The combinatorial essence of supercompactness[☆]

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Abstract

We introduce combinatorial principles that characterize strong compactness and supercompactness for inaccessible cardinals but also make sense for successor cardinals. Their consistency is established from what is supposedly optimal. Utilizing the failure of a weak version of square, we show that the best currently known lower bounds for the consistency strength of these principles can be applied.

Keywords: Ineffable, Slender, Strongly Compact, Supercompact, Thin
2000 MSC: 03E05, 03E35, 03E55, 03E65

1. Introduction

It is a well-known theorem that a cardinal κ is weakly compact if and only if it is inaccessible and the κ-tree property holds, that is, there are no κ-Aronszajn trees. By [2], the ω_2-tree property can be forced from a weakly compact cardinal and implies ω_2 is weakly compact in L. The tree property thus captures the combinatorial essence of weak compactness, even for successor cardinals. Similarly, the property that there is no special κ-Aronszajn tree captures the essence of Mahlo, see [3, (1.9)].

In the present work, we introduce principles $\mathsf{TP}(\kappa, \lambda)$ and $\mathsf{SP}(\kappa, \lambda)$ as well as $\mathsf{ITP}(\kappa, \lambda)$ and $\mathsf{ISP}(\kappa, \lambda)$ that achieve the same for strong compactness and supercompactness respectively. We present the ideals associated to the principles $\mathsf{ITP}(\kappa, \lambda)$ and $\mathsf{ISP}(\kappa, \lambda)$, prove the consistency of $\mathsf{ISP}(\omega_2, \lambda)$, the strongest of the

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principles, from a $\lambda^{<\kappa}$-ineffable cardinal, and show $\mathsf{ITP}(\kappa, \lambda)$ implies the failure of a weak form of square, giving lower bounds on its consistency strength.

Notation

The notation used is mostly standard. Ord denotes the class of all ordinals. For $A \subset \mathrm{Ord}$, $\mathrm{Lim}\, A$ denotes the class of limit points of A. Lim stands for Lim Ord. If a is a set of ordinals, otp a denotes the order type of a. For a regular cardinal δ, cof δ denotes the class of all ordinals of cofinality δ, and $\mathrm{cof}(< \delta)$ denotes those of cofinality less than δ.

For forcings, we write $p < q$ to mean p is stronger than q. Names either carry a dot above them or are canonical names for elements of V, so that we can confuse sets in the ground model with their names.

The phrases *for large enough* θ and *for sufficiently large* θ will be used for saying that there exists a θ' such that the sentence's proposition holds for all $\theta \geq \theta'$.

If $\kappa \subset X$, then

$$P'_\kappa X := \{x \in P_\kappa X \mid \kappa \cap x \in \mathrm{Ord},\ \langle x, \in \rangle \prec \langle X, \in \rangle\}$$

is club. For $x \in P'_\kappa X$ we set $\kappa_x := \kappa \cap x$. For $f : P_\omega X \to P_\kappa X$ let $\mathrm{Cl}_f := \{x \in P_\kappa X \mid \forall z \in P_\omega x\ f(z) \subset x\}$. Cl_f is club, and it is well known that for any club $C \subset P_\kappa X$ there is an $f : P_\omega X \to P_\kappa X$ such that $\mathrm{Cl}_f \subset C$.

If $X \subset X'$, $R \subset P_\kappa X$, $U \subset P_\kappa X'$, then the projection of U to X is $U \restriction X := \{u \cap X \mid u \in U\} \subset P_\kappa X$ and the lift of R to X' is $R^{X'} := \{x' \in P_\kappa X' \mid x' \cap X \in R\} \subset P_\kappa X'$.

For sections 2, 3, and 4, κ and λ are assumed to be cardinals, $\kappa \leq \lambda$, and κ is regular and uncountable.

2. Combinatorial principles for strong compactness and supercompactness

Let us call a sequence $\langle d_a \mid a \in P_\kappa \lambda \rangle$ a $P_\kappa \lambda$-*list* if $d_a \subset a$ for all $a \in P_\kappa \lambda$.

Definition 2.1. Let $D = \langle d_a \mid a \in P_\kappa \lambda \rangle$ be a $P_\kappa \lambda$-list.

- D is called *thin* if there is a club $C \subset P_\kappa \lambda$ such that $|\{d_a \cap c \mid c \subset a \in P_\kappa \lambda\}| < \kappa$ for every $c \in C$.

- D is called *slender* if for every sufficiently large θ there is a club $C \subset P_\kappa H_\theta$ such that $d_{M \cap \lambda} \cap b \in M$ for all $M \in C$ and all $b \in M \cap P_{\omega_1} \lambda$.[1]

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[1]Note that this definition is slightly weaker than the one from [1] as "for all $b \in M \cap P_\kappa \lambda$" was replaced by "for all $b \in M \cap P_{\omega_1} \lambda$." However, the proofs in [1] work for this weaker definition and the resulting stronger principle ISP just the same.

Proposition 2.2. *Let D be a $P_\kappa\lambda$-list. If D is thin, then it is slender.*

Proof. Let $C \subset P_\kappa\lambda$ be a club that witnesses $D = \langle d_a \mid a \in P_\kappa\lambda \rangle$ is thin. Define $g : C \to P_\kappa H_\theta$ by $g(c) := \{d_a \cap c \mid c \subset a \in P_\kappa\lambda\}$. Let $\bar{C} := \{M \in C^{H_\theta} \mid \forall b \in M \cap P_\kappa\lambda \, \exists c \in M \cap C \, b \subset c, \, \forall c \in M \cap C \, g(c) \subset M\}$. Then $\bar{C}$ is club. Let $M \in \bar{C}$ and $b \in M \cap P_{\omega_1}\lambda$. Then there is $c \in M \cap C$ such that $b \subset c$, so $d_{M \cap \lambda} \cap b = d_{M \cap \lambda} \cap c \cap b \in M$ as $d_{M \cap \lambda} \cap c \in g(c) \subset M$. Therefore $\bar{C}$ witnesses $\langle d_a \mid a \in P_\kappa\lambda \rangle$ is slender. $\qquad\square$

Definition 2.3. Let $D = \langle d_a \mid a \in P_\kappa\lambda \rangle$ be a $P_\kappa\lambda$-list and $d \subset \lambda$.

