Biharmonic Riemannian maps

Bayram Şahin

Inonu University, Department of Mathematics, 44280, Malatya-Turkey.
E-mail:bsahin@inonu.edu.tr

Abstract. We give necessary and sufficient conditions for Riemannian maps to be biharmonic. We also define pseudo umbilical Riemannian maps as a generalization of pseudo-umbilical submanifolds and show that such Riemannian maps put some restrictions on the base manifolds.

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1.Introduction

Smooth maps between Riemannian manifolds are useful for comparing geometric structures between two manifolds. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds and they are characterized by their Riemannian metrics and Jacobian matrices. More precisely, a smooth map $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called an isometric immersion if F_* is injective and

$$g_2(F_*X, F_*Y) = g_1(X, Y) \tag{1.1}$$

for X, Y vector fields tangent to M_1, here F_* denotes the derivative map.

On the other hand, Riemannian submersions between Riemannian manifolds were initiated by B. O'Neill [O] and A. Gray [G], see also [FIP] and [YK]. A smooth map $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ is called Riemannian submersion if F_* is onto and it satisfies the equation(1.1) for vector fields tangent to the horizontal space $(kerF_*)^\perp$. For Riemannian submersions between various manifolds, see: [FIP] and [YK].

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [F] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{m, n\}$, where $dimM_1 = m$ and $dimM_2 = n$. Then we denote the kernel space of F_* by $kerF_*$ and consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^\perp$ to $kerF_*$. Then the tangent bundle of M_1 has the following decomposition

$$TM_1 = kerF_* \oplus \mathcal{H}.$$

We denote the range of F_* by $rangeF_*$ and consider the orthogonal complementary space $(rangeF_*)^\perp$ to $rangeF_*$ in the tangent bundle TM_2 of M_2. Since $rankF < min\{m,n\}$, we always have $(rangeF_*)^\perp \neq \{0\}$. Thus the tangent bundle TM_2 of M_2 has the following decomposition

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called a Riemannian map at $p_1 \in M$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^\perp \longrightarrow (rangeF_{*p_1})$ is a linear isometry between the inner product spaces $((kerF_{*p_1})^\perp, g_1(p_1) \mid_{(kerF_{*p_1})^\perp})$ and $(rangeF_{*p_1}, g_2(p_2) \mid_{(rangeF_{*p_1})})$, $p_2 = F(p_1)$. Therefore Fischer stated in [F] that a Riemannian map is a map which is as isometric as it can be. In other words, F_* satisfies the equation(1.1) for X, Y vector fields tangent to $\mathcal{H} = (kerF_*)^\perp$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion. One of the main properties of Riemannian maps is that Riemannian maps satisfy the eikonal equation which is a link between geometric optics and physical optics. For Riemannian maps and their applications, see: [RK].

A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. Harmonic maps between Riemannian manifolds provide a rich display of both differential geometric and analytic phenomena, and they are closely related to the theory of stochastic processes and to the theory of liquid crystals in material science. On the other hand, the biharmonic maps are the critical points of the bienergy functional and, from this point of view, generalize harmonic maps. The notion of biharmonic map was suggested by Eells and Sampson [ES]. The first variation formula and, thus, the Euler-Lagrange equation associated to the bienergy was obtained by Jiang in [J1], [J2]. But biharmonic maps have been extensively studied in the last decade and there are two main research directions. In differential geometry, many authors have obtained classification results and constructed many examples. Biharmonicity of immersions was obtained in [CI], [CM], [OC] and biharmonic Riemannian submersions were studied in [OC], for a survey on biharmonic maps, see:[MO]. From the analytic point of view, biharmonic maps are solutions of fourth order strongly elliptic semilinear partial differential equations. It is known that plane elastic problems can be expressed in terms of the biharmonic equation. On the other hand, the wave maps are harmonic maps on Minkowski spaces and the biwave maps are biharmonic maps on Minkowski spaces. The wave maps arise in the analysis of the more difficult hyperbolic Yang-Mills equations either as special cases or as equations for certain families of gauge transformations. Such equations arise in general relativity for spacetimes with two Killing vector fields. Bi-Yang-Mills fields, which generalize Yang-Mills fields, have been introduced by Bejan and Urakawa [BU] recently. For relations between the biwave maps and the bi-Yang-Mills equations, see [IIU] and [JC]. Moreover, in geometric optics[D], one can obtain the eikonal equation by using the wave equation.

