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Multi-moment maps

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Abstract

We introduce a notion of moment map adapted to actions of Lie groups that preserve a closed three-form. We show existence of our multi-moment maps in many circumstances, including mild topological assumptions on the underlying manifold. Such maps are also shown to exist for all groups whose second and third Lie algebra Betti numbers are zero. We show that these form a special class of solvable Lie groups and provide a structural characterisation. We provide many examples of multi-moment maps for different geometries and use them to describe manifolds with holonomy contained in G_2 preserved by a two-torus symmetry in terms of tri-symplectic geometry of four-manifolds.

Contents

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1 Introduction

One illuminating example of the interplay between mathematics and physics is the relation between symplectic geometry and mechanics. A symplectic manifold is characterised by a closed, non-degenerate form of degree two. In modern physics higher degree forms play an important role too. While some authors have looked at extensions of field theories, closed three-forms appear to be particularly relevant in supersymmetric theories with Wess-Zumino terms, string theory and one-dimensional quantum mechanics [35, 40, 21, 3]. They have been studied mathematically in a number of contexts including stable forms [29], strong geometries with torsion [20], gerbes [8] and generalised geometry [30, 27].

One construction illustrating the link between symplectic geometry and physics is that of moment maps. A moment map is an equivariant map from a symplectic manifold into the dual of the Lie algebra of a Lie group acting by symplectomorphisms. It captures the concepts of linear and angular momentum from mechanics. The main purpose of this paper is to explain that a similar type of map exists when we are given a manifold M with a closed three-form c and a Lie group G that acts on M preserving c. We shall call the pair (M, c) a *strong geometry*, and we refer to the Lie group G as a *group of symmetries*. We write $\mathfrak{g}$ for the Lie algebra of G.

An important feature of our construction is that the resulting multi-moment map is a map from M to a vector subspace $\mathcal{P}^*_{\mathfrak{g}}$ of $\Lambda^2 \mathfrak{g}^*$, with $\mathcal{P}^*_{\mathfrak{g}}$ independent of M. This is in contrast to previous considerations [10, 24] of so-called covariant moment maps $\sigma\colon M \to \Omega^1(M, \mathfrak{g}^*)$, which are defined via the relation

$$d\langle \sigma, \mathsf{X}\rangle = X \lrcorner\, c, \qquad \text{for all } \mathsf{X} \in \mathfrak{g}, \tag{1.1}$$

where X is the vector field on M generated by $\mathsf{X} \in \mathfrak{g}$. Here the target space $\Omega^1(M, \mathfrak{g}^*)$ is an infinite-dimensional space depending both on M and on $\mathfrak{g}$. We also note that finding covariant moment maps can be hard; equation (1.1) has a solution $\langle \sigma, \mathsf{X}\rangle$ only if the cohomology class $[X \lrcorner\, c]$ vanishes in $H^2(M)$. Thus, existence of covariant moment maps often requires some non-trivial topological assumption such as $b_2(M) = 0$.

In contrast, we will show that multi-moment maps exist under mild topological assumptions: if M is simply-connected and either G is compact or M is compact with G-invariant volume form. This is analogous to symplectic moment maps, and enables us to give many examples. As one application, we will use multi-moment maps to study seven-manifolds with holonomy contained in G_2, when these have a free isometric action of a two-torus. We find that the geometry is determined by a conformal structure on a four-manifold specified by a certain triple of symplectic two-forms. This extends the work of Apostolov & Salamon [2] and fits with the perspective of Donaldson [15].

In the symplectic case, there is also a general existence theorem for moment maps in the case that the symmetry group is semi-simple; it is a result that does not require any topological assumptions on the manifold. Note that semi-simplicity of a Lie group is characterised algebraically by the vanishing of the first and second Betti numbers of the Lie algebra cohomology. In this direction, we prove that multi-moment maps exist whenever the second and third Betti numbers $b_2(\mathfrak{g})$ and $b_3(\mathfrak{g})$ of the Lie algebra cohomology of G vanish. We call Lie algebras of this type $(2,3)$-*trivial*. The weaker setting of Lie algebras with $b_2(\mathfrak{g}) = 0$, where multi-moment maps are unique if defined, provides many examples of homogeneous strong geometries, including examples that are 2-plectic in the terminology of [3].

As far as we know, $(2,3)$-trivial algebras have not been studied before. We show that these are solvable Lie algebras, that are not products of smaller dimensional algebras. Their derived algebra is of codimension one, and is necessarily nilpotent. From this one may classify the low-dimensional examples, and further study leads to a characterisation of the allowed solvable extensions of nilpotent algebras. The structure theory shows that many examples exist, including some that are unimodular. On the other hand one finds that some nilpotent algebras can not be realised as the derived algebra of a $(2,3)$-trivial algebra.

This paper is organised as follows. In section 2 we give the fundamental calculations that lead to the definition of multi-moment map and introduce the Lie kernel $\mathcal{P}_\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. We then consider topological and algebraic criteria for existence and uniqueness of multi-moment maps in section 3. As discussed above $(2,3)$-trivial Lie algebras play a natural role and section 4 is devoted to an algebraic study of this class and the description of a number of examples. We then return to strong geometries and their multi-moment maps. The basic example is provided by the total space $\Lambda^2 T^*N$ of the second exterior power of the cotangent bundle of a manifold N. Homogeneous strong geometries with multi-moment maps are closely tied to orbits in the dual $\mathcal{P}_\mathfrak{g}^*$ of the Lie kernel and we develop a Kirillov-Kostant-Souriau type theory, pointing out links with nearly Kähler and hypercomplex geometry. The final section of the paper is devoted to an investigation of torsion-free G_2-manifolds with an isometric action of a two-torus. We show how multi-moment maps lead to a description of such metrics via tri-sympletic geometry of four-manifolds.

Some of the algebraic material of this paper is supplemented by our work in [33]. Future work will address extensions of the final section providing multi-moment map approaches to torsion-free $Spin(7)$-structures with T^3-symmetry.

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2 Main definitions

Let (M, c) be a strong geometry, meaning that M is a smooth manifold and that c is a closed three-form on M. Note that unlike the symplectic case there is no one canonical form for c, not even pointwise on M. In general, we do not require any non-degeneracy of c. However, when necessary we will use the terminology of [3] that c is *2-plectic* if $X \lrcorner c = 0$ at $x \in M$ only when $X = 0$ in $T_x M$.

Remark 2.1. Since c is closed, $\ker c = \{ X \in TM : X \lrcorner c = 0 \}$ is integrable. Thus if $\ker c$ is of constant rank and has closed leaves, c induces a 2-plectic structure on $M / \ker c$. △

Remark 2.2. One could consider strongly non-degenerate three-forms c, meaning that $c(X, Y, \cdot) \neq 0$ for all $X \wedge Y \neq 0$. However, by [34] such c exist only in dimensions 3 and 7. The former case is given by a volume form, the latter by a G-structure with $G = G_2$ or its non-compact dual. △

Let G be a group of symmetries for (M, c), meaning that G acts on M preserving the three-form c. Thus for each $\mathsf{X} \in \mathfrak{g}$ we have $\mathcal{L}_X c = 0$, where X is the vector field generated by X. As $dc = 0$, this gives

$$0 = \mathcal{L}_X c = d(X \lrcorner c) + X \lrcorner dc = d(X \lrcorner c), \tag{2.1}$$

so the two-form $X \lrcorner c$ is closed. Suppose $\mathsf{Y} \in \mathfrak{g}$ commutes with X. Then we have

$$0 = \mathcal{L}_Y (X \lrcorner c) = d(Y \lrcorner X \lrcorner c) = d((X \wedge Y) \lrcorner c),$$

showing that the one form $(X \wedge Y) \lrcorner c = c(X, Y, \cdot)$ is closed. If for example, $b_1(M) = 0$, we may then write

$$(X \wedge Y) \lrcorner c = d\nu_{\mathsf{X} \wedge \mathsf{Y}}$$

for some smooth function $\nu_{\mathsf{X} \wedge \mathsf{Y}} \colon M \to \mathbb{R}$. This is the basis of the construction of the multi-moment map. However, the set of decomposable elements $\mathsf{X} \wedge \mathsf{Y}$ in $\Lambda^2 \mathfrak{g}$ for which X and Y commute is a complicated variety. It is more natural to consider the following submodule of $\Lambda^2 \mathfrak{g}$.

Definition 2.3. The *Lie kernel* $\mathcal{P}_\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the $\mathfrak{g}$-module

$$\mathcal{P}_\mathfrak{g} := \ker \left(L \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g} \right),$$

where L is the linear map induced by the Lie bracket.

The previous calculation may now be extended to elements of the Lie kernel. For a bivector $p = \sum_{j=1}^k X_j \wedge Y_j$ we write

$$p \lrcorner c := \sum_{j=1}^k c(X_j, Y_j, \cdot).$$

Lemma 2.4. *Suppose G is a group of symmetries of a strong geometry (M, c). Let $\mathsf{p} = \sum_{j=1}^k \mathsf{X}_j \wedge \mathsf{Y}_j$ be an element of the Lie kernel $\mathcal{P}_{\mathfrak{g}}$ and let $p = \sum_{j=1}^k X_j \wedge Y_j$ be the corresponding bivector on M. Then*

$$d(p \lrcorner c) = 0. \tag{2.2}$$

Proof. The condition that p lies in $\mathcal{P}_{\mathfrak{g}}$ is that $0 = L(\mathsf{p}) = \sum_{j=1}^k [\mathsf{X}_j, \mathsf{Y}_j]$. This together with (2.1) and $dc = 0$ gives

$$\begin{aligned} 0 &= \sum_{j=1}^k [Y_j, X_j] \lrcorner c = \sum_{j=1}^k \Big([\mathcal{L}_{Y_j}, X_j \lrcorner] c \Big) \\ &= \sum_{j=1}^k d(Y_j \lrcorner X_j \lrcorner c) + Y_j \lrcorner d(X_j \lrcorner c) - X_j \lrcorner d(Y_j \lrcorner c) - X_j \lrcorner Y_j \lrcorner dc \\ &= \sum_{j=1}^k d(Y_j \lrcorner X_j \lrcorner c) = d(p \lrcorner c), \end{aligned}$$

as required. □

Thus if for example $b_1(M) = 0$, there is a smooth function $\nu_p \colon M \to \mathbb{R}$ with $d\nu_p = p \lrcorner c$ for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$.

We are now able to define the main object to be studied in this paper.

Definition 2.5. Let (M, c) be a strong geometry with a symmetry group G. A *multi-moment map* is an equivariant map $\nu \colon M \to \mathcal{P}_{\mathfrak{g}}^*$ satisfying

$$d\langle \nu, \mathsf{p} \rangle = p \lrcorner c \tag{2.3}$$

for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$.

