BAR COMPLEXES AND EXTENSIONS OF CLASSICAL EXPONENTIAL FUNCTORS

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ABSTRACT. We compute Ext-groups between classical exponential functors (i.e. symmetric, exterior or divided powers) and their Frobenius twists. Our method relies on bar constructions, and bridges these Ext-groups with the homology of Eilenberg-Mac Lane Spaces.

Together with [T3], this article provides an alternative approach to classical Ext-computations [FS, FFSS, C1, C2] in the category of strict polynomial functors over fields, and it corrects some mistakes in [C2]. We also obtain significant Ext-computations for strict polynomial functors over the integers.

1. Introduction

1.1. Let $\mathbb{k}$ be a commutative ring, and let $GL_{n,\mathbb{k}}$ be the general linear group over $\mathbb{k}$. If $\mathbb{k}$ is not a field of characteristic zero, the rational representations of $GL_{n,\mathbb{k}}$ do not split as direct sums of simple representations. This gives birth [J, I, Chap 4] to nontrivial extension groups between representations of $GL_{n,\mathbb{k}}$, which measure the various ways of pasting them together. Unfortunately, these extension groups are often quite difficult to compute, even when one restricts to the most basic representations of $GL_{n,\mathbb{k}}$. In this article, we deal with the classical (but challenging) problem of computing extension groups between symmetric, exterior or divided powers of the standard representation $\mathbb{k}^n$ of $GL_{n,\mathbb{k}}$.

A first partial answer to this problem was obtained in [A], where the extension groups $\mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(S^*(\mathbb{k}^n),\Lambda^*(\mathbb{k}^n))$ between symmetric and exterior powers of the standard representation are computed over a field $\mathbb{k}$. Actually, Akin uses the Schur algebras $S(n,d)$ to perform this computation. Indeed, $S^*(\mathbb{k}^n)$ and $\Lambda^*(\mathbb{k}^n)$ are $S(n,d)$-modules and there is an isomorphism

$$\mathrm{Ext}^*_{GL_{n,\mathbb{k}}}(S^d(\mathbb{k}^n),\Lambda^d(\mathbb{k}^n))\simeq \mathrm{Ext}^*_{S(n,d)}(S^d(\mathbb{k}^n),\Lambda^d(\mathbb{k}^n))\ .$$

The interest of this approach lies in the fact that extension between $S(n,d)$-representations are often easier to compute than representations between $GL_{n,\mathbb{k}}$-representations.

Further progress on the problem was made using the category $\mathcal{P}_{\mathbb{k}}$ of 'strict polynomial functors' introduced by Friedlander and Suslin [FS]. These strict polynomial functors are a stabilized version of modules over Schur algebras (see appendix 8 for a summary of the theory of strict polynomial functors), and they are more powerful than modules over Schur algebras for computations. Examples of strict polynomial functors are the symmetric powers

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functors $V \mapsto S^d(V)$, the exterior powers functors $V \mapsto \Lambda^d(V)$ and the divided powers functors $V \mapsto \Gamma^d(V) = (V^{\otimes d})^{\mathfrak{S}_d}$ (where the variable V takes values in finitely generated projective $\Bbbk$-modules). We call these examples 'classical exponential functors' (the reason for this is given in paragraph 1.2.3 below). Extensions between these functors give informations between extension between the corresponding $GL_{n,\Bbbk}$ representations, e.g. there is an isomorphism:

$$\bigoplus_{i,j} \mathrm{Ext}^*_{GL_{n,\Bbbk}}(S^i(\Bbbk^n), \Lambda^j(\Bbbk^n)) \simeq \bigoplus_{i,j\le n} \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^i, \Lambda^j)\ .$$

Over a field $\Bbbk$ of positive characteristic, extension groups between classical exponential functors were all computed in the innovative articles [FFSS] and [C2]. In fact, these authors do not only compute extension groups between classical exponential functors. Because of links [FFSS] with the representation theory of the finite groups $GL_n(\mathbb{F}_q)$ and with the category of unstable modules over the Steenrod algebra, they compute more generally all the extension groups of the form $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^{*\,(r)}, Y^{*\,(s)})$, where X^* and Y^* are classical exponential functors and $X^{*\,(r)}$ and $Y^{*\,(s)}$ denote the precompositions of X^* and Y^* by the Frobenius twist functors $I^{(r)}$ and $I^{(s)}$ $(r, s \in \mathbb{N})$.

In this article, we study the extension groups between classical exponential functors, and also between their twisted versions if $\Bbbk$ is a field of positive characteristic. Our methods are different from the methods used in [FFSS, C2], and they also allow explicit computations over the integers. We obtain the following results.

(1) We get new independent proofs of all the computations performed in [FFSS] (the results are stated in theorems 7.19 and 7.20).
(2) We get new independent computations all the extension groups computed in [C2]. We explain why some results of [C2] are false and we correct them (in theorems 7.21, 7.22 and 7.23).
(3) We also obtain completely new results, namely we compute extension groups between classical exponential functors over the ring $\Bbbk = \mathbb{Z}$ (in theorem 6.11).

To perform our computations, we use bar complexes to show that extension groups between classical exponential functors are nothing but the singular homology of some Eilenberg-Mac Lane spaces under disguise. Then we elaborate on Cartan's computation [Car] of the homology of Eilenberg-Mac Lane spaces to get explicit results. On our way, we obtain the following computation, of independent interest.

(4) We compute (theorems 5.7, 5.13 and remark 5.9) the homology of iterated bar constructions of the divided power algebra $\Gamma^*(V)$, over any field. (This was done only for *prime fields* by Cartan [Car]).

Finally, over fields of positive characteristic, we show that the extension groups between twisted exponential functors can quite easily deduced from the extension groups between untwisted exponential functors. The latter fact is proved once again via bar complexes, together with the techniques developed in [T3].

1.2. Before undertaking computations, we wish to clarify the meaning of 'computing extension groups'. So we now recall various structures (functoriality, products, coproducts...) which equip extension groups between classical exponential functors and their Frobenius twists, and which ones it is important to compute. Full details are given in section 2.

1.2.1. *Strict polynomial structures.* Let $\Bbbk$ be a commutative ring and let V be a finitely generated projective $\Bbbk$-module. If F is a strict polynomial functor, we denote by F^V the strict polynomial functor $U \mapsto F(\mathrm{Hom}_{\Bbbk}(V,U))$. Throughout the article, we shall write for short:

$$\mathbb{E}^*(F,G;V) := \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F^V,G)\ , \quad \mathbb{H}(F,G;V) := \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F^V,G)\ .$$

Thus $\mathbb{E}^*(F,G;\Bbbk)$ equals $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G)$. Actually $V \mapsto \mathbb{E}^*(F,G;V)$ is a strict polynomial functor.

In this article, we systematically study extension groups of the form $\mathbb{E}^*(F,G;V)$ instead of $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G)$. This yields more general results, and also reveals some hidden useful structures (e.g. exponential structures, cf. 1.2.3).

1.2.2. *Products and gradings.* Let A^* be a graded strict polynomial algebra (that is, the A^i are strict polynomial functors, and the multiplication $A^i \otimes A^j \to A^{i+j}$ is a morphism of strict polynomial functor) and let C^* be strict polynomial coalgebra. Then the extension groups $\mathbb{E}^*(C^*,A^*;V)$ are equipped with a product defined as the composite

$$\mathbb{E}^i(C^k,A^m;V)\otimes\mathbb{E}^j(C^\ell,A^n;V) \to E^{i+j}(C^k\otimes C^\ell,A^m\otimes A^n;V) \to E^{i+j}(C^{k+\ell},A^{m+n};V)$$

where the first map is induced by tensor products and the second one by the multiplication of A^* and the comultiplication of C^*.

So the extension groups between (twisted) classical exponential functors, e.g. $\mathbb{E}^*(S^{*(3)},\Lambda^{*(5)};V)$, form a trigraded strict polynomial algebra, and this algebra structure greatly helps to organize the computations and the results.

In fact, it is not necessary to compute extensions between (twisted) classical exponential functors as *trigraded* algebras. Indeed, there are no extensions between homogeneous strict polynomial functors of different strict polynomial degree, so the last two gradings carry the same information. For example, $\mathbb{E}^*(S^{*(3)},\Lambda^{*(5)};V)$ actually equals $\bigoplus_{h,d\geq 0}\mathbb{E}^h(S^{dp^2\,(3)},\Lambda^{d\,(5)};V)$. Therefore we only compute the extension groups as bigraded algebras in this article.

1.2.3. *Exponential structures.* Actually, symmetric powers, exterior powers, divided powers and their Frobenius twists are not only algebras and coalgebras: they are exponential functors. A (multigraded) exponential functor E^* is a functor satisfying a (multigraded) 'exponential isomorphism': $E^*(V\oplus W)\simeq E^*(V)\otimes E^*(W)$.

When $\Bbbk$ is a field, extensions between graded exponential functors inherit an exponential structure:

$$\mathbb{E}^*(X^*,Y^*;V\oplus W)\simeq \mathbb{E}^*(X^*,Y^*;V)\otimes\mathbb{E}^*(X^*,Y^*;W)\ .$$

The product described in section 1.2.2 coincides with the composite

$$\mathbb{E}^*(X^*,Y^*;V)^{\otimes 2}\simeq \mathbb{E}^*(X^*,Y^*;V^{\oplus 2}) \xrightarrow{\mathbb{E}^*(X^*,Y^*;\Sigma_2)} \mathbb{E}^*(X^*,Y^*;V)\ ,$$

where $\Sigma_2 : V^{\oplus 2} \to V$ sends (x,y) to $x+y$. The exponential structure also yields a coproduct (where $\Delta_2 : V \to V^{\oplus 2}$ is the diagonal map):

$$\mathbb{E}^*(X^*,Y^*;V) \xrightarrow{\mathbb{E}^*(X^*,Y^*;\Delta_2)} \mathbb{E}^*(X^*,Y^*;V^{\oplus 2}) \simeq \mathbb{E}^*(X^*,Y^*;V)^{\otimes 2} \;.$$

In [FFSS, C2], both the algebra structure and the coalgebra structures are computed. But in fact the coalgebra structure is not so interesting. Indeed, it is not hard to see that the algebra structure of $\mathbb{E}^*(X^*,Y^*;V)$ determines the exponential structure, hence also the coalgebra structure.

That's why we limit ourselves to computing $\mathbb{E}^*(X^*,Y^*;V)$ (for X^*,Y^* (twisted) exponential functors) as strict polynomial algebras in this article.

1.3. Let us indicate what is known so far about extensions between classical exponential functors S^*, Γ^* and Λ^* over a commutative ring $\Bbbk$.

- The algebras $\mathbb{E}^*(\Gamma^*,X^*;V)$, for $X=\Gamma,\Lambda$ or S are very easy to compute. Indeed $\Gamma^{d\,V}$ is projective in $\mathcal{P}_\Bbbk$ for all $d\ge 0$ and the Yoneda lemma (see appendix 8.1.3(4) yields a bigraded algebra isomorphism, natural in V:

$$\bigoplus_{i\ge 0,d\ge 0} \mathbb{E}^i(\Gamma^d,X^d;V) \simeq \bigoplus_{d\ge 0}\mathbb{E}^0(\Gamma^d,X^d;V) \simeq \bigoplus_{d\ge 0} X^d(V)\;.$$

- Now Kuhn duality $\mathbb{E}^*(X^d,S^d;V) \simeq \mathbb{E}^*(\Gamma^d,X^{d\,\sharp};V)$ gives for free bigraded algebra isomorphisms, natural in V:

$$\bigoplus_{i\ge 0,d\ge 0} \mathbb{E}^i(X^d,S^d;V) \simeq \bigoplus_{d\ge 0}\mathbb{E}^0(X^d,S^d;V) \simeq \bigoplus_{d\ge 0} X^{d\,\sharp}(V)\;.$$

  (Recall that $\Lambda^{d\,\sharp}=\Lambda^d$ and $S^{d\,\sharp}=\Gamma^d$.)
- Finally, it is well known (and we shall give a proof in remark 4.6) that $\mathbb{E}^i(\Lambda^d,\Lambda^d;V)=0$ for i positive. Now an elementary computation gives a bigraded algebra isomorphism (natural in V):

$$\bigoplus_{i\ge 0,d\ge 0} \mathbb{E}^i(\Lambda^d,\Lambda^d;V) \simeq \bigoplus_{d\ge 0}\mathbb{E}^0(\Lambda^d,\Lambda^d;V) \simeq \bigoplus_{d\ge 0} \Gamma^d(V)\;.$$

So, the only non-trivial extension groups to be computed are:

$$\mathbb{E}^*(\Lambda^*,\Gamma^*;V)\simeq \mathbb{E}^*(S^*,\Lambda^*;V) \quad\text{ and }\quad \mathbb{E}^*(S^*,\Gamma^*;V)\;.$$

The first ones were first computed (when $\Bbbk$ is a field and $V=\Bbbk$) by Akin in [A], but only as graded vector spaces, that is, without the algebra structure. Recently, all these extension groups were computed as algebras (still when $\Bbbk$ is a field) by Chałupnik in [C2]

The purpose of section 4 is to compute these extension groups, as algebras, for $V=\Bbbk^m$, over commutative rings $\Bbbk$. Our method bridges strict polynomial functors with classical algebraic topology computations. Indeed, we show that the algebras $\mathbb{E}^*(S^*,\Lambda^*;\Bbbk^m)$ and $\mathbb{E}^*(S^*,\Gamma^*;\Bbbk^m)$ are respectively linked with the homology algebra of the Eilenberg-Mac Lane spaces $K(\mathbb{Z}^m,3)$ and $K(\mathbb{Z}^m,4)$. Our method generalizes the main theorem of [A].

In section 5, we elaborate on Cartan's computations [Car] to obtain explicit results over fields of prime characteristic. In fact, the careful reader will notice that not all our results agree with the computations previously made by Chałupnik [C2]. So, for sake of completeness, we give in section

5.3 elementary examples which show that our results seem to be the right ones.

Finally, we give explicit computations over $\Bbbk = \mathbb{Z}$ in section 6 (still based on the computations of [Car]).

1.4. Assume now that $\Bbbk$ is a field of positive characteristic p. We are interested in the algebras $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$, for $r, s \geq 0$ and when X^* and Y^* are one of the classical exponential functors S^*, Λ^* or Γ^*. These algebras were computed in [FFSS] and [C2]. To be more specific, let us order the classical exponential functors from the most projective one to the most injective one: $\Gamma^* < \Lambda^* < S^*$. Then the algebras $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; \Bbbk)$ for $X^* \leq Y^*$ were computed in [FFSS], using hypercohomology spectral sequence associated to De Rham complexes. The algebras for $X^* > Y^*$ were computed in [C2] (but once again with mistakes), relying on the same techniques, together with Koszul duality.

In section 7, we give an independent approach to compute these algebras. Namely, we explain how one can recover the algebra $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ from the algebra $\mathbb{E}^*(X^*, Y^*; V)$ in a simple way. To get rid of the Frobenius twists, we essentially combine an analysis of bar constructions, and the use of Troesch complexes as in [T3].

Thus, the present article together with [T3] provide a complete alternative approach to classical computations in the category of strict polynomial functors [FS, FFSS, C1, C2], independent from these articles and relying on different techniques.

2. The structure of the extension groups $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$

Definition 2.1. Let $\Bbbk$ be a commutative ring. A graded strict polynomial algebra consists of the following data.

- For all i, a strict polynomial functor A^i. If $\Bbbk$ is not a field, we allow A^i to have values in arbitrary $\Bbbk$-modules.
- For all i, j, a morphism of strict polynomial functors $A^i \otimes A^j \to A^{i+j}$,
- A morphism of strict polynomial functors $\Bbbk \to A^0$,

These data are required to satisfy the usual axioms of graded algebras. A morphism of graded strict polynomial algebras $A^* \to B^*$ is a set of morphisms of strict polynomial functors $f^i : A^i \to B^i$ which commute with products.

Similarly, one defines multigraded strict polynomial algebras, strict polynomial CDGA algebras (i.e. Commutative Differential Graded Augmented), etc.

Definition 2.2. Let $\Bbbk$ be a commutative ring. A (multi)graded strict polynomial exponential functor is a (multi)graded strict polynomial algebra E^* with $E^0(0) = \Bbbk$, and such that for all $V, W \in \mathcal{V}_\Bbbk$, the following composite (where ι_V and ι_W denote the canonical inclusions of V, W in $V \oplus W$) is an isomorphism:

$$E^*(V) \otimes E^*(W) \xrightarrow{E^*(\iota_V)\otimes E^*(\iota_W)} E^*(V \oplus W)^{\otimes 2} \xrightarrow{\text{mult}} E^*(V \oplus W)\ .$$

Let X^* and Y^* be classical exponential functors (i.e. S^*, Λ^* or Γ^*) and let $X^{*\,(r)}$ and $Y^{*\,(s)}$ denote their precompositions by the Frobenius twist functors $I^{(r)}$ and $I^{(s)}$ (For positive values of r, s, these Frobenius twist functors are defined only if $\Bbbk$ is a field of positive characteristic. For arbitrary rings $\Bbbk$, take $r = s = 0$. Since $I^{(0)}$ is the identity functor, this still makes sense). The purpose of this section is to establish that the extension groups $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V) = \bigoplus_{i,j,k\geq 0} \mathbb{E}^i(X^{j\,(r)}, Y^{k\,(s)}; V)$ are trigraded strict polynomial algebras, and even better, trigraded exponential functors if $\Bbbk$ is a field. We begin with the study of the functors $V \mapsto \mathbb{H}(F, G; V)$.

2.1. **Parameterized** Hom **groups.** Let d be a positive integer. We first work in the category $\mathcal{P}_{d,\Bbbk}$ of homogeneous strict polynomial functors of degree d over a commutative ring $\Bbbk$. This category identifies with the category of $\Bbbk$-linear functors from $\Gamma^d\mathcal{V}_\Bbbk$ to $\mathcal{V}_\Bbbk$, where $\mathcal{V}_\Bbbk$ is the category of finitely generated projective $\Bbbk$-modules, and $\Gamma^d\mathcal{V}_\Bbbk$ is the category with the same objects as $\mathcal{V}_\Bbbk$ but whose morphisms sets are $\mathfrak{S}_d$-equivariant maps: $\mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(V, W) = \mathrm{Hom}_{\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d})$.

For all $F \in \mathcal{P}_{d,\Bbbk}$, we define families $\{F_V\}_{V\in\mathcal{V}_\Bbbk}$ and $\{F^V\}_{V\in\mathcal{V}_\Bbbk}$ of strict polynomial functors parameterized by finitely projective $\Bbbk$-modules:

$$F_V : W \mapsto F(V \otimes W) \quad \text{and} \quad F^V : W \mapsto F(\mathrm{Hom}_\Bbbk(V, W))\ .$$

Morphisms $f \in \mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(V, V')$ induce morphisms of strict polynomial functors $F_V \to F_{V'}$ and $F^{V'} \to F^V$ (in a way which respects composition). As a consequence, for all $F, G \in \mathcal{P}_{d,\Bbbk}$ we have functors:

$$\begin{array}{ccc} \Gamma^d\mathcal{V}_\Bbbk & \to & \Bbbk\mathrm{-Mod} \\ V & \mapsto & \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V, G) \end{array}\ , \qquad \begin{array}{ccc} \Gamma^d\mathcal{V}_\Bbbk & \to & \Bbbk\mathrm{-Mod} \\ V & \mapsto & \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G_V) \end{array}\ .$$

Observe that these two functors have values in the category $\Bbbk-\mathrm{Mod}$ of $\Bbbk$-modules. To prove that they define genuine strict polynomial functors (i.e. elements of $\mathcal{P}_{d,\Bbbk}$), we must prove that they have values in the subcategory $\mathcal{V}_\Bbbk$ of finitely generated projective $\Bbbk$-modules. This is indeed the case if $\Bbbk$ is a Dedekind ring (e.g. $\Bbbk$ is a field or $\mathbb{Z}$), as the following lemma shows it.

Lemma 2.3. *Let $\Bbbk$ be a Dedekind ring. Then for all $F, G \in \mathcal{P}_{d,\Bbbk}$, $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(F, G)$ is a finitely generated projective $\Bbbk$-module.*

Proof. If $F = \Gamma^{d,V}$, the $\Bbbk$-module $\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\Gamma^{d,V}, G)$ is isomorphic to $G(V)$, hence finitely generated and projective. For a general, F, there is an epimorphism from a finite sum of $\Gamma^{d,\Bbbk^d}$ onto F, hence $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(F, G)$ is a submodule of a finite direct sum of $G(\Bbbk^d)$. Over Dedekind rings, submodules of finitely generated projective $\Bbbk$-modules are also finitely generated and projective. Whence the result. □

Over arbitrary commutative rings $\Bbbk$, Hom-groups need not have values in $\mathcal{V}_\Bbbk$. So the two functors above are only elements of the category $\widetilde{\mathcal{P}_{d,\Bbbk}}$ of strict polynomial functors with values in arbitrary $\Bbbk$-modules. See appendix 8 for further details about this category.

Lemma 2.4. *The two strict polynomial functors*

$$V \mapsto \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V, G) \quad \textit{and} \quad V \mapsto \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G_V)$$

are canonically isomorphic. We denote them by $\mathbb{H}(F,G;V)$.

Proof. Let us first take $F=\Gamma^{d,U}$. Then $(\Gamma^{d,U})^V=\Gamma^{d,U\otimes V}$ and the Yoneda lemma yields an isomorphism:

$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}((\Gamma^{d,U})^V,G)\simeq G(U\otimes V)\simeq \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\Gamma^{d,U},G_V)\,,$$

natural with respect to G, to $f\in\mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(V,V')$ and $g\in\mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(U,U')$ (or equivalently to $g\in\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\Gamma^{d,U'},\Gamma^{d,U})$. Since the $\Gamma^{d,U}$, $U\in\mathcal{V}_\Bbbk$, form a projective generator of $\mathcal{P}_{d,\Bbbk}$, we can take presentations of F to extend this isomorphism to all $F\in\mathcal{P}_{d,\Bbbk}$. □

For $d=0$, the category $\mathcal{P}_{0,\Bbbk}$ of homogeneous strict polynomial functors of degree 0 identifies with the category of constant functors with finitely generated projective values (that is with $\mathcal{V}_\Bbbk$). So the discussion above is trivial and lemma 2.4 also holds for $d=0$.