- d is called a *cofinal branch of D* if for all $a \in P_\kappa\lambda$ there is $z_a \in P_\kappa\lambda$ such that $a \subset z_a$ and $d \cap a = d_{z_a} \cap a$.

- d is called an *ineffable branch of D* if there is a stationary set $S \subset P_\kappa\lambda$ such that $d \cap a = d_a$ for all $a \in S$.

Combining these two definitions, we can define the following four combinatorial principles.

Definition 2.4.   • $\mathsf{TP}(\kappa, \lambda)$ holds if every thin $P_\kappa\lambda$-list has a cofinal branch.

- $\mathsf{SP}(\kappa, \lambda)$ holds if every slender $P_\kappa\lambda$-list has a cofinal branch.

- $\mathsf{ITP}(\kappa, \lambda)$ holds if every thin $P_\kappa\lambda$-list has an ineffable branch.

- $\mathsf{ISP}(\kappa, \lambda)$ holds if every slender $P_\kappa\lambda$-list has an ineffable branch.

Remark 2.5. The reader should note that the principle $\mathsf{TP}(\kappa, \kappa)$ is just the tree property for κ. Also, if κ is an inaccessible cardinal, then every $P_\kappa\lambda$-list is thin. Therefore $\mathsf{TP}(\kappa, \lambda)$ and $\mathsf{SP}(\kappa, \lambda)$ as well as $\mathsf{ITP}(\kappa, \lambda)$ and $\mathsf{ISP}(\kappa, \lambda)$ are equivalent if κ is inaccessible. Furthermore this means an inaccessible cardinal κ is weakly compact if and only if $\mathsf{TP}(\kappa, \kappa)$ holds, and it is ineffable if and only if $\mathsf{ITP}(\kappa, \kappa)$ holds.

Remark 2.6. The following implications hold.

1. $\mathsf{ISP}(\kappa, \lambda)$ implies $\mathsf{SP}(\kappa, \lambda)$,
2. $\mathsf{ISP}(\kappa, \lambda)$ implies $\mathsf{ITP}(\kappa, \lambda)$,
3. $\mathsf{ITP}(\kappa, \lambda)$ implies $\mathsf{TP}(\kappa, \lambda)$,
4. $\mathsf{SP}(\kappa, \lambda)$ implies $\mathsf{TP}(\kappa, \lambda)$.

We will see that 1 and 3 can not be reversed. For if κ is a strongly compact cardinal that is not supercompact, then by Theorem 2.9 $\mathsf{SP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$, but by Theorem 2.10 we have that $\mathsf{ITP}(\kappa, \lambda)$ cannot hold for all $\lambda \geq \kappa$. This is also true for smaller κ. One can show that the Mitchell collapse preserves $\mathsf{SP}(\kappa, \lambda)$. However, by Theorem 5.7, if the Mitchell collapse produces a model in which $\mathsf{ITP}(\kappa, \lambda)$ holds, then also in the ground model $\mathsf{ITP}(\kappa, \lambda)$ holds, so that again collapsing a strongly compact cardinal that is not supercompact yields a model in which $\mathsf{SP}(\kappa, \lambda)$ holds but $\mathsf{ITP}(\kappa, \lambda)$ fails. Furthermore implication 2 can not be reversed. This follows from the fact that the forcing axiom $\mathsf{PFA}(\Gamma_\Sigma)$ from [4] can be seen to imply $\mathsf{ITP}(\omega_2, \lambda)$ for all $\lambda \geq \omega_2$. The paper also shows that $\mathsf{PFA}(\Gamma_\Sigma)$ is consistent with the approachability property holding for ω_1. It is easily seen that this contradicts $\mathsf{ISP}(\omega_2, \omega_2)$, so that in any model of $\mathsf{PFA}(\Gamma_\Sigma)$ + "the approachability property holds for ω_1" $\mathsf{ITP}(\omega_2, \lambda)$ holds for all $\lambda \geq \omega_2$ but $\mathsf{ISP}(\omega_2, \omega_2)$ fails.

Jech [5] was the first to consider generalizations of the concept of a tree to $P_\kappa \lambda$-lists. He gave the following characterization of strong compactness.

Theorem 2.7. *The following are equivalent.*

1. *κ is strongly compact.*
2. *For every $\lambda \geq \kappa$, every $P_\kappa \lambda$-list has a branch.*

Shortly after, Magidor [6] extended Jech's result to supercompactness with the next theorem.

Theorem 2.8. *The following are equivalent.*

1. *κ is supercompact.*
2. *For every $\lambda \geq \kappa$, every $P_\kappa \lambda$-list has an ineffable branch.*

By Remark 2.5 we can rephrase Theorems 2.7 and 2.8 in the following way.

Theorem 2.9. *Suppose κ is inaccessible. Then κ is strongly compact if and only if $\mathsf{TP}(\kappa, \lambda)$ holds for every $\lambda \geq \kappa$.*

Theorem 2.10. *Suppose κ is inaccessible. Then κ is supercompact if and only if $\mathsf{ITP}(\kappa, \lambda)$ holds for every $\lambda \geq \kappa$.*

The advantage of these new formulations is that $\mathsf{TP}(\kappa, \lambda)$ and $\mathsf{ITP}(\kappa, \lambda)$ are not limited to inaccessible cardinals, as we will see in section 5.

3. The corresponding ideals

The principles $\mathsf{ITP}(\kappa, \lambda)$ and $\mathsf{ISP}(\kappa, \lambda)$ have ideals canonically associated to them.

Definition 3.1. Let $A \subset P_\kappa \lambda$ and let $D = \langle d_a \mid a \in P_\kappa \lambda \rangle$ be a $P_\kappa \lambda$-list. D is called *A-effable* if for every $S \subset A$ that is stationary in $P_\kappa \lambda$ there are $a, b \in S$ such that $a \subset b$ and $d_a \neq d_b \cap a$. D is called *effable* if it is $P_\kappa \lambda$-effable.