In this paper, we mainly investigate the biharmonicity of Riemannian maps from Riemannian manifolds to space forms. In section 2, we introduce notations and give fundamental formulas of the bitension field, then we obtain some preparatory results of Riemannian maps in section 3. We also define pseudo umbilical Riemannian maps as a generalization of pseudo umbilical isometric immersions, obtain a necessary and sufficient condition for a Riemannian map to be pseudo umbilical and give a method how to construct pseudo-umbilical Riemannian maps. In section 4, we find necessary and sufficient conditions for Riemannian maps to be harmonic and observe that pseudo-umbilical Riemannian maps from Riemannian manifolds M_1 to space forms $M_2(c)$ with additional conditions must be either harmonic or $c > 0$.

2.Preliminaries

In this section we recall some basic materials from [BW] and [MO]. Let (M, g_M) be a Riemannian manifold and $\mathcal{V}$ be a $q-$ dimensional distribution on M. Denote its orthogonal distribution $\mathcal{V}^{\perp}$ by $\mathcal{H}$. Then, we have

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{2.1}$$

$\mathcal{V}$ is called the vertical distribution and $\mathcal{H}$ is called the horizontal distribution. We use the same letters to denote the orthogonal projections onto these distributions.

By the unsymmetrized second fundamental form of $\mathcal{V}$, we mean the tensor field $A^{\mathcal{V}}$ defined by

$$A^{\mathcal{V}}_E F = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F), \quad E, F \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. The symmetrized second fundamental form $B^{\mathcal{V}}$ of $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(E, F) = \frac{1}{2}\{A^{\mathcal{V}}_E F + A^{\mathcal{V}}_F E\} = \frac{1}{2}\{\mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{H}(\nabla_{\mathcal{V}F}\mathcal{V}E)\} \tag{2.3}$$

for any $E, F \in \Gamma(TM)$. The integrability tensor of $\mathcal{V}$ is the tensor field $I^{\mathcal{V}}$ given by

$$I^{\mathcal{V}}(E, F) = A^{\mathcal{V}}_E F - A^{\mathcal{V}}_F E - \mathcal{H}([\mathcal{V}E, \mathcal{V}F]). \tag{2.4}$$

Moreover, the mean curvature vector field of $\mathcal{V}$ is defined by

$$\mu^{\mathcal{V}} = \frac{1}{q} Trace B^{\mathcal{V}} = \frac{1}{q}\sum_{i=1}^{q} \mathcal{H}(\nabla_{e_r} e_r), \tag{2.5}$$

where $\{e_1, ..., e_q\}$ is a local frame of $\mathcal{V}$. By reversing the roles of $\mathcal{V}$, $\mathcal{H}$, $B^{\mathcal{H}}$, $A^{\mathcal{H}}$ and $I^{\mathcal{H}}$ can be defined similarly. For instance, $B^{\mathcal{H}}$ is defined by

$$B^{\mathcal{H}}(E, F) = \frac{1}{2}\{\mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{V}(\nabla_{\mathcal{H}F}\mathcal{H}E)\} \tag{2.6}$$

and, hence we have

$$\mu^{\mathcal{H}} = \frac{1}{m-q} Trace B^{\mathcal{H}} = \frac{1}{m-q}\sum_{s=1}^{m-q} \mathcal{V}(\nabla_{E_s} E_s), \tag{2.7}$$

where $E_1, ..., E_{m-q}$ is a local frame of $\mathcal{H}$. A distribution $\mathcal{D}$ on M is said to be minimal if, for each $x \in M$, the mean curvature vector field vanishes.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth map between them. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N, p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.8}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. Let $\varphi : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds and assume M is compact, then its energy is

$$E(\varphi) = \int_M e(\varphi)\, v_g = \frac{1}{2}\int_M |d\varphi|^2\, v_g.$$

The critical points of E are called harmonic maps. Standard arguments yield the associated Euler-Lagrange equation, the vanishing of the tension field:$\tau(\varphi) = trace(\nabla \varphi_*)$. Let $\varphi : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. Define its bienergy as

$$E^2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g.$$

Critical points of the functional E^2 are called biharmonic maps and its associated Euler-Lagrange equation is the vanishing of the bitension field

$$\tau^2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - trace_{g_M} R^N(d\varphi, \tau(\varphi))d\varphi, \tag{2.9}$$

where $\Delta^{\varphi}\tau(\varphi) = -trace_{g_M}(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})$ is the Laplacian on the sections of $\varphi^{-1}(TN)$ and R^N is the Riemann curvature operator on (N, g_N). A map between two Riemannian manifolds is said to be proper biharmonic if it is a non-harmonic biharmonic map.