Note that for G Abelian $\mathcal{P}_{\mathfrak{g}} = \Lambda^2 \mathfrak{g}$. On the other hand if G is a compact simple Lie group then the Lie kernel is a module familiar from a special class of Einstein manifolds. Indeed Wolf [41, Corollary 10.2] (cf. [4, Proposition 7.49]) showed that in this case $\Lambda^2 \mathfrak{g} = \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}$ as a sum of irreducible modules, so $SO(\dim G)/G$ is an isotropy irreducible space.

3 Existence and uniqueness

As mentioned in the introduction, one of the principal advantages of multi-moment maps over covariant moment maps is that one can prove that multi-moment maps are guaranteed to exist under a wide range of circumstances.

We start first with topological criteria.

Theorem 3.1. *Let (M, c) be a strong geometry with a symmetry group G and assume that $b_1(M) = 0$. If either*

(i) *G is compact, or*

(ii) *M is compact and orientable, and G preserves a volume form on M,*

then there exists a multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.

Proof. Working component by component, we may assume that M is connected. As noted after Lemma 2.4 the condition $b_1(M) = 0$ ensures that there are functions ν_{p} with $d\nu_p = p \lrcorner c$ for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. However, each of these functions may be adjusted by adding a real constant. To build a multi-moment map ν via $\langle \nu, \mathsf{p} \rangle = \nu_p$ we need to ensure equivariance. In the two cases above this may be achieved by either averaging over G or over M. In the second case, one chooses ν_p with mean value 0. In the first case, one chooses a basis (p_i) of $\mathcal{P}_{\mathfrak{g}}$ and puts $\nu(m) = \int_G \sum_i \mathrm{Ad}_g^*(\nu_{p_i}(g \cdot m)) \,\mathrm{vol}_G$. In both cases equation (2.3) is satisfied, and ν is a multi-moment map. □

As we saw in the above proof, one crucial point is making a canonical choice of function ν_p. The following situation occurs in many examples and provides a differential geometric criterion for a construction of multi-moment maps.

Proposition 3.2. *Suppose G is a group of symmetries of a strong geometry (M, c) and that there exists a G-invariant 2-form $b \in \Omega^2(M)$ such that $db = c$. Then $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ given by*

$$\langle \nu, \mathsf{p} \rangle = b(p) \tag{3.1}$$

is a multi-moment map.

Proof. The map ν is equivariant, since b is invariant. We have $\nu_p = b(p)$ with $d(b(p)) = d(p \lrcorner b) = p \lrcorner dc$ by the calculation in Lemma 2.4, so equation (2.3) is satisfied, as required. □

Let us now turn to algebraic criteria for multi-moment maps. This involves study of the Lie kernel. The dual of the exact sequence

$$0 \longrightarrow \mathcal{P}_{\mathfrak{g}} \xrightarrow{\iota} \Lambda^2 \mathfrak{g} \xrightarrow{L} \mathfrak{g}$$

is the sequence

$$\mathfrak{g}^* \xrightarrow{d} \Lambda^2 \mathfrak{g}^* \xrightarrow{\pi} \mathcal{P}_{\mathfrak{g}}^* \longrightarrow 0, \tag{3.2}$$

which is also exact. Hence the dual $\mathcal{P}_{\mathfrak{g}}^*$ of the Lie kernel can be identified with the quotient space $\Lambda^2 \mathfrak{g}^* / d(\mathfrak{g}^*)$. As $B^1(\mathfrak{g}) = d(\mathfrak{g}^*)$ is a subspace of $Z^2(\mathfrak{g}) = \ker(d\colon \Lambda^2 \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^*)$, we have an induced linear map

$$d_{\mathcal{P}} \colon \mathcal{P}_{\mathfrak{g}}^* \to \Lambda^3 \mathfrak{g}^* .$$

More concretely given $\beta \in \mathcal{P}_{\mathfrak{g}}^*$, we choose $\tilde{\beta} \in \pi^{-1}(\beta)$ and then $d_{\mathcal{P}}\beta = d\tilde{\beta}$.

Let $b_n(\mathfrak{g})$ denote the dimension of the nth Lie algebra cohomology group, so $b_n(\mathfrak{g}) = \dim H^n(\mathfrak{g}) = \dim Z^n(\mathfrak{g}) - \dim B^n(\mathfrak{g})$. The next result follows directly from the above discussion.

Proposition 3.3. *The linear map $d_{\mathcal{P}} \colon \mathcal{P}_{\mathfrak{g}}^* \to \Lambda^3 \mathfrak{g}^*$ is a $\mathfrak{g}$-morphism with image contained in $Z^3(\mathfrak{g})$. It is injective if and only if $b_2(\mathfrak{g}) = 0$. If this condition holds then $d_{\mathcal{P}}$ is an isomorphism from $\mathcal{P}_{\mathfrak{g}}^*$ onto $Z^3(\mathfrak{g})$ if and only if $b_3(\mathfrak{g}) = 0$.* □

We will see that this distinguishes a class of Lie groups and Lie algebras that play a special role in the theory of multi-moment maps analogous to the role of semi-simple groups in symplectic geometry. We therefore make a definition.

Definition 3.4. A connected Lie group G or its Lie algebra $\mathfrak{g}$ that satisfies $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$ will be called *(cohomologically)* $(2,3)$*-trivial*.

Theorem 3.5. *Let (M, c) be a strong geometry with connected $(2,3)$-trivial symmetry group G acting nearly effectively. Then there exists a unique multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.*

More generally, if just $b_2(\mathfrak{g}) = 0$, then multi-moment maps for nearly effective actions of G are unique when they exist.

Proof. The invariant three-form c determines a G-equivariant map $\Psi\colon M \to Z^3(\mathfrak{g})$, given by

$$\langle \Psi, \mathsf{X} \wedge \mathsf{Y} \wedge \mathsf{Z} \rangle = c(X, Y, Z) \tag{3.3}$$

for $\mathsf{X}, \mathsf{Y}, \mathsf{Z} \in \mathfrak{g}$. When $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$, for each $m \in M$ there is a unique element $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ satisfying $d_{\mathcal{P}}\nu(m) = \Psi(m)$. Since $d_{\mathcal{P}}$ is a G-morphism, it follows that $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ is also a G-equivariant.

We claim that ν is a multi-moment map. Note that, in general $d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to Z^3(\mathfrak{g}) \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}})^*$. The assumption $b_2(\mathfrak{g}) = 0$, gives that the dual map $d_{\mathcal{P}}^*$ is a surjection $Z^3(\mathfrak{g})^* \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}}) \to \mathcal{P}_{\mathfrak{g}}$. This dual map is given as minus the adjoint action, since

$$\begin{aligned} \langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \mathsf{p} \rangle &= \langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \sum_{i=1}^{k} \mathsf{X}_i \wedge \mathsf{Y}_i \rangle \\ &= -\sum_{i=1}^{k} (\alpha([\mathsf{Z}, \mathsf{X}_i], \mathsf{Y}_i) + \alpha([\mathsf{X}_i, \mathsf{Y}_i], \mathsf{Z}) + \alpha([\mathsf{Y}_i, \mathsf{Z}], \mathsf{X}_i)) = -\langle \alpha, \mathrm{ad}_{\mathsf{Z}}(\mathsf{p}) \rangle, \end{aligned} \tag{3.4}$$

for $\mathsf{Z} \in \mathfrak{g}$, $\mathsf{p} = \sum_{i=1}^{k} \mathsf{X}_i \wedge \mathsf{Y}_i \in \mathcal{P}_{\mathfrak{g}}$. Hence we may write any $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$ in the form $\mathsf{p} = \sum_{i=1}^{r} \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i)$, with $\mathsf{Z}_i \in \mathfrak{g}$ and $\mathsf{q}_i \in \mathcal{P}_{\mathfrak{g}}$. Now the function

$$\nu_{\mathsf{p}} = -\sum_{i=1}^{r} \langle \Psi, \mathsf{Z}_i \wedge \mathsf{q}_i \rangle = -\sum_{i=1}^{r} c(Z_i \wedge q_i)$$

satisfies $d\nu_{\mathsf{p}} = -\sum_{i=1}^{r} \mathcal{L}_{Z_i}(q_i \lrcorner c) = p \lrcorner c$, since $d(q_i \lrcorner c) = 0$ by (2.2). Moreover we have that

$$\nu_{\mathsf{p}}(m) = -\sum_{i=1}^{r} \langle d_{\mathcal{P}} \nu(m), \mathsf{Z}_i \wedge \mathsf{q}_i \rangle = \sum_{i=1}^{r} \langle \nu(m), \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i) \rangle = \langle \nu(m), \mathsf{p} \rangle.$$

Thus ν is a multi-moment map.

For the last part of the Theorem, note that a multi-moment map ν defines elements $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ and the above calculations show that $d_{\mathcal{P}}(\nu(m)) = \Psi(m)$. However, $b_2(\mathfrak{g}) = 0$ implies that there is at most one solution $\nu(m)$ to this equation, so ν is then unique. □

Note that any semi-simple Lie group G has $b_1(\mathfrak{g}) = 0 = b_2(\mathfrak{g})$. Also any reductive group G with one-dimensional centre still has $b_2(\mathfrak{g}) = 0$; in particular this applies to $G = U(n)$. So when multi-moment maps for these group actions exist, they are unique. However, any simple Lie group G has $b_3(\mathfrak{g}) = 1$, so there can be obstructions to existence.

4 (2,3)-trivial Lie algebras

In this section we give a structural description of the $(2,3)$-trivial Lie algebras, list them in low dimensions and show that there are many examples.

Theorem 4.1. *Any non-trivial finite-dimensional Lie algebra $\mathfrak{g} \neq \mathbb{R}, \mathbb{R}^2$ satisfying $b_3(\mathfrak{g}) = 0$ is solvable and not nilpotent. If in addition we have that $b_2(\mathfrak{g}) = 0$ then $\mathfrak{g}$ cannot be a direct sum of two non-trivial subalgebras, and its derived algebra is a codimension one ideal.*

Proof. To verify the first statement, we consider $\mathfrak{r}$, the solvable radical of $\mathfrak{g}$. This is the maximal solvable ideal of $\mathfrak{g}$ and the quotient $\mathfrak{g} / \mathfrak{r}$ is semi-simple. By [31], the cohomology of $\mathfrak{g}$ is given by

$$H^k(\mathfrak{g}) \cong \sum_{i+j=k} H^i(\mathfrak{g} / \mathfrak{r}) \otimes H^j(\mathfrak{r})^{\mathfrak{g}},$$

where $V^{\mathfrak{g}}$ is the set of fixed points of the action $\mathfrak{g}$ on V. We thus have $b_3(\mathfrak{g}) \geqslant b_3(\mathfrak{g} / \mathfrak{r})$. As any non-trivial semi-simple Lie algebra has non-trivial third cohomology group, we deduce that $b_3(\mathfrak{g}) = 0$ implies $\mathfrak{g} = \mathfrak{r}$, so that $\mathfrak{g}$ is solvable. It is necessarily non-nilpotent since it is known [13] that non-Abelian nilpotent Lie algebras are of dimension greater than two and have $b_i \geqslant 2$ for any $0 < i < \dim \mathfrak{g}$, whereas the only non-Abelian three-dimensional nilpotent algebra has $b_3(\mathfrak{g}) = 1$.