Definition 3.2. We let

$$I_{\mathrm{IT}}[\kappa, \lambda] := \{A \subset P_\kappa \lambda \mid \text{there exists a thin } A\text{-effable } P_\kappa \lambda\text{-list}\},$$

$$I_{\mathrm{IS}}[\kappa, \lambda] := \{A \subset P_\kappa \lambda \mid \text{there exists a slender } A\text{-effable } P_\kappa \lambda\text{-list}\}.$$

By $F_{\mathrm{IT}}[\kappa, \lambda]$ and $F_{\mathrm{IS}}[\kappa, \lambda]$ we denote the filters associated to $I_{\mathrm{IT}}[\kappa, \lambda]$ and $I_{\mathrm{IS}}[\kappa, \lambda]$ respectively.

Note that $\mathsf{ITP}(\kappa, \lambda)$ and $\mathsf{ISP}(\kappa, \lambda)$ now say that $I_{\mathrm{IT}}[\kappa, \lambda]$ and $I_{\mathrm{IS}}[\kappa, \lambda]$ are proper ideals respectively. By Proposition 2.2 we have $I_{\mathrm{IT}}[\kappa, \lambda] \subset I_{\mathrm{IS}}[\kappa, \lambda]$.

Proposition 3.3. $I_{\mathrm{IT}}[\kappa, \lambda]$ *and* $I_{\mathrm{IS}}[\kappa, \lambda]$ *are normal ideals on* $P_\kappa \lambda$.

Proof. Suppose $D \subset P_\kappa \lambda$ and $g : D \to \lambda$ is regressive. Set $A_\gamma := g^{-1}{}''\{\gamma\}$. Let $f : \lambda \times \lambda \to \lambda$ be bijective, and define $f_{\alpha_1} : \lambda \to \lambda$ by $f_{\alpha_1}(\alpha_0) := f(\alpha_0, \alpha_1)$. We show that if $A_\gamma \in I_{\mathrm{IT}}[\kappa, \lambda]$ for all $\gamma < \lambda$, then $D \in I_{\mathrm{IT}}[\kappa, \lambda]$, and that if $A_\gamma \in I_{\mathrm{IS}}[\kappa, \lambda]$ for all $\gamma < \lambda$, then $D \in I_{\mathrm{IS}}[\kappa, \lambda]$.

In the thin case, that is, if $A_\gamma \in I_{\mathrm{IT}}[\kappa, \lambda]$ for all $\gamma < \lambda$, let $\langle d_a^\gamma \mid a \in P_\kappa \lambda \rangle$ be a thin A_γ-effable $P_\kappa \lambda$-list for $\gamma < \lambda$. Let $C^\gamma \subset P_\kappa \lambda$ be a club witnessing $\langle d_a^\gamma \mid a \in P_\kappa \lambda \rangle$ is thin. Set $C := \Delta_{\gamma < \lambda} C^\gamma$. We may assume that for all $a \in C$ and all $\alpha_0, \alpha_1 < \lambda$

$$f(\alpha_0, \alpha_1) \in a \leftrightarrow \alpha_0, \alpha_1 \in a. \tag{1}$$

For $a \in C \cap D$ set

$$d_a := f''_{g(a)} d_a^{g(a)},$$

and set $d_a := \emptyset$ for $a \in P_\kappa \lambda - (C \cap D)$. If $c \in C$ and $a \in C \cap D$ are such that $c \subset a$ and $g(a) \notin c$, then

$$d_a \cap c = \emptyset. \tag{2}$$

For if $g(a) \notin c$, then by (1) we have $d_a \cap c = f''_{g(a)} d_a^{g(a)} \cap c \subset \operatorname{rng} f_{g(a)} \cap c = \emptyset$. Thus for fixed $c \in C$ we have $\{d_a \cap c \mid c \subset a \in C \cap D\} \subset \{f''_\gamma d_a^\gamma \cap c \mid \gamma \in c, \ c \subset a \in C \cap A_\gamma\} \cup \{\emptyset\}$.

5

For $\gamma \in c$ we have $c \in C^\gamma$ and thus, as C^γ witnesses $\langle d_a^\gamma \mid a \in P_\kappa\lambda \rangle$ is thin, $|\{d_a^\gamma \cap c \mid c \subset a \in C \cap A_\gamma\}| < \kappa$. Therefore $|\{d_a \cap c \mid c \subset a \in P_\kappa\lambda\}| < \kappa$, which shows $\langle d_a \mid a \in P_\kappa\lambda \rangle$ is thin.

If $A_\gamma \in I_{\mathrm{IS}}[\kappa, \lambda]$ for all $\gamma < \lambda$, let $\langle d_a^\gamma \mid a \in P_\kappa\lambda \rangle$ be a slender A_γ-effable $P_\kappa\lambda$-list for $\gamma < \lambda$. Let $C^\gamma \subset P_\kappa' H_\theta$ be a club witnessing $\langle d_a^\gamma \mid a \in P_\kappa\lambda \rangle$ is slender, where θ is some large enough cardinal. Set $C := \Delta_{\gamma<\lambda} C^\gamma$. We can again assume that for all $M \in C$ and $\alpha_0, \alpha_1 < \lambda$ $f(\alpha_0, \alpha_1) \in M \leftrightarrow \alpha_0, \alpha_1 \in M$. In addition, we may require that

$$\langle M, \in, f \restriction (M \times M) \rangle \prec \langle H_\theta, \in, f \rangle \tag{3}$$

for every $M \in C$. As above we define $d_a := f''_{g(a)} d_a^{g(a)}$ for $a \in (C \restriction \lambda) \cap D$ and let $d_a := \emptyset$ otherwise. By the same argument that led to (2), we have

$$d_a \cap b = \emptyset \tag{4}$$

if $b \in P_\kappa\lambda$, $a \in (C \restriction \lambda) \cap D$, $b \subset a$, and $g(a) \notin b$. To show $\langle d_a \mid a \in P_\kappa\lambda \rangle$ is slender, let $M \in C$ and $b \in M \cap P_{\omega_1}\lambda$. Set $a := M \cap \lambda$. If $M \notin D$, then $d_a \cap b \subset d_a = \emptyset \in M$, so assume $M \in D$. Then $d_a \cap b = f''_{g(a)} d_a^{g(a)} \cap b = f''_{g(a)} (d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b)$. If $g(a) \notin b$, then by (4) $d_a \cap b = \emptyset \in M$, so suppose $g(a) \in b$. Then $f_{g(a)}^{-1}{}'' b = b$, so by the slenderness of $\langle d_{\tilde{a}}^{g(a)} \mid \tilde{a} \in P_\kappa\lambda \rangle$ we have $d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b \in M$. Thus, as $g(a) \in b \subset M$, by (3) $d_a \cap b = f''_{g(a)} (d_a^{g(a)} \cap f_{g(a)}^{-1}{}'' b) \in M$.