3.Riemannian maps

In this section, we obtain some new results which will be using in the next section. First note that in [S2] we showed that the second fundamental form $(\nabla F_*)(X, Y)$, $\forall X, Y \in \Gamma((kerF_*)^{\perp})$, of a Riemannian map has no components in $rangeF_*$.

Lemma 3.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((kerF_*)^{\perp}). \tag{3.1}$$

As a result of Lemma 3.1, we have

$$(\nabla F_*)(X, Y) \in \Gamma((rangeF_*)^{\perp}), \forall X, Y \in \Gamma((kerF_*)^{\perp}). \tag{3.2}$$

Also from [S1], we have the following.

Lemma 3.2 *Let $F : (M, g_M) \longrightarrow (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then the tension field τ of F is*

$$\tau = -m_1 F_*(\mu^{kerF_*}) + m_2 H_2, \tag{3.3}$$

where $m_1 = dim(kerF_*), m_2 = rankF$, μ^{kerF_*} and H_2 are the mean curvature vector fields of the distributions of $kerF_*$ and $rangeF_*$, respectively.

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F. Then according to [N], for any vector field X on M_1 and any section V of $(rangeF_*)^{\perp}$, where $(rangeF_*)^{\perp}$ is the subbundle of $F^{-1}(TM_2)$ with fiber $(F_*(T_pM))^{\perp}$-orthogonal complement of $F_*(T_pM)$ for g_2 over p, we have $\nabla^{F\perp}_X V$ which is the orthogonal projection of $\nabla^2_X V$ on $(F_*(TM))^{\perp}$. In [N], the author also showed that $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F\perp} g_2 = 0$. We now define A_V as

$$\nabla^2_{F_*X} V = -A_V F_* X + \nabla^{F\perp}_X V, \tag{3.4}$$

where $A_V F_* X$ is the tangential component (a vector field along F) of $\nabla^2_{F_*X} V$. It is easy to see that $A_V F_* X$ is bilinear in V and $F_* X$ and $A_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(A_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)), \tag{3.5}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma((rangeF_*)^\perp)$. Since (∇F_*) is symmetric, it follows that A_V is a symmetric linear transformation of $rangeF_*$.

We now define pseudo-umbilical Riemannian maps as a generalization of pseudo-umbilical isometric immersions. Pseudo-umbilical Riemannian maps will be useful when we deal with the biharmonicity of Riemannian maps.

Definitio 3.1. Let $F : (M, g_1) \longrightarrow (M_2, g_2)$ be a Riemannian map between Riemannian manifolds M_1 and M_2. Then we say that F is a pseudo-umbilical Riemannian map if

$$A_{H_2} F_*(X) = \lambda F_*(X) \tag{3.6}$$

for $\lambda \in C^\infty(M_1)$ and $X \in \Gamma((kerF_*)^\perp)$.

Here we present an useful formula for pseudo umbilical Riemannian maps by using(3.5) and(3.6).

Proposition 3.1. *Let $F : (M, g_1) \longrightarrow (M_2, g_2)$ be a Riemannian map between Riemannian manifolds M_1 and M_2. Then F is pseudo-umbilical if and only if*

$$g_2((\nabla F_*)(X, Y), H_2) = g_1(X, Y) g_2(H_2, H_2) \tag{3.7}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. Let $\{\tilde{e}_1, ..., \tilde{e}_{m_1}, e_1, ..., e_{m_2}\}$ be an orthonormal basis of $\Gamma(TM_1)$ such that $\{\tilde{e}_1, ..., \tilde{e}_{m_1}\}$ is an orthonormal basis of $kerF_*$ and $\{e_1, ..., e_{m_2}\}$ is an orthonormal basis of $(kerF_*)^\perp$. Then since F is a Riemannian map we have

$$\sum_{i=1}^{m_2} g_2(A_{H_2} F_*(e_i), F_*(e_i)) = m_2 \lambda.$$

Using(3.5), we get

$$\sum_{i=1}^{m_2} g_2(\frac{1}{m_2} (\nabla F_*)(e_i, e_i), H_2) = \lambda.$$

Thus we obtain

$$\lambda = g_2(H_2, H_2). \tag{3.8}$$

Then, from (3.5), (3.6) and(3.8) we have(3.7). The converse is clear.