For the second statement of the theorem, suppose $\mathfrak{g}$ is a direct sum $\mathfrak{h} \oplus \mathfrak{k}$ of Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. Using the Künneth formula, we obtain

$$b_2(\mathfrak{g}) = b_2(\mathfrak{h}) + b_2(\mathfrak{k}) + b_1(\mathfrak{h})b_1(\mathfrak{k}),$$
$$b_3(\mathfrak{g}) = b_3(\mathfrak{h}) + b_3(\mathfrak{k}) + b_2(\mathfrak{h})b_1(\mathfrak{k}) + b_1(\mathfrak{h})b_2(\mathfrak{k}).$$

This immediately gives $b_2(\mathfrak{h}) = 0 = b_2(\mathfrak{k})$ and $b_3(\mathfrak{h}) = 0 = b_3(\mathfrak{k})$. It also follows that either $b_1(\mathfrak{h}) = 0$ or $b_1(\mathfrak{k}) = 0$. Reordering the factors, we can assume that $b_1(\mathfrak{h}) = 0$. Thus $\mathfrak{h}$ has $b_1(\mathfrak{h}) = 0 = b_2(\mathfrak{h})$ and so is semi-simple. But now the number of simple factors of $\mathfrak{h}$ is equal to $b_3(\mathfrak{h})$ which is 0. So $\mathfrak{h} = \{0\}$, and $\mathfrak{g}$ is not a non-trivial direct sum.

Now we consider the last assertion of the theorem. Note that $b_1(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the derived algebra. As $\mathfrak{g}$ is solvable, we get $b_1(\mathfrak{g}) > 0$. Suppose $b_1(\mathfrak{g}) \geqslant 2$. Then there are two linearly independent elements e_1, e_2 in $Z^1(\mathfrak{g})$. As $e_{12} := e_1 \wedge e_2 \in Z^2(\mathfrak{g})$ and $b_2(\mathfrak{g}) = 0$, we can find an element e_3 with $de_3 = e_{12}$. Note that we have $\dim\langle e_1, e_2, e_3\rangle = 3$. Inductively, we may find $e_4, \dots, e_n$ with $de_j = e_{1,j-1}$ such that $e_1, \dots, e_n$ is a basis for $\mathfrak{g}$. But, now $e_{1n} \in Z^2(\mathfrak{g})$ can not be exact, contradicting $b_2(\mathfrak{g}) = 0$. Thus, we must have $b_1(\mathfrak{g}) = 1$. □

We will refine this result later, but it is already sufficient to list the smallest examples of $(2,3)$-trivial Lie algebras. In dimension one, the only Lie algebra is Abelian and is automatically $(2,3)$-trivial. In dimension two a Lie algebra is either Abelian or isomorphic to the $(2,3)$-trivial algebra $(0,21)$. These first two examples are uninteresting from the point of view of multi-moment maps since they have $\mathcal{P}_\mathfrak{g} = \{0\}$. However, in dimensions three and four we may use the known classification of solvable Lie algebras [1] to obtain more interesting examples. Note that for any Lie algebra of dimension n, we have

$$\dim \mathcal{P}_\mathfrak{g} = b_1(\mathfrak{g}) + \tfrac{1}{2}n(n-3),$$

since the kernel of left most map in (3.2) is $H^1(\mathfrak{g}) = Z^1(\mathfrak{g})$. Thus a $(2,3)$-trivial algebra has $\dim \mathcal{P}_\mathfrak{g} = (n-1)(n-2)/2$, which is non-zero for $n \geqslant 3$.

Proposition 4.2. *The inequivalent* $(2,3)$*-trivial Lie algebras in dimensions three and four are listed in the Tables 4.1 and 4.2.*

To explain the notation, consider the example $\mathfrak{h}_4 = (0, 21+31, 31, 2.41+32)$. This means there is a basis $e_1, \dots, e_4$ for $\mathfrak{h}_4^*$ such that $de_1 = 0$, $de_2 = e_{21} + e_{31}$, $de_3 = e_{31}$ and $de_4 = 2e_{41} + e_{32}$.

We will sketch a proof of this Proposition that is independent of the classification lists, using the following more detailed structure result. Full

$\mathfrak{r}_3$	$(0, 21+31, 31)$	
$\mathfrak{r}_{3,\lambda}$	$(0, 21, \lambda.31)$	$\lambda \in (-1,1] \setminus \{0\}$
$\mathfrak{r}'_{3,\lambda}$	$(0, \lambda.21+31, -21+\lambda.31)$	$\lambda > 0$

Table 4.1. The inequivalent three-dimensional $(2,3)$-trivial Lie algebras.

$\mathfrak{r}_4$	$(0, 21+31, 31+41, 41)$	
$\mathfrak{r}_{4,\lambda}$	$(0, 21, \lambda.31+41, \lambda.41)$	$\lambda \neq -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{4,\mu,\lambda}$	$(0, 21, \mu.31, \lambda.41)$	$(\mu,\lambda) \in \mathscr{R}$
$\mathfrak{r}'_{4,\mu,\lambda}$	$(0, \mu.21, \lambda.31+41, -31+\lambda.41)$	$\mu > 0,\ \lambda \neq -\frac{\mu}{2}, 0$
$\mathfrak{d}_{4,\lambda}$	$(0, \lambda.21, (1-\lambda).31, 41+32)$	$\lambda \geqslant \frac{1}{2},\ \lambda \neq 1, 2$
$\mathfrak{d}'_{4,\lambda}$	$(0, \lambda.21+31, -21+\lambda.31, 2\lambda.41+32)$	$\lambda > 0$
$\mathfrak{h}_4$	$(0, 21+31, 31, 2.41+32)$	

Table 4.2. The inequivalent four-dimensional $(2,3)$-trivial Lie algebras. The set $\mathscr{R}$ consists of the $\mu, \lambda \in (-1,1] \setminus \{0\}$ with $\lambda \geqslant \mu$ and $\mu + \lambda \neq 0, -1$.

details of the classification and its extension to five-dimensional algebras are given in [33].

Theorem 4.3. *A Lie algebra $\mathfrak{g}$ with derived algebra $\mathfrak{k} = \mathfrak{g}'$ is $(2,3)$-trivial if and only if $\mathfrak{g}$ is solvable, $\mathfrak{k}$ is nilpotent of codimension 1 in $\mathfrak{g}$ and $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$.*

Proof. The derived algebra $\mathfrak{k} = \mathfrak{g}'$ of a solvable algebra $\mathfrak{g}$ is always nilpotent, so Theorem 4.1 implies that it only remains to check the assertions on the $\mathfrak{g}$-invariant part of the cohomology of $\mathfrak{k}$. For this, as $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, we may use the spectral sequence of Hochschild & Serre [31] that has $E_2^{j,i} \cong H^j(\mathfrak{g} / \mathfrak{k}, H^i(\mathfrak{k}))$. Now the codimension one condition means that we may write $\mathfrak{g} / \mathfrak{k} = \mathbb{R}A$ for some element A. Note that $H^i(\mathfrak{k})$ is a $\mathfrak{g} / \mathfrak{k}$ module. For any $\mathfrak{g} / \mathfrak{k}$-module M, the cohomology groups $H^j(\mathbb{R}A, M)$ are defined from the chain groups $C^j(\mathbb{R}A, M) = \Lambda^j(\mathbb{R}A)^* \otimes M = \operatorname{Hom}(\mathbb{R}A, M)$. These can only be non-zero for $j = 0, 1$ and in both cases they are isomorphic to M. The chain map is $d_{\mathbb{R}}$ which on C^0 is $(d_{\mathbb{R}} f)(A) = A \cdot f$. Thus $E_2^{0,i} = \ker d_{\mathbb{R}} = M^A$ and $E_2^{1,1} = M / \operatorname{im} d_{\mathbb{R}} \cong \ker d_{\mathbb{R}} = M^A$. We see that the E_2-term of our spectral sequence is

$$E_2^{j,i} \cong \begin{cases} H^i(\mathfrak{k})^{\mathfrak{g}} & \text{for } j = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the spectral sequence degenerates at the E_2-term and we conclude that

$$H^2(\mathfrak{g}) \cong H^2(\mathfrak{k})^{\mathfrak{g}} + H^1(\mathfrak{k})^{\mathfrak{g}}, \quad H^3(\mathfrak{g}) \cong H^3(\mathfrak{k})^{\mathfrak{g}} + H^2(\mathfrak{k})^{\mathfrak{g}},$$

from which the result follows. □

Proof (Sketch proof of Proposition 4.2). Let $\mathfrak{g}$ be a $(2,3)$-trivial algebra of dimension three. Then $\mathfrak{k} = \mathfrak{g}'$ is nilpotent and two-dimensional, so $\mathfrak{k} \cong \mathbb{R}^2$. The element A of Theorem 4.3 acts on $\mathbb{R}^2$ invertibly and the induced action on $H^2(\mathbb{R}^2) \cong \Lambda^2\mathbb{R}^2 \cong \mathbb{R}$ is also invertible. So either A is diagonalisable over $\mathbb{C}$ with non-zero eigenvalues whose sum is non-zero, giving cases $\mathfrak{r}_{3,\lambda}$ and $\mathfrak{r}'_{3,\lambda}$, or A acts with Jordan form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, $\lambda \neq 0$, giving case $\mathfrak{r}_3$. The particular structure coefficients are obtained by replacing A by a non-zero multiple.

For $\mathfrak{g}$ of dimension four, we have $\mathfrak{k} \cong \mathbb{R}^3$ or the Heisenberg algebra $\mathfrak{h}_3 = (0,0,12)$. The former gives the algebras from the $\mathfrak{r}$-series when one enforces that no sum of one, two or three eigenvalues of A is zero. The latter gives the remaining algebras; we have $H^1(\mathfrak{h}_3) \cong \langle e_1, e_2\rangle$, $H^2(\mathfrak{h}_3) \cong \langle e_{13}, e_{23}\rangle$, $H^3(\mathfrak{h}_3) \cong \langle e_{123}\rangle$, A acts invertibly on these spaces and its action in e_3 is determined by its action on e_1 and e_2. □

Theorem 4.3 enables us to generate many examples of $(2,3)$-trivial Lie algebras in higher dimensions. Say that a nilpotent algebra $\mathfrak{k}$ is *positively graded* if there is a vector space direct sum decomposition $\mathfrak{k} = \mathfrak{k}_1 + \dots + \mathfrak{k}_r$ with $[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$ for all i, j.