In both cases we arrived at a $P_\kappa\lambda$-list $\langle d_a \mid a \in P_\kappa\lambda \rangle$ such that for a club $C \subset P_\kappa\lambda$ that is closed under f and f^{-1} we have

$$d_a = f''_{g(a)} d_a^{g(a)}$$

for every $a \in C \cap D$, and $d_a = \emptyset$ for $a \in P_\kappa\lambda - (C \cap D)$. Suppose that $D \notin I_{\mathrm{IT}}[\kappa, \lambda]$ for the thin case or $D \notin I_{\mathrm{IS}}[\kappa, \lambda]$ for the slender case. Then there are $S \subset C \cap D$ stationary in $P_\kappa\lambda$ and $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in S$. Since g is regressive we may assume $S \subset A_\gamma$ for some $\gamma < \lambda$. But then for $\tilde{d} := f_\gamma^{-1}{}'' d$ and $a \in S$ it holds that

$$d_a^\gamma = f_\gamma^{-1}{}'' f''_\gamma d_a^\gamma = f_\gamma^{-1}{}'' d_a = f_\gamma^{-1}{}'' (d \cap a) = f_\gamma^{-1}{}'' d \cap f_\gamma^{-1}{}'' a = \tilde{d} \cap a,$$

contradicting $\langle d_a^\gamma \mid a \in P_\kappa\lambda \rangle$ being effable. $\qquad \square$

It is standard to verify that if $\lambda < \lambda'$, then $I_{\mathrm{IT}}[\kappa, \lambda] \subset \{A' \restriction \lambda \mid A' \in I_{\mathrm{IT}}[\kappa, \lambda']\}$ and $I_{\mathrm{IS}}[\kappa, \lambda] \subset \{A' \restriction \lambda \mid A' \in I_{\mathrm{IS}}[\kappa, \lambda']\}$. This implies the following proposition.

Proposition 3.4. *Suppose $\lambda \leq \lambda'$. Then* $\mathsf{ITP}(\kappa, \lambda')$ *implies* $\mathsf{ITP}(\kappa, \lambda)$, *and* $\mathsf{ISP}(\kappa, \lambda')$ *implies* $\mathsf{ISP}(\kappa, \lambda)$.

It is easy to check $\operatorname{cof} \omega \cap \kappa \in I_{\mathrm{IT}}[\kappa, \kappa]$. The following theorem is the two cardinal analog of this observation.

Theorem 3.5. *Suppose* $\operatorname{cf} \lambda \geq \kappa$. *Then*

$$\{a \in P_\kappa \lambda \mid \operatorname{Lim} a \cap \operatorname{cof} \omega \subset a\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

Proof. Let $A := \{a \in P_\kappa \lambda \mid \exists \eta_a \in \operatorname{Lim} a - a \;\; \operatorname{cf} \eta_a = \omega\}$ and for $a \in A$ let η_a be a witness. For $\delta \in \operatorname{cof} \omega \cap \lambda$ let $\langle d_\nu^\delta \mid \nu < \tau_\delta \rangle$ be an enumeration of $\{d \subset \delta \mid \operatorname{otp} d = \omega, \; \sup d = \delta\}$. For $a \in P_\kappa \lambda$ and $\delta \in \operatorname{Lim} a \cap \operatorname{cof} \omega$ let

$$\nu_a^\delta := \min\{\nu < \tau_\delta \mid \sup(d_\nu^\delta \cap a) = \delta\}.$$

For $a \in A$ set

$$d_a := d_{\nu_a^{\eta_a}}^{\eta_a} \cap a,$$

and for $a \in P_\kappa \lambda - A$ let $d_a := \emptyset$.

Then $\langle d_a \mid a \in P_\kappa \lambda \rangle$ is A-effable, for suppose there were a cofinal $U \subset A$ and a $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in U$. Let $a \in U$. Since $\operatorname{cf} \lambda \geq \kappa$ there exists $b \in U$ such that $a \cup \operatorname{Lim} a \subset b$. But then $\operatorname{otp}(d_b \cap a) < \omega$, contradicting $d_b \cap a = d_a$.

$\langle d_a \mid a \in P_\kappa \lambda \rangle$ is also thin, for let $a \in P_\kappa \lambda$. Let

$$B_a := \{d_{\nu_a^\delta}^\delta \cap a \mid \delta \in \operatorname{Lim} a \cap \operatorname{cof} \omega\} \cup P_\omega a.$$

Then $|B_a| < \kappa$. Let $b \in A$ with $a \subset b$, and suppose $d_b \cap a \notin P_\omega a$. Since $a \subset b$, we have $\nu_b^\delta \leq \nu_a^\delta$ for all $\delta \in \operatorname{Lim} a \cap \operatorname{cof} \omega$. Because $|d_b \cap a| = \omega$ we also have that $d_{\nu_b^{\eta_b}}^{\eta_b} \cap a = d_b \cap a$ is unbounded in η_b. Therefore $\nu_a^{\eta_b} \leq \nu_b^{\eta_b}$, so that $\nu_a^{\eta_b} = \nu_b^{\eta_b}$. But this means $d_b \cap a = d_{\nu_a^{\eta_b}}^{\eta_b} \cap a \in B_a$. $\qquad\square$

When κ is inaccessible, the filter $F_{\mathrm{IT}}[\kappa, \lambda]$ has some additional simple but helpful properties. These will be used in section 5.

