J(\tau))^3}$$

to find that on $U_\alpha \subset \mathbf{T}^*$ we have

$$\frac{dz_\alpha}{d\tau_\alpha} = s\eta^4(\tau_\alpha), \qquad s = 2k\pi\frac{\sqrt[3]{2}}{\sqrt{3}} \tag{3.5}$$

and k is a global constant given by a 6-th root of -1. Hence, on any intersection $U_\alpha \cap U_\beta$ on $\mathbf{T}^*$ we have

$$dz_\alpha = s\eta^4(\tau_\alpha)d\tau_\alpha = s\eta^4(\tau_\beta)d\tau_\beta = dz_\beta \tag{3.6}$$

as expected. We see that the Dedekind eta function provides the transition between the local flat coordinates $z_\alpha(x')$ and the hyperbolic $\tau_\alpha(x')$ coordinates on $\mathbf{T}^*$. In other words it plays the role of a bridge between the euclidean geometry on $\mathbf{T}^* \cong H/\Gamma'$ and its natural hyperbolic geometry. Moreover, from the formula (3.6), we obtain that

$$\wp(z_\alpha, L_0)dz_\alpha^2 = \frac{k}{3(2\pi)^3} g_2(\tau_\alpha)d\tau_\alpha^2 \tag{3.7}$$

Let q be the holomorphic quadratic differential on $\mathbf{T}^*$ that is determined by the Eisenstein series $g_2(\tau)$ i.e. with respect to the atlas $\{(U_\alpha, \tau_\alpha)\}_\alpha$ it can be written as $q = (\frac{k}{3(2\pi)} g_2(\tau_\alpha)d\tau_\alpha^2)_\alpha$. Now, with respect to the atlas $\{(U_\alpha, z_\alpha)\}_\alpha$, q takes the form:

$$q = (\wp(z_\alpha, L_0)dz_\alpha^2)_\alpha \tag{3.8}$$

Similarly, it is easy to check that

$$t(\tau_\alpha) = \frac{2\pi\sqrt{3}}{\sqrt[3]{2}} \frac{du(\tau_\alpha)}{d\tau_\alpha} \eta^{-4}(\tau_\alpha) \tag{3.9}$$

and hence on each U_α we can write

$$g_3(\tau_\alpha) = \frac{(2\pi)^7}{3\sqrt[3]{2}} e^{-\frac{i\pi}{6}} \eta^8(\tau_\alpha) \frac{du(\tau_\alpha)}{d\tau_\alpha} \tag{3.10}$$

Let ξ denote the holomorphic differential on $\mathbf{T}^*$ wich is determined by $g_3(\tau)$. The above formulae allow us to write

$$\xi = (\wp'(z_\alpha, L_0)dz_\alpha^3)_\alpha = (\frac{2}{(2\pi)^9} e^{\frac{i\pi}{2}} g_3(\tau_\alpha)d\tau_\alpha^3)_\alpha \tag{3.11}$$

In other words we have shown the following:

Lemma 3. *The Γ'-automorphic forms on H corresponding to the differentials $q = (\wp(z_\alpha, L_0)dz_\alpha^2)_\alpha$ and $\xi = (\wp'(z_\alpha, L_0)dz_\alpha^3)_\alpha$ on $\boldsymbol{T}^*$ are exactly ones determined by the standard Eisenstein series $g_2(\tau)$ and $g_3(\tau)$ respectively. More precisely we have*

$$\frac{k^2}{3(2\pi)^2} g_2(\tau) = \frac{k^2\sqrt[3]{4}}{3} (2\pi)^2 \eta^8(\tau)u(\tau) \tag{3.12}$$

and

$$\frac{2k^3}{(2\pi)^3} g_3(\tau) = \frac{k^3\sqrt[3]{2}}{3} (2\pi)^2 \eta^8(\tau) \frac{du(\tau)}{d\tau} \tag{3.13}$$

respectively.

From the relations (3.12) and (3.13) we obtain (after differentiating the first equation):

$$\frac{dg_2(\tau)}{d\tau} = 8\frac{\eta'(\tau)}{\eta(\tau)} g_2(\tau) + \frac{3k}{\pi} g_3(\tau) \tag{3.14}$$

We notice that when we choose the 6-th root k of -1 as $k = -i$ then the latter formula is equivalent to the Serre derivative of the modular form $E_4(\tau) = \frac{3}{2\pi^2} g_2(\tau)$.

3.2. **Some Hecke Operators.** Let us introduce (see [7]) the operator $T_{\langle g\rangle,k}$ of weight $k \in \mathbb{Z}$ acting on the space of functions $f : H \to \mathbb{C}$ as follows

$$(T_{\langle g\rangle,k}f)(\tau) = \sum_{r=1}^{3} {j_{g^r}}^{-k} f(g^r\tau) \tag{3.15}$$

where $j_\gamma(\tau) = c\tau + d$ for any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2\mathbb{R}$. Let $\mathcal{A}_k(G)$ denote the space of G-automorphic forms of weight k for a Fuchsian group G. Since we have

$$T_{\langle g\rangle,k} : \mathcal{A}_k(\Gamma') \to \mathcal{A}_k(\Gamma_c) \tag{3.16}$$

and

$$T_{\langle g\rangle,k} : \mathcal{A}_k(\Gamma(2)) \to \mathcal{A}_k(\Gamma_c) \tag{3.17}$$

we may find some relations between k-forms for $\Gamma(2)$ and k-forms for Γ'. Namely, these Hecke operators together with the projections π'_c and $\pi_2^c : X(2) \to X_c$ allow us to transform $\frac{k}{2}$-differentials on $\mathbf{X}'$ into $\frac{k}{2}$-differentials on $\mathbf{X}(2)$ and vice versa. Let us denote the composition of $(\pi_2^c)^*$ and of $T_{\langle g\rangle,k}$ as the operator $\widehat{H}_k$. Let us check what are the images of $\widehat{H}_0$ produced by the Γ'-automorphic functions $u(\tau)$ and $t(\tau)$. Since $L(g\tau) = ie^{\frac{i\pi}{6}}L(\tau)$ and $L(g^2\tau) = -ie^{-\frac{i\pi}{6}}L(\tau)$ we have $g_2(L(g\tau)) = \rho^2 g_2(L(\tau))$ and $g_2(L(g^2\tau)) = \rho g_2(L(\tau))$. Hence

$$\widehat{H}_0 u(\tau) = 0 \qquad \textit{and} \qquad \widehat{H}_0 t(\tau) = 3t(\tau)$$

