TWO FAMILIES OF MAXIMAL CURVES WHICH ARE NOT GALOIS SUBCOVERS OF THE HERMITIAN CURVE

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ABSTRACT. We show that the generalized Giulietti-Korchmáros curve and the maximal curve with equation $x^{q^2} - x = y^{(q^n+1)/(q+1)}$ defined over $\mathbb{F}_{q^{2n}}$, for $n \geq 3$ odd and $q \geq 3$, are not Galois subcovers of the Hermitian curve over $\mathbb{F}_{q^{2n}}$. For $q = 2$, we show that the generalized GK curve is covered by the Hermitian curve.

1. Introduction and Statements of Results

Let $\mathbb{F}_{q^2}$ be the finite field with q^2 elements, and let $\mathcal{X}$ be a projective, nonsingular, geometrically irreducible curve (hereafter referred to as a *curve*) defined over $\mathbb{F}_{q^2}$. We say that $\mathcal{X}$ is $\mathbb{F}_{q^2}$-maximal if the number of its rational points attains the Hasse-Weil upper bound

$$\left|\mathcal{X}(\mathbb{F}_{q^2})\right| = q^2 + 1 + 2g(\mathcal{X})q,$$

where $g(\mathcal{X})$ is the genus of $\mathcal{X}$.

A curve $\mathcal{C}$ is called a *subcover* of $\mathcal{X}$ over $\mathbb{F}_{q^2}$ (or equivalently $\mathcal{X}$ is a *cover* of $\mathcal{C}$) if there exists a surjective map $\phi : \mathcal{X} \to \mathcal{C}$ with $\mathcal{C}$, $\mathcal{X}$ and ϕ defined over $\mathbb{F}_{q^2}$. More information about maximal curves can be found in [2, 3, 7, 9, 13, 16, 17] and their references. The most important example of a maximal curve is the Hermitian curve $\mathcal{H}$, which is defined over $\mathbb{F}_{q^2}$ by the equation

$$y^q + y = x^{q+1}.$$

It has genus $\frac{1}{2}q(q-1)$. By the work of [8, 15, 26], the genus of any maximal curve $\mathcal{X}$ satisfies

$$g(\mathcal{X}) \in [0, (q-1)^2/4] \cup \{q(q-1)/2\}.$$

Therefore the Hermitian curve has the largest possible genus that a maximal curve can have. It is shown in [21] that the Hermitian curve is the unique maximal curve having genus $\frac{1}{2}q(q-1)$.

By a result known as Serre's theorem (see [18, Proposition 6]), any subcover of a $\mathbb{F}_{q^2}$-maximal curve is $\mathbb{F}_{q^2}$-maximal. Most of the known maximal curves are subcovers of the Hermitian curve $\mathcal{H}$, and systematic studies on subcovers of $\mathcal{H}$ can be found in [4, 5, 12]. The first example of a maximal curve which is not a Galois subcover of $\mathcal{H}$ is discovered by Garcia and Stichtenoth [11]. This curve has defining equation

$$y^9 - y = z^7$$

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over $\mathbb{F}_{3^6}$, and is a special case with $q = 3$, $n = 3$ of the curve $\mathcal{X}_n$ with defining equation

$$y^{n^2} - y = z^{\frac{q^n+1}{q+1}} \tag{1.1}$$

over $\mathbb{F}_{q^{2n}}$, for $q \geq 2$ and odd $n \geq 3$. The curve $\mathcal{X}_n$ was shown to be maximal in [1]. On the other hand, an unpublished result by Rains and Zieve states that the Ree curve over $\mathbb{F}_{3^6}$ is not a Galois subcover of the Hermitian curve over the same field.

In [13], Giulietti and Korchmáros give an example of a maximal curve, now called the GK curve, that is not covered by the Hermitian curve. The GK curve has been generalized by Garcia, Güneri and Stichtenoth in [10]. These generalized curves, called the generalized GK curves $\mathcal{C}_n$, are maximal curves over $\mathbb{F}_{q^{2n}}$ for a prime power q and odd $n \geq 3$ ([10], see also [6]). They have defining equations

$$\begin{aligned} x^q + x &= y^{q+1} \\ y^{q^2} - y &= z^{\frac{q^n+1}{q+1}}. \end{aligned} \tag{1.2}$$

It is shown in [10] that the curves with $n = 3$ are isomorphic to those given originally by Giulietti and Korchmáros. It is not known whether these generalized GK curves $\mathcal{C}_n$ are covered by the Hermitian curve for $n \geq 5$. In this paper, we give a partial answer to this problem by showing that $\mathcal{C}_n$ is not a Galois subcover of the Hermitian curve over the same finite field for any odd $n \geq 3$ and $q \geq 3$. More precisely, we prove the following.