Proposition 3.6. *Let κ be inaccessible. Then*

$$\{a \in P_\kappa' \lambda \mid \kappa_a \text{ inaccessible}\} \in F_{\mathrm{IT}}[\kappa, \lambda].$$

Proof. As κ is inaccessible, $\{a \in P'_\kappa \lambda \mid \kappa_a \text{ strong limit}\}$ is club. So it remains to show $A := \{a \in P'_\kappa \lambda \mid \kappa_a \text{ singular}\} \in I_{\text{IT}}[\kappa, \lambda]$. Suppose $A \notin I_{\text{IT}}[\kappa, \lambda]$, and for $a \in A$ let $d_a \subset a$ be such that $\sup d_a = \kappa_a$, $\text{otp}\, d_a = \text{cf}\, \kappa_a$. Then there exists a stationary $S \subset A$ such that $d_a = d \cap a$ for all $a \in S$. We may assume $\kappa_a = \delta$ for some $\delta < \kappa$ and all $a \in S$. But if $a, b \in S$ are such that $a \subset b$ and $\kappa_a < \kappa_b$, then $\text{otp}\, d_b > \delta$, a contradiction. $\qquad\square$

Proposition 3.7. *Let κ be inaccessible. Let $g : P_\kappa \lambda \to P_\kappa \lambda$. Then*

$$\{a \in P'_\kappa \lambda \mid \forall z \in P_{\kappa_a} a\ g(z) \subset a\} \in F_{\text{IT}}[\kappa, \lambda].$$

Proof. Suppose not. Then

$$B := \{a \in P'_\kappa \lambda \mid \exists z_a \in P_{\kappa_a} a\ g(z_a) \not\subset a\} \notin I_{\text{IT}}[\kappa, \lambda].$$

So let $S \subset B$ be stationary and $z \subset \lambda$ be such that $z_a = z \cap a$ for all $a \in S$. For all $a \in S$ we have $\mu_a := |z_a| < \kappa_a$, so there are a stationary $S' \subset S$ and $\mu < \kappa$ such that $\mu_a = \mu$ for all $a \in S'$.

Suppose $|z| > \mu$. Then there is $y \subset z$ such that $|y| = \mu^+ < \kappa$. But $S'' := \{a \in S' \mid y \subset a\}$ is stationary and for every $a \in S''$ we have $z_a = z \cap a \supset y \cap a = y$, which implies $\mu = \mu_a = |z_a| \geq |y| = \mu^+$, a contradiction.

Since S' is cofinal, there is an $a \in S'$ such that $z \cup g(z) \subset a$. But then $z_a = z \cap a = z$ and $g(z_a) = g(z) \subset a$, so that $a \notin B$, contradicting $S' \subset B$. $\qquad\square$

4. The failure of a weak version of square

We define a weak variant of the square principle that is natural for our application. It is a "threaded" version of Schimmerling's two cardinal square principle that is only defined on a subset E of λ.

Definition 4.1. A sequence $\langle C_\alpha \mid \alpha \in \text{Lim} \cap E \cap \lambda \rangle$ is called a $\square_E(\kappa, \lambda)$-*sequence* if it satisfies the following properties.

  (i) $0 < |C_\alpha| < \kappa$ for all $\alpha \in \text{Lim} \cap E \cap \lambda$,
 (ii) $C \subset \alpha$ is club for all $\alpha \in \text{Lim} \cap E \cap \lambda$ and $C \in C_\alpha$,
(iii) $C \cap \beta \in C_\beta$ for all $\alpha \in \text{Lim} \cap E \cap \lambda$, $C \in C_\alpha$ and $\beta \in \text{Lim}\, C$,
 (iv) there is no club $D \subset \lambda$ such that $D \cap \delta \in C_\delta$ for all $\delta \in \text{Lim}\, D \cap E \cap \lambda$.

We say that $\square_E(\kappa, \lambda)$ holds if there exists a $\square_E(\kappa, \lambda)$-sequence. $\square(\kappa, \lambda)$ stands for $\square_\lambda(\kappa, \lambda)$.

Note that $\square_{\tau, <\kappa}$ implies $\square(\kappa, \tau^+)$ and that $\square(\lambda)$ is $\square(2, \lambda)$.

Theorem 4.2. *Suppose* cf $\lambda \geq \kappa$ *and* $\square_{\mathrm{cof}(<\kappa)}(\kappa, \lambda)$ *holds. Then* $\neg\mathsf{ITP}(\kappa, \lambda)$.

Proof. Let $A := \{a \in P_\kappa\lambda \mid \mathrm{Lim}\, a \cap \mathrm{cof}\, \omega \subset a\}$. By Theorem 3.5, $A \in F_{\mathrm{IT}}[\kappa, \lambda]$. So it remains to show $A \in I_{\mathrm{IT}}[\kappa, \lambda]$. We may assume $\sup a \notin a$ for all $a \in A$. Let $\langle C_\gamma \mid \gamma \in \mathrm{Lim} \cap \mathrm{cof}(<\kappa) \cap \lambda \rangle$ be a $\square_{\mathrm{cof}(<\kappa)}(\kappa, \lambda)$-sequence. For $\gamma \in \mathrm{Lim} \cap \mathrm{cof}(<\kappa) \cap \lambda$ let $C_\gamma \in \mathcal{C}_\gamma$, and set $d_a := C_{\sup a} \cap a$ for $a \in A$, otherwise $d_a := \emptyset$. Then, since $\mathrm{Lim}\, a \cap \mathrm{cof}\, \omega \subset a$,

$$\sup d_a = \sup a \tag{5}$$

for every $a \in A$.

$\langle d_a \mid a \in P_\kappa\lambda \rangle$ is thin, for let $a \in P_\kappa\lambda$. Set

$$B_a := \{(C \cap a) \cup h \mid \exists \eta \in \mathrm{Lim}\, a \; C \in \mathcal{C}_\eta, \; h \in P_\omega a\} \cup P_\omega a.$$

Then $|B_a| < \kappa$. Let $b \in A$, $a \subset b$, and suppose $d_b \cap a \notin P_\omega a$. Let $\eta := \max \mathrm{Lim}(d_b \cap a)$. Then $\eta \in \mathrm{Lim}\, C_{\sup b}$, so there is a $C \in \mathcal{C}_\eta$ such that $d_b \cap \eta = C_{\sup b} \cap b \cap \eta = C \cap b$, so $d_b \cap a \cap \eta = C \cap a$. Since $|d_b \cap a - \eta| < \omega$, this means $d_b \cap a = (C \cap a) \cup (d_b \cap a - \eta) \in B_a$.