Now let us turn to the category $\mathcal{P}_\Bbbk$ of strict polynomial functors. Then $\mathcal{P}_\Bbbk$ splits as the direct sum of its full subcategories $\mathcal{P}_{d,\Bbbk}$. In particular, if F,G are strict polynomial functors, they split as finite direct sums $F=\bigoplus F_d$ and $G=\bigoplus G_d$, where F_d and G_d are homogeneous of degree d, and the functors

$$\begin{aligned} V&\mapsto \mathrm{Hom}_{\mathcal{P}_\Bbbk}(F^V,G)=\bigoplus \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F_d^V,G_d)\\ V&\mapsto \mathrm{Hom}_{\mathcal{P}_\Bbbk}(F,G_V)=\bigoplus \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F_d,(G_d)_V)\end{aligned}$$

are canonically isomorphic strict polynomial functors, which we still denote by $\mathbb{H}(F,G;V)$. The following lemma summarizes the main properties of parameterized Hom groups.

Lemma 2.5. *Let* $\Bbbk$ *be a commutative ring. Parameterized* Hom *groups yield a bifunctor:*

$$\begin{array}{ccc}\mathcal{P}_\Bbbk^{\mathrm{op}}\times\mathcal{P}_\Bbbk & \to & \widetilde{\mathcal{P}_\Bbbk}\\ (F,G) & \mapsto & \mathbb{H}(F,G;V)\end{array}.$$

If F, G *are homogeneous of degree* d, *then so is* $\mathbb{H}(F,G;V)$. *Moreover:*

1. *Kuhn duality yields an isomorphism of strict polynomial functors* $\mathbb{H}(F,G;V)\simeq\mathbb{H}(G^\sharp,F^\sharp;V)$, *natural in* F,G.
2. *If* G *is homogeneous of degree* d, *there is an isomorphism* $\mathbb{H}(\Gamma^d,G;V)\simeq G(V)$, *natural in* G.
3. *Tensor products induce a morphism of strict polynomial functors:*

$$\mathbb{H}(F,G;V)\otimes\mathbb{H}(F',G';V)\xrightarrow{\otimes}\mathbb{H}(F\otimes F',G\otimes G';V)\,.$$

Finally, if $\Bbbk$ *is a Dededkind ring (e.g. a field of* $\mathbb{Z}$*), then* $\mathbb{H}(F,G;V)$ *actually belongs to* $\mathcal{P}_\Bbbk$. *Thus,* $(F,G)\mapsto\mathbb{H}(F,G;V)$ *defines an intern* Hom *in* $\mathcal{P}_\Bbbk$.

Proof. The first part of lemma 2.5 follows from lemma 2.4. To prove (1), we can assume that F,G are homogeneous of degree d. Since $(F^V)^\sharp=(F^\sharp)_V$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V,G)\simeq\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(G^\sharp,(F^V)^\sharp)=\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(G^\sharp,(F^\sharp)_V)\,,$$

natural in F,G and $f\in\mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(V,V')$. Whence the result. (2) is a reformulation of the Yodeda lemma $\mathrm{Hom}_{\mathcal{P}_{\Bbbk,d}}(\Gamma^{d,V},G)\simeq G(V)$ (see appendix

8.1.3(4)). For (3), we we can assume that F, G (resp. F', G') are homogeneous of degree d (resp. e). The map:

$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V, G) \otimes \mathrm{Hom}_{\mathcal{P}_{e,\Bbbk}}((F')^W, G') \xrightarrow{\otimes} \mathrm{Hom}_{\mathcal{P}_{d+e,\Bbbk}}(F^V \otimes (F')^W, G \otimes G')$$

is natural with respect to $f \in \mathrm{Hom}_{\Gamma^d \mathcal{V}_\Bbbk}(V, V')$ and $g \in \mathrm{Hom}_{\Gamma^e \mathcal{V}_\Bbbk}(W, W')$ (i.e. it is a morphism of strict polynomial bifunctors). Hence it becomes a morphism of strict polynomial functors if on takes $V = W$. The last statement follows from lemma 2.3. □

2.2. **Parameterized extension groups.** The category of strict polynomial functors over a commutative ring $\Bbbk$ is an exact category with enough injectives and projectives. So we can define Ext groups as usual. Strictly speaking, the extension groups $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G)$ are only defined up to an isomorphism (corresponding to a choice of injective or projective resolution). But in the category of strict polynomial functors there are natural projective resolutions $\mathbb{P}(F)$, and we define $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G)$ as the homology of the complex $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\mathbb{P}(F), G)$, which fixes this problem.

The following proposition follows directly from lemmas 2.4 and 2.5 by taking resolutions.

Proposition 2.6. *Let $\Bbbk$ be a commutative ring and let F, G be strict polynomial functors over $\Bbbk$. For all $i \geq 0$, the functors*

$$V \mapsto \mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(F^V, G) \quad \text{and} \quad V \mapsto \mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(F, G_V)$$

are isomorphic strict polynomial functors (with values in arbitrary $\Bbbk$-modules). We denote them by $\mathbb{E}^i(F, G; V)$. This yields bifunctors:

$$\begin{array}{ccc} \mathcal{P}_\Bbbk^{\mathrm{op}} \times \mathcal{P}_\Bbbk & \to & \widetilde{\mathcal{P}_\Bbbk} \\ (F, G) & \mapsto & \mathbb{E}^i(F, G; V) \end{array} .$$

The homogeneous part of degree d of the strict polynomial functor $\mathbb{E}^i(F, G; V)$ equals $\mathbb{E}^i(F_d, G_d; V)$, where F_d and G_d denote the homogeneous parts of F and G of degree d. Moreover, Kuhn duality induces an isomorphism $\mathbb{E}^i(F, G; V) \simeq \mathbb{E}^i(G^\sharp, F^\sharp; V)$, and tensor products induce a morphism of strict polynomial functors

$$\mathbb{E}^i(F, G; V) \otimes \mathbb{E}^j(F', G'; V) \to \mathbb{E}^{i+j}(F \otimes F', G \otimes G'; V) .$$

Finally, if $\Bbbk$ is a field, then $(F, G) \mapsto \mathbb{E}^i(F, G; V)$ actually has values in $\mathcal{P}_\Bbbk$.

Corollary 2.7. *Let $\Bbbk$ be a commutative ring. Let C^*, resp. A^*, be a graded strict polynomial coalgebra, resp. algebra. The extension groups*

$$\mathbb{E}^*(C^*, A^*; V) = \bigoplus_{i,j,k \geq 0} \mathbb{E}^i(C^j, A^k; V)$$

form a trigraded strict polynomial algebra, whose product equals the composite (where the last map is induced by the comultiplication of C^* and the multiplication of A^*):

$$\mathbb{E}^*(C^*, A^*; V)^{\otimes 2} \xrightarrow{\otimes} \mathbb{E}^*(C^{*\otimes 2}, A^{*\otimes 2}; V) \to \mathbb{E}^*(C^*, A^*; V)$$

In this article, we shall study the strict polynomial algebras $\mathbb{E}^*(C^*, A^*; V)$ when C^* and A^* are classical exponential functors or their Frobenius twists. Observe that being a strict polynomial algebra is stronger than being an algebra natural with respect to V. Indeed, strict polynomial functors with values in arbitrary $\Bbbk$-modules can be thought of as genuine functors $\mathcal{V}_\Bbbk \to \Bbbk-\mathrm{Mod}$ equipped with an additional 'strict polynomial structure'. Hence there is a forgetful functor (see also appendix 8)

$$\mathcal{U} : \widetilde{\mathcal{P}_\Bbbk} \to \mathcal{F}ct(\mathcal{V}_\Bbbk, \Bbbk-\mathrm{Mod}) .$$

Thus, a strict polynomial algebra $A^*(V)$ yield an algebra $(\mathcal{U}A^*)(V)$, natural in V. However, non isomorphic strict polynomial algebras $A^*(V)$ and $B^*(V)$ may have the same underlying natural algebras $(\mathcal{U}A^*)(V) \simeq (\mathcal{U}B^*)(V)$. For example, take $\Bbbk = \mathbb{F}_p$. Then $S^*(V)$ and $S^*(V^{(1)})$ are non isomorphic strict polynomial algebras, but they become equal after applying $\mathcal{U}$. This reflects the fact that the algebras $S^*(V)$ and $S^*(V^{(1)})$ are isomorphic as algebras, the isomorphism is compatible with the action of the finite group $GL_n(\mathbb{F}_p)$ ($n = \dim(V)$), but not with the action of the group scheme GL_n. See section 5.1 for further details about the difference between algebras functorial in V and strict polynomial algebras.

2.3. **Exponential functors.** In this paragraph, we recall well-known facts about strict polynomial exponential functors. Then we prove that if $\Bbbk$ is a field, the extension groups between strict polynomial exponential functors are strict polynomial exponential functors. Most of the material presented here is already contained under a slightly different form in [FFSS].

Lemma 2.8. *Let E^* be a (multi)graded strict polynomial functor, with $E^0(0) = \Bbbk$. The following statements are equivalent.*

(i) *E^* is a (multi)graded strict polynomial exponential functor.*
(ii) *E^* is equipped with an isomorphism of multigraded strict polynomial bifunctors $E^*(V) \otimes E^*(W) \simeq E^*(V \oplus W)$.*
(iii) *E^* is a (multi)graded strict polynomial coalgebra such that for all $V, W \in \mathcal{V}_\Bbbk$, the following composite is a $\Bbbk$-linear isomorphism:*

$$E^*(V \oplus W) \xrightarrow{E^*(\Delta_2)} E^*(V \oplus W)^{\otimes 2} \xrightarrow{E^*(\pi_V) \otimes E^*(\pi_W)} E^*(V) \otimes E^*(W) .$$

Proof. Let us prove (i)⇔(ii), (ii)⇔(iii) is similar. First, (i)⇒(ii) is trivial. So let us assume (ii). Then we define a multiplication as the composite:

$$E^*(V)^{\otimes 2} \simeq E^*(V \oplus V) \xrightarrow{E^*(\Sigma_2)} E^*(V) ,$$

where $\Sigma_2(x, y) = x + y$. By naturality of E_*, the composite

$$E^*(V) \otimes E^*(W) \to E^*(V \oplus W)^{\otimes 2} \to E^*(V \oplus W)$$

equals the isomorphism $E^*(V) \otimes E^*(W) \simeq E^*(V \oplus W)$, so the strict polynomial algebra E^* is actually an exponential functor. □

The following lemma is proved exactly in the same fashion as lemma 2.8.

Lemma 2.9. *Let E_1^* and E_2^* be (multi)graded strict polynomial exponential functors, and let $f^* : E_1^* \to E_2^*$ be a morphism of (multi)graded strict polynomial functors. The following statements are equivalent.*

(i) *f^* is a morphism of strict polynomial algebras.*
(ii) *f^* is a morphism of strict polynomial exponential functors (that is, f^* commutes with the exponential isomorphisms).*
(iii) *f^* is a morphism of strict polynomial coalgebras.*

Lemma 2.10. *Let E^* be a graded strict polynomial exponential functor, and let F, G be strict polynomial functors. Assume that for all i, the* Hom*-groups $\mathbb{H}(E^i, F; V)$ and $\mathbb{H}(E^i, G; V)$ are $\Bbbk$-projective. Then the composite (where the last map is induced by the comultiplication of E^*)*

$$\mathbb{H}(E^*, F; V) \otimes \mathbb{H}(E^*, G; V) \xrightarrow{\otimes} \mathbb{H}(E^* \otimes E^*, F \otimes G; V) \to \mathbb{H}(E^*, F \otimes G; V)$$

is a graded isomorphism (take the total degree on the left handside).

Proof. **Step 1: Projectivity of $\mathbb{H}(E^i, J; V)$, J injective.** Observe first that if J is injective, then $\mathbb{H}(E^i, J; V)$ is always finitely generated and projective as a $\Bbbk$-module. Indeed, since functors of the form S^d_U form an injective cogenerator of $\mathcal{P}_\Bbbk$, all injectives are direct summands of (finite sums of) such injectives, so the proof reduces to the case $J = S^d_U$. By lemmas 2.4, 2.5(1) and 2.5(2), $\mathbb{H}(E^i, S^d_U; V)$ is isomorphic to $E^{i\,\sharp}(U \otimes V)$ which is finitely generated and projective since $E^{i\,\sharp} \in \mathcal{P}_\Bbbk$.

Step 2: Reduction to the injective case. For $X \in \mathcal{P}_\Bbbk$, let $X \hookrightarrow J^0_X \to J^1_X$ denote an injective copresentation of X, and let $[X]$ denote $\mathbb{H}(E^*, X; V)$. By left exactness of $[-]$, $[X]$ is the kernel of $[J^0_X] \to [J^1_X]$. For arbitrary F, G, with $[F]$ projective, $F \otimes G$ is the kernel of the map $J^0_F \otimes J^0_G \to J^1_F \otimes J^0_G \oplus J^0_F \otimes J^1_G$, and since $[F]$, $[J^0_G]$ and J^1_G are projective, $[F] \otimes [G]$ equals the kernel of the map $[J^0_F] \otimes [J^0_G] \to [J^1_F] \otimes [J^0_G] \oplus [J^0_F] \otimes [J^1_G]$. Now the maps of lemma 2.10 fit into a commutative diagram with exact rows:

$$\begin{array}{ccccc}
[F] \otimes [G] & \hookrightarrow & [J^0_F] \otimes [J^0_G] & \longrightarrow & [J^1_F] \otimes [J^0_G] \oplus [J^0_F] \otimes [J^1_G] \; . \\
\downarrow & & \downarrow & & \downarrow \\
[F \otimes G] & \hookrightarrow & [J^0_F \otimes J^0_G] & \longrightarrow & [J^1_F \otimes J^0_G] \oplus [J^0_F \otimes J^1_G]
\end{array}$$

So to prove lemma 2.10, it suffices to prove the case when F and G are injective. Since all injectives are direct summand (finite sums of) of functors of the form S^d_U, we can even assume that $F = S^d_U$ and $G = S^e_W$.

Step 3: Case $F = S^d_U$ and $G = S^e_W$. We are going to treat all values of d and e at the same time. Let us write $\langle E^*, S^*_U\rangle$ for $\bigoplus_{i,d\geq 0} \mathbb{H}(E^i, S^d_U; V)$. It suffices to show that the following composite is an isomorphism:

$$\langle E^*, S^*_U\rangle \otimes \langle E^*, S^*_W\rangle \to \langle E^* \otimes E^*, S^*_U \otimes S^*_W\rangle \to \langle E^*, S^*_U \otimes S^*_W\rangle.$$

By duality, this is equivalent to proving that the composite:

$$\langle \Gamma^{*\,U}, E^{\sharp\,*}\rangle \otimes \langle \Gamma^{*\,W}, E^{\sharp\,*}\rangle \to \langle \Gamma^{*\,U} \otimes \Gamma^{*\,W}, E^{\sharp\,*} \otimes E^{\sharp\,*}\rangle \to \langle \Gamma^{*U} \otimes \Gamma^{*W}, E^{\sharp\,*}\rangle$$

is an isomorphism. This composite fits into a commutative diagram:

$$\begin{array}{ccccc}
\langle \Gamma^{*\,U}, E^{\sharp\,*}\rangle \otimes \langle \Gamma^{*\,W}, E^{\sharp\,*}\rangle & \longrightarrow & \langle \Gamma^{*\,U} \otimes \Gamma^{*\,W}, E^{\sharp\,*} \otimes E^{\sharp\,*}\rangle & \longrightarrow & \langle \Gamma^{*U} \otimes \Gamma^{*W}, E^{\sharp\,*}\rangle \\
\big\downarrow{\scriptstyle (1)} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\
\langle \Gamma^{*\,U\oplus W}, E^{\sharp\,*}\rangle \otimes \langle \Gamma^{*\,U\oplus W}, E^{\sharp\,*}\rangle & \xrightarrow{(2)} & \langle \Gamma^{*\,U\oplus W}, E^{\sharp\,*} \otimes E^{\sharp\,*}\rangle & \xrightarrow{(3)} & \langle \Gamma^{*U\oplus W}, E^{\sharp\,*}\rangle
\end{array}$$

where the maps are as follows. The vertical isomorphisms are induced by the exponential formula for divided powers. The map (1) is induced by the canonical inclusions $\iota_U : U \hookrightarrow U \oplus W$ and $\iota_W : W \hookrightarrow U \oplus W$. The map (2) is induced by tensor products and the comultiplication of $\Gamma^{*\,U\oplus W}$

(i.e. it is nothing but the map of lemma 2.10 with 'E^*'= $\Gamma^{*\,U\oplus W}$, and 'F'= 'G'= $E^{\sharp\,*}$). The map (3) is induced by the multiplication of $\mathbb{E}^{\sharp\,*}$.

Now, we can use the Yoneda lemma (see appendix 8.1.3(4)). The map (1) readily identifies through the Yoneda isomorphism with the map $\mathbb{E}^{\sharp\,*}_V(U)\otimes \mathbb{E}^{\sharp\,*}_V(W)\to\mathbb{E}^{\sharp\,*}_V(U\oplus W)^{\otimes 2}$ induced by ι_U and ι_W. The map (2) readily identifies through the Yoneda isomorphism with the identity map of $\mathbb{E}^{\sharp\,*}_V(U\oplus W)^{\otimes 2}$, and the map (3) identifies with the multiplication $\mathbb{E}^{\sharp\,*}_V(U\oplus W)^{\otimes 2}\to \mathbb{E}^{\sharp\,*}_V(U\oplus W)$. Since $\mathbb{E}^{\sharp\,*}$ is an exponential functor, so is $\mathbb{E}^{\sharp\,*}_V$, so that the composite of (1), (2) and (3) identifies with the exponential isomorphism $\mathbb{E}^{\sharp\,*}_V(U)\otimes\mathbb{E}^{\sharp\,*}_V(W)\simeq\mathbb{E}^{\sharp\,*}_V(U\oplus W)$. Thus the first row of the diagram is an isomorphism, which finishes the proof. □

Remark 2.11. Alternatively, one could prove lemma 2.10 by using first the sum-diagonal adjunction as in the proof of [FFSS, Thm 1.7], and then identifying the isomorphism obtained, as in [T2, Lemma 5.13]. We observe that our proof does not use bifunctors, and only relies on the Yoneda lemma.

Lemma 2.12 (Compare [FFSS, Thm 1.7])**.** *Let $\Bbbk$ be a field, let E^* be a graded strict polynomial exponential functor, and let F,G be strict polynomial functors. The composite:*

$$\mathbb{E}^*(E^*,F;V)\otimes\mathbb{E}^*(E^*,G;V)\xrightarrow{\otimes}\mathbb{E}^*(E^*\otimes E^*,F\otimes G;V)\to\mathbb{E}^*(E^*,F\otimes G;V)$$

yields a bigraded isomorphism.

Proof. Let I_F and I_G be injective coresolutions of F and G. Since $\Bbbk$ is a field, we can use the Künneth formula to identify $\mathbb{E}^*(E^*,F;V)\otimes\mathbb{E}^*(E^*,G;V)$ with the homology of the complex $\mathbb{H}(E^*,I_F;V)\otimes\mathbb{H}(E^*,I_G;V)$. Now the result follows from lemma 2.10. □

Proposition 2.13. *Let $\Bbbk$ be a field, let X^*,Y^* be graded strict polynomial exponential functors. Then $\mathbb{E}^*(X^*,Y^*;V)$ is a trigraded strict polynomial exponential functor.*

Proof. We know by lemma 2.8 and corollary 2.7 that $\mathbb{E}^*(X^*,Y^*;V)$ is a trigraded strict polynomial algebra. The composite

$$\mathbb{E}^*(X^*,Y^*;V)\otimes\mathbb{E}^*(X^*,Y^*;W)\to\mathbb{E}^*(X^*,Y^*;V\oplus W)^{\otimes 2}\to\mathbb{E}^*(X^*,Y^*;V\oplus W)$$

is an isomorphism. Indeed, it equals the composite

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^*,Y^*_V)\otimes\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^*,Y^*_W)\to\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^*,Y^*_V\otimes Y^*_W)\to\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^*,Y^*_{V\oplus W}),$$

where the first map is the isomorphism of lemma 2.12 and the second one is induced by the isomorphism $Y^*_V\otimes Y^*_W\simeq Y^*_{V\oplus W}$. □

We make no use of proposition 2.13 in this article. We have stated it only to justify that it is *a priori* not worthy to to care about the coalgebra structure on $\mathbb{E}^*(X^*,Y^*;V)$ (as claimed in the introduction). Indeed, by lemma 2.8, if we know the algebra structure, we automatically know the coalgebra structure (the obvious candidate is the good one!). Observe that one cannot prove that the algebra structure determines the coalgebra structure if one restricts to computing the unparameterized extension groups $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^*,Y^*)=\mathbb{E}^*(X^*,Y^*;\Bbbk)$.

3. The signed product on $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$

Let X^*, Y^* be classical exponential functors, an let r, s be nonnegative integers. In this section, we modify the strict polynomial algebras $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ by introducing a sign on the products. We denote by $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ the resulting algebras. In the remainder of the article, we shall work with the strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ rather than the strict polynomial algebras $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$. Indeed, the sign introduced simplifies the computations (see e.g. lemma 4.2) and yields more readable results. We first introduce signed algebras.

3.1. **Signed algebras and graded commutativity.**

Let $\Bbbk$ be a commutative ring. In what follows, we always assume that algebras, coalgebras, etc. are nonnegatively (bi)graded, and defined over $\Bbbk$.