Corollary 4.4. *Let $\mathfrak{k}$ be any positively graded nilpotent Lie algebra. Then there is a $(2,3)$-trivial Lie algebra whose derived algebra is $\mathfrak{k}$.*

Proof. Let $\mathfrak{g} = \langle A\rangle + \mathfrak{k}$ where ad_A acts as multiplication by i on $\mathfrak{k}_i$. Then $\mathfrak{g}$ is a solvable algebra. Moreover $(\Lambda^s \mathfrak{k})^{\mathfrak{g}} = \{0\}$ for $s \geqslant 1$, so the cohomological condition of Theorem 4.3 is satisfied and $\mathfrak{g}$ is as required. □

The algebras constructed in this way are completely solvable, meaning that each ad_X, for $X \in \mathfrak{g}$, has only real eigenvalues on $\mathfrak{g}$.

Example 4.5. It may be checked directly that every nilpotent Lie algebra of dimension at most six can be positively graded. The classification of these nilpotent algebras (see [38]) then gives over 30 different $(2,3)$-trivial algebras in dimension 7, see [33]. ◊

Example 4.6. Another class of positively graded algebras is given as follows. Let $\mathrm{Der}(\mathfrak{k})$ be the algebra of derivations of $\mathfrak{k}$. A *maximal torus* $\mathfrak{t}$ for $\mathfrak{k}$ is a maximal Abelian subalgebra of the semi-simple elements of $\mathrm{Der}(\mathfrak{k})$. The nilpotent Lie algebra $\mathfrak{k}$ is said to have *maximal rank* if $\dim \mathfrak{t} = \dim(\mathfrak{k} / \mathfrak{k}')$. Favre [17] showed that there are only finitely many systems of weights for such algebras and following [39] a number of classification results have been obtained via Kac-Moody techniques, see [19] and the references therein. There is a large number (thousands) of families of such algebras. From the general theory, one knows [17, p. 83] that there is a positive grading of each maximal rank nilpotent Lie algebra $\mathfrak{k}$. This grading satisfies $\sum_{i=s+1}^{r} \mathfrak{k}_i = \mathfrak{k}^{(s)} = [\mathfrak{k}, \mathfrak{k}^{(s-1)}]$. Thus each of these distinct nilpotent algebras of maximal rank arises as the derived algebra of non-isomorphic $(2,3)$-trivial Lie algebras. ♢

We note that in the construction of Corollary 4.4, ad_A is a semi-simple derivation of $\mathfrak{k}$. Generally, if $\mathfrak{g}$ is solvable, then $A \in \mathfrak{g} \setminus \mathfrak{g}'$ acts on $\mathfrak{k} = \mathfrak{g}'$ as a derivation. For $\mathfrak{g}$ to be $(2,3)$-trivial, Theorem 4.3 implies that this action is not nilpotent on $H^k(\mathfrak{k})$ for $k = 1,2,3$. For $\dim \mathfrak{g} \geqslant 5$, this condition has most force since these three cohomology groups have dimension at least 2 [13].

Now a nilpotent Lie algebra $\mathfrak{k}$ is said to be *characteristically nilpotent* if $\mathrm{Der}(\mathfrak{k})$ acts on $\mathfrak{k}$ by nilpotent endomorphisms. It is known that this is equivalent to $\mathrm{Der}(\mathfrak{k})$ being a nilpotent Lie algebra. For a characteristically nilpotent algebra $\mathfrak{k}$, any solvable extension will act nilpotently on the cohomology of $\mathfrak{k}$. Theorem 4.3 thus gives the following result.

Corollary 4.7. *If $\mathfrak{k}$ is a characteristically nilpotent Lie algebra, then $\mathfrak{k}$ is never the derived algebra of a $(2,3)$-trivial algebra.* □

Example 4.8. The first example of a characteristically nilpotent Lie algebra was constructed by Dixmier and Lister [14] in dimension eight. However, there are seven-dimensional examples with the same property and even continuous families [25] including:

$$(0,0,12,13,23,14+25+\alpha.23,16+25+35+\alpha.24), \qquad \alpha \neq 0.$$

Thus no member of this family of algebras can occur as the derived algebra of any $(2,3)$-trivial Lie algebra. ♢

A Lie algebra $\mathfrak{g}$ is called *unimodular* if the Lie algebra homomorphism $\chi\colon \mathfrak{g} \to \mathbb{R}$ given by $\chi(x) = \mathrm{Tr}(\mathrm{ad}(x))$ has trivial image. Such Lie algebras are interesting since unimodularity is a necessary condition for the existence of a co-compact discrete subgroup [36].

Corollary 4.9. *The simply-connected $(2,3)$-trivial Lie groups of dimension four or below are not unimodular. In particular they do not admit a compact quotient by a lattice.*

Proof. An n-dimensional Lie algebra $\mathfrak{g}$ is unimodular if and only if $b_n(\mathfrak{g}) = 1$. Moreover, one may show that unimodular algebras satisfy Hodge duality $b_k(\mathfrak{g}) = b_{n-k}(\mathfrak{g})$. For $\mathfrak{g}$ a $(2,3)$-trivial Lie algebra of dimension three, we have $b_3(\mathfrak{g}) = 0$, so $\mathfrak{g}$ is not unimodular. For $\mathfrak{g}$ of dimension four, unimodularity implies $b_1(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$. But $(2,3)$-trivial algebras have $b_1(\mathfrak{g}) = 1$, so they can not be unimodular in dimension four. □

Example 4.10. It can be shown that in dimension five and above there are unimodular $(2,3)$-trivial Lie algebras, see [33]. Moreover one may verify that there are solvmanifolds of the form G/Γ, where G is $(2,3)$-trivial. Indeed using [5, Proposition 7.2.1(i)] one may see that there are $(2,3)$-trivial Lie groups which admit a lattice. One such example has Lie algebra

$$(0, \lambda_1.12, \lambda_2.13, \lambda_3.14, \lambda_4.15),$$

where $\exp(\lambda_i) \approx 0.1277, 0.6297, 2.797, 4.446$ are the four roots of the polynomial $s^4 - 8s^3 + 18s^2 - 10s + 1$. As this Lie algebra is completely solvable it follows from Hattori's Theorem [28] that one has an isomorphism $H^*_{\mathrm{dR}}(G/\Gamma) \cong H^*(\mathfrak{g})$. In particular the five-dimensional solvmanifold constructed in this way has vanishing second and third de Rham cohomology groups. ♢

5 Examples and applications

As strong geometry has no analogue of the Darboux Theorem, the theory of multi-moment maps is in some senses less rigid than that for symplectic moment maps and there is a wider variety of types of example.

5.1 Second exterior power of the cotangent bundle

In symplectic geometry one of the fundamental examples is provided by the cotangent bundle of a manifold, which in mechanics may be interpreted as a phase space. In strong geometry, an analogous example is provided by the second exterior power $M = \Lambda^2 T^* N$ of a base manifold N. This carries a canonical two-form b, given by

$$b_\alpha(W_1, W_2) = \alpha(\pi_* W_1, \pi_* W_2), \qquad W_1, W_2 \in T_\alpha M,$$

where $\pi\colon \Lambda^2 T^* N \to N$ is the bundle projection. From this one defines a closed three-form c on M, via

$$c = db.$$

This form is 2-plectic: in local coordinates $(q^1, \dots, q^n)$ on N we have $\alpha = \sum_{i<j} p_{ij} dq^i \wedge dq^j$ defining local coordinates (q^i, p_{ij}) on M in which $c = \sum_{i<j} dp_{ij} \wedge dq^i \wedge dq^j$. This is the fundamental example in [3, 10].

If G is a group of diffeomorphisms of N, then there is an induced action on $M = \Lambda^2 T^* N$ which preserves b and hence c. As $c = db$, Proposition 3.2

gives that there is a multi-moment map ν determined by (3.1), which here reads

$$\langle \nu(\alpha), \mathsf{p} \rangle = \alpha(p_N)$$

where p_N is the field of bivectors on N determined by $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. To summarise

Proposition 5.1. *If a Lie group G acts on a smooth manifold N, then the induced action on $M = \Lambda^2 T^* N$ admits a multi-moment map with respect to the canonical 2-plectic structure.* □

Remark 5.2. Suppose N^n carries an H-structure, i.e., a reduction of the structure group of N to $H \leqslant GL(n, \mathbb{R})$. Then at each point of $q \in N$ we have a canonical decomposition $\Lambda^2_q T^* N = \oplus_i V_i(q)$ into isotypical H-modules. If the action of G preserves the H-structure then the induced map of $\Lambda^2 T^* N$ preserves the subbundles V_i. Each bundle V_i carries a strong geometry via the pull-back of c on $M = \Lambda^2 T^* N$, and the action of G again admits a multi-moment map. For example, if N is an oriented four-manifold and G preserves the orientation, then there are multi-moment maps $\nu_\pm$ defined on the 2-plectic seven-manifolds $\Lambda^2_\pm$. The particular case of $SO(4) = Sp(1)_+ Sp(1)_-$ acting on $N = \mathbb{R}^4 = \mathbb{H}$ via $(A, B) \cdot q = Aq\overline{B}$ has multi-moment map on $\Lambda^2_+ N \cong \mathbb{H} + \operatorname{Im} \mathbb{H}$ given by $\langle \nu_+(q, p), a \otimes b \rangle = \frac{1}{2} \operatorname{Re}(paqb\overline{q})$, for $q \in \mathbb{H}$, $p \in \operatorname{Im} \mathbb{H}$, $a \otimes b \in \mathfrak{sp}(1)_+ \otimes \mathfrak{sp}(1)_- = \operatorname{Im} \mathbb{H} \otimes \operatorname{Im} \mathbb{H} \cong \mathcal{P}_{\mathfrak{sp}(1)_+ + \mathfrak{sp}(1)_-}$. △

5.2 Homogeneous strong geometries

If G acts transitively on a strong manifold M, then we may define $\Psi \colon M \to Z^3(\mathfrak{g})$ via (3.3), and the image will be a G-orbit in $Z^3(\mathfrak{g})$. Conversely, formula (3.3) can be used to define strong geometries that map to a given orbit in $Z^3(\mathfrak{g})$: given $\Psi \in Z^3(\mathfrak{g})$, let be K_Ψ denote the connected subgroup generated by $\ker \Psi = \{ X \in \mathfrak{g} : X \lrcorner \Psi = 0 \}$; for any closed group H of G with $H \subset K_\Psi$, equation (3.3) defines a closed three-form c on the homogeneous space G/H and this strong geometry maps to $G \cdot \Psi \subset Z^3(\mathfrak{g})$.