$\langle d_a \mid a \in P_\kappa\lambda \rangle$ is also A-effable. For suppose there were a cofinal $U \subset A$ and $d \subset \lambda$ such that $d_a = d \cap a$ for all $a \in U$. Then d is unbounded in λ by (5). Let $\delta \in \mathrm{Lim}\, d \cap \mathrm{cof}(<\kappa) \cap \lambda$. We will show $d \cap \delta \in \mathcal{C}_\delta$, which contradicts the fact that $\langle \mathcal{C}_\alpha \mid \alpha \in \mathrm{Lim} \cap \mathrm{cof}(<\kappa) \cap \lambda \rangle$ is a $\square_{\mathrm{cof}(<\kappa)}(\kappa, \lambda)$-sequence, thus finishing the proof. For every $a \in U$ such that $\delta \in \mathrm{Lim}(d \cap a)$ we have $C_{\sup a} \cap a = d_a = d \cap a$, and thus $\delta \in \mathrm{Lim}\, C_{\sup a}$, so that there is a $C_a \in \mathcal{C}_\delta$ such that $d \cap a \cap \delta = C_a \cap a$. But since $|\mathcal{C}_\delta| < \kappa$, there is a cofinal $U' \subset \{a \in U \mid \delta \in \mathrm{Lim}(d \cap a)\}$ such that $C_a = C$ for some $C \in \mathcal{C}_\delta$ and all $a \in U'$. But then we have $d \cap \delta \cap a = C \cap a$ for all $a \in U'$, which means $d \cap \delta = C \in \mathcal{C}_\delta$. $\square$

As a corollary, we get a well-known result originally due to Solovay [7].

Corollary 4.3. *Suppose* κ *is supercompact. Then* $\neg\square_{\mathrm{cof}(<\kappa)}(\kappa, \lambda)$ *for all* $\kappa \leq \lambda$ *with* cf $\lambda \geq \kappa$. *In particular* $\neg\square(\lambda)$ *for all* $\lambda \geq \kappa$ *with* cf $\lambda \geq \kappa$.

Proof. This follows directly from Theorem 2.10 and Theorem 4.2. $\square$

5. Consistency results

Definition 5.1. Let $V \subseteq W$ be a pair of transitive models of ZFC.

- (V, W) satisfies the μ-covering property if the class $P_\mu^V V$ is cofinal in $P_\mu^W V$, that is, for every $x \in W$ with $x \subset V$ and $|x| < \mu$ there is $z \in P_\mu^V V$ such that $x \subset z$.

- (V, W) satisfies the μ-approximation property if for all $x \in W$, $x \subset V$, it holds that if $x \cap z \in V$ for all $z \in P_\mu^V V$, then $x \in V$.

A forcing $\mathbb{P}$ is said to satisfy the μ-covering property or the μ-approximation property if for every V-generic $G \subset \mathbb{P}$ the pair $(V, V[G])$ satisfies the μ-covering property or the μ-approximation property respectively.

The following theorem was originally discovered by Mitchell [2]. We cite [8], where it is presented in the more modern way we use. The reader should note we use the convention that conditions are only defined on their support.

Theorem 5.2. *Let κ be inaccessible, $\tau < \kappa$ be regular and uncountable. Then there exists an iteration $\langle \mathbb{P}_\nu \mid \nu \leq \kappa \rangle$ such that forcing with $\mathbb{P}_\kappa$ preserves all cardinals less than or equal to τ, $\Vdash_\kappa \kappa = \tau^+$ and for $\eta = 0$ and every inaccessible $\eta \leq \kappa$*

(i) *$\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and η-cc,*
(ii) *if $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_\eta \dot{\mathbb{Q}}$ satisfies the ω_1-approximation property,*
(iii) *for every $\nu < \eta$, $\mathbb{P}_\nu$ is definable in H_η from the parameters τ and ν,*
(iv) *$\mathbb{P}_\eta$ satisfies the ω_1-covering property.*

The next is a standard lemma which we will need.

Lemma 5.3. *Let $\kappa > \omega$ be regular, $\mathbb{P}_\kappa$ be the direct limit of an iteration $\langle \mathbb{P}_\nu \mid \nu < \kappa \rangle$. Suppose $\mathbb{P}_\kappa$ is κ-cc. Let $p \in \mathbb{P}_\kappa$ and $\dot{x} \in V^{\mathbb{P}_\kappa}$ such that $p \Vdash \dot{x} \in P_\kappa V$. Then there is $\rho < \kappa$ such that $p \Vdash \dot{x} \in V[\dot{G}_\rho]$.*

Recall from [5] that κ is called λ-*ineffable* if every $P_\kappa \lambda$-list has an ineffable branch.

Theorem 5.4. *Let κ, λ be cardinals, τ regular uncountable, $\tau < \kappa \leq \lambda$, and $\langle \mathbb{P}_\nu \mid \nu \leq \kappa \rangle$ be an iteration such that for all inaccessible $\eta \leq \kappa$*

(i) *$\mathbb{P}_\eta$ is the direct limit of $\langle \mathbb{P}_\nu \mid \nu < \eta \rangle$ and η-cc,*

(ii) *if $\mathbb{P}_\kappa = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_\eta \dot{\mathbb{Q}}$ satisfies the ω_1-approximation property,*

(iii) *for every $\nu < \eta$, $\mathbb{P}_\nu$ is definable in H_η from the parameters τ and ν,*

(iv) *$\mathbb{P}_\eta$ satisfies the ω_1-covering property.*

Suppose κ is $\lambda^{<\kappa}$-ineffable. Then $\Vdash_\kappa \mathsf{ISP}(\kappa, \lambda)$.

Proof. Let $G \subset \mathbb{P}_\kappa$ be V-generic and work in $V[G]$. Let $\langle d_a \mid a \in P_\kappa \lambda \rangle$ be a slender $P_\kappa \lambda$-list, and let $C' \subset P_\kappa H_\theta$ be a club witnessing the slenderness of $\langle d_a \mid a \in P_\kappa \lambda \rangle$ for some large enough θ.

For $x \in P_\kappa \lambda$ by Lemma 5.3 there is $\rho_x < \kappa$ such that $x \in V[G_{\rho_x}]$. Thus $C := \{M \in C' \mid \forall x \in P_\kappa \lambda \cap M \; \rho_x \in M\}$ is such that $P_\kappa \lambda \cap M \subset V[G_{\kappa_M}]$ for all $M \in C$.