It is known that the composition of a Riemannian submersion and an isometric immersion is a Riemannian map [F].Using this we have the following.

Theorem 3.1. *Let $F_1 : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a Riemannian submersion and $F_2 : (M_2, g_2) \longrightarrow (M_3, g_3)$ a pseudo-umbilical isometric immersion. Then the map $F_2 \circ F_1$ is a pseudo umbilical Riemannian map.*

Proof. From the second fundamental form of the composite map $F_2 \circ F_1$ [BW], we have

$$(\nabla(F_2 \circ F_1)_*(X,Y) = F_{2*}((\nabla F_{1*})(X,Y)) + (\nabla F_{2*})(F_{1*}X, F_{1*}Y)$$

for $X, Y \in \Gamma((kerF_{1*})^{\perp})$. Then proof follows from the definition of pseudo-umbilical submanifolds.

Remark. 3.1. We note that above theorem gives a method to find examples of pseudo umbilical Riemannian maps. It also tells that if one has an example of pseudo-umbilical submanifolds, it is possible to find an example of pseudo umbilical Riemannian maps. For examples of pseudo umbilical submanifolds, see: [C].

4.Biharmonicity of Riemannian maps

In this section we obtain the biharmonicity of Riemannian maps between Riemannian manifolds. We also show that pseudo-umbilical biharmonic Riemannian maps put some restrictions on the total manifold of such maps.

Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then the adjoint map ${}^*F_*$ of F_* is characterized by $g_1(x, {}^*F_{*p_1}y) = g_2(F_{*p_1}x, y)$ for $x \in T_{p_1}M_1$, $y \in T_{F(p_1)}M_2$ and $p_1 \in M_1$. Considering F^h_* at each $p_1 \in M_1$ as a linear transformation

$$F^h_{*p_1} : ((kerF_*)^{\perp}(p_1), g_{1_{p_1((kerF_*)^{\perp}(p_1))}}) \longrightarrow (rangeF_*(p_2), g_{2_{p_2(rangeF_*)(p_2))}}),$$

we will denote the adjoint of F^h_* by ${}^*F^h{}_{*p_1}$. Let ${}^*F_{*p_1}$ be the adjoint of F_{*p_1} : $(T_{p_1}M_1, g_{1_{p_2}}) \longrightarrow (T_{p_2}M_2, g_{2_{p_2}})$. Then the linear transformation

$$({}^*F_{*p_1})^h : rangeF_*(p_2) \longrightarrow (kerF_*)^{\perp}(p_1)$$

defined by $({}^*F_{*p_1})^h y = {}^*F_{*p_1}y$, where $y \in \Gamma(rangeF_{*p_1}), p_2 = F(p_1)$, is an isomorphism and $(F^h_{*p_1})^{-1} = ({}^*F_{*p_1})^h = {}^*(F^h_{*p_1})$.

We also recall that the curvature tensor R of a space form $(M(c), g)$ is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{4.1}$$

We are now ready to prove the following theorem which gives necessary and sufficient conditions for a Riemannian map to be biharmonic.

Theorem 4.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a space form $(M_2(c), g_2)$. Then F is biharmonic if and only if*

$$\begin{aligned}
&m_1 trace A_{(\nabla F_*)(.,\mu^{kerF_*})} F_*(.) - m_1 trace F_*(\nabla_{(.)}\nabla_{(.)}\mu^{kerF_*}) \\
&-m_2 trace F_*(\nabla_{(.)}{}^*F_*(A_{H_2}F_*(.))) - m_2 trace A_{\nabla^{F\perp}_{F_*(.)}H_2} F_*(.) \\
&-m_1 c(m_2 - 1)F_*(\mu^{kerF_*}) = 0
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&m_1 trace \nabla^{F\perp}_{F_*(.)}(\nabla F_*)(., \mu^{kerF_*}) + m_1 trace(\nabla F_*)(., \nabla_{(.)}\mu^{kerF_*}) \\
&+m_2 trace(\nabla F_*)(., {}^*F_*(A_{H_2}F_*(.))) - m_2 \Delta^{R^{\perp}} H_2 \\
&-m_2^2 c H_2 = 0.
\end{aligned} \tag{4.3}$$