Now suppose that $\Psi = d_{\mathcal{P}} \beta$ for some $\beta \in \mathcal{P}^*_{\mathfrak{g}}$. If the map $d_{\mathcal{P}}$ is injective, then the orbits $G \cdot \Psi$ and $G \cdot \beta$ are identified and the map $\Psi \colon M \to Z^3(\mathfrak{g})$ may now be interpreted as a map $\nu \colon M \to \mathcal{P}^*_{\mathfrak{g}}$. Injectivity of $d_{\mathcal{P}}$ is guaranteed by the condition $b_2(\mathfrak{g}) = 0$. When this holds, the proof of Theorem 3.5 shows that ν is a multi-moment map for the action of G.

Theorem 5.3. *Suppose G is a connected Lie group with $b_2(\mathfrak{g}) = 0$. Let $\mathcal{O} = G \cdot \beta \subset \mathcal{P}^*_{\mathfrak{g}}$ be an orbit of G acting on the dual of the Lie kernel. Then there are homogeneous strong manifolds $(G/H, c)$, with c corresponding to $\Psi = d_{\mathcal{P}} \beta$, such that $\mathcal{O}$ is the image of G/H under the (unique) multi-moment map ν.*

The strong geometry may be realised on the orbit $\mathcal{O}$ itself if and only if

$$\operatorname{stab}_{\mathfrak{g}} \beta = \ker(d_{\mathcal{P}} \beta). \tag{5.1}$$

In this situation, the orbit is 2-plectic and ν is simply the inclusion $\mathcal{O} \hookrightarrow \mathcal{P}_{\mathfrak{g}}^*$.

Proof. It only remains to prove the assertions of the last paragraph of the Theorem. We have $\mathcal{O} = G/K$ with $K = \mathrm{stab}_G \beta$, a closed subgroup of G. Now equation (3.4), shows that K has Lie algebra $\ker(d_{\mathcal{P}}\beta)$, so the component of the identity K^0 of K is $K^0 = K_\Psi$ for $\Psi = d_{\mathcal{P}}\beta$. In particular, Ψ vanishes on elements of $\mathfrak{k}$ and induces a well-defined form on $T_\beta\mathcal{O} = \mathfrak{g}\,/\,\mathfrak{k}$. The result now follows. □

Example 5.4. Suppose G is a $(2,3)$-trivial Lie group. Then, taking $H = \{e\}$, we see that every $\Psi \in Z^3(\mathfrak{g})$ gives rise to a strong geometry on G with multi-moment map whose image is diffeomorphic to the G-orbit of Ψ. ♢

Example 5.5. For G a $(2,3)$-trivial group of dimension at most four, the Lie kernel contains strong orbits exactly when $\mathfrak{g}' = \mathfrak{h}_3$. In this case, $\mathcal{P}_{\mathfrak{g}}$ has dimension 3, the orbits are open and the strong structure is a left-invariant volume form. ♢

Example 5.6. Consider $G = U(2) \cong (S^1 \times SU(2))/\{\pm(1,1)\}$. We have $\mathcal{P}_{\mathfrak{u}(2)} = \mathsf{T} \wedge \mathfrak{su}(2)$, where T generates the Lie algebra of S^1. The orbits of $\mathcal{P}_{\mathfrak{u}(2)}$ are thus two-dimensional and can not admit (non-trivial) strong geometries. On the other hand, suppose we write e_1, e_2, e_3 for a standard basis of $\mathfrak{su}(2)^*$ with $de_1 = -e_{23}$. Then the element $\beta = dt \wedge e_1 \in \mathcal{P}_{\mathfrak{u}(2)}^*$, has $d_{\mathcal{P}}\beta = -dt \wedge e_{23}$, defining $\Psi \in Z^3(\mathfrak{u}(2))$. This β does not satisfy condition (5.1) even though $d_{\mathcal{P}}$ identifies the orbits of β and Ψ. However, Ψ defines strong geometries on $U(2)$ and on $U(2)/\operatorname{diag}(e^{i\theta}, e^{-i\theta}) \cong S^1 \times S^2$ with multi-moment map the projection to S^2. Note that $\nu\colon U(2) \to S^2$ is essentially the Hopf fibration. ♢

Example 5.7. Consider $\mathfrak{g} = \mathfrak{su}(3)$ as a Lie algebra of complex matrices. Write E_{pq} for the elementary 3×3-matrix with 1 at position (p,q). Then $\mathfrak{su}(3)$ has a basis $A_j = i(E_{jj} - E_{j+1,j+1})$, $B_{k\ell} = E_{k\ell} - E_{\ell k}$, $C_{k\ell} = i(E_{k\ell} + E_{\ell k})$, for $j,k = 1,2$, $k < \ell = 2,3$. Let $a_1, a_2, b_{12}, \dots, c_{23}$ denote the dual basis.

The element $\beta_1 = b_{12} \wedge b_{13} - c_{12} \wedge c_{13}$ lies in $\mathcal{P}_{\mathfrak{su}(3)}^*$. One has $d_{\mathcal{P}}\beta_1 = 3a_1(b_{12}c_{13} - b_{13}c_{12})$, where we have omitted wedge signs. Direct calculation shows that $\ker d_{\mathcal{P}}\beta_1 = \langle A_2, B_{23}, C_{23}\rangle = \mathrm{stab}_{\mathfrak{su}(3)}\beta_1$. Thus, by Theorem 5.3, the $SU(3)$-orbit $\mathcal{O}_1$ of β_1 is 2-plectic with multi-moment map given by the inclusion in $\mathcal{P}_{\mathfrak{su}(3)}^*$. As the above stabiliser is isomorphic to $\mathfrak{su}(2)$, we see that up to finite covers $\mathcal{O}_1$ is $SU(3)/\,SU(2) = S^5$.

Similarly, one may realise $F_{1,2}(\mathbb{C}^3) = SU(3)/T^2$ as a 2-plectic manifold by considering the orbit of $\beta_2 = c_{12}b_{12} + b_{13}c_{13} + c_{23}b_{23} \in \mathcal{P}_{\mathfrak{su}(3)}^*$.

It is interesting to note that $F_{1,2}(\mathbb{C}^3)$ carries a nearly Kähler structure. Such a geometry may be specified by a two-form σ and a three-form ψ_+ whose pointwise stabiliser in $GL(6,\mathbb{R})$ is isomorphic to $SU(3)$. The nearly

Kähler condition is then $d\sigma = \psi_+$, $d\psi_- = -\frac{1}{2}\sigma^2$, where $\psi_+ + i\psi_- \in \Lambda^{3,0}$. Direct check shows that each homogeneous strict nearly Kähler six-manifold $G/H = F_{1,2}(\mathbb{C}^3)$, $\mathbb{C}P(3)$, $S^3 \times S^3$ and S^6, as classified by Butruille [9], may be realised as a 2-plectic orbit $G \cdot \beta$ in $\mathcal{P}_{\mathfrak{g}}^*$. Moreover this may done in such a way that $\Psi = d_{\mathcal{P}}\beta$ induces $c = \psi_+$ via (3.3) and β induces σ in a corresponding way. Further details may be found in [33]. ♢

To characterise the homogeneous geometries of Theorem 5.3, we introduce the following terminology.

Definition 5.8. Let G be a group of symmetries of a strong geometry (M, c). We say that the action is *weakly $\mathcal{P}_{\mathfrak{g}}$-transitive* if G acts transitively on M and for each non-zero $X \in T_xM$, there is a $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$ such that $c(X \wedge p)$ is non-zero.

Corollary 5.9. *If G is $(2,3)$-trivial, then the weakly $\mathcal{P}_{\mathfrak{g}}$-transitive 2-plectic geometries with symmetry group G are discrete covers of orbits $\mathcal{O} = G \cdot \beta$ in $\mathcal{P}_{\mathfrak{g}}^*$ satisfying condition* (5.1).

More generally, if G is a Lie group with $b_2(\mathfrak{g}) = 0$, then the orbits $\mathcal{O} = G \cdot \beta \subset \mathcal{P}_{\mathfrak{g}}^*$ satisfying (5.1) *are, up to discrete covers, the weakly $\mathcal{P}_{\mathfrak{g}}$-transitive 2-plectic geometries that admit a multi-moment map.*

Proof. The differential $\nu_* \colon T_xM \to \mathcal{P}_{\mathfrak{g}}^*$ of the multi-moment map is given by $\langle \nu_*(X), \mathsf{p}\rangle = (X \lrcorner c)(p)$. As G acts weakly $\mathcal{P}_{\mathfrak{g}}$-transitively, we see that $\nu_*(X)$ is non-zero for each non-zero X. Thus ν_* is injective and ν has discrete fibres. Its image is an orbit $G \cdot \beta$ and the proof of Theorem 3.5 shows that the 3-form c on M is induced by $\Psi = d_{\mathcal{P}}\beta$. As ν is a local diffeomorphism and c is 2-plectic it follows that (5.1) is satisfied. Conversely, any orbit $\mathcal{O} = G \cdot \beta$ satisfying (5.1) is 2-plectic with injective multi-moment map ν. Since ν_* is injective, the equation $\langle \nu_*(X), \mathsf{p}\rangle = c(X \wedge p)$ shows that the action is weakly $\mathcal{P}_{\mathfrak{g}}$-transitive. □

5.3 Compact Lie groups with bi-invariant metric

Let G be a compact semi-simple Lie group. Its Lie algebra $\mathfrak{g}$ admits an inner product $\langle \cdot, \cdot \rangle$ invariant under the adjoint representation, which is proportional to minus the Killing form. The left- and right-invariant Cartan one-forms $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ are given by $\theta^L(X) = (L_{g^{-1}})_*(X)$, $\theta^R(X) = (R_{g^{-1}})_*(X)$, where $L_g, R_g \colon G \to G$ denote left- and right-multiplication by g. A bi-invariant, and hence closed, three-form is defined on G by

$$c(X, Y, Z) = \langle [\theta^L(X), \theta^L(Y)], \theta^L(Z)\rangle, \qquad \text{for } X, Y, Z \in \Gamma(TG). \tag{5.2}$$

