

RECENT PROGRESS ON SINGULARITIES OF LAGRANGIAN MEAN CURVATURE FLOW

ANDRÉ NEVES

Dedicated to Professor Richard Schoen on his sixtieth birthday.

ABSTRACT. We survey some of the state of the art regarding singularities in Lagrangian mean curvature flow. Some open problems are suggested at the end.

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1. INTRODUCTION

Since Yau's solution to the Calabi Conjecture, Calabi-Yau manifolds and minimal Lagrangians (called special Lagrangians) have acquired a central role in Geometry and Mirror Symmetry over the last 30 years. Unfortunately, the most basic question one can ask about special Lagrangians, whether they exist in a given homology or Hamiltonian isotopy class, is still largely open. Special Lagrangians are area-minimizing and so one could approach the existence problem by trying to minimize area among all Lagrangians in a given class. Schoen–Wolfson [25] studied the minimization problem and showed that, when the real dimension is four, a Lagrangian minimizing area among all Lagrangians in a given class exists, is smooth everywhere except finitely many points, but not necessarily a minimal surface. Later Wolfson [42] found a Lagrangian sphere Σ with nontrivial homology on a given K3 surface for which the Lagrangian which minimizes area among all Lagrangians homologous to Σ is not a special Lagrangian and the surface which minimizes area among all surfaces homologous to Σ is not Lagrangian. This shows the subtle nature of the problem and that variational methods

do not seem to be very effective. For this reason there has been increased interest in evolving a given Lagrangian submanifold by the gradient flow for the area functional (Lagrangian mean curvature flow) and hope to obtain convergence to a special Lagrangian.

Initially there was a source of optimism and, under the assumption that the tangent planes of the initial Lagrangian lie in some convex subset of the Grassmanian bundle, Smoczyk, Tsui, and Wang [30, 32, 36, 37] proved that the Lagrangian mean curvature flow exists for all time and converge to a special Lagrangian. Similar results were also obtained in the symplectic or graphical setting by Chen, Li, Smoczyk, Tian, Tsui, and Wang [11, 30, 31, 32, 34, 35, 36, 37, 38]. Unfortunately, the minimal surfaces which were produced by this method were already known to exist which means that, in order to find new special Lagrangians, one should drop the convexity assumptions on the image of the Gauss map. The drawback in doing so is that long-time existence can no longer be assured and, as a matter of fact, the author showed [20] that finite-time singularities do occur for very “well-behaved” initial conditions.

Theorem 1.1. *There is $L \subset \mathbb{C}^2$ Lagrangian, asymptotic to two planes at infinity, and with arbitrarily small oscillation of the Lagrangian angle so that the solution to mean curvature flow develops finite time singularities.*

These examples all live in \mathbb{C}^2 and so it was a natural open question whether, in a compact Calabi-Yau, one could have “good” initial conditions which develop finite time singularities under the flow. As a matter of fact, Thomas and Yau [33] proposed a notion of “stability” for the flow (see either [33] or [23] for the details) and conjectured that Lagrangian mean curvature flow of “stable” initial conditions will exist for all time and converges to a special Lagrangian. Unfortunately, their stability condition is in general hard to check and it seems to be a highly nontrivial statement the existence of Lagrangians which are “stable” in their sense and not special Lagrangian. Thus Wang [39] simplified the Thomas-Yau conjecture to become

Conjecture. *Let L be a Lagrangian in a Calabi-Yau manifold which is embedded and Hamiltonian isotopic to a special Lagrangian Σ . Then the Lagrangian mean curvature flow exists for all time and converges to Σ .*

Schoen and Wolfson [26] constructed solutions to Lagrangian mean curvature flow which become singular in finite time and where the initial condition is homologous to a special Lagrangian. On the other hand, we remark that the flow *does* distinguish between isotopy class and homology class. For instance, on a two dimensional torus, a curve γ with a single self intersection which is homologous to a simple closed geodesic will develop a finite time singularity under curve shortening flow while if we make the more restrictive assumption that γ is isotopic to a simple closed geodesic, Grayson’s Theorem [13] implies that the curve shortening flow will exist for all time and sequentially converge to a simple closed geodesic.

To this end, the author has recently shown [23, Theorem A] that Wang’s conjecture is false.

Theorem 1.2. *Let M be a four real dimensional Calabi-Yau and Σ an embedded Lagrangian. There is L Hamiltonian isotopic to Σ so that the*

Lagrangian mean curvature flow starting at L develops a finite time singularity.

In any case the upshot is that it will be hard to avoid singularities for Lagrangian mean curvature flow and so it is important to understand how singularities form if one expects to use the flow to produce special Lagrangians. The subject is still in its infancy and so the purpose of this survey is to collect some of the basic techniques that have been used to tackle singularity formation and exemplify how they can be applied in simple cases. For more on long time existence and convergence results the reader is encouraged to read [39, 40].

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2. PRELIMINARIES

Let J and ω denote, respectively, the standard complex structure on \mathbb{C}^n and the standard symplectic form on \mathbb{C}^n . We consider the closed complex-valued n -form given by

$$\Omega \equiv dz_1 \wedge \dots \wedge dz_n$$

and the Liouville form given by

$$\lambda \equiv \sum_{i=1}^n x_i dy_i - y_i dx_i, \quad d\lambda = 2\omega,$$

where $z_j = x_j + iy_j$ are complex coordinates of \mathbb{C}^n . We set

$$B_S = \{x \in \mathbb{C}^n \mid |x| < S\} \quad \text{and} \quad A(R, S) = \{x \in \mathbb{C}^n \mid R < |x| < S\}.$$

Given $f \in C^1(\mathbb{C}^n)$, Df denotes its gradient in \mathbb{C}^n and ∇f its gradient in L .

A smooth n -dimensional submanifold L in \mathbb{C}^n is said to be *Lagrangian* if $\omega_L = 0$ and a simple computation shows that

$$\Omega_L = e^{i\theta} \text{vol}_L,$$

where vol_L denotes the volume form of L and θ is some multivalued function called the *Lagrangian angle*. When the Lagrangian angle is a single valued function the Lagrangian is called *zero-Maslov class* and if

$$\cos \theta \geq \varepsilon$$

for some positive ε , then L is said to be *almost-calibrated*. Furthermore, if $\theta \equiv \theta_0$, then L is calibrated by

$$\text{Re} \left(e^{-i\theta_0} \Omega \right)$$

and hence area-minimizing. In this case, L is referred as being *special Lagrangian*.

For a smooth Lagrangian, the relation between the Lagrangian angle and the mean curvature is given by the following remarkable property (see for instance [33])

$$H = J\nabla\theta.$$

A Lagrangian L_0 is said to be *rational* if for some real number a

$$\lambda(H_1(L_0, \mathbb{Z})) = \{a2k\pi \mid k \in \mathbb{Z}\}.$$

Any Lagrangian having $H_1(L_0, \mathbb{Z})$ finitely generated can be perturbed in order to become rational and so this condition is not very restrictive. When $a = 0$ the Lagrangian is called *exact* and this means there is $\beta \in C^\infty(L_0)$ for which $d\beta = \lambda$. Furthermore, if L_0 is also zero-Maslov class, it was shown in [20, Section 6] that the rational condition is preserved by Lagrangian mean curvature flow.

Let L_0 be a smooth Lagrangian in \mathbb{C}^n with area ratios bounded above, meaning there is C_0 so that

$$\mathcal{H}^n(L_0 \cap B_R(x)) \leq C_0 R^n \text{ for all } R > 0 \text{ and } x \in \mathbb{C}^n.$$

Under suitable conditions, bounded area ratios, Lagrangian, zero-Maslov class, and almost-calibrated are conditions which are preserved by the flow. All solutions to Lagrangian mean curvature flow considered in this survey are assumed to have polynomial area growth, bounded Lagrangian angle, and a primitive for the Liouville form with polynomial growth as well.

A submanifold L of Euclidean space is called a *self-expander* if $H = x^\perp/2$ and what this means is that $M_t = \sqrt{t}M$ is a smooth solution to mean curvature flow for all $t > 0$. If L is an exact and zero-Maslov class Lagrangian in \mathbb{C}^n then

$$H = \frac{x^\perp}{2} \implies 2J\nabla\theta = -J\nabla\beta \implies \nabla(\beta + 2\theta) = 0$$

and so $\beta + 2\theta$ is constant.

Given any (x_0, T) in $\mathbb{R}^{2n} \times \mathbb{R}$, we consider the backwards heat kernel

$$\Phi(x_0, T)(x, t) = \frac{\exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{(4\pi(T-t))^{n/2}}.$$

3. BASIC TECHNIQUES

I will describe the main technical tools that have been used to understand singularities.