So, the Weierstrass function $\wp$ on $\mathbf{T}^*$ produces the zero function on $\mathbf{X}(2)$ but the Weierstrass function $\wp'$ produces a multiple of the lifting of the absolute invariant J_c from $\mathbf{X}_c$ to $\mathbf{X}(2)$. We already know that the regular quadratic differential $(dz_\alpha^2)_\alpha$ on $\mathbf{T}^*$ corresponds to the Γ'-automorphic form $s^2\eta^8(\tau)$ on H . It occurs that the image under the operator $\widehat{H}_4 = (\pi_2^c)^* \circ T_{\langle g\rangle,4}$ (transforming $\mathcal{A}_4(\Gamma')$ into $\mathcal{A}_4(\Gamma(2))$) of $\eta^8(\tau)$ vanishes. Although $\widehat{H}_0 u(\tau) = 0$ and $\widehat{H}_4\eta^8(\tau)$=0 the operator $\widehat{H}_4$ acts on their product $u(\tau)\eta^8(\tau)$ by multiplication by 3. This is because the product is a Γ-automorphic form i.e. $u(\tau)\eta^8(\tau) \in \mathcal{A}_4(\Gamma) \subset \mathcal{A}_4(\Gamma')$. Generally we have

Lemma 4. *For any $\varphi \in \mathcal{A}_k(\Gamma)$ and for any $f \in \mathcal{A}_0(\Gamma')$ we have $\widehat{H}_k(f\varphi) = \varphi\widehat{H}_0(f)$.*

Proof. Simple □

We have exactly the same properties when we replace $\mathcal{A}_k(\Gamma')$ by $\mathcal{A}_k(\Gamma(2))$ and the operator $\widehat{H}_k$ by the the operator $\widetilde{H}_k$ defined as the composition ${\pi'_c}^* \circ T_{\langle g\rangle,k}$ and transforming $\mathcal{A}_k(\Gamma(2))$ into $\mathcal{A}_k(\Gamma')$. Since we have

$$T_{\langle g\rangle,4}\vartheta_3(\tau)^8 = \frac{3}{(2\pi)^4} g_2(\tau) \tag{3.18}$$

the image by $\widetilde{H}_4$ of the differential on $\mathbf{X}(2)$ determined by $\vartheta_3(\tau)^8 \in \mathcal{A}_4(\Gamma(2))$ produces the differential

$$(\frac{1}{k^2}(\frac{3}{2\pi})^2\wp(z_\alpha, L_0)dz_\alpha^2)_\alpha = (\frac{3}{(2\pi)^4} g_2(\tau_\alpha)d\tau_\alpha^2)_\alpha \tag{3.19}$$

on $\mathbf{T}^*$. (Here $\vartheta_3(\tau) \equiv \vartheta_3(0,\tau)$ is the standard theta function on H.) However when we start with $g_2(\tau)$ as Γ'-automorpfic form then, using the operatoe $\widehat{H}_4$ we will not return to $\vartheta_3(\tau)^8 \in \mathcal{A}_4(\Gamma(2))$. Instead of we obtain $\widehat{H}_4 g_2(\tau) = 3g_2(\tau)$ as an element of $\mathcal{A}_4(\Gamma(2))$.

3.3. **Bers embedding.** Since the differential q given by (3.8) has a pole of order 2 at the puncture of $\mathbf{T}^*$ it is not integrable so, although it is holomorphic on $\mathbf{T}^*$, it does not correspond to any element of the Banach space $\mathfrak{B}_2(L, \Gamma')$ (the space of all holomorphic Nehari-bounded forms on the lower half-plane L of weight 4, [8]). This means that we cannot use q to construct the Bers embedding $\mathcal{T}_{1,1} \to \mathfrak{B}_2(L, \Gamma')$. However, we see from (3.6) that the holomorphic differential on $\mathbf{T}^*$

$$\varphi = (dz_\alpha^2)_\alpha = (s^2 \eta^8(\tau_\alpha) d\tau_\alpha^2)_\alpha \tag{3.20}$$

corresponds to $\Phi = \phi(\tau) d\tau^2$ with $\varphi(\tau) = s^2 \eta^8(\tau)$ and hence it corresponds to an element of $\mathfrak{B}_2(L, \Gamma')$ which may be used to find a concrete Bers embedding. However now, the space $\mathcal{T}_{1,1}$ must have its origin at $\mathbf{T}^*$. This means that we must find the domain of complex numbers b such that the Schwarzian differential equation

$$\{w, \tau\} = b\varphi(\tau) \tag{3.21}$$

has a schlicht solution w which has a quasiconformal extention $\widehat{w}$ to all $\mathbb{C}$ compatible with Γ'. Since a shlicht solution w of (3.21) can be written as the quotient $\frac{y_1}{y_2}$ of two linearly independent solutions of $y''(\tau) + \frac{1}{2} b\varphi(\tau) y(\tau) = 0$ to find the values of b for which $\widehat{w}\Gamma'\widehat{w}^{-1}$ is a quasi-Fuchsian of signature $(1; 1)$ we should consider the linear differential equation

$$y''(\tau) + \frac{bs^2}{2} \eta^8(\tau) y(\tau) = 0 \qquad \tau \in L \tag{3.22}$$