Theorem 1.1. *The generalized GK curve, defined by* (1.2) *over* $\mathbb{F}_{q^{2n}}$, *is not a Galois subcover of the Hermitian curve over* $\mathbb{F}_{q^{2n}}$ *for any odd* $n \geq 3$ *and* $q \geq 3$.

For $q = 2$, the situation is completely different. We prove that the generalized GK curve $\mathcal{C}_n$ over $\mathbb{F}_{2^{2n}}$ is covered by the Hermitian curve over $\mathbb{F}_{2^{2n}}$, and the degree of the covering is given.

Theorem 1.2. *The generalized GK curve, defined by* (1.2) *with* $q = 2$ *over* $\mathbb{F}_{2^{2n}}$, *is covered by the Hermitian curve over the same finite field for any odd* $n \geq 3$. *The degree of the covering is* $d = (2^n + 1)/3$.

Next, we consider the curve $\mathcal{X}_n$ defined over $\mathbb{F}_{q^{2n}}$ by (1.1), which is the second equation in the definition of the generalized GK curve. This curve is maximal and has genus $g(\mathcal{X}_n) = \frac{1}{2}(q-1)(q^n - q)$. It is proved in [1] that for $q = 2$, the curve $\mathcal{X}_n$ is covered by the Hermitian for any odd $n \geq 3$. We prove that for $q \geq 3$ and odd $n \geq 3$, the curve $\mathcal{X}_n$ is not a Galois subcover of the Hermitian curve $\mathcal{H}_n$, which answers the first question raised in [11].

Theorem 1.3. *The family of curves* $\mathcal{X}_n$, *defined by* (1.1) *over* $\mathbb{F}_{q^{2n}}$, *is not a Galois subcover of the Hermitian curve over* $\mathbb{F}_{q^{2n}}$ *for any odd* $n \geq 3$ *and* $q \geq 3$.

Finally, we note that the GK curve for $n = 3$ is not a subcover of the Hermitian [13]. By our arguments in Section 4, we have the following corollary, which answers the second question in [11].

Corollary 1.4. *The curve* $\mathcal{X}_3$ *with* $q = 3$, *defined by the equation*

$$y^9 - y = x^7$$

over $\mathbb{F}_{3^6}$ *is not covered by the Hermitian over the same finite field.*

We will prove Theorem 1.1 in Section 3, and then obtain the other theorems and corollaries by a simple argument in Section 4. We remark that for $n \geq 5$, we do not know whether the curves $\mathcal{C}_n$ and $\mathcal{X}_n$ are non-Galois covered by the Hermitian curve or not.

2. Preliminaries

From now on, we consider both the Hermitian curve $\mathcal{H}_n$ and the generalized GK curve $\mathcal{C}_n$ over $\mathbb{F}_{q^{2n}}$, where n is odd (we do not restrict q at this moment). The genus and the number of $\mathbb{F}_{q^{2n}}$-rational points on the Hermitian curve over $\mathbb{F}_{q^{2n}}$ are given by

$$g(\mathcal{H}_n) = \frac{1}{2}q^n(q^n-1), \qquad N(\mathcal{H}_n) = q^{2n} + 1 + 2g(\mathcal{H}_n)q^n. \tag{2.1}$$

and the corresponding quantities for the generalized GK curve are given by (see [10])

$$g(\mathcal{C}_n) = \frac{1}{2}(q-1)(q^{n+1}+q^n-q^2), \qquad N(\mathcal{C}_n) = q^{2n} + 1 + 2g(\mathcal{C}_n)q^n. \tag{2.2}$$

Suppose there is a covering $\phi : \mathcal{H}_n \to \mathcal{C}_n$ of degree d. From the Hurwitz genus formula (see [25, Theorem 3.4.13]), we have

$$2g(\mathcal{H}_n) - 2 \geq d \cdot (2g(\mathcal{C}_n) - 2),$$

and from the splitting of points we have

$$N(\mathcal{H}_n) \leq d \cdot N(\mathcal{C}_n).$$

By substituting the values of (2.1) and (2.2) into the above equations and using the division algorithm, we get

$$q^{n-2} + 1 \leq d \leq q^{n-2} + q^{n-4} + \ldots + q^3 + q \tag{2.3}$$

for $q \geq 3$, and

$$q^{n-2} + 1 \leq d \leq q^{n-2} + q^{n-4} + \ldots + q^3 + q + 1 \tag{2.4}$$

for $q = 2$. It is immediate from (2.3) that the range of d is empty when $n = 3$ and $q \geq 3$. In particular, we recover the known result that the GK curve $\mathcal{C}_3$ is not a subcover of the Hermitian curve $\mathcal{H}_3$ [13]. For $n \geq 5$, the ranges in (2.3) and (2.4) are nontrivial.