3.1. White's Regularity Theorem. Let $(M_t)_{t \geq 0}$ be a smooth solution to mean curvature flow of k -submanifolds in \mathbb{R}^n . Consider the local Gaussian density ratios given by

$$\Theta_t(x_0, l) = \int_{M_t} \Phi(x_0, l)(x, 0) d\mathcal{H}^k.$$

The following theorem is proven in [41]. Its content is that if the local Gaussian density ratios are very close to one, the submanifolds enjoy a priori estimates on a slightly smaller set.

Theorem 3.1 (White's Regularity Theorem). *There are $\varepsilon_0 = \varepsilon_0(n, k)$, $C = C(n, k)$ so that if $\partial M_t \cap B_{2R} = \emptyset$ and*

$$\Theta_t(x, l) \leq 1 + \varepsilon_0 \text{ for all } l \leq R^2, x \in B_{2R}, \text{ and } t \leq R^2,$$

then the $C^{2,\alpha}$ -norm of M_t in B_R is bounded by C/\sqrt{t} for all $t \leq R^2$.

3.2. Monotonicity Formulas. In [15] Huisken proved the following fundamental identity.

Theorem 3.2 (Huisken's monotonicity formula). *Let f_t be a smooth family of functions on L_t . Then, assuming all quantities are finite,*

$$\begin{aligned} \frac{d}{dt} \int_{L_t} f_t \Phi(x_0, T) d\mathcal{H}^n &= \int_{L_t} \left(\frac{df_t}{dt} - \Delta f_t \right) \Phi(x_0, T) d\mathcal{H}^n \\ &\quad - \int_{L_t} f_t \left| H + \frac{(\mathbf{x} - x_0)^\perp}{2(T - t_0)} \right|^2 \Phi(x_0, T) d\mathcal{H}^n. \end{aligned}$$

The next lemma determines test functions to be used in Huisken's monotonicity formula.

Lemma 3.3. *Let $(L_t)_{t \geq 0}$ be a zero-Maslov class smooth solution to Lagrangian mean curvature flow. Then*

i) *There is a smooth family of functions $\theta_t \in C^\infty(L_t)$ such that*

$$H = J\nabla\theta_t \quad \text{and} \quad \frac{d}{dt}\theta_t^2 = \Delta\theta_t^2 - 2|H|^2;$$

ii) *Assume that L_0 is also exact. There is a smooth family of functions $\beta_t \in C^\infty(L_t)$ with $d\beta_t = \lambda$ and*

$$\frac{d}{dt}(\beta_t + 2(t - T)\theta_t)^2 = \Delta(\beta_t + 2(t - T)\theta_t)^2 - 2|2(t - T)H - x^\perp|^2;$$

iii) *If $\mu \in C^\infty(\mathbb{C}^n)$ is such that the one parameter family of diffeomorphisms $(\phi_s)_{s \geq 0}$ generated by $JD\mu$ is in $SU(n)$, then*

$$\frac{d}{dt}\mu^2 = \Delta\mu^2 - 2|\nabla\mu|^2;$$

iv) *If $n = 2$ and $\mu(z_1, z_2) = x_1y_2 - x_2y_1$, then*

$$\frac{d}{dt}\mu^2 = \Delta\mu^2 - 2|\nabla\mu|^2;$$

Remark 3.4. If L is special Lagrangian, the third identity implies that μ is harmonic in L , a fact which was observed by Joyce in [19, Lemma 3.4]. The geometric interpretation is that μ is obtained from the moment map of some group action.

Proof. The first two equations can be found in [20, Section 6]. We now show the third identity. It suffices to show that

$$\frac{d\mu}{dt} = \Delta\mu.$$

For each fixed t consider the family $L_{s,t} = \phi_s(L_t)$. It is simple to see that $L_{s,t}$ is Lagrangian for all s and the Lagrangian angle $\theta_{s,t}$ satisfies (see [33, Lemma 2.3])

$$\frac{d}{ds}\theta_{s,t} = \Delta\mu.$$

On the other hand, each $\phi_s \in SU(n)$, which means that $\theta_{s,t} = \theta_t \circ \phi_s^{-1}$ and thus $\frac{d}{ds}|_{s=0}\theta_{s,t} = -\langle \nabla\theta_t, Z \rangle$. Therefore

$$\frac{d\mu}{dt} = \langle H, D\mu \rangle = -\langle \nabla\theta_t, Z \rangle = \frac{d}{ds}|_{s=0}\theta_{s,t} = \Delta\mu.$$

To show the last identity one can either argue that the one parameter family of diffeomorphisms generated by $Z = JD\mu$ is in $SU(2)$ or see directly that, because each coordinate function evolves by the linear heat equation, we have

$$\frac{d\mu}{dt} = \Delta\mu - 2\langle X_1^\top, Y_2^\top \rangle + 2\langle Y_1^\top, X_2^\top \rangle,$$

where $X_i = Dx_i, Y_i = Dy_i$ for $i = 1, 2$ and

$$\begin{aligned} \langle X_1^\top, Y_2^\top \rangle - \langle Y_1^\top, X_2^\top \rangle &= -\langle (JY_1)^\top, Y_2 \rangle - \langle Y_1^\top, X_2 \rangle \\ &= -\langle JY_1^\perp, Y_2 \rangle - \langle Y_1^\top, X_2 \rangle = -\langle Y_1^\perp + Y_1^\top, X_2 \rangle = -\langle Y_1, X_2 \rangle = 0. \end{aligned}$$

□

This lemma can be combined with Theorem 3.2 to show

Corollary 3.5.

- i) *A smooth zero-Maslov class Lagrangian which is a self-shrinker must be a plane.*
- ii) *If $(L_t)_{t>0}$ is an exact and smooth zero-Maslov class solution to Lagrangian mean curvature flow with area ratios bounded below and such that L_{ε_i} converges in the varifold sense to a cone L_0 when ε_i tends to zero then, for all $t > 0$,*

$$L_t = \sqrt{t}L_1.$$

- iii) *Let μ be a function satisfying the conditions of Lemma 3.3 iii) or iv). If $(L_t)_{t>0}$ is a smooth solution to Lagrangian mean curvature flow such that, when t tends to zero, L_t tends, in the Radon measure sense, to a measure supported in $\mu^{-1}(0)$, then $L_t \subset \mu^{-1}(0)$ for all t .*

Remark 3.6.

- a) Assuming almost-calibrated, the first statement was proven by Wang in [35] (see also [9] for a similar result in the symplectic case). The second statement was proven in [22].
- b) It is important in i) that we assume L to have bounded Lagrangian angle and no boundary. Otherwise, as it was pointed out by Joyce, the universal cover of a circle or half circle in \mathbb{C} would be counterexamples.
- c) It is important in ii) that we assume L_t to be smooth for all $t > 0$ because otherwise the result would not be true. For instance, for curve shortening flow, σ_t could be $\{(x, y) \mid xy = 0\}$ for all $t \leq 2$ and $\sigma_t = \sqrt{t-2}\sigma_3$ for all $t > 2$, where σ_3 is a self-expander asymptotic to σ_2 .

Proof. To prove i) set $L_t = \sqrt{-t}L$ which is a smooth solution to Lagrangian mean curvature flow for $t < 0$. Choose $(x_0, T) = (0, 0)$ and consider

$$\theta(t) = \int_{L_t} \theta_t^2 \Phi(x_0, T) d\mathcal{H}^n.$$

Scale invariance implies that $\theta(t)$ is constant as a function of t and so its derivative must be zero. Hence, combining Theorem 3.2 with Lemma 3.3 i) we have that L has $H = 0$. Moreover, L is a self-shrinker and so it must

have $x^\perp + 2H = 0$, which means that L is a smooth minimal cone. Thus, L must be a plane.

To prove the second statement note that the function β_t can be defined as

$$\beta_t(x) = \int_{\gamma(p_t, x)} \lambda + \beta_t(p_t),$$

where p_t belongs to L_t and $\gamma(p_t, x)$ is any path in L_t connecting p_t to x .

Because L_0 is a varifold with $x^\perp = 0$ we have that $\lambda = 0$ when restricted to L_0 and thus, from varifold convergence and the fact area ratios are bounded below, we have that when t_i tends to zero β_{t_i} converges uniformly to a constant which we can assume to be zero. As a result, we obtain that

$$\gamma(t) = \int_{L_t} (2t\theta_t + \beta_t)^2 \Phi(0, 1) d\mathcal{H}^n$$

has $\gamma(t_i)$ tending to zero. Furthermore, we have from Theorem 3.2 and Lemma 3.3 ii) that

$$\frac{d}{dt} \gamma(t) \leq -2 \int_{L_t} |2tH - x^\perp|^2 \Phi(0, 1) d\mathcal{H}^n$$

which means that $\gamma(t)$ is non-increasing and so it must be zero for all t . Therefore $2tH - x^\perp = 0$ on L_t and this implies $L_t = \sqrt{t}L_1$.