Till now, to find a Bers embedding of $\mathcal{T}_{1,1}$ we take the the Teichmueller space $\mathcal{T}_{1,1}$ originating at the punctured torus, usually $\mathbb{C} - L_i/L_i$, and we are looking for the values $b \in \mathbb{C}$ for which the Lame equation

$$y'' + \frac{1}{2}(\frac{1}{2}\wp(z, L_i) + b) y = 0 \tag{3.23}$$

has a purely parabolic monodromy group (which is the commutator subgroup of the quasi-Fuchsian group $\widehat{w}\Gamma_i\widehat{w}^{-1}$ of signature $(1; 1)$) Thus, the relation (3.6) allows us to consider the equation (3.22) instead of (3.23) and, since till now the equation (3.22) had not been investigated (to the author's knowledge), there is a possibility that we obtain new, more transparent understanding of the domain of Bers embedding of the Teichmueller space $\mathcal{T}_{1,1}$ that originates at $\mathbf{T}^* = H/\Gamma'$ (instead of at $\mathbf{T}_i^*$)

4. Fundamental Domains for Γ'

The standard quadrilateral fundamental domains $\mathfrak{F}'_4$ for Γ' and $\mathfrak{F}(\Gamma(2))$ for $\Gamma(2)$ have the same underlying set $\mathfrak{F}$, given by the quadrilateral $(-1, 0, 1, \infty)$, and hence we may choose the same set of their coset representatives in $\widetilde{\Gamma} = PSL_2\mathbb{Z}$. We may decompose the set $\mathfrak{F}$ into copies of a fundamental region $F(\Gamma) = (i - 1, \rho, i, \infty)$

or into copies of a fundamental region $F_\Gamma = (0, \rho+1, \infty)$ of the modular group according to

$$\mathfrak{F} = \mathfrak{S}_1 F(\Gamma) = \mathfrak{S}_2 F_\Gamma \tag{4.1}$$

where $\mathfrak{S}_1 = \{I, g, g^2, T, Tg, Tg^2\}$ and $\mathfrak{S}_2 = \{I, a, a^2, S, Sa, Sa^2\}$ are two sets of coset representatives in $\widetilde{\Gamma}$. When we start with the set $\mathfrak{F}$, to determine whether we have $\Gamma(2)$ or Γ' quadrilateral domain, we have to use either geometric or algebraic considerations. Geometrically, we have different identifications on the border $\partial\mathfrak{F}$ given by the generators of $\Gamma' = \langle A, B\rangle$ and of $\Gamma(2) = \left\langle -I, T^2, U\right\rangle$ respectively.

Algebraically, the free generators S and g of $\widetilde{\Gamma} = \langle S\rangle * \langle g\rangle$ determine distinct permutations of the copies of fundamental domains for $\widetilde{\Gamma}$ depending whether their union forms $\mathfrak{F}(\Gamma(2))$ or $\mathfrak{F}'_4$. More precisely, following the Millington construction [9], both S and g determine permutations μ and σ of a set of coset representatives. A permutation group $\Sigma = \left\langle \mu, \sigma | \mu^2 = \sigma^3 = I\right\rangle$ acts transitively on a set of cosets and the disjoint cycle decomposition of μ, σ and of their product $\mu\sigma$ provides the genus and inequivalent cusp widths for an appropriate subgroup of $\widetilde{\Gamma}$.

For example, if we consider cosets represented by elements of $\mathfrak{S}_1$ and if we denumerate its elements as $\{I, T, g^2, Tg^2, g, Tg\} \Leftrightarrow \{0, 1, 2, 3, 4, 5\}$ respectively then, the permutation $\mu = (03)(14)(25) \in S_6$ for the both subgroups $\Gamma(2)$ and Γ' of $\widetilde{\Gamma}$. However the motion g produces the permutation $\sigma' = (042)(153)$ for Γ' and hence $\mu\sigma' = (0, 1, 2, 3, 4, 5)$. The corresponding permutation group $\Sigma(\Gamma') = \langle \mu, \sigma'\rangle$ tells us that Γ' has genus 1, no elliptic elements and the single cusp of width 6 (equal to the lenght of the cycle $\mu\sigma'$). For $\Gamma(2)$, $g = ST$ generates the permutation $\sigma = (042)(135)$. So the product $\mu\sigma = (01)(23)(45)$ and we have three inequivalent cusps of width equal to 2 each.

We notice that the cycle structures of the generators of $\Sigma(\Gamma')$ and of $\Sigma(\Gamma(2))$ are the same but the permutations given by the products of the generators introduce distinction in the properties of cusps for Γ' and for $\Gamma(2)$ respectively. Of course we could take different enumeration of cosets and different decompositions of the fundamental region of a given subgroup of $\widetilde{\Gamma}$. The permutation group obtained by using these new data will have generators (i.e.permutations representing S and g) that are simultaneously conjugate in S_6 either to $\{\mu, \sigma'\}$ (in the case of Γ') or to $\{\mu, \sigma\}$ (for $\Gamma(2)$).

Thus, when we start with the quadrilateral region $\mathfrak{F} = (-1, 0, 1, \infty)$ then we must perform some operations for the cusp of width 6 of Γ' to be seen. However, the hexagonal fundamental domain $\mathfrak{F}'_6 = (\rho-2, \rho-1, \rho, \omega, \omega+1, \omega+2, \omega+3, \infty)$ has the parabolic vertex of index 6 already. Moreover, if we choose the following fundamental standard domains: $R = (\rho, \omega, \infty)$ for $\widetilde{\Gamma}$, $F(\Gamma_c) = T^{-2}R \cup T^{-1}R$ for Γ_c and $F(\Gamma^+_{ns}(3)) = T^{-2}R \cup T^{-1}R \cup R$ for $\Gamma^+_{ns}(3)$ then we have immediately the relations between appropriate sets given by

$$\mathfrak{F}'_6 = (I \cup T^3)F(\Gamma^+_{ns}(3)) = (I \cup T^2 \cup T^4)F(\Gamma_c) \tag{4.2}$$

These relations immediately describe the ramifications of $\mathbf{X}' \to \mathbf{X}^+_{ns}(3)$ and of $\mathbf{X}' \to \mathbf{X}_c$ at ∞ respectively.