Now we suppose that the covering $\phi : \mathcal{H}_n \to \mathcal{C}_n$ is Galois with Galois group G, with $|G| = d$. Then G can be realized as a subgroup of $\mathrm{Aut}(\mathcal{H}_n) = PGU(3, q^n)$ (see [20, 24]), and $\mathcal{C}_n$ is the quotient curve of $\mathcal{H}_n$ by G. To understand the ramification in a Galois covering, we will use the Hilbert different formula (see the proof in [25, Theorem 3.8.7]), which we state here for the sake of completeness. Let $\mathcal{X}' \to \mathcal{X}$ be a Galois covering of curves with Galois group G, and let P and P' be points on $\mathcal{X}$ and $\mathcal{X}'$ respectively (which need not lie in the field of definition of the covering) so that P maps to P' under the covering. Then the different exponent $d(P|P')$ is

$$d(P|P') = \sum_{\substack{1 \neq \sigma \in G \\ \sigma(P) = P}} i_P(\sigma), \tag{2.5}$$

where $i_P(\sigma) = v_P(\sigma(t) - t)$ with t a local uniformizer at P. Note that if the ramification of P over P' is tame, then $i_P(\sigma) = 1$ for any σ that fixes P, and in that case $d(P|P')$ is the number of elements $\sigma \neq 1$ in G that fix P. Combining

(2.5) with the Hurwitz genus formula, we get the following proposition which we will rely on heavily.

Proposition 2.1. *Suppose $\mathcal{X} \to \mathcal{X}'$ is a Galois covering of degree d with Galois group G, then*

$$2g(\mathcal{X}) - 2 = d(2g(\mathcal{X}') - 2) + \deg R,$$

where R is the ramification divisor given by

$$R = \sum_{1 \neq \sigma \in G} \sum_{P \in \mathcal{X}} i_P(\sigma) P,$$

with $i_P(\sigma) = 0$ if $\sigma(P) \neq P$.

3. Proof of Theorem 1.1

In this section, we apply 2.1 with $\mathcal{X} = \mathcal{H}_n$ and $\mathcal{X}' = \mathcal{C}_n$. To do this, we need to understand the quantity

$$i(\sigma) := \sum_{P \in \mathcal{X}} i_P(\sigma) \deg P$$

for each $\sigma \in PGU(3, q^n)$. An element in $PGU(3, q^n)$ either fixes no points on $\mathcal{H}_n$, or it fixes a point of degree one, or fixes a point of degree three (see [12]). If σ fixes no points on $\mathcal{H}_n$, then $i(\sigma) = 0$. If it fixes a point of degree three, then it fixes only that point. Since any such σ has order dividing $q^2 - q + 1$, which is relatively prime to q, the ramification is tame. Hence $i(\sigma) = 3$. The case when σ fixes a point of degree one has several subcases. Since the action of $PGU(3, q^n)$ on the points of degree one on $\mathcal{H}_n$ is transitive (see for example [14]), and $i(\sigma)$ is unchanged under conjugation (see for example [23, Chapter IV]), we may assume that the degree one point fixed is the point at infinity P_∞ when $\mathcal{H}_n$ is given by the equation $x^{q^n} + x = y^{q^n+1}$. Following [12], we write $\sigma = [a, b, c]$, where

$$\sigma(x) = ax + b, \qquad \sigma(y) = a^{q^n+1} y + ab^{q^n} x + c,$$

with $a \in \mathbb{F}_{q^{2n}} \backslash \{0\}$, $b \in \mathbb{F}_{q^{2n}}$, $c^{q^n} + c = b^{q^n+1}$. There are 2 cases:

(1) $a = 1$, then σ fixes P_∞ to a high order. In this case we have

$$i(\sigma) = \begin{cases} 2 & , \text{ if } a = 1, b \neq 0, \\ q^n + 2 & , \text{ if } a = 1, b = 0, c \neq 0. \end{cases}$$