To show iii) note that from Lemma 3.3 iii) and Theorem 3.2 we have for all $t < T$

$$\frac{d}{dt} \int_{L_t} \mu^2 \Phi(0, T) d\mathcal{H}^n \leq 0.$$

The result follows because

$$\lim_{t \rightarrow 0} \int_{L_t} \mu^2 \Phi(0, T) d\mathcal{H}^n = 0.$$

□

3.3. Poincaré type Lemma. In order to study blow-ups of singularities it is important to have a criteria which implies that a function α_i on N^i with L^2 norm of the gradient converging to zero must converge to a constant. It is simple to construct a sequence N^i (not necessarily Lagrangian) converging (in some suitable weak sense) to a disjoint union of two spheres S_1, S_2 and a sequence of functions α_i with L^2 norm of the gradient converging to zero so that α_i tends to 1 on S_1 and -1 on S_2 . The next proposition gives conditions which rule out this possibility.

Lemma 3.7. *Let (N^i) and (α_i) be a sequence of smooth k -submanifolds in \mathbb{R}^n and smooth functions on N^i respectively, such that the following properties hold for some $R > 0$:*

a) *There exists a constant D_0 such that*

$$\mathcal{H}^k(N^i \cap B_{3R}) \leq D_0 R^k \text{ for all } i \in \mathbb{N}$$

and

$$\left(\mathcal{H}^k(A) \right)^{(k-1)/k} \leq D_0 \mathcal{H}^{k-1}(\partial A)$$

for every open subset A of $N^i \cap B_{3R}$ with rectifiable boundary.

b) *There exists a constant D_1 such that for all $i \in \mathbb{N}$*

$$\sup_{N^i \cap B_{3R}} |\nabla \alpha_i| + R^{-1} \sup_{N^i \cap B_{3R}} |\alpha_i| \leq D_1.$$

c)

$$\lim_{i \rightarrow \infty} \int_{N^i \cap B_{3R}} |\nabla \alpha_i|^2 d\mathcal{H}^k = 0.$$

d) $\partial N^i \cap B_{3R} = \emptyset$ and $N^i \cap B_{2R}$ contains only one connected component which intersects B_R .

There is α such that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \sup_{N^i \cap B_R} |\alpha_i - \alpha| = 0.$$

Remark 3.8. A version of this lemma with stronger hypothesis was proven in [20, Proposition A.1]. Hypothesis a) is needed so that we have some control on the sequence N^i . Note that it rules out the example, described above, of N^i degenerating into two spheres. Hypothesis b) is also needed because if $N^i = \{(z, w) \in \mathbb{C}^2 \mid zw = 1/i\}$, it is not hard to construct a sequence α_i for which c) is true but α_i does not tend to a constant function. Finally, the last hypothesis is needed because otherwise the lemma would fail for trivial reasons.

Proof. Throughout this proof, $K = K(D_0, D_1, k)$ denotes a generic constant depending only on the mentioned quantities. Choose any sequence (x_i) in $N^i \cap B_R$. After passing to a subsequence, we have

$$\lim_{i \rightarrow \infty} x_i = x_0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \alpha_i(x_i) = \alpha$$

for some $x_0 \in B_R$ and $\alpha \in \mathbb{R}$. Furthermore, consider a sequence (ε_j) converging to zero and define

$$N^{i, \alpha, j} = \alpha_i^{-1}([\alpha - \varepsilon_j, \alpha + \varepsilon_j]).$$

The sequence (ε_j) can be chosen so that, for all $j \in \mathbb{N}$,

$$\lim_{i \rightarrow \infty} \mathcal{H}^{k-1}(\partial N^{i, \alpha, j} \cap B_{3R}) = 0$$

because, by the coarea formula, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{-\infty}^{\infty} \mathcal{H}^{k-1}(\{\alpha_i = s\} \cap B_{3R}) ds &= \lim_{i \rightarrow \infty} \int_{N^i \cap B_{3R}} |\nabla \alpha_i| d\mathcal{H}^k \\ &\leq \lim_{i \rightarrow \infty} KR^{k/2} \left(\int_{N^i \cap B_{3R}} |\nabla \alpha_i|^2 d\mathcal{H}^k \right)^{1/2} = 0. \end{aligned}$$

Lemma 3.9. *For every $j \in \mathbb{N}$*

$$\liminf_{i \rightarrow \infty} \mathcal{H}^k(N^{i, \alpha, j} \cap B_R(x_0)) \geq KR^k.$$

Proof. Given $y_i \in N^i$, denote by $\hat{B}_r(y_i)$ the *intrinsic* ball in N^i of radius r . We start by showing that $\mathcal{H}^k(\hat{B}_r(y_i)) \geq Kr^k$ for all $y_i \in B_{2R} \cap N^i$ and $r < R$. Set

$$\psi(r) = \mathcal{H}^k(\hat{B}_r(x_i))$$

which has, for all $r < R$, derivative given by

$$\psi'(r) = \mathcal{H}^{k-1} \left(\partial \hat{B}_r(y_i) \right) \geq K(\psi(r))^{(k-1)/k}.$$

Hence, integration implies $\psi(r) \geq Kr^k$ and the claim follows. From hypothesis b) there is $s_j = s(j, k, D_0, D_1, R) < R$ such that, for all i sufficiently large, $\hat{B}_{s_j}(x_i) \subset N^{i, \alpha, j}$ and thus

$$\mathcal{H}^k(B_s(x_i) \cap N^{i, \alpha, j}) \geq Ks^k \quad \text{for all } s \leq s_j.$$

Set

$$\psi_{i,j}(s) = \mathcal{H}^k(N^{i, \alpha, j} \cap B_s(x_i))$$

which has, by the coarea formula, derivative satisfying

$$\begin{aligned} \psi'_{i,j}(s) &= \oint_{\partial B_s(x_i) \cap N^{i, \alpha, j}} \frac{|x - x_i|}{|(x - x_i)^\top|} d\mathcal{H}^{k-1} \geq \mathcal{H}^{k-1}(\partial B_s(x_i) \cap N^{i, \alpha, j}) \\ &= \mathcal{H}^{k-1}(\partial(B_s(x_i) \cap N^{i, \alpha, j})) - \mathcal{H}^{k-1}(B_s(x_i) \cap \partial N^{i, \alpha, j}) \\ &\geq K \left(\mathcal{H}^k(B_s(x_i) \cap N^{i, \alpha, j}) \right)^{(k-1)/k} - \mathcal{H}^{k-1}(\partial N^{i, \alpha, j} \cap B_{3R}) \\ &= K(\psi_{i,j}(s))^{(k-1)/k} - \mathcal{H}^{k-1}(\partial N^{i, \alpha, j} \cap B_{3R}) \end{aligned}$$

for almost all s . Integration implies

$$\psi_{i,j}^{1/k}(R) \geq K(R - r_j) - \mathcal{H}^{k-1}(\partial N^{i, \alpha, j} \cap B_{3R}) \int_{r_j}^R \psi_{i,j}^{(1-k)/k}(t) dt,$$

where $r_j = \min\{s_j, KR/2\}$. Note the integral term is bounded independently of i for all i sufficiently large and so

$$\liminf_{i \rightarrow \infty} \psi_{i,j}^{1/k}(R) \geq K(R - r_j) \geq KR/2.$$

This proves Lemma 3.9. □

Suppose there is $y_i \in N^i \cap B_R$ converging to $y_0 \in B_R$ so that $\alpha_i(y_i)$ tends to $\bar{\alpha}$ distinct from α . Repeating the same type of arguments we find a closed interval I disjoint from $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$ such that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \mathcal{H}^k(\alpha_i^{-1}(I) \cap B_R(y_0)) \geq KR^k.$$

Given any positive integer p , pick disjoint closed intervals

$$I_1, \dots, I_p$$

lying between I and $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$. Hypothesis d) implies that, for all i sufficiently large, $\alpha_i^{-1}(I_l) \cap B_{2R}$ is not empty. Hence, arguing as before, we find y_1, \dots, y_p in B_{2R} such that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \mathcal{H}^k(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \geq KR^k,$$

for all l in $\{1, \dots, p\}$. This implies

$$\begin{aligned} \lim_{i \rightarrow \infty} \mathcal{H}^k(N^i \cap B_{2R}) &\geq \lim_{i \rightarrow \infty} \sum_{l=1}^p \mathcal{H}^k(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \\ &\geq pKR^k. \end{aligned}$$

Choosing p sufficiently large we get a contradiction. This proves Lemma 3.7. \square

The next result gives conditions which guarantee Lemma 3.7 a) holds.