Let $\sigma := (\lambda^{<\kappa})^V$. Let $\bar{M} \in V$ be such that $\bar{M} \prec H_\theta^V$, $\lambda \cup P_\kappa^V \lambda \subset \bar{M}$, $|\bar{M}|^V = \sigma$. Let $C_0 := C \upharpoonright \bar{M}$. Since $\mathbb{P}_\kappa$ is κ-cc, there is a $C_1 \in V$ such that $C_1 \subset C_0$ and $V \models C_1 \subset P_\kappa \bar{M}$ club.

Let

$$E := \{M \in C_1 \mid \tau < \kappa_M, \; \kappa_M \text{ inaccessible in } V, \; P_\tau^V(M \cap \lambda) \subset M\}.$$

Claim 5.4.1. *If $M \in E$, then $d_{M \cap \lambda} \in V[G_{\kappa_M}]$.*

Proof. Let $z \in P_{\omega_1}^{V[G_{\kappa_M}]}(M \cap \lambda)$. $\mathbb{P}_{\kappa_M}$ satisfies the ω_1-covering property by (iv), so there is $b \in P_{\omega_1}^V(M \cap \lambda)$ such that $z \subset b$. Let $M' \in C$ be such that $M = M' \cap \bar{M}$. Then $b \in M \subset M'$. Therefore, by the slenderness of $\langle d_a \mid a \in P_\kappa \lambda \rangle$, $d_{M \cap \lambda} \cap b = d_{M' \cap \lambda} \cap b \in P_\kappa \lambda \cap M' \subset V[G_{\kappa_{M'}}] = V[G_{\kappa_M}]$ and thus $d_{M \cap \lambda} \cap z = d_{M \cap \lambda} \cap b \cap z \in V[G_{\kappa_M}]$.

Let $\mathbb{P}_\kappa = \mathbb{P}_{\kappa_M} * \dot{\mathbb{Q}}$. Then $\dot{\mathbb{Q}}^{G_{\kappa_M}}$ satisfies the ω_1-approximation property by (ii), so since z was arbitrary we get $d_{M \cap \lambda} \in V[G_{\kappa_M}]$. $\dashv$

For $M \in E$ we have $\mathbb{P}_{\kappa_M} \subset M$ by (i) and (iii). By Claim 5.4.1 there is $\dot{d}_M \in V^{\mathbb{P}_{\kappa_M}}$ such that $\dot{d}_M^{G_{\kappa_M}} = d_{M \cap \lambda}$. Let

$$D_M := \{\langle p, \alpha, n \rangle \mid p \in \mathbb{P}_{\kappa_M}, \; \alpha \in M \cap \lambda, \; (n = 0 \wedge p \Vdash_{\kappa_M} \alpha \notin \dot{d}_M) \vee (n = 1 \wedge p \Vdash_{\kappa_M} \alpha \in \dot{d}_M)\}.$$

Then $\langle D_M \mid M \in E \rangle \in V$ and $D_M \subset M$.

Work in V. Let $f : \bar{M} \to \sigma$ be a bijection. If $\lambda > \kappa$, additionally choose f such that $f \upharpoonright \kappa = \mathrm{id} \upharpoonright \kappa$. If $\kappa = \lambda$, then $\{M \in C_1 \mid f''M = \kappa_M\}$ is club, and we may assume it is C_1. By Propositions 3.6 and 3.7

$$F := \{m \in P_\kappa' \sigma \mid \kappa_m \text{ inaccessible}, \; P_\tau(m \cap f''\lambda) \subset m\} \in F_{\mathrm{IT}}[\kappa, \sigma].$$

11

As κ is σ-ineffable, there exist a stationary $S' \subset F$ and $d' \subset \sigma$ such that $f''D_{f^{-1}''m} = d' \cap m$ for all $m \in S'$ such that $f^{-1}''m \in E$. But $E = \{f^{-1}''m \mid m \in F\} \cap C_1$ by our choice of f or the additional assumption on C_1, so for $S := \{f^{-1}''m \mid m \in S' \cap F\} \cap C_1$ and for $D := f^{-1}''d'$ we have $D_M = D \cap M$ for all $M \in S$.

Back in $V[G]$, let $T := S \upharpoonright \lambda$ and

$$d := \{\alpha < \lambda \mid \exists p \in G \, \langle p, \alpha, 1 \rangle \in D\}.$$

Claim 5.4.2. *If $a \in T$, then $d_a = d \cap a$.*

Proof. If $a \in T$, then $a = M \cap \lambda$ for some $M \in S$. But then for $\alpha \in a$, if $\alpha \in d_a = d_{M \cap \lambda} = \dot{d}_M^{G_{\kappa_M}}$, then there is $p \in G_{\kappa_M}$ such that $p \Vdash_{\kappa_M} \alpha \in \dot{d}_M$. Thus $\langle p, \alpha, 1 \rangle \in D_M = D \cap M$, so that $\alpha \in d$ by the definition of d.

By the same argument, if $\alpha \notin d_a$, then $\alpha \notin d$. $\dashv$

T is stationary in V, so it is also stationary in $V[G]$ since $\mathbb{P}_\kappa$ is κ-cc. Therefore, by Claim 5.4.2, $\langle d_a \mid a \in P_\kappa \lambda \rangle$ is not effable. $\square$

Note that if κ is λ-ineffable and cf $\lambda \geq \kappa$, then by [9] it follows that $\lambda^{<\kappa} = \lambda$. So in this case, Theorem 5.4 shows $\mathsf{ISP}(\kappa, \lambda)$ is forced from the more natural condition that κ is λ-ineffable.

Corollary 5.5. *If the theory ZFC + "there is an ineffable cardinal" is consistent, then the theory ZFC + $\mathsf{ISP}(\omega_2, \omega_2)$ is consistent.*

Proof. Taking $\tau = \omega_1$, this follows immediately from Theorem 5.2, Theorem 5.4, and Remark 2.5. $\square$

Corollary 5.6. *If the theory ZFC + "there exists a supercompact cardinal" is consistent, then the theory ZFC + "$\mathsf{ISP}(\omega_2, \lambda)$ holds for every $\lambda \geq \omega_2$" is consistent.*

Proof. This follows from Theorems 5.2, 5.4, and 2.10. $\square$

In Corollaries 5.5 and 5.6, ω_2 only serves as the minimal cardinal for which the theorems hold true. One can of course take successors of larger regular cardinals instead.