This is 2-plectic but is zero on elements of $\mathcal{P}_{\mathfrak{g}}$ for G acting on the left. Instead for $H, K \leqslant G$, let $H \times K$ act on G by

$$(h, k) \cdot g = L_h \circ R_{k^{-1}}(g) = hgk^{-1}.$$

An element $\mathsf{X} = (\mathsf{X}^H, \mathsf{X}^K) \in \mathfrak{h} \oplus \mathfrak{k}$ induces a vector field X on G given by $X_g = \frac{d}{dt} \exp(t\mathsf{X}^H) g \exp(-t\mathsf{X}^K)|_{t=0} = (R_g)_*\mathsf{X}^H - (L_g)_*\mathsf{X}^K$. For $\mathsf{p} = \sum_{j=1}^k \mathsf{X}_j \wedge \mathsf{Y}_j \in \mathcal{P}_{\mathfrak{h}\oplus\mathfrak{k}}$, we have that $\sum_{j=1}^k [X_j^H, Y_j^H] = 0$ and $\sum_{j=1}^k [X_j^K, Y_j^K] = 0$, and claim that

$$\langle \nu(g), \mathsf{p} \rangle = \sum_{j=1}^k \big(\langle \mathsf{X}_j^H, \mathrm{Ad}_g(\mathsf{Y}_j^K) \rangle - \langle \mathsf{Y}_j^H, \mathrm{Ad}_g(\mathsf{X}_j^K) \rangle \big),$$

defines a multi-moment map $\nu \colon G \to \mathcal{P}^*_{\mathfrak{h}\oplus\mathfrak{k}}$. This follows from the following computation for $A_g = (R_g)_*\mathsf{A}$:

$$\begin{aligned} d\langle \nu, \mathsf{p} \rangle (A)_g &= \frac{d}{dt} \langle \nu(\exp(t\mathsf{A})g), \mathsf{p} \rangle \Big|_{t=0} \\ &= \langle \mathsf{X}_j^H, [\mathsf{A}, \mathrm{Ad}_g(\mathsf{Y}_j^K)] \rangle - \langle \mathsf{Y}_j^H, [\mathsf{A}, \mathrm{Ad}_g(\mathsf{X}_j^K)] \rangle \\ &= -\langle [\mathrm{Ad}_{g^{-1}} \mathsf{X}_j^H, \mathsf{Y}_j^K] + [\mathsf{X}_j^K, \mathrm{Ad}_{g^{-1}} \mathsf{Y}_j^H], \theta^L(A)_g \rangle = (p \lrcorner c)(A)_g, \end{aligned}$$

since $\theta^L(A)_g = \mathrm{Ad}_{g^{-1}} \mathsf{A}$. By considering $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}\oplus\mathfrak{k}}$ of the form $\mathsf{p} = (\mathsf{X}^H, 0) \wedge (0, \mathsf{Y}^K)$ with $\mathsf{X}^H \in \mathfrak{h}$ and $\mathsf{Y}^K \in \mathfrak{k}$ arbitrary, one finds that

$$\ker(\nu_*)_g = (L_g)_*[\mathrm{Ad}_{g^{-1}} \mathfrak{h}, \mathfrak{k}]^\perp.$$

In the case that $\mathfrak{h} = \mathfrak{g}$, the set $\ker(\nu_*)_e$ is a subalgebra of $\mathfrak{g}$ and the image of ν is an orbit.

One example is given by $\mathfrak{h} = \mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{k} = \mathfrak{u}(1) = \mathrm{diag}(ia, -ia, 0)$. Then $\ker(\nu_*)_e = \mathfrak{u}(2)$ and the multi-moment map ν is the projection from $SU(3)$ to $\mathbb{C}P(2) = SU(3)/U(2)$. Now $\mathbb{C}P(2)$ is quaternionic Kähler, and $SU(3)$ carries a hypercomplex structure [32]. The bi-invariant metric on $SU(3)$ realises the hypercomplex structure as a strong HKT manifold whose torsion-three form c is given by (5.2) [26]. The symmetry group of this HKT structure is precisely $H \times K = SU(3) \times U(1)$ and the map ν realises $SU(3)$ as a twisted associated bundle over $\mathbb{C}P(2)$ [37].

5.4 Strong geometries from symplectic manifolds

Let us show how the theory of multi-moment maps for strong geometries subsumes that of symplectic moment maps. Given a symplectic manifold (N, ω) one has a strong geometry on $M = S^1 \times N$ with $c = \phi \wedge \omega$, where ϕ is the invariant one-form dual to the circle action on S^1. This geometry is 2-plectic. If N comes with a symplectic action of a Lie group H, then $G = S^1 \times H$ is a symmetry group for the strong geometry on M. The corresponding Lie kernel is given by

$$\mathcal{P}_{\mathbb{R}+\mathfrak{h}} \cong \mathcal{P}_{\mathfrak{h}} + \mathbb{R} \otimes \mathfrak{h}\,.$$

Proposition 5.10. *Let (N, ω) be a symplectic manifold with a Hamiltonian action of H, moment map $\mu\colon N \to \mathfrak{h}^*$. Then $M = S^1 \times N$ carries a strong geometry with symmetry group $G = S^1 \times H$ and this has a multi-moment map ν that may be identified with μ.*

Proof. We first claim that $p \lrcorner \omega = 0$, for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}} \subset \mathcal{P}_{\mathfrak{g}}$. Writing $\mathsf{p} = \sum_{j=1}^{k} \mathsf{X}_j \wedge \mathsf{Y}_j \in \mathcal{P}_{\mathfrak{h}}$, we have

$$\omega(p) = \sum_{j=1}^{k} \omega(X_j, Y_j) = \sum_{j=1}^{k} Y_j \lrcorner d\langle \mu, \mathsf{X}_j \rangle = \sum_{j=1}^{k} \mathcal{L}_{Y_j} \langle \mu, \mathsf{X}_j \rangle.$$

But μ is equivariant, so $\mathcal{L}_Y \langle \mu, \mathsf{X} \rangle = \langle \mu, [X, Y] \rangle$. As $\sum_{j=1}^{k} [\mathsf{X}_j, \mathsf{Y}_j] = 0$ it follows that $\omega(p) = 0$, as claimed.

Now we may define $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ by

$$\langle \nu, \mathsf{p} \rangle = 0, \qquad \langle \nu, \mathsf{T} \wedge \mathsf{X} \rangle = \langle \mu, \mathsf{X} \rangle,$$

for $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}}$ and $\mathsf{X} \in \mathfrak{h}$, where T is the generator of the S^1 action on the first factor of $M = S^1 \times G$. Now $d\langle \nu, \mathsf{p} \rangle = 0 = p \lrcorner c$ and

$$d\langle \nu, \mathsf{T} \wedge \mathsf{X} \rangle = X \lrcorner \mu = (T \wedge X) \lrcorner c,$$

so equation (2.3) is satisfied. As the definition of ν is equivariant, we have that ν is a multi-moment map. ☐

6 Reduction of torsion-free G_2-manifolds

Let us recall the fundamental aspects of G_2-geometry from [6]. On $\mathbb{R}^7$ we consider the three-form ϕ_0 given by

$$\phi_0 = e_{123} + e_1(e_{45} + e_{67}) + e_2(e_{46} - e_{57}) - e_3(e_{47} + e_{56}), \tag{6.1}$$

where $e_1, \dots, e_7$ is the standard dual basis and wedge signs have been omitted. The stabiliser of ϕ_0 is the compact 14-dimensional Lie group

$$G_2 = \{\, g \in GL(7, \mathbb{R}) : g^* \phi_0 = \phi_0 \,\}.$$

This group preserves the standard metric on $g_0 = \sum_{i=1}^{7} e_i^2$ on $\mathbb{R}^7$ and the volume form $\mathrm{vol}_0 = e_{1234567}$. These tensors are uniquely determined by ϕ_0 via the relation $6 g_0(X, Y) \mathrm{vol}_0 = (X \lrcorner \phi_0) \wedge (Y \lrcorner \phi_0) \wedge \phi_0$. The Hodge $*$-operator gives a four-form

$$*\phi_0 = e_{4567} + e_{23}(e_{67} + e_{45}) + e_{13}(e_{57} - e_{46}) - e_{12}(e_{56} + e_{47}).$$

A G_2-structure on a seven-manifold Y is given by a three-form $\phi \in \Omega^3(Y)$ which is linearly equivalent at each point to ϕ_0. It determines a metric g, an volume form vol and a four-form $*\phi$ on Y. The G_2-structure is called *torsion-free* if both of the forms ϕ and $*\phi$ are closed. This happens precisely when $\nabla^{\mathrm{LC}}\phi = 0$ [18]. One then calls (Y, ϕ) a torsion-free G_2-manifold. In this situation the metric g has holonomy contained in G_2.

Since a torsion-free G_2-geometry comes equipped with a closed three-form, we may study multi-moment maps for such manifolds. Let us assume that (Y, ϕ) has a two-torus symmetry with a non-constant multi-moment map $\nu\colon Y \to \mathcal{P}^*_{\mathbb{R}^2} \cong \mathbb{R}$. Choosing generating vector fields U and V for the T^2-action, we have $d\nu = \phi(U, V, \cdot)$. The latter is non-zero if and only if U and V are linearly independent. So T^2 acts locally freely on some open set $Y_0 \subset Y$.

We may define three two-forms on Y_0 by

$$\omega_0 = V \lrcorner U \lrcorner *\phi, \quad \omega_1 = U \lrcorner \phi \quad \text{and} \quad \omega_2 = V \lrcorner \phi.$$

To relate these to the G_2-structure consider the positive function h and one-forms θ_i given by

$$\begin{gathered} (g_{UU}g_{VV} - g_{UV}^2)\, h^2 = 1 \\ \theta_1 = h^2(g_{VV}U^\flat - g_{UV}V^\flat), \quad \theta_2 = h^2(g_{UU}V^\flat - g_{UV}U^\flat), \end{gathered}$$

where $U^\flat = g(U, \cdot)$ and $g_{UU} = g(U, U)$, etc. Note that h is well-defined on Y_0, and that (θ_1, θ_2) is dual to (U, V).