Definition 3.1. A bigraded strict polynomial algebra $A^{*,*}$ is $(1,\epsilon)$-commutative (with $\epsilon \in \{0,1\}$) if the following diagrams commute up to a $(-1)^{ij+\epsilon k\ell}$ sign:

$$\begin{array}{ccc} A^{i,k}\otimes A^{j,\ell} & \xrightarrow{\text{mult}} & A^{i+j,k+\ell} \\ \downarrow{\simeq} & & \| \\ A^{j,\ell}\otimes A^{i,k} & \xrightarrow{\text{mult}} & A^{i+j,k+\ell} \end{array}.$$

Thus, a bigraded strict polynomial algebra is bigraded commutative in the usual sense if it is $(1,1)$ commutative. One defines $(1,\epsilon)$-commutative bigraded coalgebras similarly. One easily checks that if $E^{*,*}$ is a bigraded strict polynomial exponential functor, then $E^{*,*}$ is $(1,\epsilon)$-commutative as an algebra if and only if it is $(1,\epsilon)$-commutative as a coalgebra.

Definition 3.2. Let $A^{*,*}$ be a $(1,\epsilon)$-commutative bigraded strict polynomial algebra. The 'signed algebra' $\overline{A}^{*,*}$ is the bigraded strict polynomial algebra which equals $A^{*,*}$ as a bigraded functor, and whose multiplication is defined by sending $x\otimes y \in A^{i,k}\otimes A^{j,\ell}$ onto $(-1)^{\epsilon i\ell} m(x\otimes y)$, where m is the multiplication of $A^{*,*}$.

Observe that the signed algebra $\overline{A}^{*,*}$ is still associative. One defines similarly signed coalgebras. In general, the totalization of a $(1,\epsilon)$-commutative bigraded strict polynomial algebra $A^{*,*}$ is a not graded commutative. The following elementary lemma (which we actually use in section 7.3) explains why it is sometimes easier to work with the signed algebras $\overline{A}^{*,*}$.

Lemma 3.3. *Let $A^{*,*}$ be a bigraded strict polynomial algebra. For $\alpha \in \mathbb{Z}$, we define the graded algebra $\mathrm{Tot}^{\alpha} A^{*,*}$ to be the same algebra as $A^{*,*}$, with $A^{k,\ell}$ placed in total degree $k+\alpha\ell$. If $A^{*,*}$ is $(1,\epsilon)$-commutative, then for all $i\in\mathbb{Z}$, $\mathrm{Tot}^{2i+\epsilon}(\overline{A}^{*,*})$ is graded commutative.*

A similar result holds for $(1,\epsilon)$-commutative bigraded coalgebras.

3.2. **The signed algebra $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$.**

Convention 3.4. Let $\Bbbk$ be a commutative ring, let X^*, Y^* be classical exponential functors, and let r, s be nonnegative integers (with $r = s = 0$ if

$\Bbbk$ is not a field of positive characteristic). We consider the extension groups

$$\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V) := \bigoplus_{h,d\geq 0} \mathbb{E}^h(X^{dp^s\,(r)}, Y^{dp^t\,(s)}; V)\ ,$$

as a bigraded strict polynomial algebra, with $\mathbb{E}^h(X^{dp^s\,(r)}, Y^{dp^t\,(s)}; V)$ placed in bidegree (h, dp^{s+r}) (that is, 'h' is the cohomological degree and 'dp^{s+r}' is the strict polynomial degree), and with product as defined in corollary 2.7.

By convention, $I^{(0)}$ is the identity functor of $\mathcal{V}_\Bbbk$. So, if $r = s = 0$ then $\mathbb{E}^*(X^{*\,(0)}, Y^{*\,(0)}; V)$ actually denotes $\mathbb{E}^*(X^*, Y^*; V)$. The following lemma is an easy check (or use [FFSS, Lemma 1.11]).

Lemma 3.5. *Let X^*, Y^* be a pair of classical exponential functors, and let $\epsilon \in \{0,1\}$ denote the sum $\epsilon := \epsilon(X^*)+\epsilon(Y^*)$ modulo 2, with $\epsilon(S^*) = \epsilon(\Gamma^*) = 0$ and $\epsilon(\Lambda^*) = 1$. Then $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ is a $(1,\epsilon)$-commutative bigraded strict polynomial algebra.*

Convention 3.6. We denote by $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ the signed algebra associated to the bigraded strict polynomial algebra $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$.

Example 3.7. **(1)** The bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ is defined as follows. As a bigraded strict polynomial functor we have:

$$\overline{\mathbb{E}}^*(S^*, \Lambda^*; V) := \bigoplus_{h,d\geq 0} \mathbb{E}^h(S^d, \Lambda^d; V)\ ,$$

with $\mathbb{E}^h(S^d, \Lambda^d; V)$ placed in bidegree (h, d). If x and y have respective bidegrees (i, d) and (j, e), the multiplication sends $x\otimes y$, onto $(-1)^{ie} m(x\otimes y)$, where 'm' is the product of $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ (as defined in corollary 2.7).
(2) The bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ is defined by

$$\overline{\mathbb{E}}^*(S^*, \Gamma^*; V) := \bigoplus_{h,d\geq 0} \mathbb{E}^h(S^d, \Gamma^d; V)\ ,$$

with $\mathbb{E}^h(S^d, \Gamma^d; V)$ placed in bidegree (h, d) and with multiplication as in corollary 2.7.

Observe that in the following situations, the algebras $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ and $\mathbb{E}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ are equal.

- The ring $\Bbbk$ has characteristic 2.
- The number of Λ^* in the pair (X^*, Y^*) is even.
- The extension groups $\mathbb{E}^h(X^{dp^s\,(r)}, Y^{dp^t\,(s)}; V)$ are trivial if h is odd.

As we shall see later, these three situations cover almost all cases, with the notable exception of $\overline{\mathbb{E}}^*(S^{*\,(r)}, \Lambda^{*\,(s)}; V)$ in odd characteristic. However, we keep the notation $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$ in every cases in order to give a unified treatment of all the computations.

4. Extensions between classical exponential functors: Bar constructions and $K(\mathbb{Z}^m, n)$

Let $\Bbbk$ be a commutative ring. In this section, we express the strict polynomial algebras $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ and $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ (or to be more specific, a totalization of these algebras, cf. section 4.1(4) below) in terms of bar constructions and singular homology of some Eilenberg Mac Lane spaces.

4.1. **Notations and degrees.** Before starting, we take time to describe clearly the various objects involved in our computations and their gradings.

4.1.1. *The explicit homological degree.* All the objects used in our computations bear a homological degree (and the differentials, if any, lower the homological degree by one). We now list these objects, and indicate how they are graded.

(1) If V is a $\Bbbk$-module, we denote by $V[i]$ a copy of V placed in homological degree i, for $i \geq 0$.
(2) If X^* is a classical exponential functor (i.e. S^*, Λ^*, or Γ^*), and $i > 0$, we denote by $X[i]_*$ the exponential functor:
$$X[i]_* \;:\; V \mapsto X^*(V[i]) \;.$$
In other words, $X[i]_{di}(V)$ equals $X^d(V)$ and $X[i]_n(V)$ equals zero if $d \not| n$. We denote the degree of $X[i]_*$ by a subcript to emphasize that we think of it rather as a homological degree than as a cohomological one, cf. point (4) below.
(3) Let A_* be a graded strict polynomial algebra, and let E_* be a strict polynomial exponential functor. We define the graded strict polynomial algebra $\mathbb{H}(E, A_*; V)$ (with multiplication as in corollary 2.7) by:
$$\mathbb{H}(E, A_*; V) := \textstyle\bigoplus_{i,j\geq 0} \mathbb{H}(E_i, A_j; V)\,,$$
with $\mathbb{H}(E_i, A_j; V)$ placed in homological degree j.
(4) For all $i \geq 0$, we denote by $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ and $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$ the graded strict polynomial algebras obtained by totalizing the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ and $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ in the following way:
$$\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V) = \textstyle\bigoplus_{h,d\geq 0} \overline{\mathbb{E}}^h(S^d, \Lambda^d; V)\;,$$
with $\overline{\mathbb{E}}^h(S^d, \Lambda^d; V)$ placed in homological degree $d(2i+1) - h$, and
$$\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V) := \textstyle\bigoplus_{h,d\geq 0} \overline{\mathbb{E}}^h(S^d, \Gamma^d; V)\;,$$
with $\overline{\mathbb{E}}^h(S^d, \Gamma^d; V)$ placed in homological degree $d(2i+2)-h$ (beware the signs).

Remark 4.1. The graded strict polynomial algebras $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ and $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$ are the $\mathrm{Tot}^{\epsilon-(2i+2)}$-totalizations of $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ and $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ in the sense of lemma 3.3. In particular, they are graded commutative.

4.1.2. *The implicit strict polynomial degree.* The objects listed above are strict polynomial algebras, as will be all the algebras involved in our computations. Thus they bear an implicit strict polynomial degree, e.g. the elements of $S^d(V[i]) = S[i]_{di}(V)$ have explicit homological degree di but also an implicit strict polynomial degree d, which is the degree of the functor S^d. This strict polynomial degree is usually not explicitly indicated in our computations. Indeed, it is always very easy to compute and it is automatically taken in charge by the fact that all morphisms of strict polynomial functors preserve the strict polynomial degree.

4.2. **Injective coresolutions of classical exponential functors.**

4.2.1. *Quick recollections of bar constructions.* Let $\Bbbk$ be a commutative ring, and let A_* be a Differential Graded Augmented $\Bbbk$-algebra. The (reduced, normalized) bar construction over A is the Differential Graded Coalgebra $\overline{B}(A_*)$ defined as follows.

- Let A'_* be the kernel of the augmentation $\epsilon : A_* \to \Bbbk$. Then $\overline{B}(A_*)$ equals $\bigoplus_{n\ge 0} {A'_*}^{\otimes n}$ as a $\Bbbk$-module.
- A scalar λ of $\Bbbk = {A'_*}^{\otimes 0}$ is denoted by $\lambda[]$ and has degree 0. For $n \geq 1$, an element $a_1 \otimes \dots \otimes a_n$ of $A_{k_1} \otimes \dots \otimes A_{k_n}$ is denoted by $[a_1|\dots|a_n]$ and has degree $n + \sum k_i$.
- The differential $d : \overline{B}(A_*)_k \to \overline{B}(A_*)_{k-1}$ sends an element $[a_1|\dots|a_n]$ to the sum:

$$\sum_{i=1}^{n-1}(-1)^{e_i}[a_1|\dots|a_ia_{i+1}|\dots|a_n] - \sum_{i=1}^{n}(-1)^{e_{i-1}}[a_1|\dots|d(a_i)|\dots|a_n]\ ,$$

  where $e_0 = 0$ and for $i \geq 1$, e_i equals $i + \sum_{j\le i}\deg(a_i)$.
- The coproduct $\Delta : \overline{B}(A_*) \to \overline{B}(A_*) \otimes \overline{B}(A_*)$ sends an element $[a_1|\dots|a_n]$ to the sum

$$\sum_{i=0}^{n}[a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n]\ .$$

If A_* is a CDGA $\Bbbk$-algebra (i.e. Commutative Differential Graded Augmented), we can define a 'shuffle product' on $\overline{B}(A_*)$ compatible with the differential, which makes $\overline{B}(A_*)$ into a CDGA $\Bbbk$-algebra. Hence we can iterate bar constructions, and we denote by $\overline{B}^n(A_*)$ the n-th iterated bar construction of A_*. To be more specific, let $a_i \in A_{k_i}$ for $1 \le i \le p+q$. Then the product $[a_1|\dots|a_p] * [a_{p+1}|\dots|a_{p+q}]$ of two elements equals

$$\sum \epsilon(\sigma)\,[a_{\sigma^{-1}(1)}|\dots|a_{\sigma^{-1}(p+q)}]$$

where the sum is taken over all (p,q)-shuffles σ, and $\epsilon(\sigma)$ is the Koszul sign such that $x_1 \wedge \dots \wedge x_n = \epsilon(\sigma)x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)}$ in $\Lambda(x_1,\dots,x_n)$, where each x_i has degree $k_i + 1$.

Now if $A_*(V)$ is a strict polynomial CDGA algebra, the formulas above show that the multiplication and the differential are morphisms of strict polynomial functors. So iterated bar constructions yield functors ($n \geq 0$):

$$\left\{\begin{array}{c}\text{strict polyn.}\\ \text{CDGA-alg.}\end{array}\right\} \xrightarrow{\overline{B}^n} \left\{\begin{array}{c}\text{strict polyn.}\\ \text{CDGA-alg.}\end{array}\right\}$$

Finally, recall from [ML, X Th. 11.2] that bar constructions preserve quasi isomorphisms.

4.2.2. *Bar constructions of symmetric and exterior algebras.* Let $\Bbbk$ be a commutative ring and let $V[i]$ be a finitely generated projective $\Bbbk$-module, concentrated in homological degree $i \geq 0$. If $d \geq 0$, the homogeneous part of $\overline{B}(S^*(V[i]))$ of strict polynomial degree d equals the complex:

$$\underbrace{V^{\otimes d}}_{\substack{\text{degree}\\ \text{d(i+1)}}} \to \bigoplus_{k=0}^{d-2} V^{\otimes k}\otimes S^2(V)\otimes V^{\otimes d-k-2} \to \cdots \to \begin{array}{c} S^{d-1}(V)\otimes V \\ \oplus V\otimes S^{d-1}(V)\end{array} \to \underbrace{S^d(V)}_{\substack{\text{degree}\\ \text{di+1}}} \;.$$

Similarly, one gets the homogeneous part of $\overline{B}(\Lambda^*(V[i]))$ of strict polynomial degree d by replacing symmetric powers by exterior ones in the complex above.

Assume that the algebras $S^*(V[i])$, resp. $\Lambda^*(V[i])$, are graded commutative (i.e. assume that i is even, resp. odd, if the characteristic of $\Bbbk$ is different from 2). Then the canonical inclusions $\Lambda^d(V) \hookrightarrow V^{\otimes d}$, resp. $\Gamma^d(V) \hookrightarrow V^{\otimes d}$, define morphisms of (strict polynomial) differential graded Hopf algebras:

$$\Lambda^*(V[i+1]) \hookrightarrow \overline{B}(S^*(V[i])) \quad \text{and} \quad \Gamma^*(V[i+1]) \hookrightarrow \overline{B}(\Lambda^*(V[i]))\;.$$

These morphisms are quasi-isomorphisms, see e.g. [T1, Lemma 3.19] for this classical result.

In particular, the homogeneous part of strict polynomial degree d of $\overline{B}(S^*(V[i]))$ yields a coresolution of $\Lambda^d(V)$ by symmetric powers. Since bar constructions preserve quasi-isomorphisms, the composite

$$\Gamma^*(V[i+2]) \hookrightarrow \overline{B}(\Lambda^*(V[i+1])) \hookrightarrow \overline{B}^2(S^*(V[i]))$$

is also a quasi isomorphism. The homogeneous part of strict polynomial degree d of $\overline{B}^2(S^*(V[i]))$ yields a coresolution of $\Gamma^d(V)$ by symmetric powers which has the form:

$$\underbrace{V^{\otimes d}}_{\substack{\text{degree}\\ \text{d(i+2)}}} \to \bigoplus_{k=1}^{d-2} V^{\otimes d} \to \cdots \to \begin{array}{c} S^{d-1}(V)\otimes V \\ \oplus V\otimes S^{d-1}(V)\end{array} \to \underbrace{S^d(V)}_{\substack{\text{degree}\\ \text{di+2}}} \;.$$

In the framework of strict polynomial functors, the resolutions above are particularly interesting. Indeed, symmetric powers are injective objects, so we obtain the following result.

Lemma 4.2. *Let $\Bbbk$ be a commutative ring and let i be a nonnegative integer. The graded strict polynomial algebras $\overline{\mathbb{E}}^*(S,\Lambda[2i+1]_*;V)$ and $\overline{\mathbb{E}}^*(S,\Gamma[2i+2]_*;V)$ are respectively given by the homology of $\mathbb{H}(S,\overline{B}(S[2i]_*);V)$ and $\mathbb{H}(S,\overline{B}^2(S[2i]_*);V)$.*

Proof. We first treat the case of $\overline{\mathbb{E}}^*(S,\Lambda[2i+1]_*;V)$. As explained above, the homogeneous part of strict polynomial degree d of $\overline{B}(S^*(U[2i]))$ yields a chain complex $I_{d,\bullet}(U) := C_{d(2i+1)}(U) \to \cdots \to C_0(U) = 0$ whose objects are symmetric tensors and whose homology equals $\Lambda^d(U[2i+1])$ in degree $d(2i+1)$. Thus, the n-th homology group of $\mathbb{H}(S^d, I_{d,\bullet};V)$ equals $\overline{\mathbb{E}}^{d(2i+1)-j}(S^d,\Lambda^d;V)$.

So it remains to study the products. If $C_\bullet$ is a chain complex, we denote by $C^\bullet$ the same complex, but viewed as a cochain complex with $C^i := C_{-i}$. For all $a\in\mathbb{Z}$ We also denote by $C\langle a\rangle^\bullet$ the cochain complex defined by shifting: $C\langle a\rangle^i := C^{d+i}$, and the differential of $C\langle a\rangle^\bullet$ equals the differential of $C^\bullet$ up to a $(-1)^d$ sign. Here comes a subtlety about signs: the canonical isomorphism of complexes

$$\phi : C\langle a\rangle^\bullet \otimes D\langle b\rangle^\bullet \xrightarrow{\simeq} (C\otimes D)\langle a+b\rangle^\bullet$$

sends $x\otimes y \in C\langle a\rangle^k \otimes D\langle b\rangle^\ell = C^{k+a}\otimes D^{\ell+b}$ to $(-1)^{kb}x\otimes y$. (The sign is needed for the compatibility with the differential).

Now let us apply this to $\overline{\mathbb{E}}^*(S,\Lambda[2i+1]_*;V)$. The cochain complex $I_d\langle d(2i+1)\rangle^\bullet(U)$ is an injective coresolution of $\Lambda^d(U[2i+1])$. On the cochain level, the 'usual product' of two classes c_1, c_2 (i.e. the product of the algebra $\mathbb{E}^*(S^*,\Lambda^*;V)$, as in corollary 2.7) is defined as follows. Choose two cycles z_1, z_2 in $\mathbb{H}(S^d, I_d\langle d(2i+1)\rangle^\bullet)$ and $\mathbb{H}(S^e, I_e\langle e(2i+1)\rangle^\bullet)$ representing c_1 and c_2 respectively, take their product $(z_1\otimes z_2)\circ\Delta_{d,e}$ in $\mathbb{H}(S^{d+e}, I_d\langle d(2i+1)\rangle^\bullet \otimes I_e\langle e(2i+1)\rangle^\bullet)$ (here $\Delta_{d,e}: S^{d+e}\to S^d\otimes S^e$ is the comultiplication), and send this element into $\mathbb{H}(S^{d+e}, I_{d+e}\langle (d+e)(2i+1)\rangle^\bullet)$ via a lifting of the multiplication $\Lambda^d\otimes\Lambda^e\to\Lambda^{d+e}$. Now, such a lifting is given by the composite (where '$*$' is the shuffle product):

$$I_d\langle d(2i+1)\rangle^\bullet\otimes I_e\langle e(2i+1)\rangle^\bullet \xrightarrow{\phi} (I_d\otimes I_e)\langle (d+e)(2i+1)\rangle^\bullet \xrightarrow{*} I_{d+e}\langle (d+e)(2i+1)\rangle^\bullet .$$

Thus, the 'signed product' of $\overline{\mathbb{E}}^*(S,\Lambda[2i+1]_*;V)$ is defined by sending the cycles z_1, z_2 onto the image of $(z_1\otimes z_2)\circ\Delta_{d,e}$ by the shuffle product. That is, $\overline{\mathbb{E}}^*(S,\Lambda[2i+1]_*;V)$ is computed as a strict polynomial algebra by the homology of $\mathbb{H}(S,\overline{B}(S[2i]_*))$. The case of $\overline{\mathbb{E}}^*(S,\Gamma[2i+2]_*;V)$ is similar. □

4.3. **An interchange property.** This subsection is the core of our computation. The key is the following interchange property, which is very specific to exponential functors.

Lemma 4.3. *Let $\Bbbk$ be a commutative ring, let E_* be a strict polynomial exponential functor and let A_* be a strict polynomial CDGA algebra. Assume that for all $i,j\geq 0$ the* Hom*-groups $\mathbb{H}(E_i, A_j;V)$ are $\Bbbk$-projective.*

There is an isomorphism of strict polynomial differential graded strict polynomial functors:

$$\mathbb{H}(E,\overline{B}(A_*);V)\simeq \overline{B}(\mathbb{H}(E,A_*;V)) .$$

If E_* is commutative (i.e. $x_i\cdot x_j = x_j\cdot x_i$ in the algebras $E_*(V)$), then $\mathbb{H}(E,A_*;V)$ is graded commutative and the isomorphism is an isomorphism of strict polynomial CDGA algebras.

Remark 4.4. If $\Bbbk$ is a Dedekind ring (e.g $\Bbbk$ is a field or $\mathbb{Z}$), then the assumption that the Hom-groups $\mathbb{H}(E_i,F;V)$ and $\mathbb{H}(E_i,G;V)$ are $\Bbbk$-projective is automatically satisfied by lemma 2.3.