(2) $a \neq 1$, then σ may also fix other points of degree one. In this case, if p denotes the characteristic of $\mathbb{F}_{q^{2n}}$, we have

$$i(\sigma) = \begin{cases} 1 & , \text{ if } p \text{ divides } \operatorname{ord}(\sigma), \\ q^n + 1 & , \text{ if } \operatorname{ord}(\sigma) \text{ divides } q + 1, \\ 2 & , \text{ otherwise.} \end{cases}$$

Combining all the cases, we obtain the following proposition. The significance of the proposition is that either $i(\sigma)$ is very small, or $i(\sigma)$ is very large, and nothing in the middle can happen.

Proposition 3.1. *If $\sigma \in PGU(3, q^n)$, then $i(\sigma) = 0, 1, 2, 3, q^n + 1$ or $q^n + 2$.*

Remark 3.1. Proposition 3.1 can also be verified using the Artin representation (see [22, Chapter 19], [23, Chapter VI]) of $PGU(3,q^n)$. In this case, the Artin representation is the unique irreducible representation of minimal degree $2g(\mathcal{H}_n) = q^n(q^n-1)$ (see [19, Lemma 4.1]).

Now we have the contribution of each element in $PGU(3,q^n)$ to the ramification divisor, and we are ready to finish the proof of Theorem 1.1. From now on, we assume $q \geq 3$ unless otherwise stated.

(Proof of Theorem 1.1) Suppose now that $\phi : \mathcal{H}_n \to \mathcal{C}_n$ is a Galois covering with group $G \subseteq PGU(3,q^n)$ of order $|G| = d$. Using (2.3), we write

$$d = q^{n-2} + a, \qquad 1 \leq a \leq q^{n-4} + q^{n-6} + \ldots + q^3 + q. \tag{3.1}$$

By the Hurwitz genus formula the degree of the ramification divisor R is

$$\deg R = (2g(\mathcal{H}_n) - 2) - d(2g(\mathcal{C}_n) - 2). \tag{3.2}$$

From Proposition 2.1, $\deg R = \sum_{1\neq\sigma\in G} i(\sigma)$. Proposition 3.1 gives the contribution $i(\sigma)$ for each of the $d-1$ nontrivial elements in G. The nontrivial elements divide into two groups according to $i(\sigma) = 0,1,2,3$ or $i(\sigma) = q^n+1, q^n+2$. Let $d = 1+u+v$ with $u = \#\{\sigma \neq 1 : i(\sigma) = 0,1,2,3\}$ and $v = \#\{\sigma \neq 1 : i(\sigma) = q^n+1, q^n+2\}$. We derive two inequalities for u and v with no common solution, thus proving that no Galois covering exists. For the first inequality we use that the remainder of $\deg R$ modulo q^n+1 is in $[0, 3u+v]$. On the other hand, combining (3.2) with

$$2g(\mathcal{H}_n) - 2 = (q^n-2)(q^n+1)$$
$$2g(\mathcal{C}_n) - 2 = (q^2-1)(q^n+1) - (q^3+1),$$

and using (3.1), gives $\deg R \equiv d(q^3+1) \equiv d - q + aq^3 \pmod{q^n+1}$. Since $d-q+aq^3 < q^n+1$, it is the remainder of $\deg R$ modulo q^n+1 and $d - q + aq^3 \leq 3u+v$, or $2u \geq aq^3 - q + 1$. For the second inequality we use that $\deg R \leq 3u + (q^n+2)v$. In combination with

$$2g(\mathcal{H}_n) - 2 = (q^n-2)(q^n+2) - (q^n-2)$$
$$2g(\mathcal{C}_n) - 2 = (q^2-1)(q^n+2) - (q^3-q^2),$$

and $3u < 3d < q^n+2$, we obtain $v + 1 \geq \deg R/(q^n+2) \geq (q^n-3) - d(q^2-1) = d - aq^2 - 3$, or $u \leq aq^2 + 3$. But $2(aq^2+3) < aq^3 - q + 1$ and no solution for u exists. This proves Theorem 1.1.

Remark 3.2. Note that the argument above also works for $q = 2$. This excludes every possible d in the range given by (2.4) except

$$d = 2^{n-2} + 2^{n-4} + \ldots + 2^3 + 2 + 1 = \frac{2^n+1}{3}.$$

This gives the degree in Theorem 1.2.