When we work with the quadrilateral domain $\mathfrak{F}'_4$ then, in fact, we are dealing with the subgroups of $PSL_2\mathbb{Z}/\Gamma'$ that may be identified with the finite subgroups $\langle S\rangle$ and $\langle g\rangle$ of the modular group itself. But when we consider the hexagonal fundamental domain $\mathfrak{F}'_6$ then the more natural is to view the quotient $PSL_2\mathbb{Z}/\Gamma'$

as given by $\langle T\rangle \mathrm{mod} T^6$. Although we have that T^3 is equivalent to S modulo Γ' (more precisely $T^3 = S[S^{-1}, T][S^{-1}, T^{-1}]$) and we can write

$$\widetilde{\Gamma}/\Gamma' \cong \langle S\rangle \times \langle g\rangle \cong \langle T^3\rangle \times \langle T^2\rangle mod T^6 \tag{4.3}$$

we notice that the elements S and g have finite order in $\widetilde{\Gamma}$ whereas both T^3 and T^2 are generators of infinite parabolic subgroups of $PSL_2\mathbb{Z}$. So we are dealing with transparent differences between the nature of algebraic objects that may be associated to $\mathfrak{F}_4'$ and $\mathfrak{F}_6'$ respectively and which are involved in the hidden structure of the Veech curve determined by the dynamical system of a billiard (in a $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$-triangle) and described by the error correcting Golay code G_{24} in [1]. These differences become even deeper when we consider a non-unitary representation χ of $\widetilde{\Gamma}$ in $\mathbb{C}^2 = Span\{J_c, J_n\}$. Since the Γ'-automorphic functions $u(\tau)$ and $t(\tau)$ are given by the liftings of J_n and of J_c from $\Gamma^+_{ns}(3)$ and from Γ_c to Γ' appropriately we may identify the underlying vector space $\mathbb{C}^2$ for χ with the linear span of the Weierstrass functions $\wp(p(\tau), L_0) \cong u(\tau)$ and $\wp'(p(\tau), L_0) \cong t(\tau)$ respectively. Since

$$u(\frac{-1}{\tau}) = u(\tau) \qquad u(\tau+1) = \rho u(\tau)$$

and

$$t(\frac{-1}{\tau}) = t(\tau) \qquad t(\tau+1) = -t(\tau)$$

the transformation S acts as identity. We have: $\chi(S) = I$, $\chi(T) = \begin{pmatrix} -1 & 0 \\ 0 & \rho \end{pmatrix}$ and $\chi(T^6) = I$. Thus to see χ as a representation of $\widetilde{\Gamma}/\Gamma'$ on $Span\{\wp, \wp'\}$ we must take the set $\{T^k, k = 0 \ldots 5\}$ as a set of the cosets representatives of Γ' in $\widetilde{\Gamma}$. In other words, it is the cusp of Γ' and its ramification indices over $\mathbf{X}_c$ and over $\mathbf{X}^+_{ns}(3)$ respectively that are important here, and it is the hexagonal domain $\mathfrak{F}_6'$ which immediately produces the relations (4.2).

5. Decomposition of L_0

The projection $p : H \to H/N \cong \mathbb{C} - L_0$, $N = [\Gamma', \Gamma']$ corresponds to the abelization of Γ', $\Gamma'/N \cong \mathbb{Z}^2$. More precisely, let Γ' be generated by $A = [S, T^{-1}] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and by $B = [S, T] = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Any element $\gamma \in \Gamma'$ has the abelianized form

$$\gamma = A^m B^n \mathfrak{n}, \qquad (m, n) \in \mathbb{Z}^2, \qquad \mathfrak{n} \in N \tag{5.1}$$

We usually write $\gamma = mA + nB$, [10], so that

$$p(\mathfrak{n}\tau) = p(\tau) = z \in \mathbb{C} - L_0, \qquad L_0 = [\omega_1, \omega_2] = c[1, \rho]$$

and

$$p(\gamma\tau) = p(\tau) + m\omega_1 + n\omega_2 \qquad for \qquad \gamma = A^m B^n \mathfrak{n} \in \Gamma'$$

Let the quaternion group $Q_8 = \langle \alpha, \beta | \alpha^4 = 1, \alpha^2 = \beta^2, \alpha\beta = \beta\alpha^{-1}\rangle$ be realized by the following matrices in $SL_2(3)$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \tag{5.2}$$

$$\beta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \quad \beta\alpha = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad \alpha\beta = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The group $\mathbb{Z}_3$ that occurs in (2.6) is generated by $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and acts on Q_8 by the automorphisms determined by $X\alpha X^{-1} = \beta$ and $X\beta X^{-1} = \alpha\beta$. Let r_3' denote the restriction of the homomorphism r_3 to Γ'. It maps

$$A \to \beta, \quad A^2 \to \beta^2, \quad A^3 \to \beta^3, \quad A^4 \to I \tag{5.3}$$

$$B \to \beta\alpha, \quad AB \to \alpha^3, \quad A^2B \to \alpha\beta, \quad A^3B \to \alpha$$

From now on we will use the following enumeration of the elements of Q_8:

$$\{I, \beta, \beta^2, \beta^3, \beta\alpha, \alpha^3, \alpha\beta, \alpha\} \equiv \{q_1, q_2, \ldots, q_8\} \tag{5.4}$$

respectively. Let $\sigma \in S_8$ be the permutation $\sigma = (13)(24)(57)(68)$ of $\{q_1, \ldots, q_8\}$ corresponding to the multiplication by $\alpha^2 = \beta^2 = (\alpha\beta)^2 = -I$.