Proof of lemma 4.3. Let us write for short $[X]$ instead of $\mathbb{H}(E,X;V)$. By lemma 2.10, there are isomorphisms fitting into commutative diagrams (where the vertical arrows are induced by multiplications):

$$\begin{array}{ccc}
[A_{k_1}]\otimes\dots[A_{k_i}]\otimes[A_{k_{i+1}}]\dots\otimes[A_{k_n}] & \xrightarrow[\theta]{\simeq} & [A_{k_1}\otimes\dots A_{k_i}\otimes A_{k_{i+1}}\dots\otimes A_{k_n}] \\
\downarrow & & \downarrow \\
[A_{k_1}]\otimes\dots[A_{k_i+k_{i+1}}]\dots\otimes[A_{k_n}] & \xrightarrow[\theta]{\simeq} & [A_{k_1}\otimes\dots A_{k_i+k_{i+1}}\dots\otimes A_{k_n}] .
\end{array}$$

Thus, the isomorphism of differential graded strict polynomial functors $\overline{B}([A_*]) \simeq [\overline{B}(A_*)]$ follows directly from the definitions of bar constructions given in section 4.2.1. If E_* is commutative, then for all $\sigma \in \mathfrak{S}_n$ the isomorphisms of lemma 2.10 fit into commutative diagrams :

$$\begin{array}{ccc} [A_{k_1}] \otimes \cdots \otimes [A_{k_n}] & \xrightarrow[\theta]{\simeq} & [A_{k_1} \otimes \cdots \otimes A_{k_n}] \\ \downarrow \sigma & & \downarrow [\sigma] \\ [A_{k_{\sigma^{-1}(1)}}] \otimes \cdots \otimes [A_{k_{\sigma^{-1}(n)}}] & \xrightarrow[\theta]{\simeq} & [A_{k_{\sigma^{-1}(1)}} \otimes \cdots \otimes A_{k_{\sigma^{-1}(n)}}] \end{array} .$$

In particular, if $m_{k,\ell} : A_k \otimes A_\ell \to A_{k+\ell}$ is the product of A_*, and τ is the exchange map $X \otimes Y \simeq Y \otimes X$, then $[m_{k,\ell}] \circ \theta \circ \tau = [m_{k,\ell} \circ \tau] \circ \theta = (-1)^{k\ell}[m_{k,\ell}] \circ \theta$. Hence $\mathbb{H}(E, A_*; V)$ is graded commutative. So $\overline{B}([A_*])$ and $[\overline{B}(A_*)]$ are both strict polynomial CDGA algebras. The product in $\overline{B}([A_*])$ is the shuffle product '$*$', whose restriction to $(\bigotimes_{i=1}^{p}[A_{k_i}]) \otimes (\bigotimes_{i=p+1}^{p+q}[A_{k_i}])$ equals $\prod \epsilon(\sigma)\sigma$ (product taken over all (p,q)-shuffles), and and the product in $[\overline{B}(A_*)]$ is $[*] \circ \theta$, whose restriction to $[\bigotimes_{i=1}^{p} A_{k_i}] \otimes [\bigotimes_{i=p+1}^{p+q} A_{k_i}]$ equals $\prod \epsilon(\sigma)[\sigma] \circ \theta$ (with the same signs as for $[\overline{B}(A_*)]$). Thus, by the commutative diagram above, the two products coincide. □

Let i be a positive integer. We can now compute the graded algebras $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ and $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$. Theorem 4.5 can be thought of as a generalization of [A, Thm p. 361] (Akin's theorem corresponds to the case of $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$, with $V = \Bbbk$, and without the algebra structure).

Theorem 4.5. *Let $\Bbbk$ be a commutative ring, let V be a finitely generated projective $\Bbbk$-module and let i be a nonnegative integer. The algebras $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ and $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$ are respectively computed, as graded strict polynomial algebras, by the homology of the bar constructions $\overline{B}(\Gamma^*(V[2i]))$ and $\overline{B}^2(\Gamma^*(V[2i]))$.*

Proof. By lemma 4.2, $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ equals the homology of the strict polynomial CDGA algebra $\mathbb{H}(S, \overline{B}(S[2i]_*); V)$. By lemma 2.5(1) and (2), the Hom-groups $\mathbb{H}(S, S[2i]_*; V)$ equal $\Gamma^*(V[2i])$ as an algebra. Hence they are $\Bbbk$-projective, and by lemma 4.3, $\mathbb{H}(S, \overline{B}(S[2i]_*); V)$ is isomorphic to the strict polynomial CDGA algebra $\overline{B}(\Gamma^*(V[2i]))$. The case of $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$ is similar. □

Remark 4.6. As another application of the interchange property, one can prove that $\mathbb{E}^*(\Lambda^d, \Lambda^d; V)$ equals zero if $* > 0$. Indeed, let $\mathbb{E}^*(\Lambda, \Lambda[3]_*; V)$ denote the graded strict polynomial functor $\bigoplus_{i,d \geq 0} \mathbb{E}^i(\Lambda^d, \Lambda^d; V)$ with $\mathbb{E}^i(\Lambda^d, \Lambda^d; V)$ in degree $3d - i$. Then $\mathbb{E}^*(\Lambda, \Lambda[3]_*; V)$ equals the homology of $\mathbb{H}(\Lambda, \overline{B}(S[2]_*); V)$, which equals $\overline{B}(\mathbb{H}(\Lambda, S[2]_*; V))$ by the interchange property (no algebra structure here, since Λ^* is not commutative). Now $\mathbb{H}(\Lambda, S[2]_*; V) \simeq \Lambda^*(V[2])$. So $\mathbb{E}^*(\Lambda, \Lambda[3]_*; V)$ finally equals $\Gamma^*(V[3])$ as a graded strict polynomial functor. Whence the result.

Thus, computing the graded algebras $\overline{\mathbb{E}}^*(S, \Lambda[2i+1]_*; V)$ and $\overline{\mathbb{E}}^*(S, \Gamma[2i+2]_*; V)$ reduces to computing the homology of iterated bar constructions of the divided power algebra $\Gamma^*(V[2i])$. In fact it is sufficient to do the computation for one specific value of i, as the following result shows it.

Lemma 4.7. *Let $\Bbbk$ be a commutative ring, let V be a finitely generated projective $\Bbbk$-module, and let i be an even positive integer. For all $n \geq 0$, $\overline{B}^n(\Gamma^*(V[0]))$ and $\overline{B}^n(\Gamma^*(V[i]))$ are equal as ungraded algebras. An element of homological degree h and strict polynomial degree d in $\overline{B}^n(\Gamma^*(V[0]))$ has homological degree $h+di$ and strict polynomial degree d in $\overline{B}^n(\Gamma^*(V[i]))$.*

Proof. Let $A_*(V)$ be a graded strict polynomial algebra. We define a new graded strict polynomial algebra $\widetilde{A}_*(V)$ which equals $\bigoplus_h A_h(V)$ as an ungraded object, and an element $a \in A_h(V)$ of strict polynomial degree $\deg^{\mathrm{sp}}(a) = d$ has homological degree $h+di$ in $\widetilde{A}_*(V)$. Then the homological degree of $[a_1|\dots|a_k] \in A_{h_1} \otimes \cdots \otimes A_{h_k}$ equals $k + \sum h_j$ in $\overline{B}(A_*(V))$, and it equals $k+\sum(h_j+i\deg^{\mathrm{sp}}(a_j)) = (k+\sum h_j)+i\deg^{\mathrm{sp}}([a_1|\dots|a_k])$ in $\overline{B}(\widetilde{A}_*(V))$. Now lemma 4.7 clearly holds for $n = 0$. For $n \geq 1$, one obtains the result by induction on the preceding observation (set $A_*(V) = \overline{B}^{n-1}(\Gamma^*(V[0]))$ and $\widetilde{A}_*(V) = \overline{B}^{n-1}(\Gamma^*(V[i]))$). □

We explicitly compute the homology of the iterated bar constructions of $\Gamma^*(V[2])$ in sections 5 and 6. But before this, let us give a topological interpretation of theorem 4.5.

4.4. **The homology of Eilenberg-Mac Lane spaces.** Let π be an abelian group, let n be a positive integer. The Eilenberg-Mac Lane space $K(\pi, n)$ is the topological space (more specifically the CW-complex, unique up to homotopy equivalence) whose i-th homotopy group equals 0 if $i \neq n$, and π if $i = n$. Since $K(\pi, n)$ is an H-space, its singular homology with coefficients in a commutative ring $\Bbbk$ is a graded commutative algebra. By [EML], the homology of the iterated bar constructions $\overline{B}^n(\Bbbk\pi)$ compute the singular homology algebras $H_*(K(\pi, n), \Bbbk)$, naturally in π (here $\Bbbk\pi$ is the group algebra of π over $\Bbbk$).

For $n = 1$, $H_*(K(\pi, 1), \Bbbk)$ equals [Br, I.4] the homology algebra $H_*(\pi, \Bbbk) = \mathrm{Tor}_*^{\Bbbk\pi}(\Bbbk, \Bbbk)$ (as a functor of the abelian group π). The first homology of an abelian group π with coefficients in a commutative ring $\Bbbk$ is isomorphic to the $\Bbbk$-module $\pi \otimes_{\mathbb{Z}} \Bbbk$. Using products, we get a $\Bbbk$-algebra morphism, natural in π:

$$\psi : \Lambda^*(\pi \otimes_{\mathbb{Z}} \Bbbk\,[1]) \to H_*(\pi, \Bbbk)\ .$$

If π is a free abelian group, this map is an isomorphism [Br, V.6 Th 6.4(ii)].

Actually one can do a little better. All the elements of $\overline{B}(\Bbbk\pi)_1$ are cycles, so that the canonical map $\Bbbk\pi = \overline{B}(\Bbbk\pi)_1 \to H_1(\pi, \Bbbk) = \pi \otimes_{\mathbb{Z}} \Bbbk$ is surjective. If π is free, then $\pi \otimes_{\mathbb{Z}} \Bbbk$ is a free $\Bbbk$-module, and we can *choose* a section of this map. Since the bar construction is graded commutative, taking products induces a map of differential graded $\Bbbk$-algebras:

$$\widetilde{\psi} : \Lambda^*(\pi \otimes_{\mathbb{Z}} \Bbbk\,[1]) \to \overline{B}(\Bbbk\pi)$$

which equals ψ after taking homology. Thus $\widetilde{\psi}$ is a quasi isomorphism. Now $\overline{B}(\Lambda^*(\Bbbk^m[1]))$ is quasi isomorphic to $\Gamma^*(\Bbbk^m[2])$ (by section 4.2.2) and bar constructions preserves quasi isomorphisms, so $\overline{B}^n(\Gamma^*(\mathbb{Z}^m \otimes_{\mathbb{Z}} \Bbbk\,[2]))$ computes $H_*(K(\mathbb{Z}^m, n+2), \Bbbk)$ for all $n \geq 2$ and $m \geq 1$. Thus, theorem 4.5 yields the following result.

Theorem 4.8. *Let* $\Bbbk$ *be a commutative ring, then we have isomorphisms of graded* $\Bbbk$*-algebras (with* $\overline{\mathbb{E}}^*(S,\Lambda[3]_*;\Bbbk^m)$ *and* $\mathbb{E}^*(S,\Gamma[4]_*;\Bbbk^m)$ *as specified in section 4.1):*

$$\overline{\mathbb{E}}^*(S,\Lambda[3]_*;\Bbbk^m) \simeq H_*(K(\mathbb{Z}^m,3),\Bbbk)\;,$$
$$\overline{\mathbb{E}}^*(S,\Gamma[4]_*;\Bbbk^m) \simeq H_*(K(\mathbb{Z}^m,4),\Bbbk)\;.$$

Remark 4.9. Theorem 4.8 expresses the extension groups as graded algebras. To recover the bigrading, we need to examine the implicit strict polynomial degree on the homology of $\overline{B}^n(\Gamma^*(\Bbbk^m[2]))$, $n=1,2$.

The homology of Eilenberg-Mac Lane spaces were computed by Cartan [Car] when $\Bbbk$ is a prime field or $\mathbb{Z}$. We use Cartan's computations to compute explicitly the bigraded algebras $\bigoplus_{h,d\ge 0}\overline{\mathbb{E}}^h(S^d,\Lambda^d;V)$ and $\bigoplus_{h,d\ge 0}\overline{\mathbb{E}}^h(S^d,\Gamma^d;V)$ in the following sections.

Remark 4.10. Observe that the quasi-isomorphism $\widetilde{\psi}$ is not natural with respect to the group $\pi=\mathbb{Z}^m$ (although ψ is). Thus, it is not clear at first sight that the isomorphism induced by $\widetilde{\psi}$

$$H_*(\overline{B}^n(\Gamma^*(\mathbb{Z}^m\otimes_{\mathbb{Z}}\Bbbk\,[2]))) \simeq H_*(K(\mathbb{Z}^m,n+2),\Bbbk)$$

is natural with respect to $\mathbb{Z}^m$. However when $\Bbbk$ is a field, this isomorphism is actually natural with respect to $\mathbb{Z}^m$.

Indeed, Cartan has build [Car] universal graded $\Bbbk$-algebras $A_*(\mathbb{Z}^m,n,\Bbbk)$ and isomorphisms which fit into a commutative triangle

$$\begin{array}{ccc} H_*(\overline{B}^n(\Gamma^*(\mathbb{Z}^m\otimes_{\mathbb{Z}}\Bbbk\,[2]))) & \xrightarrow{\simeq} & H_*(K(\mathbb{Z}^m,n+2),\Bbbk)\;. \\ & {\scriptstyle\simeq}\nwarrow \quad \nearrow{\scriptstyle\simeq} & \\ & A_*(\mathbb{Z}^m,n,\Bbbk) & \end{array}$$

The $A_*(\mathbb{Z}^m,n,\Bbbk)$ are functors in the variable $\mathbb{Z}^m$, and the isomorphisms with source $A_*(\mathbb{Z}^m,n,\Bbbk)$ are natural with respect to $\mathbb{Z}^m$. Their naturality is clear since they are built using the natural maps $A_{n+2}(\mathbb{Z}^m,n,\Bbbk)=\mathbb{Z}^m\otimes_{\mathbb{Z}}\Bbbk\simeq H_{n+2}(K(\mathbb{Z}^m,n+2),\Bbbk)$ and cohomology operations which are also natural. Further details are given in sections 5.1 and 5.2.

5. Extensions between classical exponential functors: Explicit computations over fields

In this section, we elaborate on Cartan's computation of the homology of Eilenberg-Mac Lane spaces [Car] to get explicit computations of $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)$ and $\overline{\mathbb{E}}^*(S^*,\Gamma^*;V)$ over a field $\Bbbk$ of positive characteristic (recall from the introduction that the computation is trivial in characteristic zero). We finish the section by comparing our results with previous computations of Chałupnik [C2].

5.1. **Computations over a field of odd characteristic.**

5.1.1. *Cartan's result.* We assume first that $\Bbbk = \mathbb{F}_p$ is a prime field with $p > 2$, and V is a finite dimensional $\Bbbk$-vector space. Then the homology algebras of the $\overline{B}^n(\Gamma^*(V[2]))$ are computed in [Car, Theoreme fondamental, p. 9-03]. For $n = 1$, the homology is:

$$\Gamma^*\left(\bigoplus_{k\ge 0} V_{\phi_p\gamma_p^k\sigma\sigma}[2p^{k+1}+2]\right)\otimes\Lambda^*\left(\bigoplus_{k\ge 0} V_{\sigma\gamma_p^k\sigma\sigma}[2p^k+1]\right)\;,$$

where each $V_{\phi_p\gamma_p^k\sigma\sigma}$ and each $V_{\sigma\gamma_p^k\sigma\sigma}$ is a copy of V (placed in the homological degree indicated in the brackets). For $n = 2$, the homology is given by an analogous formula:

$$\Gamma^*\left(\bigoplus_{k\ge 0} V_{\sigma\sigma\gamma_p^k\sigma\sigma}[2p^k+2]\right)\otimes\Lambda^*\left(\bigoplus_{k,\ell\ge 0} V_{\sigma\gamma_p^k\phi_p\gamma_p^\ell\sigma\sigma}[p^k(2p^{\ell+1}+2)+1]\right)$$
$$\otimes\Gamma^*\left(\bigoplus_{k,\ell\ge 0} V_{\phi_p\gamma_p^k\phi_p\gamma_p^\ell\sigma\sigma}[p^{k+1}(2p^{\ell+1}+2)+2]\right)\;.$$

To be more specific, the isomorphisms are built as follows. The letters σ, ϕ_p and γ_p refer to operations in the homology algebras of iterated bar constructions of a CDGA $\mathbb{F}_p$-algebra A_* (natural in A_*) [Car, Exposes 6 and 7], respectively to the suspension, the transpotence and the r-th divided powers:

$$\sigma : H_k(A_*) \to H_{k+1}(\overline{B}(A_*))\;,$$
$$\phi_p : H_{2k}(A_*) \to H_{p2k+2}(\overline{B}(A_*))\;,$$
$$\gamma_r : H_{2k}(\overline{B}(A_*)) \to H_{r2k}(\overline{B}(A_*))\;.$$

Each copy of V generating the homology is simply obtained by applying a suitable sequence of operations to $V_{\sigma\sigma} = \Gamma^1(V[2]) = H_2(\Gamma^*(V[2]))$ (where $\Gamma^*(V[2])$ as a CDGA algebra with trivial differential). For example, the generator $V_{\sigma\phi_p\gamma_p^3\sigma\sigma}$ is the image of $V_{\sigma\sigma}$ by the sequence of operations:

$$V_{\sigma\sigma}\xrightarrow{\gamma_p^3}H_{2p^3}(\Gamma^*(V[2]))\xrightarrow{\phi_p}H_{2p^4+2}(\overline{B}(\Gamma^*(V[2])))\xrightarrow{\sigma}H_{2p^4+3}(\overline{B}^2(\Gamma^*(V[2]))).$$

In general, the operations to be applied to $V_{\sigma\sigma}$ are the one needed to complete the word '$\sigma\sigma$' in order to obtain the word indexing the copy of V considered, starting from the right to the left.

The isomorphisms are then built from these generating copies of V, together with products and divided powers. Observe that the map $V_{\sigma\sigma}[2]\hookrightarrow\Gamma^*(V[2])$, the products and the cohomology operations are all natural in V, so that Cartan's result is natural in V.

5.1.2. *The strict polynomial structure.* Cartan's result gives the homology of $\overline{B}^n(\Gamma^*(V[2]))$ as a functor in V, but not as a strict polynomial functor in V. One must be careful about this point. Indeed, the forgetful functor

$$\mathcal{U} : \mathcal{P}_\Bbbk \to \mathcal{F}ct(\mathcal{V}_\Bbbk,\mathcal{V}_\Bbbk)$$

is not injective on objects, and two non isomorphic strict polynomial algebras A_* and B_* may have the same underlying algebra $\mathcal{U}A_* = \mathcal{U}B_*$ (see section 2.2). In this section, we determine the homology of $\overline{B}^n(\Gamma^*(V[2]))$ as a strict polynomial algebra. Our method is as follows. We first determine all the graded strict polynomial algebras A_* such that $\mathcal{U}A_*$ equals Cartan's computation. Then we use a strict polynomial degree argument to prove that only one of these A_* can correspond to the strict polynomial structure of the homology of $\overline{B}^n(\Gamma^*(V[2]))$.

Warning 5.1. The behavior of the functor $\mathcal{U}$ is subtle. In our arguments below, we only use the functoriality of $\mathcal{U}$ and the fact that $\mathcal{U}$ reflects isomorphisms, i.e. f is an isomorphism in $\mathcal{P}_\Bbbk$ if and only if $\mathcal{U}f$ is an isomorphism in $\mathcal{F}ct(\mathcal{V}_\Bbbk, \mathcal{V}_\Bbbk)$, cf. appendix 8.

We first need a few results about additive strict polynomial functors. We say that a functor $F \in \mathcal{P}_\Bbbk$ is additive if the underlying functor $\mathcal{U}F$ is additive, that is if

$$\forall V, W \in \mathcal{V}_\Bbbk \quad \forall f, g \in \mathrm{Hom}_\Bbbk(V, W) \quad F(f+g) = F(f) + F(g)$$

(read this as an equality of functions of f, g, not of polynomials).

Lemma 5.2. *Let $\Bbbk$ be a field of positive characteristic p (p even or odd).*

- *(Classification) If F is an additive functor, then F either equals zero or is a finite direct sum of Frobenius twists $I^{(r)}$ (with possibly different $r \geq 0$).*
- *(Retracts) Let F, G be additive functors, and let $f \in \mathrm{Hom}_{\mathcal{P}_\Bbbk}(F, G)$. Then (i) and (ii) are equivalent.*
  - (i) *There exists $V \in \mathcal{V}_\Bbbk$ such that the $\Bbbk$-linear map $f_V : F(V) \to G(V)$ is surjective*
  - (ii) *There exists $\iota \in \mathrm{Hom}_{\mathcal{P}_\Bbbk}(G, F)$ such that $f \circ \iota = \mathrm{Id}_F$.*

Proof. To prove the classification, we can assume that F is homogeneous of degree d. If $d = 0$, then F is constant and additive, hence $F = 0$. So let us assume that $d \geq 1$. There are two cases.

Case 1: d is not a power of p. Then the polynomial $F_{\Bbbk,\Bbbk} \in S^d(\Bbbk^\vee) \otimes \mathrm{End}_\Bbbk(F(\Bbbk))$ is of the form $F_{\Bbbk,\Bbbk}(x) = x^d \otimes v$, with $d \neq p^r$, and additive. Hence it must equal zero. Thus $0 = F_{\Bbbk,\Bbbk}(1)$ is the identity map of $F(\Bbbk)$. This is only possible if $F(\Bbbk) = 0$. By additivity of F, for all $n \geq 1$ we have $F(\Bbbk^n) \simeq F(\Bbbk)^{\oplus n} = 0$. Since $d \geq 1$ we also have $F(0) = 0$. So $F = 0$.

Case 2: $d = p^r$, for $r \geq 0$. Assume that $F \neq 0$. since evaluation on $\Bbbk^n$, (for $n \geq p^r$) yields an equivalence of categories $\mathcal{P}_{p^r,\Bbbk} \simeq S(n, p^r)-\mathrm{mod}$ (cf. appendix 8) it suffices to prove that $F(\Bbbk^n)$ is a direct sum of $(\Bbbk^n)^{(r)}$.