Lemma 3.10. *Let L be a Lagrangian in \mathbb{C}^n such that $\partial L \cap B_R = \emptyset$ and either i)*

$$\inf_{L \cap B_R} \cos \theta \geq \delta$$

or ii) $n = 2$ and for some ε small enough

$$\int_{L \cap B_R} |H|^2 d\mathcal{H}^2 \leq \varepsilon.$$

There is $D = D(\delta, n)$ so that

$$(\mathcal{H}^n(A))^{(n-1)/n} \leq D\mathcal{H}^{n-1}(\partial A)$$

for all open subsets A of $L \cap B_R$ with rectifiable boundary.

Proof. We follow [20, Lemma 7.1] and prove i). The Isoperimetric Theorem [27, Theorem 30.1] guarantees the existence of an integral current B with compact support such that $\partial B = \partial A$ and for which

$$(\mathcal{H}(B))^{(n-1)/n} \leq C\mathcal{H}^{n-1}(\partial A),$$

where $C = C(n)$. If T denotes the cone over the current $A - B$ (see [27, page 141]), then $\partial T = A - B$ and thus, because

$$\operatorname{Re} \Omega_L = \cos \theta \geq \delta,$$

we obtain

$$\begin{aligned} \mathcal{H}^n(A) &\leq \delta^{-1} \int_A \operatorname{Re} \Omega = \delta^{-1} \int_B \operatorname{Re} \Omega + \delta^{-1} \int_{\partial T} \operatorname{Re} \Omega \\ &\leq \delta^{-1} \mathcal{H}^n(B) + \delta^{-1} \int_T d\operatorname{Re} \Omega \leq \delta^{-1} (C\mathcal{H}^{n-1}(\partial A))^{n/(n-1)}. \end{aligned}$$

To prove ii) we use Michael-Simon Sobolev inequality which implies (see [27, Theorem 18.6])

$$(\mathcal{H}^2(A))^{1/2} \leq C \int_A |H| + C\mathcal{H}^1(\partial A)$$

for some universal constant C . In this case we have

$$(\mathcal{H}^2(A))^{1/2} \leq C (\mathcal{H}^2(A))^{1/2} \left(\int_A |H|^2 \right)^{1/2} + C\mathcal{H}^1(\partial A)$$

and so we get the desired result whenever

$$C^2 \int_{L \cap B_R} |H|^2 \leq 1/4.$$

\square

3.4. Compactness Result. We state a compactness result for zero-Maslov class Lagrangians with bounded Lagrangian angle. The proof can be found in [20, Proposition 5.1].

Proposition 3.11. *Let L^i be a sequence of smooth zero-Maslov class Lagrangians in \mathbb{C}^n such that, for some fixed $R > 0$, the following properties hold:*

(a) *There exists a constant D_0 for which*

$$\mathcal{H}^n(L^i \cap B_{2R}) \leq D_0 R^n \quad \text{and} \quad \sup_{L^i \cap B_{2R}} |\theta_i| \leq D_0$$

for all $i \in \mathbb{N}$.

(b)

$$\lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)) = 0$$

and

$$\lim_{i \rightarrow \infty} \int_{L^i \cap B_{2R}(0)} |H|^2 d\mathcal{H}^n = 0.$$

Then, there exist a finite set $\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$ and integral special Lagrangian currents

$$L_1, \dots, L_N$$

such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported in $B_R(0)$ and every f in $C(\mathbb{R})$

$$\lim_{i \rightarrow \infty} \int_{L^i} f(\theta_i) \phi d\mathcal{H}^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi),$$

where μ_j and m_j denote, respectively, the Radon measure of the support of L_j and its multiplicity.

Remark 3.12. With the extra assumption that L^i is almost-calibrated, a similar result to Proposition 3.11 was proven in [10, Theorem 4.1]. The proposition is optimal in the sense that given Lagrangian planes P_1, P_2 intersecting transversely at the origin and two positive integers n_1, n_2 it is possible to construct a sequence of zero-Maslov class Lagrangians L^i with L^2 norm of mean curvature converging to zero and such that L^i tends to $n_1 P_1 + n_2 P_2$ in the varifold sense.

4. APPLICATIONS I: BLOW-UPS

Let $(L_t)_{0 \leq t < T}$ be a zero-Maslov class solution to Lagrangian mean curvature flow in \mathbb{C}^n with a singularity at x_0 at time T . Pick a sequence $(\lambda_i)_{i \in \mathbb{N}}$ tending to infinity and consider the sequence of blow-ups

$$L_s^i = \lambda_i(L_{T+s\lambda_i^{-2}} - x_0) \quad \text{for all } s < 0.$$

The next theorem was proven in [20, Theorem A] and in [10] assuming an extra almost-calibrated condition.

Theorem 4.1. *There exist integral special Lagrangian current cones*

$$L_1, \dots, L_N$$

with Lagrangian angles $\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$ such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s < 0$

$$\lim_{i \rightarrow \infty} \int_{L_s^i} f(\theta_{i,s}) \phi d\mathcal{H}^n = \sum_{j=1}^N m_j f(\bar{\theta}_j) \mu_j(\phi),$$

where μ_j and m_j denote the Radon measure of associated with L_j and its multiplicity respectively.

Furthermore, the set $\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$ does not depend on the sequence of rescalings chosen.

When $n = 2$ special Lagrangian cones are simply a union of planes having the same Lagrangian angle.

Sketch of proof. Set

$$\Theta_i(s) = \int_{L_s^i} \Phi(0, 0) d\mathcal{H}^n = \int_{L_{T+s\lambda_i^{-2}}} \Phi(0, T) d\mathcal{H}^n$$

and

$$\theta_i(s) = \int_{L_s^i} \theta_s^2 \Phi(0, 0) d\mathcal{H}^n = \int_{L_{T+s\lambda_i^{-2}}} \theta_{T+s\lambda_i^{-2}}^2 \Phi(0, T) d\mathcal{H}^n.$$

From Theorem 3.2 we have for $b < a < 0$

$$(1) \quad \int_b^a \int_{L_s^i} \left| H - \frac{x^\perp}{s} \right|^2 \Phi(0, 0) d\mathcal{H}^n ds = \Theta_i(a) - \Theta_i(b)$$

and

$$(2) \quad 2 \int_b^a \int_{L_s^i} |H|^2 \Phi(0, 0) d\mathcal{H}^n ds \leq \theta_i(a) - \theta_i(b).$$

But $\int_{L_t} \Phi(0, T) d\mathcal{H}^n$ and $\int_{L_t} \theta_t^2 \Phi(0, T) d\mathcal{H}^n$ are monotone non-increasing by Theorem 3.2 and thus

$$\begin{aligned} \lim_{i \rightarrow \infty} \Theta_i(a) &= \lim_{t \rightarrow T} \int_{L_t} \Phi(0, T) d\mathcal{H}^n = \lim_{i \rightarrow \infty} \Theta_i(b) \\ \lim_{i \rightarrow \infty} \theta_i(a) &= \lim_{t \rightarrow T} \int_{L_t} \theta_t^2 \Phi(0, T) d\mathcal{H}^n = \lim_{i \rightarrow \infty} \theta_i(b). \end{aligned}$$

Therefore, we obtain from (1) and (2) that

$$(3) \quad \lim_{i \rightarrow \infty} \int_b^a \int_{L_s^i} \left(|H|^2 + |x^\perp|^2 \right) \Phi(0, 0) d\mathcal{H}^n ds = 0.$$

The result follows from combining Proposition 3.11 with some standard facts of mean curvature flow. \square

When the initial condition is rational we obtain extra structure regarding the behavior of blow-ups.

Theorem 4.2. *Assume the initial condition is rational and, in case $n > 2$, almost-calibrated. Then, for almost all s_0 , if $\Sigma_i \subseteq L_{s_0}^i$ has $\partial \Sigma_i \cap B_{3R} = \emptyset$*

and only one connected component of $\Sigma_i \cap B_{2R}$ intersects B_R then, after passing to a subsequence, we can find $j \in \{1, \dots, N\}$ so that

$$\lim_{i \rightarrow \infty} \int_{\Sigma^i} f(\theta_{i,s_0}) \phi d\mathcal{H}^n = m f(\bar{\theta}_j) \mu_j(\phi),$$

for every f in $C^2(\mathbb{R})$ and every smooth ϕ compactly supported in $B_R(0)$, where $m \leq m_j$ and μ_j denotes the Radon measure associated with the special Lagrangian cone L_j given by Theorem 4.1

This theorem is slightly different from the one stated in [20, Theorem B] but the proof is identical. We sketch the main idea.