Lemma 5. *The homomorphism $r_3' : \Gamma' \to Q_8$ induces a unique mapping $\kappa : \Gamma'/N \to Q_8 \times Q_8$ such that $\kappa(m,n) = (q_k, \sigma q_k)$ for some unique, appropriate $k \in \{1, \ldots, 8\}$*

Proof. Let $\mathcal{N} = \{1, \alpha^2\}$ denote a normal subgroup of Q_8 and let $r_{3,N}'$ denote the restriction of r_3' to the normal subgroup N of Γ'. Let K_N denote the kernel of the homomorphism $r_{3,N}'$, $K_N \lhd N$, so that we have $N = K_N \cup \mathcal{A}$ as a set, with $\mathcal{A} = {r_{3,N}'}^{-1}(\alpha^2)$. Since each coset (m,n) of N has the decomposition

$$(m,n) \equiv A^m B^n N = A^m B^n K_N \cup A^m B^n \mathcal{A} \tag{5.5}$$

and all elements of the set $\{A^m B^n K_N\}$ are mapped onto some concrete q_k whereas elements of $\{A^m B^n \mathcal{A}\}$ are all mapped onto σq_k, the lemma follows. □

For each $k \in \{1, \ldots, 8\}$ we introduce the subset $\mathcal{A}_k$ of Γ' as the union $\mathcal{A}_k = \mathfrak{A}_k \cup \mathfrak{B}_k$ with

$$\mathfrak{A}_k = \{A^m B^n \mathfrak{n} | \mathfrak{n} \in K_N; r_3'(A^m B^n) = q_k\} \tag{5.6}$$

and with

$$\mathfrak{B}_k = \{A^{m'} B^{n'} \mathfrak{n} | \mathfrak{n} \in \mathcal{A}; r_3'(A^{m'} B^{n'}) = \sigma q_k\} \tag{5.7}$$

.

Lemma 6. *The decomposition $\Gamma' = \bigcup_{k=1}^{8} \mathcal{A}_k$ determines a one-one correspondence between the set of elements of Q_8 and the set of elements of $\mathbb{Z}_4 \times \mathbb{Z}_2$*

Proof. Let us write

$$\mathfrak{A}_k = \{A^{m_k} B^{n_k}\} K_N \quad \textit{and} \quad \mathfrak{B}_k = \{A^{m_k'} B^{n_k'}\} \mathcal{A} \tag{5.8}$$

for appropriate pairs of integers (m_k, n_k) and (m_k', n_k') in $\mathbb{Z}^2$. Let s_k and t_k denote the smallest nonnegative integers such that $r_3'(A^{s_k} B^{t_k}) = q_k$. We see immediately that

$$\{(m_k, n_k)\} = \{(4m + s_k, 4n + t_k), (4m + 2 + s_k, 4n + 2 + t_k), m, n \in \mathbb{Z}\} \tag{5.9}$$

and

$$\{(m_k', n_k')\} = \{(4m + 2 + s_k, 4n + t_k), (4m + s_k, 4n + 2 + t_k), m, n \in \mathbb{Z}\} \tag{5.10}$$

Thus the set $\mathcal{A}_k \subset \Gamma'$ is uniquely determined by the pair $(s_k, t_k) \in \mathbb{Z}_4 \times \mathbb{Z}_2$ and the lemma follows. □

q_k	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
$a_k = (s_k, t_k)$	(0,0)	(1,0)	(2,0)	(3,0)	(0,1)	(1,1)	(2,1)	(3,1)

The explicit relations between Q_8 and $\mathbb{Z}_4 \times \mathbb{Z}_2$ are given in the table.

We recall that the lattice L_0 is produced by the images of $\infty \in H^*$ under Γ', $L_0 \cong p(\Gamma'(\infty))$, and that it is identified with the quotient $\Gamma'/N \cong \mathbb{Z}^2$. Our previous considerations lead us to the following:

Lemma 7. *The homomorphism $r_3' : \Gamma' \to Q_8$ determines the decomposition of the lattice L_0 into 8 disjoint sublattices.*

Proof. We have seen that we can decompose the set of all N-cosets in Γ' into 8 subsets of cosets given by $\{(m_k, n_k) \in \mathbb{Z}^2 | A^{m_k} B^{n_k} \stackrel{r_3'}{\to} q_k\}$ i.e. produced by cosets representatives $A^{m_k} B^{n_k} \in \mathfrak{A}_k$. Now, to each such subset we may uniquely associate the subset $\mathcal{L}_k \subset \mathbb{Z}^2$ of the form:

$$\mathcal{L}_k = \{a_k + 4\mathbb{Z}^2\} \cup \{a_k + (2,2) + 4\mathbb{Z}^2\} \tag{5.11}$$

Using the correspondence $(m,n) \Leftrightarrow m\omega_1 + n\omega_2$ as well as the symmetry properties of the lattice $L_0 = [\omega_1, \omega_2] = \omega_1[1, \rho]$ expressed by $[1, \rho] = [1, \omega]$ for $\omega = \rho + 1 = e^{\frac{i\pi}{3}}$ we obtain immediately that each subset $\mathcal{L}_k \subset \mathbb{Z}^2$, $k \in \{1, \ldots, 8\}$ determines a unique sublattice $\widetilde{\mathcal{L}_k}$ of L_0 given by

$$\widetilde{\mathcal{L}_k} = \widetilde{a}_k + \omega_1[4, 2\omega] \subset L_0, \qquad \widetilde{a}_k = s_k\omega_1 + t_k\omega_2 \tag{5.12}$$