It is worth noting that, when using the Mitchell forcing from Theorem 5.2, Corollary 5.6 and, when cf $\lambda \geq \kappa$, Theorem 5.4 were best possible, as shows the next theorem. Its proof can be found in [1, Theorem 2.3.5] or [10], where similar "pull back" theorems are used in a more general setting.

Theorem 5.7. *Let $V \subset W$ be a pair of models of* ZFC *that satisfies the κ-covering property and the τ-approximation property for some $\tau < \kappa$, and suppose κ is inaccessible in V. Then*

$$P_\kappa^W \lambda - P_\kappa^V \lambda \in I_{\mathrm{IT}}^W[\kappa, \lambda],$$

which furthermore implies

$$F_{\mathrm{IT}}^V[\kappa, \lambda] \subset F_{\mathrm{IT}}^W[\kappa, \lambda].$$

So in particular, if $W \models$ ITP(κ, λ), then $V \models$ ITP(κ, λ).

We proceed to give lower bounds on the consistency strength of our combinatorial principles. We first consider the one cardinal variant, showing Corollary 5.5 was best possible.

The next lemma is usually only given in its weaker version where κ is required to be weakly compact.

Lemma 5.8. *Suppose κ is regular uncountable and the tree property holds for κ. Let $A \subset \kappa$. If $A \cap \alpha \in L$ for all $\alpha < \kappa$, then $A \in L$.*

Proof. Let $\delta := \kappa + \omega$. By [2, Proposition 5.3], κ is inaccessible in $L[A]$. By the usual argument, one proves there exists a nonprincipal κ-complete ultrafilter U on $P^{L[A]}\kappa \cap L_\delta[A]$, see [2, Proof of Theorem 5.9]. Let M be the transitive collapse of the internal ultrapower of $L_\delta[A]$ by U, and let $j : L_\delta[A] \to M$ be the corresponding embedding. Then j has critical point κ. As $L_\delta[A] \models V = L[A]$, we have $M \models V = L[j(A)]$, so $M = L_\gamma[j(A)]$ for some limit ordinal $\gamma \geq \delta$. It holds that $L_\delta[A] \models \forall \alpha < \kappa \ A \cap \alpha \in L$, so $L_\gamma[j(A)] \models \forall \alpha < j(\kappa) \ j(A) \cap \alpha \in L$, so in particular $L_\gamma[j(A)] \models A = j(A) \cap \kappa \in L$. Therefore really $A \in L$. $\square$

Theorem 5.9. *Suppose κ is regular and uncountable. If* ITP(κ, κ) *holds, then $L \models \kappa$ is ineffable.*

Proof. Again by [2, Proposition 5.3], κ is inaccessible in L.

Let $\langle d_\alpha \mid \alpha < \kappa \rangle \in L$. Then $\{d_\alpha \cap \beta \mid \alpha \leq \kappa\} \subset P^L\beta$. So $\langle d_a \mid a \in P_\kappa\lambda \rangle$, where $d_a = \emptyset$ if $a \notin \kappa$, is thin as $|P^L\beta| < \kappa$. Thus by ITP(κ, κ) there is a $d \subset \kappa$ such that $d_\alpha = d \cap \alpha$ for stationarily many $\alpha < \kappa$. This also means $d \cap \gamma \in L$ for all $\gamma < \kappa$. Therefore $d \in L$ by Lemma 5.8. Since $\{\alpha < \kappa \mid d_\alpha = d \cap \alpha\} \in L$ is also stationary in L, the proof is finished. $\square$

13

The best known lower bounds for the consistency strength of $\mathsf{ITP}(\kappa, \lambda)$ are derived from the failure of square. The following theorem is due to Jensen, Schimmerling, Schindler, and Steel [11].

Theorem 5.10. *Suppose $\lambda \geq \omega_3$ is regular such that $\eta^\omega < \lambda$ for all $\eta < \lambda$. If $\neg\square(\lambda)$ and $\neg\square_\lambda$, then there exists a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

Corollary 5.11. *The consistency of ZFC + "there is a κ^+-ineffable cardinal κ" implies the consistency of ZFC + "there is a proper class of strong cardinals and a proper class of Woodin cardinals."*

Proof. If κ is κ^+-ineffable, then it is inaccessible and thus $\eta^\omega < \kappa$ for all $\eta < \kappa$. By Proposition 3.4, $\mathsf{ITP}(\kappa, \kappa)$ holds. By Theorem 4.2, $\mathsf{ITP}(\kappa, \kappa)$ and $\mathsf{ITP}(\kappa, \kappa^+)$ imply $\neg\square(\kappa)$ and $\neg\square(\kappa^+)$, so by Theorem 5.10 there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals. $\square$

Corollary 5.12. *Suppose κ is regular uncountable and $\lambda \geq \omega_3$ is such that $\mathrm{cf}\,\lambda \geq \kappa$ and $\eta^\omega < \lambda$ for all $\eta < \lambda$. If $\mathsf{ITP}(\kappa, \lambda^+)$ holds, then there exists an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

Proof. This follows from Proposition 3.4, Theorem 4.2, and Theorem 5.10. $\square$

6. Conclusion

The reader will have noted that one could also define principles corresponding to λ-almost ineffability. However, by [12] λ-ineffability and λ-almost ineffability both characterize supercompactness, so that considering these principles does not seem to give any new insights.

The main motivation behind the principles we considered is of course the quest for an inner model for a supercompact cardinal. So far the most interesting applications of the principles can be found in [10], which shows the following. Suppose κ is an inaccessible cardinal and $\mathbb{P}$ is an iteration of forcings of size less than κ that takes direct limits stationarily often. If $\mathbb{P}$ forces PFA and $\kappa = \omega_2$, then κ is strongly compact. If $\mathbb{P}$ is additionally required to be proper, then κ is necessarily supercompact. As this is the only known means of constructing models of PFA from large cardinal assumptions, it gives strong heuristic evidence on the lower bound of the consistency strength of PFA.

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