Proof. First of all, from(4.1) and (3.4) we have

$$traceR^2(F_*(.),\tau(F))F_*(.) = m_1c(m_2-1)F_*(\mu^{kerF_*}) - m_2^2cH_2, \tag{4.4}$$

where R^2 is the curvature tensor field of M_2. Let $\{\tilde{e}_1,...,\tilde{e}_{m_1},e_1,...,e_{m_2}\}$ be a local orthonormal frame on M_1, geodesic at $p\in M_1$ such that $\{\tilde{e}_1,...,\tilde{e}_{m_1}\}$ is an orthonormal basis of $kerF_*$ and $\{e_1,...,e_{m_2}\}$ is an orthonormal basis of $(kerF_*)^\perp$. At p we have

$$\begin{aligned}\Delta\tau(F) &= -\sum_{i=1}^{m_2}\overset{F}{\nabla}_{e_i}\overset{F}{\nabla}_{e_i}\tau(F)\\ &= -\sum_{i=1}^{m_2}\overset{F}{\nabla}_{e_i}\{\overset{F}{\nabla}_{e_i}(-m_1F_*(\mu^{kerF_*})+m_2H_2)\}.\end{aligned}$$

Then using(2.8),(3.2) and(3.4) we get

$$\begin{aligned}\Delta\tau(F) &= -\sum_{i=1}^{m_2}\overset{F}{\nabla}_{e_i}\{-m_1(\nabla F_*)(e_i,\mu^{kerF_*})-m_1F_*(\nabla_{e_i}\mu^{kerF_*})\\ &+ m_2(-A_{H_2}F_*(e_i)+\overset{F\perp}{\nabla}_{F_*(e_i)}H_2)\}.\end{aligned}$$

Using again (2.8),(3.2) and(3.4) we obtain

$$\begin{aligned}\Delta\tau(F) &= m_1\sum_{i=1}^{m_2}-A_{(\nabla F_*)(e_i,\mu^{kerF_*})}F_*(e_i)+\overset{F\perp}{\nabla}_{F_*(e_i)}(\nabla F_*)(e_i,\mu^{kerF_*})\\ &+ m_1\sum_{i=1}^{m_2}(\nabla F_*)(e_i,\nabla_{e_i}\mu^{kerF_*})+F_*(\nabla_{e_i}\nabla_{e_i}\mu^{kerF_*})\\ &+ m_2\sum_{i=1}^{m_2}\overset{F}{\nabla}_{e_i}A_{H_2}F_*(e_i)-m_2\sum_{i=1}^{m_2}-A_{\overset{F\perp}{\nabla}_{F_*(e_i)}H_2}F_*(e_i)\\ &+ \overset{F\perp}{\nabla}_{F_*(e_i)}\overset{F\perp}{\nabla}_{F_*(e_i)}H_2.\end{aligned}$$

On the other hand, since $A_{H_2}F_*(e_i)\in\Gamma(F_*((kerF_*)^\perp))$, we can write

$$F_*(X) = A_{H_2}F_*(e_i)$$

for $X\in\Gamma((kerF_*)^\perp)$, where

$$X = (F_*)^{-1}(A_{H_2}F_*(e_i)) = {}^*F_*(A_{H_2}F_*(e_i)).$$

Then using(2.8) we have

$$\overset{F}{\nabla}_{e_i}A_{H_2}F_*(e_i) = (\nabla F_*)(e_i,{}^*F_*(A_{H_2}F_*(e_i)))+F_*(\nabla_{e_i}{}^*F_*(A_{H_2}F_*(e_i))).$$

Thus we obtain

$$\Delta\tau(F) = m_1\sum_{i=1}^{m_2}-A_{(\nabla F_*)(e_i,\mu^{kerF_*})}F_*(e_i)+\overset{F\perp}{\nabla}_{F_*(e_i)}(\nabla F_*)(e_i,\mu^{kerF_*})$$

$$
\begin{aligned}
&+\quad m_1\sum_{i=1}^{m_2}(\nabla F_*)(e_i,\nabla_{e_i}\mu^{kerF_*})+F_*(\nabla_{e_i}\nabla_{e_i}\mu^{kerF_*})\\
&+\quad m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i,{}^*F_*(A_{H_2}F_*(e_i)))+F_*(\nabla_{e_i}{}^*F_*(A_{H_2}F_*(e_i)))\\
&-\quad m_2\sum_{i=1}^{m_2}-A_{\nabla^{F\perp}_{F_*(e_i)}H_2}F_*(e_i)+\nabla^{F\perp}_{F_*(e_i)}\nabla^{F\perp}_{F_*(e_i)}H_2. \qquad (4.5)
\end{aligned}
$$

Thus putting (4.4) and (4.5) in(2.9) and then taking the $F_*((kerF_*)^{\perp})=rangeF_*$ and $(rangeF_*)^{\perp}$ parts we have(4.2) and(4.3).