$\square$

We notice that the K-multiple (for $K = \frac{\pi}{2}\vartheta_3^2(0, \omega)$) of the lattice $[4, 2\omega]$ gives the primitive periods of the function sinus amplitudis $sn(2Kz)$. We will not pursuit this direction here. Instead of we will look at $\widetilde{\mathcal{L}_k}$ as the sublattice $L_k = \widetilde{a}_k + 4L_0$ of L_0 together with its halfpoints $\{h_k\} = \widetilde{a}_k + 2(\omega_1, \omega_2) + 4L_0$. We see immediately that although the decomposition of L_0 into mutually disjoint subsets $\widetilde{\mathcal{L}_k}$, $k = \{1, \ldots, 8\}$ is uniquely determined by q_k's, the realization of each $\widetilde{\mathcal{L}_k}$ as a sublattice of L_0 (more precisely a 4-dilate of L_0) together with appropriate half points is not a canonical one. Namely we can do this in three distinct ways. To analyse the situation let us introduce the lattices:

$$\Lambda_3 = [\omega_1^3, \omega_2^3] = 4[\omega_1, \omega_2] \tag{5.13}$$

$$\Lambda_4 = [\omega_1^4, \omega_2^4] = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \circ \Lambda_3 = [\omega_2^3, -\omega_1^3 - \omega_2^3] \tag{5.14}$$

$$\Lambda_2 = [\omega_1^2, \omega_2^2] = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \circ \Lambda_3 = [-\omega_1^3 - \omega_2^3, \omega_1^3] \tag{5.15}$$

Equivalently we could write $\Lambda_4 = g \circ \Lambda_3$ and $\Lambda_2 = g^2 \circ \Lambda_3$, $g = ST$. Although all these three lattices Λ_i's, $i = 2, 3, 4$ are equivalent, the realizations of each subsets $\widetilde{\mathcal{L}_k}$ by distinct Λ_i's requires distinct half points of these lattices. Thus we have:

$$I. \quad \widetilde{\mathcal{L}_k} = \{\widetilde{a}_k + \Lambda_3\} \cup \{\widetilde{a}_k + \frac{\omega_1^3 + \omega_2^3}{2} + \Lambda_3\} \tag{5.16}$$

or

$$II. \quad \widetilde{L}_k = \{\widetilde{a}_k + \Lambda_4\} \cup \{\widetilde{a}_k + \frac{\omega_2^4}{2} + \Lambda_4\} \tag{5.17}$$

or

$$III. \quad \widetilde{\mathcal{L}_k} = \{\widetilde{a}_k + \Lambda_2\} \cup \{\widetilde{a}_k + \frac{\omega_1^2}{2} + \Lambda_2\} \tag{5.18}$$

We observe that the realizations I, II and III are associated to the pairs (Q_8, I), (Q_8, g) and to (Q_8, g^2) respectively.

6. Conclusions

We have seen that we needed both homomorphisms, modulo 2 and modulo 3, to find that the Weierstrass functions $\wp$ and $\wp'$ on $p(H) \cong \mathbb{C} - L_0$ have liftings to H given by the absolute invariants $J_n(\tau) = (J(\tau))^{\frac{1}{3}}$ (for $\Gamma^+_{ns}(3)$) and $J_c(\tau) = (J(\tau) - 1)^{\frac{1}{2}}$, (for Γ_c) respectively. Further, we have obtained that the homomorphism r_3' determines the decomposition of Γ' into subsets $\mathcal{A}_k = {r'}_3^{-1}(q_k)$, $k = 1, \ldots, 8$ (equivalently into the cosets of a normal subgroup $\Gamma' \cap \Gamma(3)$ in Γ'). Then we decomposed each $\mathcal{A}_k \subset \Gamma'$ as $\mathcal{A}_k = \mathfrak{A}_k \cup \mathfrak{B}_k$ according to (5.6) and (5.7). We have noticed that the set of pairs $(m_k, n_k) \in \mathbb{Z}^2$ with $A^{m_k}B^{n_k} \in \mathfrak{A}_k$ determines a sublattice $\widetilde{\mathcal{L}_k}$ of the lattice $L_0 \cong \mathbb{Z}^2$. Although the sublattice $\widetilde{\mathcal{L}_k}$ is not a dilate of L_0 we may view it as given by a lattice equivalent to L_0 together with the set of all of its appropriate half points. Such realization is not a canonical one and we have three ways, I, II and III, to do this. In other words, we obtain the decomposition of L_0 into eight disjoint subsets, $L_0 = \bigcup_{k=1}^{8} \widetilde{\mathcal{L}_k}$, each of which can be seen as

$$\widetilde{\mathcal{L}_k} = \{\widetilde{a}_k + \Lambda_l\} \cup \{\widetilde{a}_k + h(l) + \Lambda_l\} \qquad k = 1, \ldots, 8 \tag{6.1}$$

for $l = 2, 3, 4$ (here $\Lambda_l = g^l \circ \Lambda_3$ and $h(l)$ is an appropriate, depending on l, half-point of Λ_l). All lattices Λ_l's are 4-dilates of L_0 and the essential differences between I, II and III lie in the different positions of half-points. These three realizations correspond to the elements of the group $\langle g \rangle < SL_2\mathbb{Z}$ involved in the formulae (5.13) – (5.15). More precisely, although the three lattices $\Lambda_l = [\omega_1^l, \omega_2^l] = g^l \circ \Lambda_3$, $l = 2, 3, 4$ coincide (and are all 4-dilates of the lattice L_0) the fact that $g \notin \Gamma(2)$ implies that the half-points of the lattices $g^l \circ \Lambda_3$'s are not preserved. Since $r_3(\langle g \rangle) \cong SL_2(3)/Q_8$ we may view the group $Q_8 \triangleleft SL_2(3)$ as producing the decomposition $L_0 = \bigcup_{k=1}^{8} \widetilde{\mathcal{L}}_k$ and we may view the quotient $SL_2(3)/Q_8$ (which is associated to the symmetries of the lattice L_0 described by the cyclic group $\langle g \rangle$) as responsible for the three realizations given by (5.16), (5.17) and (5.18) respectively. We see that:

- The realization I is associated to $(\Lambda_3, \frac{\omega_1^3 + \omega_2^3}{2})$ and involves the half points that correspond to the zeros of $\vartheta_3(v, \rho)$
- The realization II is associated to $(\Lambda_4, \frac{\omega_2^4}{2})$ and involves the half poins that correspond to the zeros of $\vartheta_4(v, \rho)$
- The realization III is associated to $(\Lambda_2, \frac{\omega_1^2}{2})$ and involves the half points that correspond to the zeros of $\vartheta_2(v, \rho)$

Here $v = \frac{z}{4\omega_1}$, $L_0 = [\omega_1, \omega_2]$, $z = p(\tau)$ for $\tau \in H$ and we use exactly the same subindex l for a lattice Λ_l and for the corresponding even theta function. Moreover, the relations $\Lambda_4 = g \circ \Lambda_3$ and $\Lambda_2 = g^2 \circ \Lambda_3$ are parallel to the following relations respectively:

$$\vartheta_4^8(0, \tau) = (j_g(\tau))^{-4} \vartheta_3^8(0, g\tau), \qquad \vartheta_2^8(0, \tau) = (j_{g^2}(\tau))^{-4} \vartheta_3^8(0, g^2\tau)$$

Now, the global section $L(x')$ of lattices over $\mathbf{T}^*$ introduced earlier leads to the fiber space over $\mathbf{T}^*$ whose fiber at any point $x' \in \mathbf{T}^*$ is a complex torus $\mathbf{T}_{x'}$ given by

$\mathbb{C}/L(x')$ and attached to x' at the origin. However although for each point $x' \in \mathbf{T}^*$ the lattice $L(x')$ is well defined it is not equipped with any concrete basis and hence its half points are determined only up to permutations. The situation will change when we restrict ourselves to a single map (U_α, τ_α), that is to $x' \in U_\alpha \subset \mathbf{T}^*$. Now we can write

$$L(x') = L(\tau_\alpha) = \mu(\tau_\alpha)[1, \tau_\alpha] = [\omega_1^\alpha, \omega_2^\alpha]$$

and the half points are given by $h_1^\alpha = \frac{\omega_1^\alpha}{2}$, $h_2^\alpha = \frac{\omega_2^\alpha}{2}$ and by $h_3^\alpha = \frac{\omega_1^\alpha + \omega_2^\alpha}{2}$ respectively. Since the decomposition of L_0 corresponds to the decomposition of $\mathbb{Z}^2$ given by (5.11) the realization I defines the decompositions of each $L(\tau_\alpha(x'))$, $x' \in U_\alpha$ onto eight sublattices together with their half points as follows:

$$\widetilde{\mathcal{L}}_k^\alpha = \{\widetilde{a_k^\alpha} + 4L(\tau_\alpha)\} \cup \{\widetilde{a}_k^\alpha + 4h_3^\alpha + 4L(\tau_\alpha)\}, \quad k = 1, \dots, 8 \tag{6.2}$$

where $\widetilde{a_k^\alpha} = s_k\omega_1^\alpha + t_k\omega_2^\alpha$. Each $\widetilde{\mathcal{L}}_k^\alpha$ produces torus isomorphic to $\mathbf{T}(\tau_\alpha(x'))$ attached at the origin to x' together with well defined half-point on it (corresponding to the zero of $\vartheta_3(z, \tau_\alpha) : U_\alpha \times \mathbb{C} \to \mathbb{C}$). Since $k = 1, \dots, 8$, we may consider (on a set U_α) the field $(4L(\tau_\alpha), 4h_3^\alpha + 4L(\tau_\alpha))^{\otimes 8}$ and hence we naturally obtain the function $\vartheta_3^8(0, \tau_\alpha)$ (on U_α). Similarly, starting with the realization II or III we arrive to the functions $\vartheta_4^8(0, \tau_\alpha)$ or to $\vartheta_2^8(0, \tau_\alpha)$ respectively. None of these functions $\vartheta_l^8(0, \tau_\alpha)$, $l = 2, 3, 4$ can be (using the atlas $\{(U_\alpha, \tau_\alpha)\}_\alpha$) extended to the whole $\mathbf{T}^*$ to define any meaningful object on it.

The existence of the three pictures I, II and III of each $\widetilde{L}_k$, $k = 1, \dots, 8$ comes from the symmetry properties of the lattice $p(\Gamma'\infty) = L_0$. Since the group $\langle g \rangle$ is responsible for the existence of these three realizations we may naturally involve the Hecke operators $T_{\langle g \rangle, k}$ introduced in the subsection (3.2). Thus, for $l = 2, 3, 4$ on each $U_\alpha \subset \mathbf{T}^*$ we obtain

$$T_{\langle g \rangle, 4}\vartheta_l^8(0, \tau_\alpha) = \vartheta_3^8(0, \tau_\alpha) + \vartheta_4^8(0, \tau_\alpha) + \vartheta_2^8(0, \tau_\alpha)$$

Since for $x' \in U_\alpha \cap U_\beta$ we have $\tau_\beta(x') = \gamma\tau_\alpha(x')$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$ and

$$T_{\langle g \rangle, 4}\vartheta_l^8(0, \tau_\beta) = (c\tau_\alpha + d)^4 T_{\langle g \rangle, 4}\vartheta_l^8(0, \tau_\alpha)$$

the family $\{T_{\langle g \rangle, 4}\vartheta_l^8(0, \tau_\alpha)\}_\alpha$ forms well defined quadratic differential on $\mathbf{T}^*$ which is exactly the same for each $l = 2, 3, 4$. The necessity of applying on U_α the Hecke operator $T_{\langle g \rangle, 4}$ to $\vartheta_l^8(0, \tau_\alpha)$ (or equivalently, the necessity of taking equally weighted sum $\sum_{l=2}^4 \vartheta_l^8(0, \tau_\alpha)$ on U_α) reflects the fact that each one of these three realizations is equally important. Thus, the explicite forms of the Thetanullverte $\vartheta_l(0, \tau_\alpha)$, $l = 2, 3, 4$ on U_α (which result from these all three realizations) provide