So let $n \geq p^r$ and let $\Bbbk_i$ denote the vector space $\Bbbk$ acted on by the torus $\mathbb{G}_m^{\times n}$ by $(\lambda_1, \dots, \lambda_n) \cdot x = \lambda_i \cdot x$. Then $F(\Bbbk_i)$ is acted on by $\mathbb{G}_m^{\times n}$ by $(\lambda_1, \dots, \lambda_n) \cdot x = \lambda_i^{p^r} \cdot x$. Now by additivity of F, there is a $\mathbb{G}_m^{\times n}$-equivariant isomorphism $\bigoplus_{i \leq n} F(\Bbbk_i) \simeq F(\bigoplus_{i \leq n} \Bbbk_i)$. In particular, all the weights of the $S(n, p^r)$-module $F(\Bbbk^n)$ are of the form $(\mu_1, \dots, \mu_n)$ with all $\mu_i = 0$ but one which equals p^r.

As a consequence let $S_1(\Bbbk^n), \dots, S_N(\Bbbk^n)$ be the composition series of $F(\Bbbk^n)$. Then the $S_i(\Bbbk^n)$ are finite direct sums of simples with highest weight $(p^r, 0, \dots, 0)$, that is of $(\Bbbk^n)^{(r)}$.

Now we know (see e.g. [FS]) that $\mathrm{Ext}^1((\Bbbk^n)^{(r)}, (\Bbbk^n)^{(r)}) = 0$. Thus there cannot be nontrivial extensions between finite direct sums of $(\Bbbk^n)^{(r)}$. This implies that the compositions series of $F(\Bbbk^n)$ has length $N = 1$. That is, $F(\Bbbk^n)$ is a finite direct sum of copies of $(\Bbbk^n)^{(r)}$.

Finally, let us prove the characterization of retracts. We can assume that F, G are homogeneous of degree p^r, $r \geq 1$. The identity map is a basis of $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)})$, so tensor products yield isomorphisms for $k, \ell \geq 1$:

$$\mathrm{Hom}_{\Bbbk}(\Bbbk^k, \Bbbk^\ell) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I^{(r)} \otimes \Bbbk^k, I^{(r)} \otimes \Bbbk^\ell), \quad f \mapsto \mathrm{Id} \otimes f,$$

and the result follows. □

Let us now explain the link between additive (strict polynomial) functors and (strict polynomial) exponential functors. If A_* is a multigraded strict polynomial augmented algebra, we denote by $Q(A_*)$ the indecomposables of A_*, that is $Q(A_*)$ is the cokernel of the multiplication $A'_* \otimes A'_* \to A'_*$, where A'_* is the augmentation ideal of A_*. Then $Q(A_*)$ is a multigraded strict polynomial functor and $\mathcal{U}Q(A_*) = Q(\mathcal{U}A_*)$. Similarly the primitives $P(C_*)$ of a multigraded strict polynomial coaugmented coalgebra C_* form a multigraded strict polynomial subfunctor of C_* and $\mathcal{U}P(C_*) = P(\mathcal{U}C_*)$. With these definitions we have the following result.

Lemma 5.3. *Let $\Bbbk$ be a field, and let E_* be a (multi)graded strict polynomial exponential functor. The strict polynomial functors $P(E_*)$ and $Q(E_*)$ are additive. Moreover, if there exists $V \in \mathcal{V}_{\Bbbk}$ such that the composite $P(E_*)(V) \hookrightarrow E_*(V) \twoheadrightarrow Q(E_*)(V)$ is surjective, then $Q(E_*)$ is a direct summand in E_*.*

Proof. Let E'_* be the augmentation ideal of E_*. Since E_* is exponential, we have $E'_*(V \oplus W)$ is isomorphic to $E'_*(V) \otimes \Bbbk \oplus \Bbbk \otimes E'_*(W) \oplus E'_*(V) \otimes E'_*(W)$. Moreover, the multiplication $E'_*(V \oplus W)^{\otimes 2} \to E'_*(V \oplus W)$ identifies through this decomposition with the direct sum of three maps (which are induced by multiplications):

(1) $(E'_*(V) \otimes \Bbbk)^{\otimes 2} \to E'_*(V) \otimes \Bbbk$,

(2) $(\Bbbk \otimes E'_*(W))^{\otimes 2} \to \Bbbk \otimes E'_*(W)$,

(3) $(E'_*(V) \otimes \Bbbk) \otimes (\Bbbk \otimes E'_*(W)) \oplus \begin{array}{c}\text{other summands}\\ \text{of } E'_*(V \oplus W)^{\otimes 2}\end{array} \to E'_*(V) \otimes E'_*(W)$.

The first two maps have respective cokernels $Q(E_*)(V)$ and $Q(E_*)(W)$ and the last one is surjective. This shows that $Q(E_*)$ is additive. The proof that the primitives are additive is similar.

Finally, if the map $P(E_*)(V) \to Q(E_*)(V)$ is surjective for some $V \in \mathcal{V}_{\Bbbk}$, then by lemma 5.2, it admits a section ι. So the composite $Q(E^*) \xrightarrow{\iota} P(E_*) \hookrightarrow E_*$ is a section of $E_* \twoheadrightarrow Q(E^*)$. □

We are now ready to determine all the candidates for a strict polynomial version of Cartan's computation

Lemma 5.4. *Let $\Bbbk$ be a field of positive characteristic p (p even or odd), and for all $d \geq 0$, let $F_d \in \mathcal{F}ct(\mathcal{V}_{\Bbbk}, \mathcal{V}_{\Bbbk})$ be a finite direct sum of n_d copies of the identity functor. The graded strict polynomial algebras A_* such that $\mathcal{U}A_* = \Gamma^*(F_{\mathrm{even}}) \otimes \Lambda^*(F_{\mathrm{odd}})$ are of the form:*

$$A_* = \Gamma^*(G_{\mathrm{even}}) \otimes \Lambda^*(G_{\mathrm{odd}}) ,$$

where each G_d is a direct sum of n_d Frobenius twists $I^{(r)}$ (with possibly different $r \geq 0$).

Proof. **Step 1: Duality.** Let E_* be a graded exponential (non strict polynomial) functor. Then finding the graded strict polynomial algebras A_* such that $\mathcal{U}A_* = E_*$ is equivalent to finding the graded strict polynomial algebras $A_*^\sharp$ such that $\mathcal{U}A_*^\sharp = E_*^\sharp$.

Indeed, if $\mathcal{U}A_* = E_*$, then for all $V, W \in \mathcal{V}_\Bbbk$ the composite

$$A_*(V) \otimes A_*(W) \to A_*(V \oplus W)^{\otimes 2} \to A_*(V \oplus W)$$

is an isomorphism (indeed, this is true for $\mathcal{U}A_*$). Thus A_* is an exponential functor, and $\mathcal{U}A_*$ coincides with E_* as an exponential functor. Equivalently, $\mathcal{U}A_*^\sharp$ coincides with $E_*^\sharp$ as an exponential functor. This is in turn equivalent to the fact that $A_*^\sharp$ is a graded strict polynomial algebra such that $\mathcal{U}A_*^\sharp$ coincides with $E_*^\sharp$ as an algebra.

So, to prove lemma 5.4, it suffices to prove that the graded strict polynomial algebras B_* such that $\mathcal{U}B_* = S^*(F_{\text{even}}) \otimes \Lambda^*(F_{\text{odd}})$ are of the form $S^*(G_{\text{even}}) \otimes \Lambda^*(G_{\text{odd}})$ with G_* as indicated.

Step 2: Indecomposables. If B_* is as indicated in step 1, then the indecomposables of B_* are a direct summand in B_*. Indeed, since $\mathcal{U}B_* = S^*(F_{\text{even}}) \otimes \Lambda^*(F_{\text{odd}})$, there exists $V \in \mathcal{V}_\Bbbk$, e.g. $V = \Bbbk$, such that the composite $P(B_*)(V) \to B_*(V) \to Q(B_*)(V)$ is surjective. Then one applies lemma 5.3.

Now, the indecomposables of B_* are an additive strict polynomial functor with $\mathcal{U}Q(B_*) = Q(\mathcal{U}B_*) = F_*$. So by lemma 5.2, $Q(B_*)_d$ is a finite direct sum of n_d Frobenius twists for all $d \geq 0$.

Step 3: Universal property. The morphism of graded strict polynomial functors induces $Q(B_*) \hookrightarrow B_*$ induces a morphism of strict polynomial algebras $S^*(Q(B_*)_{\text{even}}) \otimes \Lambda^*(Q(B_*)_{\text{odd}}) \to B_*$. For all $V \in \mathcal{V}_\Bbbk$, this morphism is an isomorphism after evaluation on V. Hence, it is an isomorphism. Thus B_* is of the form $S^*(G_{\text{even}}) \otimes \Lambda^*(G_{\text{odd}})$ with G_* as indicated in lemma 5.4, which concludes the proof. □

Now we determine which strict polynomial structure corresponds to the homology of $\overline{B}^n(\Gamma^*(V[2]))$. To do this this, we examine further properties of the operations σ, ϕ_p and γ_r for strict polynomial CDGA algebras.

Lemma 5.5. *Let $\Bbbk = \mathbb{F}_p$ with p odd, let $A_*(V)$ be a strict polynomial CGDA $\mathbb{F}_p$-algebra and let k be a positive integer.*

1. *The suspension $\sigma : H_k(A_*(V)) \to H_{k+1}(\overline{B}(A_*(V)))$ is a morphism of strict polynomial functors. In particular, it preserves the strict polynomial degree.*
2. *If $\alpha \in H_{2k}(\overline{B}(A_*(V)))$ has strict polynomial degree d, then for all $r \geq 1$, its r-th divided power $\gamma_r(\alpha) \in H_{2kr}(\overline{B}(A_*(V)))$ has strict polynomial degree rd.*
3. *Assume that $a^p = 0$ in the graded algebra $A_*(V)$. If $\alpha \in H_{2k}(A_*(V))$ has strict polynomial degree d, the transpotence ϕ_p sends it to an element of strict polynomial degree pd of $H_{2kp+2}(\overline{B}(A_*(V)))$.*

Proof. Recall from [Car, Exposes 3 et 4] the properties which characterize the 'construction' $(A_*, \overline{B}(A_*), B(A_*))$.

(i) $B(A_*) := A_* \otimes \overline{B}(A_*)$ as a graded strict polynomial algebra. Thus, both A_* and $\overline{B}(A_*)$ can be viewed as graded subalgebras of $B(A_*)$.

(ii) $B(A_*)$ is equipped with a differential ∂, such that following maps are morphisms of strict polynomial CDGA $\mathbb{F}_p$-algebras:

$$A_* \xrightarrow{\mathrm{Id}\otimes 1} B(A_*) \xrightarrow{\epsilon\otimes\mathrm{Id}} \overline{B}(A_*)\ .$$

(B) For all $k \geq 1$, the composite $\overline{B}(A_*)_{k+1} \hookrightarrow B(A_*)_{k+1} \xrightarrow{\partial} B(A_*)_k$ is injective and induces an isomorphism $\overline{B}(A_*)_{k+1} \simeq ZB(A_*)_k$ onto the cycles of degree k of $B(A_*)$.

Let us prove (1). Let ZA_k denote the degree k cycles of A_*, $k \geq 1$. Then σ is defined on ZA_k as the composite [Car, Expose 6, p. 6-01]:

$$ZA_k \xrightarrow{\mathrm{Id}\otimes 1} ZB(A_*)_k \simeq \overline{B}(A_*)_{k+1}\ .$$

This is a map of strict polynomial functors. By (ii), it has values in the cycles of the CDGA $\overline{B}(A_*)$, and it takes boundaries to boundaries. So, it induces a morphism of strict polynomial functors $\sigma : H_k(A_*) \to H_{k+1}(\overline{B}(A_*))$.

To prove (2), recall from [Car, Theoreme 1, p. 7-02] that the divided powers γ_r, $r \geq 1$ are actually defined in $\overline{B}(A_*)$ and satisfy $\gamma_1(a) = a$ and $\partial\gamma_r(a) = \partial a * \gamma_{r-1}(a)$ for $r \geq 1$. Since ∂ preserves the strict polynomial degree we have $\deg^{\mathrm{sp}}(\gamma_r(a)) = \deg^{\mathrm{sp}}(a) + \deg^{\mathrm{sp}}(\gamma_{r-1}(a))$ for $r \geq 1$ and the result follows by induction on r.

Finally, ϕ_p is defined in the following way [Car, p. 6-05]. Let $a \in ZA_{2k} \subset ZB(A_*)_{2k}$ representing α. There is an x such that $\partial x = a$. Now the element $y \in Z\overline{B}(A_*)_{r2k+2}$ representing $\phi_p(\alpha)$ satisfies $\partial y = a^{p-1} * \partial x$. So we have $\deg^{\mathrm{sp}}(\phi_p(\alpha)) = \deg^{\mathrm{sp}}(a^{p-1}) + \deg^{\mathrm{sp}}(x) = p\deg^{\mathrm{sp}}(\alpha)$. □

Remark 5.6. Lemmas 5.5(1) and (2) remain valid over an arbitrary commutative ring $\Bbbk$, with the additional hypothesis that $A_*(V)$ is strictly graded commutative (i.e. $a \cdot a = 0$ for all $a \in A_{\mathrm{odd}}(V)$). This hypothesis ensures that $H_*(\overline{B}(A_*))$ has divided powers [Car, Theoreme 1 p. 7-01], and the proofs carry without change. The transpotence ϕ_2 also exists over a field of characteristic 2 [Car, p. 6-05] , and lemma 5.5(3) is valid without change in this setting.

By lemma 5.5, each V_w appearing in Cartan's computation of the homology of $\overline{B}^n(\Gamma^*(V[2]))$ is formed by elements of strict polynomial degree p^k, where k is the number of occurrences of γ_p and ϕ_p in the word w. This indicates which strict polynomial algebra from lemma 5.4 is the right one. Thus we obtain the following result.

Theorem 5.7. *Let $\Bbbk = \mathbb{F}_p$ be a prime field of odd characteristic, and let V be a finite dimensional $\Bbbk$-vector space. Then the homology of $\overline{B}(\Gamma^*(V[2]))$ equals the graded strict polynomial algebra:*

$$\Gamma^*\left(\bigoplus\nolimits_{k\geq 0} V^{(k+1)}[2p^{k+1}+2]\right) \otimes \Lambda^*\left(\bigoplus\nolimits_{k\geq 0} V^{(k)}[2p^k+1]\right)\ ,$$

and the homology of $\overline{B}^2(\Gamma^*(V[2]))$ equals the graded strict polynomial algebra:

$$\Gamma^*\left(\bigoplus\nolimits_{k\geq 0} V^{(k)}[2p^k+2]\right) \otimes \Lambda^*\left(\bigoplus\nolimits_{k,\ell\geq 0} V^{(k+\ell+1)}[p^k(2p^{\ell+1}+2)+1]\right)$$
$$\otimes\Gamma^*\left(\bigoplus\nolimits_{k,\ell\geq 0} V^{(k+\ell+2)}[p^{k+1}(2p^{\ell+1}+2)+2]\right)\ .$$

Corollary 5.8. *The computations of theorem 5.7 hold over any field* $\mathbb{k}$ *of odd characteristic.*

Proof. Contrarily to ordinary functors, the strict polynomial functors have an exact base change. That is, if $\mathbb{k}$ is a field of characteristic p, there is an exact functor [SFB, Prop 2.6]: $\otimes_{\mathbb{F}_p}\mathbb{k} : \mathcal{P}_{\mathbb{F}_p} \to \mathcal{P}_{\mathbb{k}}$. This base change functor commutes with tensor products and sends divided powers to divided powers and Frobenius twists to Frobenius twists. Whence the result. □

Remark 5.9. The method developed in this section allows more generally to obtain the homology of the iterated bar constructions $\overline{B}^n(\Gamma^*(V[2]))$ as strict polynomial algebras, for any n. The recipe is simple. In Cartan's theorem [Car, Theoreme fondamental, p. 9-03], replace each copy of V attached to a word w (to be more specific, w is a 'mot admissible de hauteur $n+2$ et de première espèce'), by $V^{(k)}$, k being the number of letters of w equal to γ_p or ϕ_p. Then what you have written is the description of the homology of $\overline{B}^n(\Gamma^*(V[2]))$, as a strict polynomial algebra, which is valid for all base field $\mathbb{k}$ of characteristic p. This recipe is also true in characteristic $p = 2$, as we will see in section 5.2.

5.1.3. *Computation of extension groups.* We can now compute the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ and $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ (with bigradings and products specified in example 3.7). The comparison with [C2, Prop 3.1, Thm 3.2, Cor 4.5] is made in section 5.3. Before stating the result, let us introduce one more notation.

Notation 5.10. Let A and B be multigraded $\mathbb{k}$-algebras. We denote by $A \overset{\circ}{\otimes} B$ their naive product, that is $A \overset{\circ}{\otimes} B = A \otimes B$ as multigraded $\mathbb{k}$-modules, equipped with the multiplication $(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb')$, without sign.

If A and B are graded $\mathbb{k}$-algebras, with either A or B concentrated in even degrees, then the usual tensor product (i.e. with a Koszul sign) and the naive tensor product coincide. For example, the tensor products in theorem 5.7 could be interpreted as usual ones, or as naive ones. In general, the two tensor products are different.

Theorem 5.11 (Main computation I)**.** *Let* $\mathbb{k}$ *be a field of odd characteristic* p*, and let* V *be a finite dimensional* $\mathbb{k}$*-vector space. For all* $k \geq 0$ *and* $i \geq 0$ *we consider* $V^{(k)}[i]$ *with bigrading* (i, p^k) *(i.e. 'i' is the explicit homological grading and '*p^k*' is the implicit strict polynomial grading of* $V^{(k)}[i]$*). We have isomorphisms of bigraded strict polynomial algebras:*

$$\bigoplus_{h,d\geq 0} \overline{\mathbb{E}}^h(S^d, \Lambda^d; V) \simeq \Lambda^*\left(\bigoplus_{k\geq 0} V^{(k)}[p^k - 1]\right) \overset{\circ}{\otimes} \Gamma^*\left(\bigoplus_{k\geq 0} V^{(k+1)}[p^{k+1} - 2]\right),$$

$$\bigoplus_{h,d\ge 0} \overline{\mathbb{E}}^h(S^d,\Gamma^d;V) \simeq$$

$$\Gamma^*\left(\bigoplus_{k\ge 0} V^{(k)}[2p^k-2]\right) \overset{\circ}{\otimes} \Lambda^*\left(\bigoplus_{k\ge 0,\ell\ge 0} V^{(k+\ell+1)}[2p^{k+\ell+1}-2p^k-1]\right)$$

$$\overset{\circ}{\otimes} \Gamma^*\left(\bigoplus_{k\ge 0,\ell\ge 0} V^{(k+\ell+2)}[2p^{k+\ell+2}-2p^{k+1}-2]\right).$$

Proof. According to theorem 4.5, there is an isomorphism of strict polynomial algebras $\bigoplus_{h,d\ge 0}\overline{\mathbb{E}}^h(S^d,\Lambda^d;\Bbbk^m) \simeq H_*(\overline{B}(\Gamma^*(V[2]))$. The bigrading (h,d) can be read on the right handside as follows: an element of $H_i(\overline{B}(\Gamma^*(V[2]))$ with strict polynomial degree d is placed in bidegree $(h,d) := (3d-i,d)$. Now, $H_*(\overline{B}(\Gamma^*(V[2]))$ is computed in theorem 5.7. The generators $V^{(k)}[i]$ are in $H_i(\overline{B}(\Gamma^*(V[2]))$, and have strict polynomial degree p^r. Hence they have bigrading $(3p^r-i,p^r)$ in the bigraded algebra $\bigoplus_{h,d\ge 0}\overline{\mathbb{E}}^h(S^d,\Lambda^d;V)$. Whence the result. The computation of $\bigoplus_{h,d\ge 0}\overline{\mathbb{E}}^h(S^d,\Gamma^d;V)$ is similar. □

5.2. **Computations over a field of characteristic 2.** Assume first that $\Bbbk=\mathbb{F}_2$ and let V be a finite dimensional vector space. The homology of the iterated bar constructions $\overline{B}^n(\Gamma^*(V[2]))$ are computed in [Car, Theoreme fondamental, p. 10-02]. In particular, the homology of $\overline{B}(\Gamma^*(V[2]))$ and $\overline{B}^2(\Gamma^*(V[2]))$ equal respectively

$$\Gamma^*\left(\bigoplus\nolimits_{k\ge 0} V_{\sigma\gamma_2^k\sigma\sigma}[2^{k+1}+1]\right) \text{ and } \Gamma^*\left(\bigoplus\nolimits_{k,\ell\ge 0} V_{\sigma\gamma_2^k\sigma\gamma_2^\ell\sigma\sigma}[2^k(2^{\ell+1}+1)+1]\right),$$

where the $V_{\sigma\gamma_2^k\sigma\sigma}$ and the $V_{\sigma\gamma_2^k\sigma\gamma_2^\ell\sigma\sigma}$ are copies of V. To be more specific, each copy is obtained from $V_{\sigma\sigma}=\Gamma^1(V[2])=H_2(\Gamma^*(V[2]))$ by applying a suitable sequence of supensions σ and divided power operations γ_2. Similarly to the case $\Bbbk=\mathbb{F}_p$ with p odd, the sequence of σ and γ_2 to be applied is the one which enables to complete the word '$\sigma\sigma$' into the word indexing the copy of V considered, going from the right to the left. For example, $V_{\sigma\gamma_2\sigma\sigma}=\sigma(\gamma_2(V_{\sigma\sigma}))$.

Since the map $V_{\sigma\sigma}\to H_2(\Gamma^*(V[2]))$ as well as the suspension and the divided powers are natural with respect to V, these results yield the computation of the $\overline{B}^n(\Gamma^*(V[2]))$, $n=1,2$ as algebras natural in V. But not as strict polynomial algebras. However, one can retrieve the strict polynomial structure exactly as in the case $\Bbbk=\mathbb{F}_p$ with p odd. We first classify all possible strict polynomial structures.