Proposition 6.1. *On Y_0, the three-form ϕ and the four-form $*\phi$ are*

$$\begin{aligned} \phi &= h^2\omega_0 \wedge d\nu + \omega_1 \wedge \theta_1 + \omega_2 \wedge \theta_2 + d\nu \wedge \theta_2 \wedge \theta_1, \\ *\phi &= \omega_0 \wedge \theta_1 \wedge \theta_2 + h^2\bigl(g_{VV}\omega_1 \wedge \theta_2 \wedge d\nu - g_{UU}\omega_2 \wedge \theta_1 \wedge d\nu \\ &\qquad + g_{UV}(\omega_1 \wedge \theta_1 - \omega_2 \wedge \theta_2) \wedge d\nu + \tfrac{1}{2}\omega_0 \wedge \omega_0\bigr). \end{aligned}$$

Proof. Working locally at a point and using the T^2-action we may write the first two standard basis elements of $\mathbb{R}^7$ as $E_1 = aU = U/g_{UU}^{1/2}$, $E_2 = bU + cV = hg_{UU}^{1/2}(V - g_{UV}g_{UU}^{-1}U)$. We then have $\theta_1 = ae_1 + be_2$ and $\theta_2 = ce_2$. Now using (6.1) we get $ac\, d\nu = e_3$, $ac\, \omega_0 = -(e_{56} + e_{47})$, $a\, \omega_1 = e_{23} + e_{45} + e_{67}$ and

$$ac\, \omega_2 = -a(e_{13} - e_{46} + e_{57}) - b(e_{23} + e_{45} + e_{67}).$$

The given expressions now follow. □

Now suppose that $t \in \nu(Y_0) \subset \mathbb{R}$ is a regular value for $\nu\colon Y_0 \to \mathbb{R}$. Then $\mathcal{X}_t = \nu^{-1}(t)$ is a smooth hypersurface with unit normal $N = h(d\nu)^\sharp$. This inherits an $SU(3)$-structure $(\sigma, \psi_\pm)$ given by

$$\begin{aligned}\sigma = N \lrcorner \phi = h\omega_0 + h^{-1}\theta_1 \wedge \theta_2, \quad \psi_+ = \iota^*\phi = \iota^*\omega_1 \wedge \theta_1 + \iota^*\omega_2 \wedge \theta_2,\\ \psi_- = -N \lrcorner *\phi = h\big(g_{VV}\iota^*\omega_1 \wedge \theta_2 - g_{UU}\iota^*\omega_2 \wedge \theta_1 \\ + g_{UV}(\iota^*\omega_1 \wedge \theta_1 - \iota^*\omega_2 \wedge \theta_2)\big),\end{aligned} \tag{6.2}$$

where $\iota\colon X_t \to Y_0$ is the inclusion. As shown in [11], oriented hypersurfaces in torsion-free G_2-manifolds are *half-flat*, meaning that

$$\sigma \wedge d\sigma = 0 \quad \text{and} \quad d\psi_+ = 0. \tag{6.3}$$

Suppose T^2 acts freely on $\mathcal{X}_t = \nu^{-1}(t)$.

Definition 6.2. The T^2 *reduction* of Y at level t is the four-manifold

$$M = \nu^{-1}(t)/T^2 = \mathcal{X}_t/T^2.$$

Proposition 6.3. *The T^2 reduction M carries three pointwise linearly independent symplectic forms defining the same orientation.*

Proof. Consider the two-forms ω_0, ω_1, ω_2 on Y_0. These forms are T^2-invariant and closed, since $d\omega_0 = \mathcal{L}_V(U \lrcorner *\phi) = 0$ and $d\omega_1 = \mathcal{L}_U\phi = 0$, cf. (2.1). Furthermore, as $V \lrcorner \omega_1 = d\nu$, their pull-backs to $\mathcal{X}_t = \nu^{-1}(t)$ are basic. Thus they descend to three closed forms σ_0, σ_1 and σ_2 on M. The proof of Proposition 6.1 shows that at a point $h\sigma_0 = -(e_{56} + e_{47})$, $h\sigma_1 = c(e_{45} + e_{67})$ and $h\sigma_2 = a(e_{46} + e_{75}) - b(e_{45} + e_{67})$, with $ac = h \neq 0$. Thus σ_0, σ_1 and σ_2 are non-degenerate symplectic forms defining the same orientation. □

The expressions for the forms in this proof show that they satisfy the following relations on M:

$$\begin{gathered} h^2\,{\sigma_0}^2 = g_{UU}^{-1}\,{\sigma_1}^2 = g_{VV}^{-1}\,{\sigma_2}^2 = 2\,\mathrm{vol}_M, \\ \sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2, \quad \sigma_1 \wedge \sigma_2 = 2g_{UV}\,\mathrm{vol}_M\,. \end{gathered} \tag{6.4}$$

Here vol_M is induced by the element e_{4567} on Y, which is the volume element on directions orthogonal to the T^2-action on $\mathcal{X}_t$. Note that (θ_1, θ_2) is a connection one-form for $\mathcal{X}_t \to M$ regarded as a principal T^2-bundle.

We now consider how this construction may be inverted, producing the G_2-geometry of Y from a triple of symplectic forms on a four-manifold M. Note that the relations (6.4) show that the symplectic forms σ_i define the same orientation on M and are pointwise linearly independent. Indeed the intersection matrix $\tilde{Q} = (q_{ij})$ with $\sigma_i \wedge \sigma_j = q_{ij}\sigma_0^2$, for $i, j = 1, 2, 3$, is positive definite. As in [16], the positive three-dimensional subbundle $\Lambda^+ = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \subset \Lambda^2 T^*M$ corresponds to a unique oriented conformal structure on M.

Definition 6.4. A *coherent symplectic triple* $\mathscr{C}$ on a four-manifold M consists of three symplectic forms σ_0, σ_1, σ_2 that pointwise span a maximal positive subspace of $\Lambda^2 T^*M$ and satisfy $\sigma_0 \wedge \sigma_i = 0$ for $i = 1, 2$.

Let $Q = (q_{ij})_{i,j=1,2}$ be the lower-right 2×2 submatrix of $\tilde{Q}$. Since $\det Q$ is positive, we may write $h = \sqrt{\det Q} \in C^\infty(M)$.

Proposition 6.5. *Let $(M, \mathscr{C})$ be a coherently tri-symplectic four-manifold. Suppose $\mathcal{X}$ is a principal T^2-bundle over M with connection one-form $\Theta = (\theta_1, \theta_2)$. Then the forms σ, $\psi_\pm$ given by*

$$\begin{gathered} \sigma = h\sigma_0 + h^{-1}\theta_1 \wedge \theta_2, \quad \psi_+ = \sigma_1 \wedge \theta_1 + \sigma_2 \wedge \theta_2, \\ \psi_- = h^{-1}(q_{22}\sigma_1 \wedge \theta_2 - q_{11}\sigma_2 \wedge \theta_1 + q_{12}(\sigma_1 \wedge \theta_1 - \sigma_2 \wedge \theta_2)) \end{gathered} \tag{6.5}$$

define an $SU(3)$-structure on $\mathcal{X}$. This structure is half-flat if and only if $d\Theta^+ = (\sigma_1, \sigma_2)A$ with $\operatorname{Tr}(AQ) = 0$.

Proof. Choose a conformal basis $e_4, \dots, e_7$ of $T^*_x M$ so that $h\sigma_i$ are as in the proof of Proposition 6.3 with $c^2 = q_{11}$, $bc = -q_{12}$ and $a^2 = q_{22} - b^2$. This is consistent with the equation $ac = h$. Now inspired by the proof of Proposition 6.1 we write $\theta_1 = ae_1 + be_2$ and $\theta_2 = ce_2$. The basis $e_1, e_2, e_7, e_4, e_6, e_5$ is then an $SU(3)$-basis for $T^*\mathcal{X}$, with defining forms given via (6.2) for $g_{UU} = q_{11}/h^2$, $g_{UV} = q_{12}/h^2$ and $g_{VV} = q_{22}/h^2$.

For the final assertion we need to study the equations (6.3). Firstly, $\sigma \wedge d\sigma = \sigma_0 \wedge d\theta_1 \wedge \theta_2 + \sigma_0 \wedge d\theta_2 \wedge \theta_1$, which vanishes only if $d\Theta^+$ is orthogonal to σ_0. This implies that $d\Theta^+$ is a linear combination $(\sigma_1, \sigma_2)A$ of σ_1 and σ_2. Now $d\psi_+ = \sigma_1 \wedge d\theta_1 + \sigma_2 \wedge d\theta_2$, and the vanishing of $d\psi_+$ gives the constraint $\operatorname{Tr}(AQ) = 0$. □

Remark 6.6. The $SU(3)$-structures found here are more general than those studied in [23] since the connection one-forms are not orthonormal. △

Example 6.7. Consider $Y = \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ endowed with the usual three-form and the action of the standard diagonal maximal torus $T^2 \subset SU(3)$. Concretely, ϕ is given by

$$\phi = \tfrac{i}{2} dx \wedge (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) + \operatorname{Re}(dz_1 \wedge dz_2 \wedge dz_3),$$

and T^2 acts by $(e^{i\theta}, e^{i\varphi}) \cdot (x, z_1, z_2, z_3) = (x, e^{i\theta}z_1, e^{i\varphi}z_2, e^{-i(\theta+\varphi)}z_3)$. The action is generated by the vector fields $U = \operatorname{Re}\{i(z_1\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_3})\}$ and $V = \operatorname{Re}\{i(z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3})\}$. It follows that the multi-moment map $\nu\colon Y \to \mathbb{R}$ is given by

$$\nu(x, z_1, z_2, z_3) = -\tfrac{1}{4}\operatorname{Re}(z_1 z_2 z_3).$$

By definition, the T^2-reduction of Y at level t is the quotient space $M_t = \nu^{-1}(t)/T^2$. In this case M_0 is singular, whereas M_t is a smooth manifold for each $t \neq 0$. Indeed considering $\Phi_t\colon M_t \to \mathbb{R}^4$ given by

$$\begin{aligned}\Phi_t(x,z_1,z_2,z_3) &= \big(x, \tfrac{1}{2}(\|z_1\|^2 - \|z_3\|^2), \tfrac{1}{2}(\|z_2\|^2 - \|z_3\|^2), \mathrm{Im}(z_1z_2z_3)\big)\\ &=: (x,u,v,w)\end{aligned}$$

we have global smooth coordinates on M_t for $t \neq 0$.

In this smooth case, writing $4\eta_u = h^2(g_{VV}du - g_{UV}dv)$ and $4\eta_v = h^2(g_{UU}dv - g_{UV}du)$, the two-forms σ_0, σ_1, σ_2 are given by

$$\begin{gathered}4\sigma_0 = dx\wedge dw + dv\wedge du, \quad 2\sigma_1 = dx\wedge du + dw\wedge\eta_v,\\ 2\sigma_2 = dx\wedge dv + \eta_u\wedge dw.\end{gathered}$$

These forms depend (implicitly) on t via the relations $4g_{UU} = \|z_1\|^2 + \|z_3\|^2$, $4g_{VV} = \|z_2\|^2 + \|z_3\|^2$, $4g_{UV} = \|z_3\|^2$ and $z_1z_2z_3 = -4t + iw$. In particular, g_{UV} is a non-constant function, so the coherent triple does not specify a hyperKähler a structure. The (oriented) conformal class has representative metric

$$dx^2 + \frac{h^2}{16}dw^2 + 4g_{UU}\eta_u^2 + 4g_{VV}\eta_v^2 + 4g_{UV}(\eta_u\eta_v + \eta_v\eta_u).$$

The curvature of the principal bundle $\nu^{-1}(t) \to M_t$ is given by

$$\begin{aligned}4d\theta_1 &= th^4 dw\wedge\big((2g_{VV} - g_{UV})\eta_u + (g_{VV} - 2g_{UV})\eta_v\big)\\ 4d\theta_2 &= th^4 dw\wedge\big((g_{UU} - 2g_{UV})\eta_u + (2g_{UU} - g_{UV})\eta_v\big).\end{aligned}$$

In the singular case $t = 0$, the two-torus collapses in two ways: to a point along the real axis $\mathbb{R}\times\{0\} \subset \mathbb{R}\times\mathbb{C}^3$ and to a circle away from $\mathbb{R}\times\{0\}$ along $z_1 = z_2 = 0$, $z_1 = z_3 = 0$ or $z_2 = z_3 = 0$. The collapsing happens when $w = 0$ and u, v satisfy one of the following three constraints: $(u = v \leqslant 0)$, $(u = 0,\ v \geqslant 0)$ or $(u \geqslant 0,\ v = 0)$. ◇

Studying a certain Hamiltonian flow, Hitchin [29] developed a relationship between torsion-free G_2-metrics and half-flat $SU(3)$-manifolds. In particular, he derived evolution equations that describe the one-dimensional flow of a half-flat $SU(3)$-manifold along its unit normal in a torsion-free G_2-manifold. When the flow equations have a solution, this determines a torsion-free G_2-metric from a half-flat $SU(3)$-manifold. In inverting our construction, one could use Hitchin's flow on the half-flat structure of Proposition 6.5. However, Hitchin's flow does not preserve the level sets of the multi-moment map: the unit normal is $h(d\nu)^\sharp$, but $\partial/\partial\nu = h^2(d\nu)^\sharp$. It is thus more natural for us to determine the flow equations associated to the latter vector field.