$$T_{\langle g \rangle, 4}\vartheta_l^8(0, \tau_\alpha) = \sum_{\underline{n} \in \mathbb{Z}^8} q_\alpha^{\underline{n}^2} + \sum_{\underline{n} \in \mathbb{Z}^8} (-1)^{\underline{n} \cdot \underline{1}} q_\alpha^{\underline{n}^2} + \sum_{\underline{n} \in \mathbb{Z}^8} q_\alpha^{(\underline{n} - \underline{e})^2} \tag{6.3}$$

and is further equal to the following expresion

$$\sum_{\underline{n} \cdot \underline{1} \in 2\mathbb{Z}} q_\alpha^{\underline{n}^2} + \sum_{\underline{n} \in \mathbb{Z}^8} q_\alpha^{(\underline{n} - \underline{e})^2} = 2\Theta_{E_8}(\tau_\alpha)$$

Here $q_\alpha = e^{i\pi\tau_\alpha}$, $\underline{e} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{Q}^8$, and $\underline{1} = 2\underline{e}$.

Another argument which leads to the sum of all $\vartheta_l^8(0, \tau_\alpha)$, $l = 2, 3, 4$ comes from the subsection (3.2). Namely, for any holomorphic atlas on $Y(2) \cong H/\Gamma(2)$ the transition functions preserve (pointwise) all half-points of each nonsingular fiber of the modular elliptic surface over $X(2)$. Hence $\vartheta_3^8(0,\tau) = \sum_{\underline{n}\in\mathbb{Z}^8} q^{\underline{n}^2}$ is a $\Gamma(2)$-automorphic form of weight 4. Roughly speaking, the existence of a global section of half-points over the moduli space $Y(2)$ allows us to consider only the first part of the right side of (6.3) wich contains only the lattice $\mathbb{Z}^8$. When we pass to the moduli space $\mathbf{T}^*$ of complex tori, it is no longer possible and (on each U_α) we must also involve the remaining terms of the left side of (6.3), that is, we must consider the lattice E_8 instead of merely $\mathbb{Z}^8$ as for $Y(2)$.

Summerizing, the occurence of the E_8-symmetry related to the moduli space $\mathbf{T}^*$ can be seen as a consequence of the relation between Γ' and Q_8 coming from the modulo 3 homomorphism and of the existence of the equally important three realizations of the decomposition of the lattice L_0 into 8 mutually disjoint subsets.

Moreover, from local relations $\widetilde{H}_4(\vartheta_3^8(\tau)d\tau^2) \cong const\wp(z)dz^2$ coming off the subsection (3.2) we obtain

$$u(\tau_\alpha) = \wp(p(\tau_\alpha), L_0) = const \cdot \frac{\Theta_{E_8}(\tau_\alpha)}{\eta^8(\tau_\alpha)} \qquad on \qquad U_\alpha \tag{6.4}$$

and hence

$$\wp(p(\tau_\alpha), L_0) = const\cdot q_\alpha^{-\frac{1}{3}} \sum_{n=0}^{\infty}\sum_{m=0}^{n} r_{E_8}(m)p_8(n-m)q_\alpha^n \tag{6.5}$$

So, we may view the Weierstrass function on the moduli curve $\mathbf{T}^*$ as a function which encodes the information about the decompositions of L_0.

Let us notice the difference between the Jacobi and our approach. Although Jacobi forms involve both euclidean variable z and hyperelliptic variable τ (in particular, the ratio of the Jacobi-Eisenstein forms of index 1 and weight 10 and 12 respectively gives a constant multiple of the Weierstrass $\wp$-function for each L_τ, $\tau \in H$) in the Jacobi picture we must work with meromorphic functions on $H\times\mathbb{C}$ satisfying some concrete conditions. In our approach we simply translate the hyperbolic objects for Γ', Γ_c, $\Gamma^+_{ns}(3)$, etc. into the euclidean objects on $\mathbb{C} - L_0$ and vice versa and this is a reason for the appearance of 8 sublattices of L_0 together with their appropriate half-points and further the appearance of $\Theta_{E_8}(q_\alpha)$ on U_α.

We have shown that the bridge between the hyperbolic structure of the universal covering space H of $\mathbf{T}^* \cong H/\Gamma'$ and the euclidean structure of $\mathbb{C} - L_0$, ($\mathbf{T}^* \cong \mathbb{C} - L_0/L_0$) is given by the function $\eta^4(\tau)$. Now, rewriting the formula (6.4) as

$$\wp(p(\tau_\alpha), L_0)\eta^8(\tau_\alpha) = const \cdot \Theta_{E_8}(\tau_\alpha) \tag{6.6}$$

we may view $\eta^8(\tau_\alpha)$ as a bridge between 2-periodic,with respect to L_0, function $\wp$ and the theta function of the lattice E_8 (which may be produced by the decomposition of L_0 into 8 sublattices together with appropriate half points in three distict ways respectively).

Let us also notice that the function $\eta^8(\tau)$ provides very strong interrelation between the groups Γ and Γ'. It is expressed by the fact that the ring of modular forms for Γ, that is the ring $\mathfrak{M}(\Gamma) = \mathbb{C}[g_2(\tau), g_3(\tau)]$ can be written as $\mathfrak{M}(\Gamma) = \mathbb{C}[\eta^8(\tau)u(\tau), \eta^8(\tau)u'(\tau)]$ and hence can be given as:

$$\mathfrak{M}(\Gamma) = \mathbb{C}[\eta^8(\tau)\wp(p(\tau), L_0), \eta^8(\tau)\wp'(p(\tau), L_0)] \tag{6.7}$$

This tells us that although the generators $g_2(\tau)$ and $g_3(\tau)$ are algebraically independent they produce differentials $\wp(z)dz^2$ and $\wp'(z)dz^3$ respectively and hence we have some "elliptic" type of differential relation between them.

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