Lemma 5.12. *Let $\Bbbk$ be a field of characteristic 2, and for all $d\ge 0$, let $F_d\in\mathcal{F}ct(\mathcal{V}_\Bbbk,\mathcal{V}_\Bbbk)$ be a finite direct sum of n_d copies of the identity functor. The graded strict polynomial algebras A_* such that $\mathcal{U}A_*=\Gamma^*(F_*)$ are of the form: $A_*=\Gamma^*(G_*)$, where each G_d is a direct sum of n_d Frobenius twists $I^{(r)}$ (with possibly different $r\ge 0$).*

Proof. Adapt the proof of lemma 5.4. □

Then we pick the right strict polynomial structure with the help of remark 5.6 and we extend the result to arbitrary fields of characteristic 2 by base change. We finally obtain the following computation.

Theorem 5.13. *Let $\Bbbk$ be a prime field of characteristic $p = 2$, and let V be a finite dimensional $\Bbbk$-vector space. Then the homology of $\overline{B}(\Gamma^*(V[2]))$ and $\overline{B}^2(\Gamma^*(V[2]))$ are respectively equal to the strict polynomial algebras:*

$$\Gamma^*\left(\bigoplus_{k\geq 0} V^{(k)}[2p^k+1]\right) \text{ and } \Gamma^*\left(\bigoplus_{k,\ell\geq 0} V^{(k+\ell)}[p^k(2p^\ell+1)+1]\right).$$

One can now use theorem 5.13 to compute the strict polynomial algebras $\bigoplus_{h,d\geq 0}\mathbb{E}^h(S^d,\Lambda^d;V)$ and $\bigoplus_{h,d\geq 0}\mathbb{E}^h(S^d,\Gamma^d;V)$ over a field $\Bbbk$ of characteristic $p=2$ exactly as in the case $p>2$. The comparison with [C2, Prop3.1, Thm 3.2, Cor 4.5] is made in section 5.3.

Theorem 5.14 (Main computation II)**.** *Let $\Bbbk$ be a field of characteristic $p = 2$, and let V be a finite dimensional $\Bbbk$-vector space. For all $k\geq 0$ and $i\geq 0$ we consider $V^{(k)}[i]$ with bigrading (i,p^k) (i.e. 'i' is the explicit homological grading and 'p^k' is the implicit strict polynomial grading of $V^{(k)}[i]$). We have isomorphisms of bigraded strict polynomial algebras:*

$$\bigoplus_{h,d\geq 0}\overline{\mathbb{E}}^h(S^d,\Lambda^d;V)\simeq\Gamma^*\left(\bigoplus_{k\geq 0}V^{(k)}[p^k-1]\right),$$

$$\bigoplus_{h,d\geq 0}\overline{\mathbb{E}}^h(S^d,\Gamma^d;V)\simeq\Gamma^*\left(\bigoplus_{k\geq 0,\ell\geq 0}V^{(k+\ell)}[2p^{k+\ell}-p^k-1]\right).$$

5.3. **Comments on the results.** We now examine the reliability of our results. We compare them with some elementary computations and with the previous computations from [C2].

5.3.1. *Even and odd characteristic.* As stated in theorems 5.11 and 5.14, the extension groups $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)$ and $\overline{\mathbb{E}}^*(S^*,\Gamma^*;V)$ have a different behavior in characteristic $p>2$ and in characteristic 2. This phenomenon is already visible on the Hom-level. Indeed, the algebra $\mathbb{H}(S^*,\Lambda^*;V)$ is the subalgebra of $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)$ generated by elements of bidegree $(0,d)$, $d\geq 0$, hence by theorems 5.11 and 5.14:

$$\mathbb{H}(S^d,\Lambda^d;V)\simeq\Lambda^d(V) \text{ if } p>2, \text{ and } \mathbb{H}(S^d,\Lambda^d;V)\simeq\Gamma^d(V) \text{ if } p=2\;.$$

The difference between even and odd characteristic is confirmed by the following elementary computation. Using the exact sequence $\Lambda^2\hookrightarrow\otimes^2\twoheadrightarrow S^2$, one sees that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(S^2,\Lambda^2)=\mathbb{H}(S^2,\Lambda^2;\Bbbk)$ is a trivial $\Bbbk$-vector space if $\Bbbk$ has characteristic $p>2$, and is one dimensional in characteristic 2, generated by the map $S^2(V)\to\Lambda^2(V)$, $vw\mapsto v\wedge w$ (which is well defined in characteristic 2 only).

5.3.2. *Comparison with* [C2] *for $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$ in odd characteristic.* We observe that our computation of $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$ in odd characteristic coincides with [C2, Prop 3.1 and Thm 3.2] only up to a sign. Indeed, [C2, Thm 3.2] asserts that $\mathbb{E}^*(S^*,\Lambda^*;V)$ is isomorphic to

$$\Lambda^*\left(\bigoplus_{k\geq 0} V^{(k)}[p^k-1]\right)\overset{\circ}{\otimes}\Gamma^*\left(\bigoplus_{k\geq 0} V^{(k+1)}[p^{k+1}-2]\right)\ ,$$

where the $V^{(k)}[p^k-1]$ are subfunctors of $\mathbb{E}^{p^k-1}(S^{p^k},\Lambda^{p^k};V)$ and the $V^{(k+1)}[p^{k+1}-2]$ are subfunctors of $\mathbb{E}^{p^{k+1}-2}(S^{p^{k+1}},\Lambda^{p^{k+1}};V)$, and the tensor product is to the naive tensor product from notation 5.10. We assert that the later algebra is isomorphic to $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)$.

We think that the signs in [C2] are wrong. Let us show on an example where the problem lies. Choose $u_1\neq u_2$ in $V^{(1)}[p-1]$ and $v\in V^{(1)}[p-2]$, let $\gamma_2(v)=v^{\otimes 2}\in\Gamma^2(V^{(1)}[p-2])$ and define

$$a_1=u_1\otimes v\in\mathbb{E}^{2p-3}(S^{2p},\Lambda^{2p};V)\ ,\ \text{and}\ \ a_2=u_2\otimes\gamma_2(v)\in\mathbb{E}^{3p-5}(S^{3p},\Lambda^{3p};V)\ .$$

Then, according to [C2, Thm 3.2], their product $a_1\cdot a_2$ in $\mathbb{E}^*(S^*,\Lambda^*;V)$ is non zero and moreover:

$$a_1\cdot a_2=(u_1\wedge u_2)\otimes v\gamma_2(v)=-(u_2\wedge u_1)\otimes\gamma_2(v)v=-a_2\cdot a_1\ .$$

The latter computation contradicts the fact that $\mathbb{E}^*(S^*,\Lambda^*;V)$ is $(1,1)$-commutative. This phenomenon does not arise with our signs.

5.3.3. *Comparison with* [C2] *for $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$ in characteristic $p=2$.* The reader can observe that our computation of $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$ does not agrees with [C2, Thm 3.2] in characteristic 2. For example, [C2, Thm 3.2] asserts that $\mathbb{H}(S^2,\Lambda^2;V)=\Lambda^2(V)\oplus V^{(1)}$ in characteristic 2, while we assert that it equals $\Gamma^2(V)$.

The following elementary computation argues in favor of our result. First, we have $\mathbb{E}^*(I^{(1)},\Gamma^2;V)=0$ for $*=0,1$ (this follows from [FS, Prop 4.4] and the long $\mathbb{E}^*(I^{(1)},-;V)$-exact sequence induced by $\Lambda^2\hookrightarrow\Gamma^2\twoheadrightarrow I^{(1)}$, or alternatively from proposition 7.1). So, the short exact sequence $I^{(1)}\hookrightarrow S^2\twoheadrightarrow\Lambda^2$ induces an isomorphism $\mathbb{H}(S^2,\Lambda^2;V)\simeq\mathbb{H}(S^2,S^2;V)$. The latter equals $\Gamma^2(V)$ by lemma 2.5(1-2).

5.3.4. *An alternative method to compute $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$ over a field.* Let $\Bbbk$ be a field of characteristic $p>0$. To compute $\overline{\mathbb{E}}(S^*,\Lambda^*,V)$, we have proved that these extension groups equal, up to a regarding, the homology of the (reduced, normalized) bar construction $\overline{B}(\Gamma^*(V[2]))$. Then we have used Cartan's computations to obtain the homology of the bar construction. We now sketch an alternative to obtain the homology of $\overline{B}(\Gamma^*(V[2]))$.

The homology of the (reduced, unnormalized) cobar construction of the algebra of polynomials $\Bbbk[V]=S^*(V^\vee[0])$ computes the rational homology $H^*(V_a,\Bbbk)$ of the additive group V_a with trivial coefficients (the symbol '$\vee$' denotes $\Bbbk$-linear duality). These homology groups were originally computed for $V=\Bbbk$ in [CPSVdK], and the result for all V is given in in [J, I Chap. 4]. They are equal, as a strict polynomial $\Bbbk$-algebra, to

$$\Lambda^*\left(\bigoplus_{r\geq 0} V^{\vee\,(r)}[1]\right)\otimes S^*\left(\bigoplus_{r\geq 1} V^{\vee\,(r)}[2]\right)\quad\text{if } p>2,$$
$$S^*\left(\bigoplus_{r\geq 0} V^{\vee\,(r)}[1]\right)\quad\text{if } p=2.$$

Let C be a cocommutative differential graded coalgebra over $\Bbbk$ with unit $\eta : \Bbbk \to C$ (e.g. $C = \Bbbk[V]$). The reduced normalized cobar construction (based on $\overline{C} = C/\mathrm{Im}\eta$) and the reduced unnormalized cobar construction (based on C) are quasi isomorphic. So we can assume that we work with the reduced normalized cobar construction, which we denote by ΩC. Moreover, the shuffle coproduct makes the cobar construction into a Hopf algebra. In fact, the computation of $H^*(\Omega\Bbbk[V])$ made in [J, I Chap. 4] is valid as a Hopf algebra (indeed, it suffices to check on the generators of the algebra that the coalgebra structures coincide, and this check is straightforward from the explicit expressions of the cycles given in [J, I 4.21 and 4.22]). Now the restricted dual of $\Omega\Bbbk[V]$ equals $\overline{B}(\Gamma^*(V[0]))$ as a Hopf algebra. Since $\Bbbk$ is a field, the homology of $\overline{B}(\Gamma^*(V[0]))$ is nothing but the dual of $H^*(\Omega\Bbbk[V])$ as a Hopf algebra. Thus we retrieve that $H_*(\overline{B}(\Gamma^*(V[0])))$ equals:

$$\Lambda^*\left(\textstyle\bigoplus_{r\geq 0} V^{\vee\,(r)}[1]\right)\otimes\Gamma^*\left(\textstyle\bigoplus_{r\geq 1} V^{\vee\,(r)}[2]\right)\quad \text{if } p>2, \tag{4}$$

$$\Gamma^*\left(\textstyle\bigoplus_{r\geq 0} V^{\vee\,(r)}[1]\right)\quad \text{if } p=2. \tag{5}$$

Now, using lemma 4.7, one easily deduces the homology of the bar construction of the divided power algebra $\Gamma^*(V[d])$ over a vector space V placed in homological degree d. One obtains the same result, but each generator space $V^{(r)}[1]$, resp. $V^{(r)}[2]$ has to be replaced by $V^{(r)}[dp^r+1]$, resp. $V^{(r)}[dp^r+2]$.

5.3.5. *Comparison with* [C2] *for* $\overline{\mathbb{E}}(S^*,\Gamma^*,V)$. The reader may observe that our computation of $\overline{\mathbb{E}}^*(S^*,\Gamma^*;V)$ does not agree with [C2, Cor 4.5] (take $i=j=0$ in Chałupnik's result), even as graded vector spaces. The following elementary computation argues in favor of our result.

Proposition 5.15. *Let $\Bbbk$ be a field of characteristic $p>0$. The extension groups $\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(S^p,\Gamma^p)$ equals $\Bbbk$ if $i=0, 2p-2$ or $2p-3$, and are trivial in the other degrees.*

Our theorems 5.11 and 5.14 agree with proposition 5.15. Indeed, $\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(S^p,\Gamma^p)$ is the component of bidegree (i,p) of $\overline{\mathbb{E}}^*(S^*,\Gamma^*,\Bbbk)$, so according to theorems 5.11 and 5.14, $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^p,\Gamma^p)$ equals:

$$\left(\Gamma^p(\Bbbk^{(0)}[0])\otimes\Bbbk\otimes\Bbbk\right)\oplus\left(\Gamma^1(\Bbbk^{(1)}[2p-2])\otimes\Bbbk\otimes\Bbbk\right)\oplus\left(\Bbbk\otimes\Lambda^1(\Bbbk^{(1+0)}[2p-3])\otimes\Bbbk\right)$$

if $p>2$, and

$$\Gamma^p(\Bbbk^{(0+0)}[0])\oplus\Gamma^1(\Bbbk^{(1+0)}[2p-2])\oplus\Gamma^1(\Bbbk^{(0+1)}[2p-3])$$

if $p=2$. In both cases, the result agrees with proposition 5.15.

On the contrary, [C2, Cor 4.5] predicts that these extension groups have dimension 2, which is not possible by proposition 5.15 (the mistake seems to come from a misprint in the computation of '$H\Theta^2(D^{*(i)})$' [C2, p. 980]).

Proof of proposition 5.15. The (elementary) proof relies on the analysis of long exact sequences. We first recall a few facts. For $0\leq k<p$, the functor $\Lambda^k\otimes S^{p-k}$ is a direct summand of the injective functor $\otimes^k\otimes S^{p-k}$, hence is injective. Moreover, one computes that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(S^p,S^p)$ is one dimensional, generated by the identity map, $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(S^p,S^1\otimes S^{p-1})$ is one dimensional,

generated by the comultiplication $S^p \to S^1 \otimes S^{p-1}$, and $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(S^p, \Lambda^k \otimes S^{p-k}) = 0$ for $k \geq 2$. Finally, the functor Γ^k is isomorphic to S^k for $k < p$.

We cut the (exact) Koszul complex:

$$\Lambda^p \hookrightarrow \Lambda^{p-1} \otimes S^1 \to \cdots \to \Lambda^1 \otimes S^{p-1} \twoheadrightarrow S^p$$

into short exact sequences and analyze the long $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^p, -)$-exact sequence associated to them. We begin on the right. Let K be the kernel of the multiplication $S^1 \otimes S^{p-1} \twoheadrightarrow S^p$. Since the composite of the comultiplication $S^p \to S^1 \otimes S^{p-1}$ and the multiplication $S^1 \otimes S^{p-1} \twoheadrightarrow S^p$ equals p times the identity map (hence zero), we obtain that $\mathrm{Ext}^i(S^p, K) \simeq \Bbbk$ if $i = 0, 1$ and 0 otherwise. For the other short exact sequences, the term $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^p, \Lambda^k \otimes S^{p-k})$ is trivial so the long exact sequence induces a shifting. We finally obtain that $\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(S^p, \Lambda^p) \simeq \Bbbk$ if $i = p-1, p-2$ and zero otherwise.

To get the computation of $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^p, \Gamma^p)$ from $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(S^p, \Lambda^p)$, we cut the (exact) Koszul complex:

$$\Gamma^p \hookrightarrow \Gamma^{p-1} \otimes \Lambda^1 \to \cdots \to \Gamma^1 \otimes \Lambda^{p-1} \twoheadrightarrow \Lambda^p$$

into short exact sequences and proceed similarly. □

6. Extensions between classical exponential functors: Explicit computations over $\mathbb{Z}$

In this section, we elaborate on [Car] to get explicit computations of $\overline{\mathbb{E}}^*(S^*, \Lambda^*; V)$ and $\overline{\mathbb{E}}^*(S^*, \Gamma^*; V)$ over the ground ring $\Bbbk = \mathbb{Z}$. Actually, we do not compute them as bigraded strict polynomial algebras, but only as graded $\mathbb{G}_m$-algebras (i.e. we do not get the naturality in V, but see conjecture 6.7).

6.1. **Graded $\mathbb{G}_m$-algebras.** We denote by $\mathbb{G}_m$ the multiplicative group scheme (i.e. $\mathbb{G}_m = GL_{1,\mathbb{Z}}$). The representation theory of $\mathbb{G}_m$ is very easy [J, I 2.11]. Indeed, if M is a $\mathbb{G}_m$-module, there is a splitting $M = \bigoplus_{d\in\mathbb{Z}} M_d$ where M_d denotes the subspace of weight d. This defines an equivalence of categories between the category of $\mathbb{G}_m$-modules and $\mathbb{G}_m$-equivariant maps and the category of $\mathbb{Z}$-graded abelian groups and graded group morphisms.

The link between $\mathbb{G}_m$ modules and strict polynomial functors is as follows. Let V be a $\mathbb{G}_m$-module, which is free and finitely generated as a $\mathbb{Z}$-module. Then evaluation on V yields a functor:

$$\mathrm{ev}_V : \begin{array}{ccc} \widetilde{\mathcal{P}_\mathbb{Z}} & \to & \mathbb{G}_m\text{-Mod} \\ F & \mapsto & F(V) \end{array} .$$

Assume that V has weight 1, that is $\mathbb{G}_m$ acts on V as the subgroup of homotheties of $GL(V)$. Then for all $F \in \widetilde{\mathcal{P}_\Bbbk}$, the decomposition of $F(V)$ induced by the weights of the action of $\mathbb{G}_m$ coincides with the decomposition induced by the splitting $F = \bigoplus_{d\geq 0} F_d$ into a direct sum of homogeneous subfunctors:

$$\textstyle\bigoplus_{d\geq 0} F(V)_d = \bigoplus_{d\geq 0} F_d(V) .$$

Thus, passing from a strict polynomial functor F to a $\mathbb{G}_m$-module $F(V)$ should be thought of as a mean of forgetting the naturality while keeping track of the strict polynomial degree (now encoded in the weight of the $\mathbb{G}_m$-action).

Definition 6.1. A graded $\mathbb{G}_m$-algebra is a graded $\mathbb{Z}$-algebra with an action of $\mathbb{G}_m$, such that the multiplication is $\mathbb{G}_m$-equivariant. A morphism of graded $\mathbb{G}_m$-algebras is a $\mathbb{G}_m$-equivariant morphism of graded algebras.

One defines similarly differential graded $\mathbb{G}_m$-algebras, etc. Evaluating a graded strict polynomial algebra on a $\mathbb{G}_m$-module yields a graded $\mathbb{G}_m$-algebra. The aim of section 6 is to compute the following graded $\mathbb{G}_m$-algebras:

(1) The homology of the differential graded $\mathbb{G}_m$-algebra $\overline{B}^n(\Gamma^*(V[2]))$, where $V[2]$ is a $\mathbb{G}_m$-module of weight 1, with homological degree 2.
(2) $\overline{\mathbb{E}}^*(S^*, Y^*; V) = \bigoplus_{h,d\geq 0} \overline{\mathbb{E}}^h(S^d, Y^d; V)$, for $Y^* = \Lambda^*$ of Γ^*. It is considered as a graded $\mathbb{G}_m$-algebra with $\overline{\mathbb{E}}^h(S^d, \Lambda^d; V)$ in homological degree h, acted on by $\mathbb{G}_m$ as a representation of weight d.

6.2. **Bar constructions of divided power algebras over $\mathbb{Z}$.** The homology algebra of the iterated bar constructions $\overline{B}^n(\Gamma^*(V[2]))$ for a free $\mathbb{Z}$-module V of finite rank are computed in [Car, expose 11] (and another description is given in [D]). Before stating Cartan's result, we need to recall elementary facts about dual Koszul and De Rham algebras.

6.2.1. *Dual Koszul and De Rham algebras.* If V is a graded $\mathbb{Z}$-module, we denote by sV its suspension ($(sV)_i := V_{i-1}$), and by s^{-1} the canonical isomorphism $sV \simeq V$ and by hs^{-1} (h in an integer) the composite $sV \simeq V \xrightarrow{h\mathrm{Id}} V$.

Definition 6.2. Let h be a positive integer. Let U be a positively graded $\mathbb{Z}$-module, concentrated in odd degree and free of finite type in each degree. The dual Koszul algebra $D\kappa^h_*(U)$ is the CDGA $\mathbb{Z}$-algebra which equals $\Gamma^*(sU)\otimes\Lambda^*(U)$ as a graded augmented algebra, and whose differential equals the composite

$$\Gamma^n(sU)\otimes\Lambda^k(U) \xrightarrow{\Delta\otimes\mathrm{Id}} \Gamma^{n-1}(sU)\otimes sU\otimes\Lambda^k(U) \xrightarrow{\mathrm{Id}\otimes hs^{-1}\otimes\mathrm{Id}} \Gamma^{n-1}(sU)\otimes U\otimes\Lambda^k(U) \xrightarrow{\mathrm{Id}\otimes m} \Gamma^{n-1}(sU)\otimes\Lambda^{k+1}(U)\ .$$

Let V be a positively graded $\mathbb{Z}$-module, concentrated in even degree and free of finite type in each degree. The dual De Rham algebra $D\Omega^h_*(V)$ is the CDGA $\Bbbk$-algebra which equals $\Gamma^*(V)\otimes\Lambda^*(sV)$ as an augmented graded algebra and whose differential equals the composite

$$\Gamma^n(V)\otimes\Lambda^k(sV) \xrightarrow{\mathrm{Id}\otimes\Delta} \Gamma^n(V)\otimes sV\otimes\Lambda^{k-1}(sV) \xrightarrow{\mathrm{Id}\otimes hs^{-1}\otimes\mathrm{Id}} \Gamma^n(V)\otimes V\otimes\Lambda^{k-1}(sV) \xrightarrow{m\otimes\mathrm{Id}} \Gamma^{n+1}(V)\otimes\Lambda^{k-1}(sV)\ .$$

If F_* is a graded strict polynomial functor concentrated in odd degree (resp. even degree), then $D\kappa^h_*(F_*(V))$ (resp. $D\Omega^h_*(F_*(V))$) is a strict polynomial CDGA $\mathbb{Z}$-algebra.