Sketch of proof. For simplicity we assume the initial condition is exact. Recall that $|\nabla \beta_{i,s}| = |x^\perp|$ and so, without loss of generality (see (3)), we can assume that, when $s = s_0$ or $s = -1$,

$$\lim_{i \rightarrow \infty} \int_{L_s^i} \left(|H|^2 + |\nabla \beta_{i,s}|^2 \right) \Phi(0,0) d\mathcal{H}^n = 0.$$

We now study the sequences Σ^i and L_{-1}^i .

From Proposition 3.11, we have that $\Sigma^i \cap B_{2R}$ converges in the varifold sense to a stationary varifold Σ with Radon measure μ_Σ , which can be represented as a sum of special Lagrangian cones with multiplicities. Furthermore in virtue of Lemmas 3.7 and 3.10 we conclude the existence of $\bar{\beta}$ so that

$$\lim_{i \rightarrow \infty} \int_{\Sigma^i} f(\beta_{i,s_0}) \phi d\mathcal{H}^n = f(\bar{\beta}) \mu_\Sigma(\phi)$$

for every f in $C^2(\mathbb{R})$ and every smooth function ϕ compactly supported in B_R .

Similar ideas to the ones use to prove Proposition 3.11 (see [20, Lemma 7.2] for details) show the existence of sets $\{\bar{\theta}_1, \dots, \bar{\theta}_Q\}$, $\{\bar{\beta}_1, \dots, \bar{\beta}_Q\}$, special Lagrangian cones L_1, \dots, L_Q , and integers m_1, \dots, m_Q so that for every smooth function ϕ compactly supported and every f in $C^2(\mathbb{R})$

$$\lim_{i \rightarrow \infty} \int_{L_{-1}^i} f(\bar{\beta}_{i,-1} - 2(s_0 + 1)\bar{\theta}_{i,-1}) \phi d\mathcal{H}^n = \sum_{j=1}^Q m_j f(\bar{\beta}_j - 2(s_0 + 1)\bar{\theta}_j) \mu_j(\phi),$$

where μ_j denotes the Radon measure of associated with L_j . Moreover, we can arrange things so that the pairs $(\bar{\theta}_j, \bar{\beta}_j)$ are all distinct and thus assume, without loss of generality, the numbers $\bar{\beta}_j - 2(s_0 + 1)\bar{\theta}_j$ are all distinct as well.

We now finish the proof. Let $f \in C^2(\mathbb{R})$ be a nonnegative cut off function which is one in $\bar{\beta}$ and zero at all but at most one element of

$$\{\bar{\beta}_j - 2(s_0 + 1)\bar{\theta}_j\}_{j=1}^Q$$

and ϕ a nonnegative function with compact support in B_R .

We have from (3) that

$$\lim_{i \rightarrow \infty} \int_{s_0}^{-1} \int_{L_s^i} \left(|H|^2 + |\nabla(\beta_{i,s} + 2(s - s_0)\theta_{i,s})|^2 \right) \Phi(0,0) d\mathcal{H}^n ds = 0$$

and so, using the evolution equation satisfied $\beta_{i,s} + 2(s - s_0)\theta_{i,s}$ (see Lemma 3.3), it is not hard to conclude that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{L_{s_0}^i} f(\bar{\beta}_{i,s_0}) \phi \, d\mathcal{H}^n &= \lim_{i \rightarrow \infty} \int_{L_{-1}^i} f(\bar{\beta}_{i,-1} - 2(s_0 + 1)\bar{\theta}_{i,-1}) \phi \, d\mathcal{H}^n \\ &= \sum_{j=1}^Q m_j f(\bar{\beta}_j - 2(s_0 + 1)\bar{\theta}_j) \mu_j(\phi). \end{aligned}$$

Therefore

$$\begin{aligned} \mu_\Sigma(\phi) &= \lim_{i \rightarrow \infty} \int_{\Sigma_i} \phi \, d\mathcal{H}^n = \lim_{i \rightarrow \infty} \int_{\Sigma_i} f(\beta_{i,s_0}) \phi \, d\mathcal{H}^n \\ &\leq \lim_{i \rightarrow \infty} \int_{L_{s_0}^i} f(\beta_{i,s_0}) \phi \, d\mathcal{H}^n = \sum_{j=1}^Q m_j f(\bar{\beta}_j - 2(s_0 + 1)\bar{\theta}_j) \mu_j(\phi). \end{aligned}$$

Because $\mu_\Sigma(\phi) > 0$ we have that $\bar{\beta} = \bar{\beta}_{j_0} - 2(s_0 + 1)\bar{\theta}_{j_0}$ for a unique j_0 and thus the inequalities above become

$$\mu_\Sigma(\phi) \leq m_{j_0} \mu_{j_0}(\phi)$$

for all $\phi \geq 0$ with compact support in B_R . This implies $\Sigma = mL_{j_0}$ for some $m \leq m_{j_0}$ in B_R , and the rest of the proof follows easily \square

The previous theorem does imply non-trivial statements regarding the blow-ups of singularities. We sketch one simple application, the details of which will appear elsewhere.

Corollary 4.3. *Assume the initial condition is rational and $n = 2$. The blow-up limit at a singularity cannot be two planes P_1, P_2 each with multiplicity one, distinct Lagrangian angles, and intersecting transversely at the origin, i.e., in Theorem 4.1 the case $N = 2$, $m_1 = m_2 = 1$, $P_1 \cap P_2 = \{0\}$, and $\bar{\theta}_1 \neq \bar{\theta}_2$ does not occur.*

Sketch of proof. We argue by contradiction and assume L_s^i converges to $P_1 + P_2$ for all $s < 0$. There is R_0 sufficiently large so that for every $0 \leq l \leq 4$ and $|x_0| > R_0/2$ we have

$$\int_{P_1 + P_2} \Phi(x_0, l)(x, 0) \, d\mathcal{H}^2 \leq 1 + \varepsilon_0/2$$

and thus, for all i sufficiently large, all $-2 \leq s < 0$, and all $0 \leq l \leq 2$, we also have

$$\Theta_s^i(x_0, l) \leq \Theta_{-2}^i(x_0, l + 2 + s) \leq 1 + \varepsilon_0,$$

where the first inequality follows from Theorem 3.2. Thus, we obtain from White's Regularity Theorem 3.1 that for any K large enough and i sufficiently large, we have uniform $C^{2,\alpha}$ bounds for L_s^i on the annulus $A(R_0, K)$ for all $-1 \leq s < 0$. Some extra work shows that, on the region $A(R_0, K)$ and for all $-1 \leq s < 0$, L_s^i can be decomposed into two connected components $\Sigma_{1,s}^i, \Sigma_{2,s}^i$ where $\Sigma_{j,s}^i$ is graphical over $P_j \cap A(R_0, K)$, with $j = 1, 2$.

We argue that $L_s^i \cap B_K$ must have two connected components for almost all $-1 \leq s < 0$. Otherwise we could apply Theorem 4.2 and conclude that the Lagrangian angle of P_1 must be identical to the Lagrangian angle of P_2 .

Some extra, but standard work, shows that $L_s^i \cap B_K$ can be decomposed into two connected components $\Sigma_{1,s}^i$ and $\Sigma_{2,s}^i$ where $\Sigma_{j,s}^i$ converges in the Radon measure sense to $P_j \cap B_K$. Hence, each $\Sigma_{j,s}^i$ is very close in the Radon measure sense to a multiplicity one disk. We can then apply White's Regularity Theorem 3.1 to each $(\Sigma_{j,s}^i)_{-1 \leq s < 0}$ and conclude, in a smaller ball centered at the origin, uniform bounds on the second fundamental form of $\Sigma_{j,s}^i$ for all $-1/2 \leq s < 0$ and all i sufficiently large. This implies uniform bounds for the second fundamental form of L_t in a neighborhood of the origin for all $t < T$ and hence no singularity occurs there. \square

5. APPLICATIONS II: SELF-EXPANDERS

Recently, Joyce, Lee, and Tsui [19] proved the following general existence theorem.

Theorem 5.1 (Joyce, Lee, Tsui). *Given any two Lagrangian planes P_1, P_2 in \mathbb{C}^n such that neither $P_1 + P_2$ nor $P_1 - P_2$ are area-minimizing, there is a Lagrangian self-expander L which is exact, zero-Maslov class with bounded Lagrangian angle, and asymptotic to $P_1 + P_2$, meaning $\sqrt{t}L$ converges, as Radon measures, to $P_1 + P_2$ when t tends to zero.*

Remark 5.2.

- i) The self-expander L found in [19] is explicit.
- ii) In [1], Anciaux found such examples assuming L is invariant under a certain $SO(n)$ action. In this case the self-expander equation reduces to an O.D.E.