Proposition 6.8. *Suppose T^2 acts freely on a connected seven-manifold Y preserving a torsion-free G_2-structure ϕ and admitting a multi-moment map ν. Let M be the topological reduction $\nu^{-1}(t)/T^2$ for any t in the image of ν. Then M is equipped with a t-dependent coherent symplectic triple $\sigma_0, \sigma_1, \sigma_2$ and $\mathcal{X}_t = \nu^{-1}(t)$ carries the half-flat $SU(3)$-structure $(\sigma, \psi_\pm)$ of Proposition 6.5. The forms on $\mathcal{X}_t$ satisfy the following system of differential equations:*

$$\begin{aligned} \psi_+' &= d(h\sigma) \\ (\tfrac{1}{2}\sigma^2)' &= -d(h\psi_-), \end{aligned} \tag{6.6}$$

where $'$ denotes differentiation with respect to t.

Moreover, given a half-flat $SU(3)$-structure on a six-manifold $\mathcal{X}_0$, the system (6.6) *has at most one solution and that solution determines a torsion-free G_2-structure on $\mathcal{X}_0 \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.*

Proof. We have

$$\phi = \sigma \wedge h d\nu + \psi_+ \quad \text{and} \quad {*\phi} = \psi_- \wedge h d\nu + \tfrac{1}{2}\sigma^2.$$

These have derivatives

$$\begin{aligned} d\phi &= (h d\sigma + dh \wedge \sigma) \wedge d\nu + d\psi_+, \\ d{*\phi} &= (h d\psi_- + dh \wedge \psi_-) \wedge d\nu + \sigma \wedge d\sigma \end{aligned}$$

Half-flatness of $(\sigma, \psi_\pm)$ gives $d\phi = 0 = d{*\phi}$ if and only if

$$0 = \frac{\partial}{\partial\nu} \lrcorner\, d\phi = -d(h\sigma) + \psi_+' \quad \text{and} \quad 0 = \frac{\partial}{\partial\nu} \lrcorner\, d{*\phi} = d(h\psi_-) + \sigma \wedge \sigma'.$$

Hence we have a torsion-free G_2-structure if and only if the evolution equations (6.6) are satisfied.

To demonstrate uniqueness of the solutions we rewrite the evolution equations as a complete set of first order differential equations for the data on M. Firstly, the derivatives of σ_0, σ_1, σ_2 and h with respect to $\partial/\partial\nu$ are:

$$\begin{gathered} \sigma_0' = 0, \quad \sigma_1' = -d\theta_2, \quad \sigma_2' = d\theta_1, \\ hh'\sigma_0^2 = (q_{11}\sigma_2 - q_{12}\sigma_1) \wedge d\theta_1 + (q_{12}\sigma_2 - q_{22}\sigma_1) \wedge d\theta_2. \end{gathered} \tag{6.7}$$

Using (6.7) and the definition of Q, we obtain the following equations:

$$q_{11}'\sigma_0^2 = -2\sigma_1 \wedge d\theta_2, \quad q_{22}'\sigma_0^2 = 2\sigma_2 \wedge d\theta_1, \quad q_{12}'\sigma_0^2 = \sigma_1 \wedge d\theta_1 - \sigma_2 \wedge d\theta_2. \tag{6.8}$$

Finally, combining (6.5) and (6.6), we obtain a relation for the derivatives of the connection one-form (θ_1, θ_2):

$$\sigma_0 \wedge \theta_1' = dq_{12} \wedge \sigma_2 - dq_{22} \wedge \sigma_1, \quad \sigma_0 \wedge \theta_2' = dq_{11} \wedge \sigma_2 - dq_{12} \wedge \sigma_1. \tag{6.9}$$

□

Remark 6.9. Modifying the arguments in the proof of [12, Theorem 2.3], one may verify that the evolution equations (6.6) together with an initial half-flat $SU(3)$-structure on $\mathcal{X}_0$ already ensure that the family consists of half-flat structures. If the initial data are analytic, we can solve the flow equations and thereby obtain a holonomy G_2-metric with T^2-symmetry. Indeed, if g_M is the time-dependent metric in the conformal class on M with volume form $\frac{1}{2}h^2\sigma_0^2$, then the G_2-metric is explicitly

$$h^2 dt^2 + g_M + h^{-2}(q_{11}\theta_1^2 + q_{22}\theta_2^2 + q_{12}(\theta_1\theta_2 + \theta_2\theta_1)).$$

Note that Bryant's study of the Hitchin flow [7] shows that non-analytic initial data can lead to an ill-posed system that has no solution. △

Summarising the results of this section we have:

Theorem 6.10. *Let (Y^7, ϕ) be a torsion-free G_2-structure with a free T^2-symmetry. Then the reduction Y at a level t is a coherently tri-symplectic four-manifold and the level set $\mathcal{X}_t$ is a T^2-bundle over M satisfying the orthogonality condition on $F_+ = d\Theta^+$ of Proposition 6.5.*

Conversely a coherently tri-symplectic four-manifold together with an orthogonal $F_+ \in \Omega^2(M, \mathbb{R}^2)$ with integral periods define a torsion-free G_2-metric with T^2-symmetry provided the flow equations (6.7), (6.8), (6.9) *admit a solution.* □

Example 6.11. Consider a complex-symplectic Kähler surface M. Let σ_0 be the Kähler form and write the complex sympletic form σ_c as $\sigma_1 + i\sigma_2$. Then $\sigma_1^2 = \sigma_2^2$ and $\sigma_1 \wedge \sigma_2 = 0 = \sigma_0 \wedge \sigma_1 = \sigma_0 \wedge \sigma_2$, so that these three forms constitute a coherent symplectic triple. The matrix Q is proportional to the identity: $h^2 = q_{11}^2 = q_{22}^2$ and $q_{12} = 0$. Let us assume that σ_1 and σ_2 have integral periods. Then we can construct a T^2-bundle $\mathcal{X}_0$ over M with curvature $(d\theta_1, d\theta_2) = (\sigma_2, -\sigma_1)$. Let us show how to solve the flow equations (6.7)–(6.9) in this case.

We look for solutions satisfying $(d\theta_1(t), d\theta_2(t)) = (\sigma_2(t), -\sigma_1(t))$ for small t. In this case the system becomes:

$$\sigma_0' = 0, \quad \sigma_1' = \sigma_1, \quad \sigma_2' = \sigma_2, \quad q_{11}' = 2q_{11}, \quad q_{22}' = 2q_{22}, \quad q_{12}' = 2q_{12},$$
$$h' = 2h, \quad \sigma_0 \wedge \theta_1' = dq_{12} \wedge \sigma_2 - dq_{22} \wedge \sigma_1, \quad \sigma_0 \wedge \theta_2' = dq_{11} \wedge \sigma_2 - dq_{12} \wedge \sigma_1.$$

This has the following solution on $\mathcal{X}_0 \times \mathbb{R}$:

$$\sigma_0(t) = \sigma_0, \quad \sigma_1(t) = e^t\sigma_1, \quad \sigma_2(t) = e^t\sigma_2, \quad h(t) = he^{2t},$$
$$q_{11}(t) = q_{22}(t) = he^{2t}, \quad q_{12}(t) = 0, \quad \theta_i(t) = \tfrac{1}{3}a_i e^{3t} + b_i, \quad \text{for } i = 1, 2,$$

where σ_1, with omission of the parameter t, denotes the initial value $\sigma_1(0)$, etc. The forms a_i and b_i are uniquely determined by the following relations:

$$\sigma_0 \wedge a_1 = -dh \wedge \sigma_1, \quad \sigma_0 \wedge a_2 = dh \wedge \sigma_2, \quad \theta_i = \tfrac{1}{3}a_i + b_i.$$

The G_2 three-form and metric corresponding to this solution are

$$\begin{aligned} \phi &= e^{2t}\sigma_0 \wedge dt + \theta_1(t) \wedge \theta_2(t) \wedge dt + e^t(\sigma_1 \wedge \theta_1(t) + \sigma_2 \wedge \theta_2(t)), \\ g &= e^{4t}dt^2 + he^{2t}g_0 + h^{-1}e^{-2t}(\theta_1^2 + \theta_2^2), \end{aligned} \tag{6.10}$$

where g_0 is the Kähler metric on M associated to σ_0. The initial factor $e^{4t}dt^2$ in g implies that the G_2-metric is incomplete.

Note that we may apply the above computations to examples such as the Fermat quartic, without having to find a hyperKähler metric (in contrast to [2, Theorem 2] and [22]). In such examples, the existence of a hyperKähler metric is given implicitly through Yau's proof of the Calabi conjecture, but no explicit expression is known. On the other hand for the Fermat quartic $(z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0) \subset \mathbb{C}P(3)$ we may instead take the natural choice of coherent symplectic triple determined by the embedding, with σ_0 the restriction of the the Fubini-Study form on $\mathbb{C}P(3)$. A local expression for the complex symplectic form $\sigma_c = \sigma_1 + i\sigma_2$ is readily derived using the Poincaré residue map. In the chart $(z_0 \neq 0)$ it is

$$\frac{dw_1 \wedge dw_2}{4w_3^3} = \frac{dw_2 \wedge dw_3}{4w_1^3} = \frac{dw_3 \wedge dw_1}{4w_2^3},$$

where $w_i = z_i/z_0$, which may be scaled so that σ_1 and σ_2 have integral periods. After this scaling, (6.10) gives explicit T^2-invariant torsion-free G_2-metrics over this base. ♢

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