For $h=1$, $D\kappa^h_*(V)$ and $D\Omega^h_*(V)$ are the graded duals of the usual Koszul and De Rham algebras. Let $\mathbb{Z}[i]$ denote a copy of $\mathbb{Z}$ placed in degree i, then $D\kappa^h_*(\mathbb{Z}[i])$ and $D\Omega^h_*(\mathbb{Z}[i])$ are the elementary complexes of type (II) from [Car, p. 11-03].

The graded algebras $D\kappa^h_*(V)$ and $D\Omega^h_*(V)$ satisfy an exponential formula. Namely, the composite

$$D\kappa^h_*(V)\otimes D\kappa^h_*(W)\to D\kappa^h_*(V\oplus W)^{\otimes 2}\xrightarrow{m} D\kappa^h_*(V\oplus W)\;,$$

where the first map is induced by the canonical inclusions of V and W into $V\oplus W$, is an isomorphism of CDGA $\mathbb{Z}$-algebras. Similarly, $D\Omega^h_*(V\oplus W)$ is isomorphic to $D\Omega^h_*(V)\otimes D\Omega^h_*(W)$.

6.2.2. *Cartan's result.* We are now ready to state [Car, Theoreme 1, p. 11-09], which computes the homology of $\overline{B}^n(\Gamma^*(V[2]))$, or equivalently the singular homology of $K(V, n+2)$, not naturally in the free $\mathbb{Z}$-module V.

Case $n=1$. Let $X_p(1)$ be the CDGA $\mathbb{Z}$-algebra:

$$X_p(1) := D\kappa^p_*\left(\bigoplus\nolimits_{k\geq 0} V[2p^{k+1}+1]\right)\;.$$

The dual Koszul complex generated by each $V[2p^{k+1}+1]$ corresponds to the tensor product of all the elementary complexes attached to the pair of words $\phi_p\gamma_p^k\sigma\sigma$, $\beta_p\phi_p\gamma_p^k\sigma\sigma$ in Cartan's denomination [Car, p. 11-08 (iii)]. Thus our $X_p(1)$ is nothing but Cartan's 'X_p' in [Car, Theoreme 1, p. 11-09]. We denote by ${}_pH_*(X_p(1))$ the p-primary part of its homology:

$${}_pH_*(X_p(1)) = \{x\in H_*(X_p(1)), \exists r\geq 1\quad p^r x = 0\}$$

It is a graded $\mathbb{Z}$-algebra without unit, concentrated in positive degrees. We make it into a unital $\mathbb{Z}$-algebra $\widehat{{}_pH_*}(X_p(1))$ in the canonical way:

$$\widehat{{}_pH_0}(X_p(1)) = \mathbb{Z}\,,\quad \widehat{{}_pH_*}(X_p(1)) = H_*(X_p(1)) \text{ for } *>0.$$

Now Cartan's result [Car, Theoreme 1, p. 11-09] yields an isomorphism of graded $\mathbb{Z}$-algebras:

$$H_*\left(\overline{B}(\Gamma^*(V[2]))\right)\simeq \Lambda^*(V[3])\otimes \bigotimes_{p\text{ prime}} \widehat{{}_pH_*}(X_p(1))\;.$$

Case $n=2$. Similarly, there is an isomorphism of graded $\mathbb{Z}$-algebras:

$$H_*\left(\overline{B}^2(\Gamma^*(V[2]))\right)\simeq \Gamma^*(V[4])\otimes \bigotimes_{p\text{ prime}} \widehat{{}_pH_*}(X_p(2))\;,$$

where $X_p(2)$ is the CDGA $\mathbb{Z}$-algebra:

$$X_p(2) := D\kappa^p_*\left(\bigoplus\nolimits_{k,\ell\geq 0} V[2p^{k+\ell+2}+2p^{k+1}+1]\right)\otimes D\Omega^p_*\left(\bigoplus\nolimits_{k\geq 0} V[2p^{k+1}+2]\right)\;.$$

Each dual Koszul complex generated by $V[2p^{k+\ell+2}+2p^{k+1}+1]$ corresponds to the tensor product of all the elementary complexes attached to the pair of words $\phi_p\gamma_p^k\phi_p\gamma_p^\ell\sigma\sigma$, $\beta_p\phi_p\gamma_p^k\phi_p\gamma_p^\ell\sigma\sigma$ in Cartan's denomination [Car, p. 11-08 (iii)]). Similarly, each dual De Rham complex generated by $V[2p^{k+1}+2]$ corresponds to the tensor product of the elementary complexes attached to the pair of words $\sigma\phi_p\gamma_p^k\sigma\sigma$, $\beta_p\sigma\phi_p\gamma_p^k\sigma\sigma$.

6.2.3. *The $\mathbb{G}_m$-action.* The homology of $\overline{B}^n(\Gamma^*(V[2]))$ is a graded strict polynomial algebra, hence a graded $\mathbb{G}_m$-algebra (we consider $V[2]$ as a $\mathbb{G}_m$-module of weight 1). We now supplement Cartan's computation by computing the $\mathbb{G}_m$-action.

Theorem 6.3. *Let V be a free finitely generated $\mathbb{Z}$-module, and let V_d denote a copy of V acted on by $\mathbb{G}_m$ with weight d. The homology of $\overline{B}(\Gamma^*(V_1[2]))$ is isomorphic, as a $\mathbb{G}_m$-graded algebra, to*

$$\Lambda^*(V_1[3]) \otimes \bigotimes_{p\ prime} \widehat{{}_pH_*}(X_p(1)) \ ,$$

where for all prime p, $X_p(1)$ denotes the CDGA $\mathbb{G}_m$-algebra

$$X_p(1) := D\kappa^p_*\left(\textstyle\bigoplus_{k\geq 0} V_{p^{k+1}}[2p^{k+1}+1]\right) .$$

The homology of $\overline{B}^2(\Gamma^*(V_1[2]))$ is isomorphic, as a $\mathbb{G}_m$-graded algebra, to

$$\Gamma^*(V_1[4]) \otimes \bigotimes_{p\ prime} \widehat{{}_pH_*}(X_p(2)) \ ,$$

where for all prime p, $X_p(2)$ denotes the CDGA $\mathbb{G}_m$-algebra

$$D\kappa^p_*\left(\textstyle\bigoplus_{k,\ell\geq 0} V_{p^{k+\ell+2}}[2p^{k+\ell+2}+2p^{k+1}+1]\right)\otimes D\Omega^p_*\left(\textstyle\bigoplus_{k\geq 0} V_{p^{k+1}}[2p^{k+1}+2]\right) .$$

Proof. Let us prove the case of $\overline{B}(\Gamma^*(V[2]))$, the case of $\overline{B}^2(\Gamma^*(V[2]))$ is similar. To compute the $\mathbb{G}_m$-action, we have to come back to the construction [Car, p. 11-07] of the morphisms of CDGA $\mathbb{Z}$-algebras $f_p : X_p(1) \to \overline{B}(\Gamma^*(V[2]))$, for p prime, and $f_0 : \Lambda^*(V[3]) \to \overline{B}(\Gamma^*(V[2]))$.

Case 1: construction of f_p, for p prime. Let (u_i) be a basis of V. Then for all basis vector u_i and all pair of words $\beta_p\phi_p\gamma_p^k\sigma\sigma$, $\phi_p\gamma_p^k\sigma\sigma$, Cartan defines a pair of elements $x_{i,k}, y_{i,k} \in \overline{B}(\Gamma^*(V[2]))$, with respective degrees $(2p^{k+1}+1)$ and $(2p^{k+1}+2)$, and such that $dy_{i,k} = px_{i,k}$ (these elements correspond to the elements denoted by x' and y' in [Car, p. 11-07], and we recall their precise definition later in the proof).

Since $\overline{B}(\Gamma^*(V[2]))$ is a CDGA $\mathbb{G}_m$-algebra, the differential is $\mathbb{G}_m$ equivariant, so $y_{i,k}$ and $x_{i,k}$ are acted on by $\mathbb{G}_m$ with the same weight. With such a pair $x_{i,k}, y_{i,k}$ at hand, products and divided powers operations yield a morphism of CDGA-algebras from the elementary complex generated by $x_{i,k}, y_{i,k}$ to $\overline{B}(\Gamma^*(V[2]))$ (cf. [Car, p. 11-03]). The elementary complex generated by $x_{i,k}, y_{i,k}$ is nothing but the dual Koszul complex on the graded $\mathbb{Z}$-module $\mathbb{Z}_{i,k}$, which is a copy of $\mathbb{Z}$ placed in degree $(2p^{k+1}+1)$. That is products and divided powers operations induce a morphism of CDGA-algebras:

$$D\kappa^p(\mathbb{Z}_{i,k}) \to \overline{B}(\Gamma^*(V[2])) \ .$$

Since $D\kappa^p(\mathbb{Z}_{i,k})$ and $\overline{B}(\Gamma^*(V[2]))$ are free as $\mathbb{Z}$-modules, the divided powers are determined by the products. Now products are $\mathbb{G}_m$-equivariant, so this morphism of algebras is $\mathbb{G}_m$-equivariant.

Now $X_p(1)$ is the tensor product of all the elementary complexes $D\kappa^p(\mathbb{Z}_{i,k})$ (and we can rewrite it as $D\kappa^p_*(\bigoplus_{k\geq 0} V[2p^{k+1}+1])$ by the exponential formula for Koszul complexes, each $V[2p^{k+1}+1]$ being the sum of the $\mathbb{Z}_{i,k}$ over all the indices i). The maps $f_p : X_p(1) \to \overline{B}(\Gamma^*(V[2]))$

are defined by tensoring the morphisms $D\kappa^p(\mathbb{Z}_{i,k}) \to \overline{B}(\Gamma^*(V[2]))$ for all i and all k. Hence they are $\mathbb{G}_m$-equivariant. So, to compute the $\mathbb{G}_m$-action on Cartan's description of the homology of $\overline{B}(\Gamma^*(V[2]))$, it suffices to determine how $\mathbb{G}_m$ acts on the complexes $D\kappa^p(\mathbb{Z}_{i,k})$. This reduces to determining how $\mathbb{G}_m$ acts on the elements $y_{i,k}$.

So we now come back to the precise definition of the $y_{i,k}$ [Car, p. 11-07]. This definition involves reduction modulo p. There is a surjective morphism of $\mathbb{G}_m$-CDGA $\mathbb{Z}$-algebras

$$\overline{B}(\Gamma^*(V[2])) \twoheadrightarrow \overline{B}(\Gamma^*(V[2])) \otimes_{\mathbb{Z}} \mathbb{F}_p = \overline{B}(\Gamma^*((V/pV)[2]))\ .$$

The element $y_{i,k}$ is a lifting of a cycle $z_{i,k}$ of $\overline{B}(\Gamma^*((V/pV)[2]))$, representing a certain homology class in the homology $c_{i,k}$ in the homology of $\overline{B}(\Gamma^*((V/pV)[2]))$ (that is in the singular homology of $K(V,3)$ with coefficients in $\mathbb{F}_p$). Since $\overline{B}(\Gamma^*(V[2])) \twoheadrightarrow \overline{B}(\Gamma^*(V[2])) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is $\mathbb{G}_m$-equivariant, $\mathbb{G}_m$ acts on $y_{i,k}$ with the same weight as it acts on $c_{i,k}$.

The class $c_{i,k}$ is defined in the following way. Recall that the index 'i' refers to a basis element u_i of the free $\mathbb{Z}$-module V and that the index 'k' refers to a word of the form $\phi_p\gamma_p^k\sigma\sigma$. Let $\overline{u_i}$ be the image of u_i through the canonical projection $V \mapsto V/pV$. We consider $\overline{u_i}$ as an element in the $\mathbb{F}_p$-vector space

$$(V/pV) = (V/pV)_{\sigma\sigma} = H_2\big(\Gamma^*((V/pV)[2])\big) \simeq H_2(K(V,2),\mathbb{F}_p)\ .$$

By [Car, Exposes 6 et 7], the homology of the $\overline{B}^n(\Gamma^*((V/pV)[2]))$, $n \geq 0$, is endowed with operations γ_r, σ and ϕ_p. So, applying the operations γ_p^k and ϕ_p to the homology class $\overline{u_i}$, we obtain a class $c_{i,k}$ (of degree $2p^{k+1}+2$) in the homology of $\overline{B}(\Gamma^*((V/pV)[2]))$. The class $\overline{u_i}$ is acted on by $\mathbb{G}_m$ with weight 1, so by lemma 5.5 and remark 5.6, $c_{i,k}$ is acted on by $\mathbb{G}_m$ with weight p^{k+1}.

So, to sum up, we have proved that Cartan's maps f_p are actually $\mathbb{G}_m$-equivariant morphisms of CDGA $\mathbb{G}_m$-algebras:

$$D\kappa^p_*(\textstyle\bigoplus_{k\geq 0} V_{p^{k+1}}[2p^{k+1}+1]) \to \overline{B}(\Gamma^*(V[2]))\ .$$

Case 2: construction of f_0. The map f_0 is simply defined in degree $3d$ by sending $\Lambda^d(V)$ into $V^{\otimes d} = (\Gamma^1(V))^{\otimes d} \subset \overline{B}(\Gamma^*(V[2]))_{3d}$ via the canonical inclusion. So if we let $\mathbb{G}_m$ act on V with weight 1, then $f_0 : \Lambda^*(V[3]) \to \overline{B}(\Gamma^*(V[2]))$ is a morphism of $\mathbb{G}_m$-CDGA $\mathbb{Z}$-algebras (we take the trivial differential on $\Lambda^*(V[3])$).

Conclusion. By [Car, Theoreme 1, p. 11-09], the map f_0 sends $\Lambda^*(V[3])$ injectively into the homology of $\overline{B}(\Gamma^*(V[2]))$, whose image is a complement of the torsion part of the homology, and the maps $f_0 \otimes f_p$ induce isomorphisms after taking the homology and restricting to the p-primary part. Since we have written the f_i as $\mathbb{G}_m$-equivariant maps, this yields our $\mathbb{G}_m$-equivariant result. □

To complete the computation of theorem 6.3, let us mention that the homology of dual Koszul and De Rham complexes is quite concrete. For example, [D, Thm 2.4.8] yields a description of these graded algebras by generators and relations. For the homology of $\overline{B}(\Gamma^*(V[2]))$, we can offer yet another nice reformulation, involving Koszul kernel algebras.

6.2.4. *A reformulation with Koszul kernel algebras.* Let us first define Koszul kernel algebras. Let U be a nonnegatively graded $\mathbb{F}_p$-vector space, which is finite dimensional in each degree. We can consider U as a graded $\mathbb{Z}$-module and we let:

$$\Gamma^0_{\mathbb{F}_p}(U)\ ,\quad \Gamma^d_{\mathbb{F}_p}(U) := (U^{\otimes d})^{\mathfrak{S}_d}\quad \text{for } d>0.$$

As usual, all tensor products are taken over the ground ring $\Bbbk$, which is here $\Bbbk = \mathbb{Z}$. We have denoted the resulting vector space by $\Gamma^d_{\mathbb{F}_p}(U)$ for two reasons. First, because the symbol '$\Gamma^d(U)$' is already commonly used to denote another object, namely the universal divided power algebra generated by U from [EML2, p.107-110] or [Car, Section 7, p.11-11]. Second, the '$\mathbb{F}_p$' index is here to remind the reader that if W is a nonnegatively graded Z-module which is degreewise free of finite rank, for $d>0$ there is a canonical isomorphism $\Gamma^d(W)\otimes\mathbb{F}_p \simeq \Gamma^d_{\mathbb{F}_p}(W\otimes\mathbb{F}_p)$.

Similarly, we let $\Lambda^*_{\mathbb{F}_p}(U)$ be the quotient of the tensor algebra $\bigoplus_{d\ge 0} U^{\otimes d}$ by the graded ideal generated by elements of the form $x\otimes x$, $x\in U$. As for divided powers, we have $\Lambda^0_{\mathbb{F}_p}(U) = U^{\otimes 0} = \mathbb{Z}$ and for $d>0$, $\Lambda^d(W)\otimes\mathbb{F}_p \simeq \Lambda^d_{\mathbb{F}_p}(W\otimes\mathbb{F}_p)$

Definition 6.4. Let p be a prime, and let U be a nonnegatively graded $\mathbb{F}_p$-vector space which is finite dimensional in each degree. We consider U as a graded $\mathbb{Z}$-module. We let $D\kappa^1_{\mathbb{F}_p,*}(U)$ be the differential graded $\mathbb{Z}$-algebra which equals $\Gamma^*_{\mathbb{F}_p}(sU)\otimes\Lambda^*_{\mathbb{F}_p}(U)$ as an algebra (as usual, the tensor product is taken over the ground ring $\mathbb{Z}$), with differential as in definition 6.2. The Koszul kernel algebra $K_*(U)$ is the graded subalgebra of $\Gamma^*_{\mathbb{F}_p}(sU)\otimes\Lambda^*_{\mathbb{F}_p}(U)$ formed by the cycles of $D\kappa^1_{\mathbb{F}_p,*}(U)$:

$$K_*(U) := \ker\left(D\kappa^1_{\mathbb{F}_p,*}(U)\xrightarrow{d} D\kappa^1_{\mathbb{F}_p,*-1}(U)\right)\ .$$

Similarly, if W is a nonnegatively graded Z-module which is degreewise free of finite rank, we denote by $K_*(W)$ the graded subalgebra of $\Gamma^*(sW)\otimes\Lambda^*(W)$ formed by the cycles of $D\kappa^1_*(W)$ as defined in definition 6.2.

Observe that if W is graded and degreewise free of finite rank, for all $d>0$ and all prime p the various Koszul kernels are linked by canonical isomorphisms:

$$K_d(W)/pK_d(W)\simeq K_d(W)\otimes\mathbb{F}_p\simeq K_d(W\otimes\mathbb{F}_p)\simeq K_d(W/pW)\ .$$

The Koszul kernel algebras $K_*(W)$ and $K_*(W/pW)$ are well known to representation theoretists. For example, if W is homogeneous, they equal the direct sum of all Weyl functors indexed by hooks [BB, Remark III.1.5].

Lemma 6.5. *Let p be a prime, and let V be a free finitely generated $\mathbb{Z}$-module. Then*

$$V\mapsto K_*\left(\textstyle\bigoplus_{k\ge 0}(V/pV)^{(k+1)}[2p^{k+1}+1]\right)$$

is a strict polynomial algebra. In particular if we consider V as a $\mathbb{G}_m$-module of weight 1, it becomes a graded $\mathbb{G}_m$-algebra. It is isomorphic to the graded $\mathbb{G}_m$-algebra (where V_d is a copy of V acted on by $\mathbb{G}_m$ with weight d)

$$\widehat{{}_pH_*}\left(D\kappa^p_*\left(\textstyle\bigoplus_{k\ge 0}V_{p^{k+1}}[2p^{k+1}+1]\right)\right)\ .$$

Proof. The first part of lemma 6.5 is obvious. Let us prove the isomorphism of graded $\mathbb{G}_m$-algebras. For all nonnegatively graded $\mathbb{G}_m$-module W, which degreewise $\mathbb{Z}$-free of finite type, the homology of $D\kappa^1_*(W)$ equals $\mathbb{Z}$, placed in degree 0 (indeed, by the exponential formula, one reduces to the trivial case where W is free of rank one). Thus, the cycles $K_*(W) \subset D\kappa^1_*(W)$ equal the boundaries in all positive degrees. So the homology of $D\kappa^p_*(W)$ equals $\mathbb{Z}$ in degree $* = 0$, and $K_*(W)/pK_*(W)$ in degrees $* > 0$. Whence an isomorphism of $\mathbb{G}_m$-algebras (in positive degrees): ${}_pH_*\left(D\kappa^p_*(W)\right) \simeq K_*(W) \otimes \mathbb{F}_p$. The latter $\mathbb{G}_m$-algebra identifies with $K_*(W \otimes \mathbb{F}_p) \simeq K_*(W/pW)$ in positive degrees, so that we finally obtain an isomorphism of graded $\mathbb{G}_m$-algebras:

$$\widehat{{}_pH_*}\left(D\kappa^p_*(W)\right) \simeq K_*(W/pW).$$

To finish the proof, observe that for $W = \bigoplus_{k\geq 0} V_{p^{k+1}}[2p^{k+1}+1]$, the graded $\mathbb{G}_m$-module W/pW coincides with $\bigoplus_{k\geq 0}(V/pV)^{(k+1)}[2p^{k+1}+1]$. □

In view of lemma 6.5, theorem 6.3 immediately gives:

Theorem 6.6. *Let V be a free $\mathbb{Z}$-module of finite rank. The homology of $\overline{B}(\Gamma^*(V[2]))$ is isomorphic to the graded algebra:*

$$\Lambda^*(V[3]) \otimes \bigotimes_{p\ prime} K_*\left(\bigoplus_{k\geq 0}(V/pV)^{(k+1)}[2p^{k+1}+1]\right) .$$

This isomorphism is a priori not natural *with respect to V, but however it preserves the strict polynomial degree.*

Conjecture 6.7. We think that theorem 6.6 actually yields a description of the homology of $\overline{B}(\Gamma^*(V[2]))$ as a strict polynomial algebra. A careful analysis of the proof of [Car, Theoreme 1, p. 11-09] seems to confirm this for the p primary part when p is odd, but there remain problems to solve for $p = 2$.

6.3. **The computation of extension groups.** We can now compute the graded $\mathbb{G}_m$-algebras

$$\overline{\mathbb{E}}^*(S^*, Y^*; V) = \bigoplus_{h,d\geq 0} \overline{\mathbb{E}}^h(S^d, Y^d; V) ,$$

where $Y^* = \Lambda^*$ or Γ^*, and where a summand $\overline{\mathbb{E}}^h(S^d, Y^d; V)$ has degree h and is acted on by $\mathbb{G}_m$ with weight d (as specified in section 6.1). By theorem 4.5, these graded algebras equal, up to a regrading, the homology of $\overline{B}^n(\Gamma^*(V[2]))$ for $n = 1, 2$. So theorem 6.6 almost gives us the result.