The next theorem shows that self-expanders are attractors for the flow in \mathbb{C}^2 . The ideas for the proof are taken from [23, Section 4] where a slightly more general version is proven.

Pick two Lagrangian planes P_1, P_2 in \mathbb{C}^2 so that $P_1 \pm P_2$ is not area minimizing and $P_1 \cap P_2 = \{0\}$. Assume $(L_t)_{t \geq 0}$ is an exact, zero-Maslov class, almost-calibrated smooth solution to Lagrangian mean curvature flow in \mathbb{C}^2 .

Theorem 5.3. *Fix S_0 and ν . There are R_0 and δ so that if L_0 is δ -close in $C^{2,\alpha}$ to $P_1 + P_2$ in $A(\delta, R_0)$, then, for all $1 \leq t \leq 2$, $t^{-1/2}L_t$ is ν -close in $C^{2,\alpha}(B_{S_0})$ to a smooth self-expander Q asymptotic to $P_1 + P_2$.*

Remark 5.4.

- i) The content of the theorem is that if the initial condition is very close, in a precise sense, to a non area-minimizing configuration of two planes and the flow exists smoothly for all $0 \leq t \leq 2$, then the flow will be very close to a smooth self-expander for all $1 \leq t \leq 2$.
- ii) The result is false if one removes the hypothesis that the flow exists smoothly for all $0 \leq t \leq 2$. For instance, there are known examples [20, Theorem 4.1] where L_0 is very close to $P_1 + P_2$ and a finite-time singularity happens at a very short time t_1 . In this case L_{t_1} can be seen as a transverse intersection of small perturbations of P_1 and P_2 (see [20, Figure 2]) and we could continue the flow past the singularity by flowing each component of L_{t_1} separately, in which

case L_1 would be very close to $P_1 + P_2$ and this is not a *smooth* self-expander.

- iii) The smoothness assumption enters the proof in Lemma 5.5. The key fact is that if $L_0 \cap B_R$ is connected and the flow exists smoothly, then $L_t \cap B_R$ will also be connected for all $0 \leq t \leq 2$ (this fails in the example described above).

Sketch of proof. Consider a sequence (R^i) converging to infinity, a sequence (δ_i) converging to zero, and a sequence of smooth flows $(L_t^i)_{0 \leq t \leq 2}$ satisfying the theorem's hypothesis with $R_0 = R^i$, $\delta = \delta^i$. From compactness for integral Brakke motions [16, Section 7.1] we know that, after passing to a subsequence, $(L_t^i)_{0 \leq t \leq 2}$ converges to an integral Brakke motion $(\bar{L}_t)_{0 \leq t \leq 2}$, where L_0^i converges to $P_1 + P_2$.

Because L_0^i converges to $P_1 + P_2$ we can assume, without loss of generality, that

$$\lim_{i \rightarrow \infty} \int_{L_0^i} (\beta_0^i)^2 \Phi(0, 4) d\mathcal{H}^2 = 0.$$

Thus, from Theorem 3.2 and Lemma 3.3 ii), we get that for every $0 < s < 4$

$$(4) \quad \lim_{i \rightarrow \infty} \int_0^s \int_{L_t^i} |2tH - x^\perp|^2 \Phi(0, 4) d\mathcal{H}^2 + \lim_{i \rightarrow \infty} \int_{L_s^i} (\beta_s^i + 2s\theta_s^i)^2 \Phi(0, 4) d\mathcal{H}^2 \\ \leq \lim_{i \rightarrow \infty} \int_{L_0^i} (\beta_0^i)^2 \Phi(0, 4) d\mathcal{H}^2 = 0,$$

which means that $H = x^\perp/2t$ on \bar{L}_t for all $t > 0$ and thus $\bar{L}_t = \sqrt{t}\bar{L}_1$ as varifolds for every $t > 0$ (see proof of [22, Theorem 3.1]). Moreover, some technical work [23, Lemma 4.4] shows that \bar{L}_t converges as Radon measures to $P_1 + P_2$ as t tends to zero i.e., \bar{L}_1 is asymptotic to $P_1 + P_2$. We are left to show that \bar{L}_1 is smooth.

Lemma 5.5. \bar{L}_1 is not stationary.

Proof. If true, then \bar{L}_1 needs to have $H = x^\perp = 0$ and thus $\bar{L}_t = \bar{L}_1$ for all t , which means (making t tend to zero) that $\bar{L}_1 = P_1 + P_2$. We will argue that \bar{L}_1 must be a special Lagrangian, which contradicts the choice of P_1 and P_2 .

Pick K large enough. Because $L_0^i \cap B_{2K}$ is connected and the flow exists smoothly, we claim $L_1^i \cap B_{2K}$ has only one connected component intersecting B_K . The details can be seen in [23, Theorem 3.1, Lemma 4.5] but the basic idea is to use the fact that L_0^i very close to $P_1 + P_2$ in $A(K/2, 3K)$ and so, like in Corollary 4.3, we conclude that for all $x_0 \in A(K/2, 3K)$, all i sufficiently large, all $0 \leq t \leq 1$, and all $0 \leq l \leq 1$, we have

$$\Theta_t^i(x_0, l) \leq 1 + \varepsilon_0.$$

White's Regularity Theorem implies we can control the $C^{2,\alpha}$ -norm of L_t^i on $A(K, 2K)$ and some long, but straightforward work, implies the desired claim.

From varifold convergence we have

$$\lim_{i \rightarrow \infty} \int_0^2 \int_{L_t^i} |x^\perp|^2 \Phi(0, 4) d\mathcal{H}^2 dt = 0$$

which combined with (4) implies that, without loss of generality,

$$\lim_{i \rightarrow \infty} \int_{L_1^i} (|H|^2 + |x^\perp|^2) \Phi(0, 4) d\mathcal{H}^2 = 0.$$

Because $L_1^i \cap B_{2K}$ has only one connected component intersecting B_K , we can use Lemma 3.7 and Lemma 3.10 to conclude the existence of $\bar{\beta}$ so that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \int_{L_1^i \cap B_K} (\beta_1^i - \bar{\beta})^2 \phi d\mathcal{H}^2 = 0.$$

Hence, from (4), we obtain

$$\lim_{i \rightarrow \infty} \int_{L_1^i \cap B_K} (\bar{\beta} + 2\theta_1^i)^2 d\mathcal{H}^2 = \lim_{i \rightarrow \infty} \int_{L_1^i \cap B_K} (\beta_1^i + 2\theta_1^i)^2 d\mathcal{H}^2 = 0$$

which means \bar{L}_1 must be a special Lagrangian cone with Lagrangian angle $-\bar{\beta}/2$. \square

Lemma 5.6. *There is C so that*

$$\int_{\bar{L}_1} \Phi(y, l)(x, 0) d\mathcal{H}^2 \leq 2 - 1/C \quad \text{for every } l \leq 2, \text{ and } y \in \mathbb{R}^4.$$

Proof. The details can be found in [23, Lemma 4.6]. Because $\bar{L}_0 = P_1 + P_2$ we obtain from Theorem 3.2

$$\begin{aligned} \int_{\bar{L}_1} \Phi(y, l)(x, 0) d\mathcal{H}^2 + \int_0^1 \int_{\bar{L}_t} \left| H + \frac{(x-y)^\perp}{2(l+1-t)} \right|^2 \Phi(y, l+1-t)(x, 0) d\mathcal{H}^2 dt \\ = \int_{P_1+P_2} \Phi(y, l+1)(x, 0) d\mathcal{H}^2 \leq 2. \end{aligned}$$

The fact that \bar{L}_1 is not stationary allows us to estimate the second term on the first line and find a constant C such that

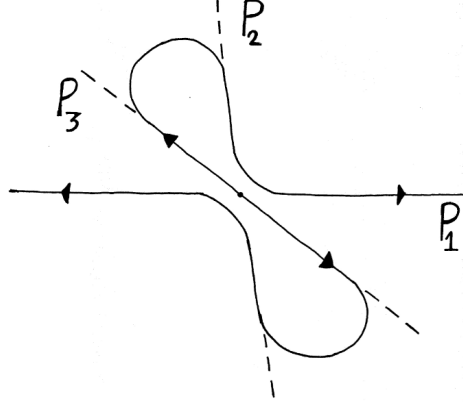
$$\int_{\bar{L}_1} \Phi(y, l)(x, 0) d\mathcal{H}^2 \leq 2 - 1/C$$

for all y and $l \leq 2$. \square

The same ideas used to show Theorem 4.1 can be modified to argue the tangent cone at any point $y \in \bar{L}_1$ must be a union of Lagrangian planes with possible multiplicities. The previous lemma implies it must be a plane with multiplicity one because otherwise

$$\lim_{r \rightarrow 0} \int_{\bar{L}_1} \Phi(y, r^2)(x, 0) d\mathcal{H}^2 \geq 2.$$

The mean curvature of \bar{L}_1 satisfies $H = x^\perp/2$ and so Allard Regularity Theorem implies uniform $C^{2,\alpha}$ bounds for \bar{L}_1 . Therefore, \bar{L}_1 is a smooth self-expander asymptotic to $P_1 + P_2$. Some extra work shows L_t^i converges strongly to $\sqrt{t}\bar{L}_1$ and this finishes the proof. \square

FIGURE 1. Curve $\gamma(\varepsilon) \cup -\gamma(\varepsilon)$.