In particular, we have the following.

Corollary 4.1. *Let F be a Riemannian map from a Riemannian manifold (M_1,g_1) to a space form $(M_2(c),g_2)$. If the mean curvature vector fields of $rangeF_*$ and $kerF_*$ are parallel, then F is biharmonic if and only if*

$$
\begin{aligned}
&m_1traceA_{(\nabla F_*)(.,\mu^{kerF_*})}F_*(.)-m_2traceF_*(\nabla_{(.)}{}^*F_*(A_{H_2}F_*(.))\\
&-m_1c(m_2-1)F_*(\mu^{kerF_*})=0
\end{aligned}
$$

and

$$
\begin{aligned}
&m_1trace\nabla^{F\perp}_{F_*(.)}(\nabla F_*)(.,\mu^{kerF_*})+m_2trace(\nabla F_*)(.,{}^*F_*(A_{H_2}F_*(.)))\\
&-m_2^2cH_2=0
\end{aligned}
$$

We also have the following result for pseudo-umbilical Riemannian maps.

Theorem 4.2. *Let F be a pseudo-umbilical biharmonic Riemannian map from a Riemannian manifold (M_1,g_1) to a space form $(M_2(c),g_2)$ such that the distribution $kerF_*$ is minimal and the mean curvature vector field H_2 is parallel. Then either F is harmonic or $c=\parallel H_2\parallel^2$.*

Proof. First note that it is easy to see that $\parallel H_2\parallel^2$ is constant. If F is biharmonic Riemannian map such that $\mu^{kerF_*}=0$ and H_2 is parallel, then from (4.3) we have

$$m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i,{}^*F_*(A_{H_2}F_*(e_i)))-m_2^2cH_2=0.$$

Since F is pseudo umbilical, we get

$$m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i,{}^*F_*(\parallel H_2\parallel^2 F_*(e_i)))-m_2^2cH_2=0.$$

On the other hand, from the linear map ${}^*F_*$ and ${}^*F_*\circ F_*=I$ (identity map), we obtain

$$m_2\sum_{i=1}^{m_2}(\nabla F_*)(e_i,\parallel H_2\parallel^2 e_i))-m_2^2cH_2=0.$$

Since the second fundamental form is also linear in its arguments, it follows that

$$m_2\parallel H_2\parallel^2\sum_{i=1}^{m_2}(\nabla F_*)(e_i,e_i))-m_2^2cH_2=0.$$

Hence we have

$$m_2^2 \parallel H_2 \parallel^2 H_2 - m_2^2 c H_2 = 0$$

which implies that

$$(\parallel H_2 \parallel^2 - c) H_2 = 0. \qquad (4.6)$$

Thus either $H_2 = 0$ or $(\parallel H_2 \parallel^2 - c) = 0$. If $H_2 = 0$, then Lemma 3.2 implies that F is harmonic, thus proof is complete.

From(4.6), we have the following result which puts some restrictions on $M_2(c)$.

Corollary 4.2. *There exist no proper biharmonic pseudo umbilical Riemannian maps from a Riemannian manifold to space forms ($M_2(c)$ with $c \leq 0$ such that the distribution $kerF_*$ is minimal and the mean curvature vector field H_2 is parallel.*

Remark 4.1. In this paper, we investigate the biharmonicity of Riemannian maps between Riemannian manifolds. Our results give some clues to investigate the biharmonicity of arbitrary maps between Riemannian manifolds. They also give a method to investigate the geometry of Riemannian maps. Since Riemannian maps are solutions of the eikonal equations which can be obtained starting from the wave equation, biharmonic maps are solutions of fourth order strongly elliptic semilinear partial differential equations and they are related to the biwave equation and bi-Yang-Mills fields, biharmonic Riemannian maps have potential for further research in terms of partial differential equations, geometric optics and mathematical physics.

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