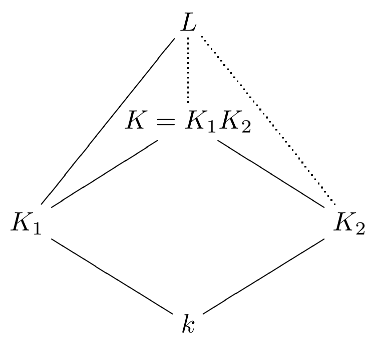
Remark 3.3. If we try the same argument on the curves $\mathcal{X}_n$ defined by (1.1), we can eliminate some d in the feasible range obtained by the upper bound and lower bound method in Section 2, but we cannot eliminate all of them. For example, we cannot eliminate $d = (q^n-1)/(q-1)$ since

$$2g(\mathcal{H}_n) - 2 = d(2g(\mathcal{X}_n) - 2) + q \cdot (q^n+1) + (d-q-1) \cdot 2.$$

4. Proof of Theorem 1.2 and 1.3

The proof will be based on the following proposition. The proof is elementary.

Proposition 4.1. *Suppose we have the following tower of fields (here dotted lines indicate that the containment is unknown):*



Then

1. *$K \subseteq L$ if and only if $K_2 \subseteq L$.*
2. *If the extension $K_2 \subseteq L$ is Galois, then $K \subseteq L$ is Galois.*
3. *If the extension $k \subseteq L$ is Galois, then $K \subseteq L$ is Galois if and only if $K_2 \subseteq L$ is Galois.*

The key observation here is that the generalized GK curve $\mathcal{C}_n$ over $\mathbb{F}_{q^{2n}}$ is the fibre product of two maximal curves over $\mathbb{F}_{q^{2n}}$, namely the curves $\mathcal{H}$, given by

$$x^q + x = y^{q+1} \tag{4.1}$$

and $\mathcal{X}_n$ given by

$$y^{q^2} - y = z^{\frac{q^n+1}{q+1}},$$

over $\mathbb{P}^1(y)$ with variable y. Denote the function field for a curve $\mathcal{Y}$ over $\mathbb{F}_{q^{2n}}$ by $\mathbb{F}_{q^{2n}}(\mathcal{Y})$, then the function field $\mathbb{F}_{q^{2n}}(\mathcal{C}_n)$ is the compositum of the function fields $\mathbb{F}_{q^{2n}}(\mathcal{H})$ and $\mathbb{F}_{q^{2n}}(\mathcal{X}_n)$ over the rational function field $\mathbb{F}_{q^{2n}}(y)$ (i.e. regard all fields that come into play as finite extensions of $\mathbb{F}_{q^{2n}}(y)$). Since n is odd, the curve $\mathcal{H}$ is a subcover of the Hermitian curve over $\mathcal{H}_n$, so we have an extension $\mathbb{F}_{q^{2n}}(\mathcal{H}) \subseteq \mathbb{F}_{q^{2n}}(\mathcal{H}_n)$. Thus we have a tower of fields as shown in Figure 1, for which Proposition 4.1 is applicable.

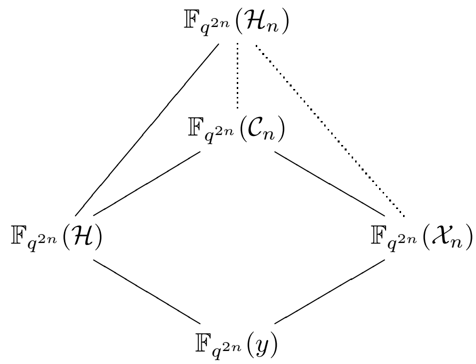


Figure 1. Tower of fields

For $q \geq 3$ and odd $n \geq 3$, the fact that $\mathcal{C}_n$ is not a Galois subcover of $\mathcal{H}_n$ together with Proposition 4.1(2) shows that $\mathcal{X}_n$ is not a Galois subcover of $\mathcal{H}_n$. This proves Theorem 1.3. For $q = 2$, Abdón, Bezerra and Quoos [1] showed that $\mathcal{X}_n$ is a subcover of $\mathcal{H}_n$. Therefore by Proposition 4.1(1), $\mathcal{C}_n$ is a subcover of $\mathcal{H}_n$. This proves Theorem 1.2. Finally, for $n = 3$, we know that $\mathcal{C}_3$ is not covered by $\mathcal{H}_3$, and again by Proposition 4.1(1), $\mathcal{X}_3$ is not covered by $\mathcal{H}_3$. This proves Corollary 1.4.

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