6. APPLICATION III: STABILITY OF SINGULARITIES

We prove a result which is related to [23, Theorem A] but, before we state it, we need to introduce some notation.

Given any curve $\gamma : I \rightarrow \mathbb{C}$, we obtain a Lagrangian surface in \mathbb{C}^2 given by

$$N = \{(\gamma(s) \cos \alpha, \gamma(s) \sin \alpha) \mid s \in I, \alpha \in S^1\}.$$

Any Lagrangian which has the same $SO(2)$ symmetry as N is called *equivariant*. If $\mu = x_1 y_2 - y_1 x_2$, it is simple to see that L is equivariant if and only if $L \subset \mu^{-1}(0)$ (see [23, Lemma 7.1]).

Let c_1 , c_2 , and c_3 be three lines in \mathbb{C} so that c_1 is the real axis (c_1^+ being the positive part and c_1^- the negative part of the real axis), c_2 , and c_3 are the positive line segments spanned by $e^{i\theta_2}$ and $e^{i\theta_3}$ respectively, with $\pi/2 < \theta_2 < \theta_3 < \pi$. These curves generate three Lagrangian planes in \mathbb{R}^4 which we denote by P_1, P_2 , and P_3 respectively.

Consider a curve $\gamma(\varepsilon) : [0, +\infty) \rightarrow \mathbb{C}$ such that (see Figure 1)

- $\gamma(\varepsilon)$ lies in the first and second quadrant and $\gamma(\varepsilon)^{-1}(0) = 0$;
- $\gamma(\varepsilon) \cap A(3, \infty) = c_1^+ \cap A(3, \infty)$ and $\gamma(\varepsilon) \cap A(\varepsilon, 1) = (c_1^+ \cup c_2 \cup c_3) \cap A(\varepsilon, 1)$;
- $\gamma(\varepsilon) \cap B_1$ has two connected components γ_1 and γ_2 , where γ_1 connects c_2 to c_1^+ and γ_2 coincides with c_3 ;
- The Lagrangian angle of γ_1 , $\arg\left(\gamma_1 \frac{d\gamma_1}{ds}\right)$, has oscillation strictly smaller than $\pi/2$.

For every ε small and R large we denote by $N(\varepsilon, R)$ the Lagrangian surface corresponding to $R\gamma(\varepsilon R)$. We remark that one can make the oscillation for the Lagrangian angle of γ_1 as small as desired by choosing θ_2 very close to $\pi/2$.

Theorem 6.1. *For all ε sufficiently small and R sufficiently large, there is δ so that if L_0 is δ -close in $C^{2,\alpha}$ to $N(\varepsilon, R)$, then the Lagrangian mean curvature flow $(L_t)_{t \geq 0}$ must have a finite time singularity.*

Remark 6.2.

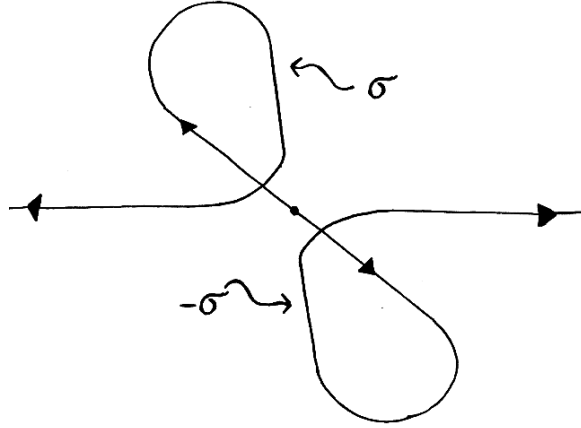


FIGURE 2. Curve $\sigma \cup -\sigma$.

- i) The ideas that go into the proof of Theorem 6.1 are exactly the same ideas that go into the proof of [20, Theorem A], with the advantage of the former having less technical details.
- ii) If L_0 is equivariant the flow reduces to an O.D.E. in which case Theorem 6.1 follows from simple barrier arguments like the ones used in [20, Section 4].
- iii) It is conceivable that a solution to mean curvature flow has a singularity at time T but there are arbitrarily small $C^{2,\alpha}$ perturbations of the initial condition which are smooth up to $T + \delta$, with δ fixed. The standard example is the dumbbell degenerate neckpinch due to Angenent and Velázquez. Thus the interest of Theorem 6.1.

Sketch of proof. Fix ε small and R large to be chosen later. The strategy is the following: If the flow $(L_t)_{t \geq 0}$ exists smoothly for all $t \leq 1$, there will be a singularity before some time $T = T(\varepsilon, R)$ and thus the flow cannot exist smoothly for all time.

Suppose the theorem does not hold. We have a sequence of smooth flows $(L_t^i)_{t \geq 0}$ where L_0^i converges to $N(\varepsilon, R)$. Compactness for integral Brakke motions [16, Section 7.1] implies that, after passing to a subsequence, $(L_t^i)_{t \geq 0}$ converges to an integral Brakke motion $(M_t)_{t \geq 0}$. Because

$$\lim_{i \rightarrow \infty} \int_{L_0^i} \mu^2 \Phi(0, 1) d\mathcal{H}^2 = 0,$$

we use Lemma 3.5 iii) and conclude that M_t lies inside $\mu^{-1}(0)$ for all t .

Let σ denote a smooth curve $\sigma : [0, +\infty) \rightarrow \mathbb{C}$ (see Figure 2) so that $\sigma^{-1}(0) = 0$, $\sigma \cup -\sigma$ is smooth at the origin, σ has a unique self intersection, and, when restricted to $[s_0, \infty)$ for some $s_0 > 0$, the curve σ can be written as the graph of a function u defined over part of the negative real axis with

$$\lim_{r \rightarrow -\infty} |u|_{C^{2,\alpha}((-\infty, r])} = 0.$$

First Claim: $M_1 = \{(\sigma(s) \cos \alpha, \sigma(s) \sin \alpha) \mid s \in [0, +\infty), \alpha \in S^1\}$.

It is a simple technical matter to find R_1 (independent of R large and ε small) so that, for all i sufficiently large, we have uniform $C_{loc}^{2,\alpha}$ bounds for L_1^i outside B_{R_1} . Moreover, for all $R_2 \geq R_1$ and i sufficiently large, $L_0^i \cap B_{R_2}$ has exactly two connected components $Q_{1,0}^i, Q_{2,0}^i$ where $Q_{1,0}^i, Q_{2,0}^i$ is arbitrarily close to $(P_1 + P_2) \cap B_{R_2}, P_3 \cap B_{R_\otimes}$ respectively. Some extra work (see [23, Theorem 3.1] for details) shows that $L_t^i \cap B_{R_2}$ can be decomposed into two connected components $Q_{1,t}^i, Q_{2,t}^i$ for all $t \leq 1$.

Fix ν small and set $S_0 = 2R_1$ in Theorem 5.3. There is R_2 so that for all R sufficiently large and ε sufficiently small we can apply Theorem 5.3 to $(Q_{1,t}^i)_{0 \leq t \leq 1}$ (more rigorously, we should actually apply [23, Theorem 4.1] because $Q_{1,t}^i$ has boundary) and conclude that, for all i large enough, $Q_{1,1}^i$ is ν close in $C^{2,\alpha}$ to a self-expander Q in B_{S_0} . In [23, Lemma 6.1] we showed Q to be unique and so we obtain uniform $C^{2,\alpha}$ bounds for $Q_{1,1}^i$ in B_{S_0} . Furthermore $Q_{2,0}^i$ is arbitrarily close to P_3 and some technical work (see [23, Theorem 3.1]) shows that $Q_{2,1}^i \cap B_{S_0}$ is also close in $C^{2,\alpha}$ to P_3 .

In sum, we have shown uniform $C^{2,\alpha}$ bounds for $L_1^i \cap B_{R_1}$ for all i sufficiently large. As a result M_1 must be smooth, equivariant, and $M_1 \cap B_{R_1}$ must have one connected component ν -close to Q and another connected component close in $C^{2,\alpha}$ to P_3 . It is now simple to see that M_1 must be described by a curve σ as claimed.

Denote by A_1 the area enclosed by the self-intersection of σ and set $T_1 = 2A_1/\pi + 1$.

Second Claim: For all i sufficiently large, L_t^i must have a singularity before T_1 .

Straightforward considerations (see [23, Theorem 5.3]) show that while $(M_t)_{t \geq 1}$ is smooth we have the existence of curves $\sigma_t : [0, +\infty) \rightarrow \mathbb{C}$ with $\sigma_t^{-1}(0) = 0$, $\sigma_t \cup -\sigma_t$ smooth,

$$M_t = \{(\sigma_t(s) \cos \alpha, \sigma_t(s) \sin \alpha) \mid s \in [0, +\infty), \alpha \in S^1\},$$

and such that σ_t evolves according to

$$\frac{dx}{dt} = \vec{k} - \frac{x^\perp}{|x|^2}.$$

While this flow is smooth all the curves σ_t have a self-intersection and so we consider A_t the area enclosed by this self-intersection and c_t the boundary of the enclosed region. From Gauss-Bonnet Theorem we have

$$\int_{c_t} \langle \vec{k}, \nu \rangle d\mathcal{H}^1 + \alpha_t = 2\pi \implies \int_{c_t} \langle \vec{k}, \nu \rangle d\mathcal{H}^1 \geq \pi,$$

where $\alpha_t \in [-\pi, \pi]$ is the exterior angle at the non-smooth point of c_t , and ν the interior unit normal. A standard formula shows that

$$\frac{d}{dt} A_t = - \int_{c_t} \left\langle \vec{k} - \frac{x^\perp}{|x|^2}, \nu \right\rangle d\mathcal{H}^1 \leq -\pi + \int_{c_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle d\mathcal{H}^1 = -\pi,$$

where the last identity follows from the Divergence Theorem combined with the fact that c_t does not contain the origin in its interior. Thus $A_t \leq A_0 - \pi t$ and so a singularity must occur at time $T < T_1$.

Angenent's work [3, 2] implies the singularity occurs because the loop of σ_t collapses, i.e. σ_T is smooth everywhere except at a cusp point and some extra work shows the curves σ_t become smooth and embedded for all $t > T$. The idea for the rest of the argument is as follows and the details can be found in [23, Theorem 5.1]. If $\hat{\theta}_t(s)$ denotes the angle that $\sigma'_t(s)$ makes with the x -axis we have

$$\theta_t(0) = 2\hat{\theta}_t(0) \quad \text{and} \quad \theta_t(\infty) = \lim_{s \rightarrow \infty} \theta_t(s) = 2\hat{\theta}_t(\infty).$$

Set $f(t) = \theta_t(\infty) - \theta_t(0) = 2(\hat{\theta}_t(\infty) - \hat{\theta}_t(0))$. Because there is a change in the topology of σ_t across T we have from the Hopf index Theorem that $f(t)$ jumps by 2π across T . On the other hand, the convergence of L_t^i to M_t is strong around the origin and outside a large compact set, which means that f must be continuous, a contradiction. \square

7. OPEN QUESTIONS

We now survey some open problems which could be relevant for the development of the field. One of the most important open questions is

Question 7.1. *Let L_0 be rational, almost-calibrated, and $(L_t)_{0 \leq t < T}$ a solution to Lagrangian mean curvature flow in \mathbb{C}^2 which becomes singular at time T . Show that L_T is smooth except at finitely many points and the tangent cone at each of these points is a special Lagrangian cone with some multiplicity.*

L_0 is required to be rational so that Theorem 4.2 can be applied and almost-calibrated because otherwise there are simple counterexamples (see [22, Example 1.1]). For instance, take a noncompact curve σ in \mathbb{C} with a unique self intersection and consider $L = \sigma \times \mathbb{R} \subset \mathbb{C}^2$ which is obviously Lagrangian. The solution to Lagrangian mean curvature flow will be $L_t = \sigma_t \times \mathbb{R}$ which will have a whole line of singularities at some time T .

If one can show that for each singular point x_0 there is $\theta(x_0)$ so that

$$(5) \quad \lim_{t \rightarrow T} \int_{L_t} (\theta_t - \theta(x_0)) \Phi(x_0, T) d\mathcal{H}^2 = 0,$$

then standard arguments prove the desired result. We now comment on the difficulty of (5). For simplicity let us assume that

$$\Theta_t(x, r^2) < 3 \text{ for all } r \leq \delta, T - \delta^2 < t < T, \text{ and } x \in \mathbb{C}^2.$$

In this case the blow up at each singular point described in Section 4 will be a union of two planes $P_1 + P_2$ by Theorem 4.1 and three cases can happen: $\dim(P_1 \cap P_2) = 2$, $\dim(P_1 \cap P_2) = 0$, or $\dim(P_1 \cap P_2) = 1$. In the first case the Lagrangian angles of P_1 and P_2 must be identical by the almost-calibrated condition and so we should have $\theta(x_0) = \theta(P_1) = \theta(P_2)$. In the second case we have from Corollary 4.3 that P_1 and P_2 must have the same Lagrangian angle and so $\theta(x_0) = \theta(P_1) = \theta(P_2)$. The third case we want to show is impossible and this is a highly non trivial matter for reasons we know explain.

In [19], Joyce, Lee, and Tsui found a Lagrangian N with small oscillation of the Lagrangian angle and such that

$$N_t = N + t\vec{v}$$

solves mean curvature flow. If we blow down this flow, i.e., consider the sequence $N_s^i = \varepsilon_i N_{s/\varepsilon_i^2}$ with ε_i tending to zero, one can easily check that, for all $s < 0$, N_s^i converges weakly to $P_1 + P_2$, where P_1, P_2 are Lagrangian planes with $P_1 \cap P_2 = \text{span}\{\vec{v}\}$. Thus, if we rule out the third case we would be ruling out Joyce, Lee, and Tsui solutions as smooth blow ups, a clearly deep fact.

In [19] the authors also found many examples of self-expanders and so a natural problem is

Question 7.2.

- Let $L \subset \mathbb{C}^n$ be a zero-Maslov class self-expander asymptotic to two planes. Show it coincides with one of the self-expanders of Joyce, Lee, and Tsui.
- Are there almost-calibrated translating solutions to Lagrangian mean curvature flow besides the ones found by Joyce, Lee, and Tsui?

Castro and Lerma [7] solved the first question when $n = 2$ assuming L is Hamiltonian stationary as well, i.e., the Lagrangian angle is harmonic. When $n > 2$ it is not known whether special Lagrangians asymptotic to two planes have to be one of the Lawlor necks, which adds interest to the proposed problem. In [23] we showed the blow-down N_s^i of translating solutions converges weakly to Σ_s with

$$\Sigma_s = m_1 P_1 + \dots + m_k P_k \text{ for all } s \leq 0 \quad \text{and} \quad \Sigma_s = \sqrt{s} \Sigma_1 \text{ for all } s > 0,$$

where the planes P_j intersect all along a line and Σ_1 is a self-expander asymptotic to $m_1 P_1 + \dots + m_k P_k$. It is easy to see that $\Sigma_1 = \sigma \times \text{line}$, where σ is a self-expander in \mathbb{C} , and so the first step towards the second problem would be to see if one can have $k = 3$ and $m_1 = m_2 = m_3 = 1$. In [8] examples were found with unbounded Lagrangian angle.

Success in Question 7.1 would make the following problem having crucial importance.

Question 7.3. Let $L \subset \mathbb{C}^n$ be zero-Maslov class, almost-calibrated, and smooth everywhere except the origin where the tangent cone is a special Lagrangian cone with multiplicity. Find $(L_t)_{\varepsilon < t < \varepsilon}$ a meaningful solution to Lagrangian mean curvature flow so that L_t converges to L when t tends to zero.

Behrndt [5] made concrete progress on this problem when the multiplicity is one and the special Lagrangian cone is stable. It would be nice to have a solution when the tangent cone is a plane with multiplicity two.

Finally, there is no result available regarding convergence of compact Lagrangians in \mathbb{C}^n . For instance, the following question can be seen as a Lagrangian analogue of Huisken's classical result for mean curvature flow of convex spheres [14].

Question 7.4. Find a condition on a Lagrangian torus in \mathbb{C}^2 , which implies that Lagrangian mean curvature flow $(L_t)_{0 < t < T}$ will become extinct at time T and, after rescale, L_t converges to the Clifford torus.

It was shown in [21, 28] that L can be Hamiltonian isotopic to a Clifford Torus and the flow still develop singularities before the optimal time. As

suggested by Joyce, the first natural thing would be to see what happens to small Hamiltonian perturbations of the Clifford torus.

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E-mail address: aneves@imperial.ac.uk

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON SW7 2AZ