To be more specific, we introduce for all $s \geq 0$ an additive 'regrading functor':

$$\mathcal{R}_s : \{\text{graded } \mathbb{G}_m\text{-modules}\} \to \{\text{graded } \mathbb{G}_m\text{-modules}\} ,$$

which sends a graded $\mathbb{G}_m$-module M of degree i and acted on by $\mathbb{G}_m$ with weight d onto the graded $\mathbb{G}_m$-module $\mathcal{R}_sM$, which is concentrated in degree $sd - i$ and which equals M as a $\mathbb{G}_m$-module. With this definition, theorem 4.5 yields isomorphisms of $\mathbb{G}_m$-algebras:

$$\overline{\mathbb{E}}^*(S^*, \Lambda^*; V) \simeq \mathcal{R}_3H_*\left(\overline{B}(\Gamma^*(V[2]))\right) , \quad \overline{\mathbb{E}}^*(S^*, \Gamma^*; V) \simeq \mathcal{R}_4H_*\left(\overline{B}(\Gamma^*(V[2]))\right) .$$

When s is even, the functor $\mathcal{R}_s$ does not affect the parity of degrees: $\mathcal{R}_sM$ is concentrated in odd degrees if and only if M is concentrated in odd degrees. So, $\mathcal{R}_s$ behaves well with differentials and tensor products of graded algebras (Koszul signs are preserved). One easily checks the following properties.

Lemma 6.8. *Let s be an even integer. Then $\mathcal{R}_s$ induces an endofunctor of differential graded (DG) $\mathbb{G}_m$-algebras:*

$$\mathcal{R}_s : \{DG\ \mathbb{G}_m\text{-}algebras\} \to \{DG\ \mathbb{G}_m\text{-}algebras\}\,,$$

compatible with tensor products. Moreover, if A is a DG $\mathbb{G}_m$-algebra, then $\mathcal{R}_sH_*(A)$ equals $H_*(\mathcal{R}_sA)$.

If s is odd, $\mathcal{R}_s$ might change the parity of degrees. As a consequence, $\mathcal{R}_s$ is not compatible with tensor products of graded algebras. To fix this problem, we have to define a skew tensor product for $\mathbb{G}_m$-algebras.

Notation 6.9. Let A and B be two graded $\mathbb{G}_m$-algebras. We denote by $A\widetilde{\otimes}B$ their skew tensor product. As a graded $\mathbb{G}_m$-module, $A\widetilde{\otimes}B$ equals $A\otimes B$. The product on $A\widetilde{\otimes}B$ is defined as follows. Let $a'_{i,d}$ be a homogeneous element of A of degree i and acted on by $\mathbb{G}_m$ with weight d, and let $b_{j,e}$ be a homogeneous element of B of degree j, and acted on $\mathbb{G}_m$ with weight e. Then the product $(a\otimes b_{j,e})\cdot(a'_{i,d}\otimes b')$ in $A\widetilde{\otimes}B$ equals $(-1)^{(i+d)(j+e)}(a\cdot a'_{i,d})\otimes(b'_{j,e}\cdot b)$.

One easily checks the following lemma.

Lemma 6.10. *Let s be an odd integer and let A, B be graded $\mathbb{G}_m$-algebras. Then the $\mathbb{G}_m$-algebra $\mathcal{R}_s(A\otimes B)$ equals $(\mathcal{R}_sA)\widetilde{\otimes}(\mathcal{R}_sB)$.*

Now we can apply the functors $\mathcal{R}_3$ and $\mathcal{R}_4$ to the results of 6.6. It is straightforward to see that $\mathcal{R}_4D\kappa^p_*(M) = D\kappa^p_*(\mathcal{R}_4M)$, $\mathcal{R}_4D\Omega^p_*(N) = D\Omega^p_*(\mathcal{R}_4N)$ and $\mathcal{R}_3K_*(U) = K_*(\mathcal{R}_3U)$ for suitable graded $\mathbb{G}_m$-modules M, N, U. Combined with lemmas 6.8 and 6.10, this gives us the following result.

Theorem 6.11 (Main computation III)**.** *Let V be a free $\mathbb{Z}$-module of finite rank. Assume that $V[i]$ and $(V/pV)^{(k+1)}[i]$ have homological degree i and are acted on by $\mathbb{G}_m$ respectively with weight 1 and p^{k+1}. There is an isomorphism of graded $\mathbb{G}_m$-algebras:*

$$\overline{\mathbb{E}}^*(S^*,\Lambda^*;V) \simeq \Lambda^*(V[0])\widetilde{\otimes}\ \widetilde{\bigotimes_{p\ prime}}\ K_*\left(\bigoplus_{k\geq 0}(V/pV)^{(k+1)}[p^{k+1}-1]\right)\,,$$

where K_* denotes the Koszul kernel algebra from definition 6.4. Let $V_d[i]$ denote a copy of V placed in homological degree i and acted on by $\mathbb{G}_m$ with weight d. There is an isomorphism of graded $\mathbb{G}_m$-algebras:

$$\overline{\mathbb{E}}^*(S^*,\Gamma^*;V) \simeq \Gamma^*(V_1[0])\otimes \bigotimes_{p\ prime} \widehat{{}_pH_*}\left(\widetilde{X}_p(2)\right)\,,$$

where for all prime p, $\widetilde{X}_p(2)$ denotes the CDGA $\mathbb{G}_m$-algebra

$$D\kappa^p_*\left(\bigoplus_{k,\ell\geq 0}V_{p^{k+\ell+2}}[2p^{k+\ell+2}-2p^{k+1}-1]\right)\otimes D\Omega^p_*\left(\bigoplus_{k\geq 0}V_{p^{k+1}}[2p^{k+1}-2]\right).$$

7. Extensions between twisted classical exponential functors

Throughout this section, $\Bbbk$ is a field of positive characteristic p. Our goal is to compute all the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V)$, where X^*, Y^* are classical exponential functors (with bidegree and product as specified in convention 3.6). Duality yields an isomorphism: $\overline{\mathbb{E}}^*(X^{*\,(r)}, Y^{*\,(s)}; V) \simeq \overline{\mathbb{E}}^*(Y^{\sharp\,*\,(s)}, X^{\sharp\,*\,(r)}; V)$, so it suffices to do the computation for $r \geq s$. Thus, our goal reduces to computing the bigraded strict polynomial algebras:

$$\overline{\mathbb{E}}^*(X^{*\,(s+t)}, Y^{p^t *\,(s)}; V)$$

for all nonnegative integers s, t and all pairs of classical exponential functors (X^*, Y^*). The principle of our proof is to express them in terms of their untwisted versions $\overline{\mathbb{E}}^*(X^*, Y^*; V)$ (which are explicitly known from the introduction or from section 5). We proceed in two steps.

(1) We show in section 7.1 that the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(t)}, Y^*; V)$ are equal, up to regrading, to $\overline{\mathbb{E}}^*(X^*, Y^*; V^{(t)})$.
(2) We show in section 7.2 that $\overline{\mathbb{E}}^*(X^{d\,(s+t)}, Y^{p^t d\,(s)}; V)$ can be expressed as a function of $\overline{\mathbb{E}}^*(X^{*\,(t)}, Y^*; V)$. This relies on the computational tools developed in [T3]. In fact, these tools only give the result up to a filtration, but with additional work, we prove that this filtration is trivial.

Our method does not depend on the computations of [FFSS, C2], where these extensions were computed first (with mistakes in the results of [C2], which we correct below).

7.1. Frobenius twists and bar constructions.

Let A^* be a strict polynomial algebra. Then for all strict polynomial functors F, G, we define an 'external product' $\mathbb{E}^*(F, A^*; V) \otimes \mathbb{E}^*(G, A^*; V) \to \mathbb{E}^*(F \otimes G, A^*; V)$ as the composite:

$$\mathbb{E}^i(F, A^k; V) \otimes \mathbb{E}^j(G, A^\ell; V) \to \mathbb{E}^{i+j}(F \otimes G, A^k \otimes A^\ell; V) \to \mathbb{E}^{i+j}(F \otimes G, A^{k+\ell}; V)$$

where the first map is induced by tensor products and the second one is induced by the multiplication of A^*. In particular, if X^*, Y^* are classical exponential functors the product on $\mathbb{E}^*(X^{*\,(t)}, Y^*; V)$ is obtained by combining the external product and the comultiplication of $X^{*\,(t)}$.

Proposition 7.1. *Let $\Bbbk$ be a field of positive characteristic p. Let F be a homogeneous strict polynomial functors of d, and let t be a nonnegative integer. For all integer i, there are isomorphisms of strict polynomial functors (with the convention that $\mathbb{E}^i(F, G, ; V) = 0$ for $i < 0$), natural in F:*

$$\mathbb{E}^i(F, S^d; V^{(t)}) \simeq \mathbb{E}^i(F^{(t)}, S^{dp^t}; V)\ ,$$
$$\mathbb{E}^i(F, \Lambda^d; V^{(t)}) \simeq \mathbb{E}^{i+(p^t-1)d}(F^{(t)}, \Lambda^{dp^t}; V)\ ,$$
$$\mathbb{E}^i(F, \Gamma^d; V^{(t)}) \simeq \mathbb{E}^{i+2(p^t-1)d}(F^{(t)}, \Gamma^{dp^t}; V)\ .$$

Moreover, for $Y^* = S^*, \Lambda^*$ or Γ^*, and for homogeneous F, G of respective degrees d, e the external product

$$\mathbb{E}^*(F, Y^d; V^{(t)}) \otimes \mathbb{E}^*(G, Y^e; V^{(t)}) \to \mathbb{E}^*(F \otimes G, Y^{d+e}; V^{(t)})$$

identifies through the isomorphism with the external product:

$$\mathbb{E}^*(F^{(t)}, Y^{dp^t}; V) \otimes \mathbb{E}^*(G^{(t)}, Y^{ep^t}; V) \to \mathbb{E}^*(F^{(t)} \otimes G^{(t)}, Y^{dp^t+ep^t}; V) \ .$$

Proof. Let us recall from [T3, Lemmas 2.2 and 2.3] an elementary computation in $\mathcal{P}_{\Bbbk}$. If $\mu = (\mu_1, \dots, \mu_n)$ is a tuple of positive integers, we denote by S^μ the tensor product $S^{\mu_1} \otimes \cdots \otimes S^{\mu_n}$, and by $\alpha\mu$ the tuple $\alpha\mu := (\alpha\mu_1, \dots, \alpha\mu_n)$, for all $\alpha \geq 0$. Then for all $G \in \mathcal{P}_{\Bbbk}$, there are isomorphisms (the first one is induced by precomposition by $I^{(t)}$, the second one by the canonical inclusion $S^{\mu\,(t)} \hookrightarrow S^{p^t\mu}$):

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G, S^\mu) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(t)}, S^{\mu\,(t)}) \ , \qquad (i)$$

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(t)}, S^{\mu\,(t)}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(t)}, S^{p^t\mu}) \ . \qquad (ii)$$

Moreover if λ is not of the form $p^t\mu$, $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(t)}, S^\lambda)$ equals zero.

If we take $G = F^{V^{(t)}}$, then $G^{(t)} = (F^{(t)})^V$, and these isomorphisms yield an isomorphism $\mathbb{H}(F, S^\mu, V^{(t)}) \simeq \mathbb{H}(F^{(t)}, S^{p^t\mu}, V)$ compatible with the external product. Since symmetric powers are injective, this proves the case of $\mathbb{E}^*(F, S^d; V^{(t)})$.

Now we prove the case of $\mathbb{E}^*(F, \Lambda^d; V^{(t)})$. We denote by $B(d)$ the homogeneous part of strict polynomial degree d of $\overline{B}(S^*(V[0]))$. By section 4.2.2, for all $F \in \mathcal{P}_{d,\Bbbk}$, $\mathbb{E}^{d-*}(F, \Lambda^d; V)$ is isomorphic to the homology of $\mathbb{H}(F, B(d)_*; V)$, and the external product is read on $\mathbb{H}(F, B(d)_*; V)$ by taking tensor products and using the multiplication of $\overline{B}(S^*(V[0]))$. Similarly, $\mathbb{E}^{dp^t-*}(F^{(t)}, \Lambda^{dp^t}; V)$ equals the homology of $\mathbb{H}(F^{(t)}, B(dp^t)_*; V)$. Now the maps $S^{d\,(t)}(V) \hookrightarrow S^{dp^t}(V)$ induce a morphism of graded algebras (concentrated in degree 0) $\overline{B}(S^{*\,(t)}(V[0])) \hookrightarrow \overline{B}(S^*(V[0]))$, whence a graded morphism compatible with external products:

$$\mathbb{H}(F, B(d)_*; V^{(t)}) \simeq \mathbb{H}(F^{(t)}, B(d)_*^{(t)}; V) \hookrightarrow \mathbb{H}(F^{(t)}, B(dp^t)_*; V) \ . \quad (iii)$$

It is actually an isomorphism. Indeed, all the summands $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F^{(t)}, S^\lambda)$ of $\mathbb{H}(F^{(t)}, B(dp^t)_*; V)$ equal zero if $\lambda \neq p^t\mu$, and if $\lambda = p^t\mu$ the corresponding summand is isomorphic to $\mathbb{H}(F, S^\mu; V^{(t)})$ by (i) and (ii). So the case of $\mathbb{E}^*(F, \Lambda^d; V^{(t)})$ follows from isomorphism (iii) by taking the homology. The case of $\mathbb{E}^*(F, \Gamma^d; V^{(t)})$ is similar. □

Corollary 7.2. *Let $\Bbbk$ be a field of positive characteristic p, let t be a positive integer, and let X^* and Y^* be classical exponential functors. There are isomorphisms of bigraded strict polynomial algebras:*

$$\mathbb{E}^*(X^*, Y^*; V^{(t)}) \simeq \mathbb{E}^*(X^{*(t)}, Y^*; V) \ , \ \textit{and}$$

$$\overline{\mathbb{E}}^*(X^*, Y^*; V^{(t)}) \simeq \overline{\mathbb{E}}^*(X^{*(t)}, Y^*; V) \ ,$$

which map $\mathbb{E}^i(X^d, Y^d; V^{(t)})$ onto $\mathbb{E}^{i+\alpha d}(X^{d(t)}, Y^{dp^t}; V)$, with $\alpha = 0$ if $Y^* = S^*$, $\alpha = p^t - 1$ if $Y^* = \Lambda^*$ and $\alpha = 2(p^t - 1)$ if $Y^* = \Gamma^*$.

Proof. The first isomorphism follows directly from proposition 7.1. If $p = 2$, the signed and the unsigned algebras are equal, so there is nothing to add to get the second isomorphism. If p is odd, for all pair of integers (i, d), the parity of i is the same as the parity of $i + \alpha d$, and the parity of d is the same as the parity of dp^t. So the first isomorphism is also compatible with signed products. □

Example 7.3. **(1)** By the Yoneda lemma, we have:

$$\overline{\mathbb{E}}^*(\Gamma^*,\Lambda^*;V)=\mathbb{E}^*(\Gamma^*,\Lambda^*;V)=\bigoplus_{d\ge 0}\mathbb{H}(\Gamma^d,\Lambda^d;V)\simeq\bigoplus_{d\ge 0}\Lambda^d(V)\;.$$

So, corollary 7.2 yields isomorphisms of bigraded strict polynomial algebras:

$$\overline{\mathbb{E}}^*(\Gamma^{*\,(t)},\Lambda^*;V)=\mathbb{E}^*(\Gamma^{*\,(t)},\Lambda^*;V)\simeq\Lambda^*(V^{(t)}[p^t-1])\;,$$

where the generator $V^{(t)}[p^t-1]$ of the right handside has bidegree (p^t-1,p^t).
(2) Assume that $p=2$. We have computed $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)=E^*(S^*,\Lambda^*;V)$ in theorem 5.14. Corollary 7.2 yields an isomorphism

$$\overline{\mathbb{E}}^*(S^{*\,(t)},\Lambda^*;V)=\mathbb{E}^*(S^{*\,(t)},\Lambda^*;V)\simeq\Gamma^*\left(\bigoplus_{k\ge 0}V^{(k+t)}[p^{k+t}-1]\right)\;,$$

where the $V^{(k+t)}[p^{k+t}-1]$ of the right handside have bidegree $(p^{k+t}-1,p^{k+t})$.
(3) Assume that $p>2$. We have computed $\overline{\mathbb{E}}^*(S^*,\Lambda^*;V)$ (which equals $E^*(S^*,\Lambda^*;V)$ only up to a sign) in theorem 5.11. By corollary 7.2, the algebra $\overline{\mathbb{E}}^*(S^{*\,(t)},\Lambda^*;V)$ is isomorphic to:

$$\Lambda^*\left(\bigoplus_{k\ge 0}V^{(k+t)}[p^{k+t}-1]\right)\overset{\circ}{\otimes}\Gamma^*\left(\bigoplus_{k\ge 0}V^{(k+t+1)}[p^{k+t+1}-2]\right)\;,$$

where the generators $V^{(k+t)}[p^{k+t}-1]$ have bidegree $(p^{k+t}-1,p^{k+t})$ and the $V^{(k+t)}[p^{k+t}-2]$ have bidegree $(p^{k+t}-2,p^{k+t})$, and where $\overset{\circ}{\otimes}$ refers to the naive tensor product from notation 5.10.

7.2. **The twisting spectral sequence and Troesch complexes.** Now we recover the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(s+t)},G^{*\,(s)};V)$ from the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(t)},Y^*;V)$. In fact, this question is already settled (up to a filtration) in [T3], by means of the 'twisting spectral sequence'. Let us describe briefly how things work.

Let us replace 'G' by $G_{V^{(s)}}$ in the statement of [T3, Thm 7.1]. Then the twisting spectral sequence can be reformulated as follows. For all $F,G\in\mathcal{P}_{\Bbbk}$ there is a spectral sequence of strict polynomial functors, natural in F,G and compatible with tensor products:

$$E_2^{i,j}(F,G,s;V)\Longrightarrow\mathbb{E}^{i+j}(F^{(s)},G^{(s)};V)\;.$$

The i-th column $E_2^{i,*}$ of the second page equals the precomposition of the strict polynomial functor $V\mapsto\mathbb{E}^i(F,G;V)$ by the *graded* strict polynomial functor $V\mapsto E_s\otimes V^{(s)}$ (E_s equals the graded vector space $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(s)},I^{(s)})$, and the partial degree denoted by '$*$' is the degree arising when one precomposes a strict polynomial functor by a graded strict polynomial functor, as explained in [T3, Section 2.5]).

When we take $F=X^{*\,(t)}$ and $G=Y^*$, with X^* and Y^* some classical exponential functors, this spectral sequence yields a spectral sequence of trigraded algebras (the extra degree comes from the fact that classical exponential functors are graded), which converges to $\mathbb{E}^*(X^{*\,(s+t)},G^{*\,(s)};V)$. Moreover, we have proved in [T3, Thm 8.11] (with the help of Troesch complexes) that the twisting spectral sequence collapses in this case. Thus, the second page actually computes, up to a filtration, the strict polynomial algebra $\mathbb{E}^*(X^{*\,(s+t)}, G^{*\,(s)}; V)$.

The only slight difficulty in using the twisting spectral sequence is to understand how the degrees are organized. In the remainder of the section we recall the explicit description of the degrees in the twisting spectral sequence. So we get in proposition 7.8 the very concrete and explicit recipe to determine $\overline{\mathbb{E}}^*(X^{*\,(s+t)}, G^{*\,(s)}; V)$ from $\overline{\mathbb{E}}^*(X^{*\,(t)}, Y^*; V)$. The result holds up to a filtration, but we shall prove in section 7.3 that the filtration involved is trivial.

7.2.1. *Evaluation on graded vector spaces.* Let E_s denote the graded vector space $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I^{(s)}, I^{(s)})$. So by [FS, Thm 4.5] or [T3, Cor 4.7], we know that E_s is a graded vector space concentrated in even degrees $2i$ for $0 \le i < p^s$, and one dimensional in these degrees. In particular E_s has dimension p^s.

Let $\mathcal{P}^*_\Bbbk$ denote the category of graded strict polynomial functors. By [T3, Section 2.5], we can define for all $s \ge 0$ an evaluation functor ev_s (compatible with tensor products)

$$\begin{array}{rccc} \mathrm{ev}_s : & \mathcal{P}_\Bbbk & \to & \mathcal{P}^*_\Bbbk \\ & F & \mapsto & F(E_s \otimes V^{(s)}) \end{array}.$$

By definition the *graded* strict polynomial functor $F(E_s \otimes V^{(s)})$ equals $F(\Bbbk^{p^s} \otimes V^{(s)})$ as an ungraded strict polynomial functor. To define the grading, we let the multiplicative group $\mathbb{G}_m$ act on each homogeneous degree i part of E_s with weight i. Thus $V \mapsto F(\Bbbk^{p^s} \otimes V^{(s)})$ is a strict polynomial functor with $\mathbb{G}_m$-action, and we define the degree of an element of $F(\Bbbk^{p^s} \otimes V^{(s)})$ as its weight under the $\mathbb{G}_m$-action. In particular, $F(E_s \otimes V^{(s)})$ is concentrated in even degrees.

For our purposes, we evaluate bigraded strict polynomial algebras on $E_s \otimes V^{(s)}$. So the resulting objects are trigraded. We decide to place the 'new degree' (i.e. the degree which pops up from the evaluation on a graded functor) in second position. We also decide to work with multigraded algebras whose last partial degree coincides with the strict polynomial degree (e.g. bigraded algebras $A^{*,*}$ such that for all (i,d) $A^{i,d} \in \mathcal{P}_{d,\Bbbk}$). Since $V \mapsto E_s \otimes V^{(s)}$ is a homogeneous strict polynomial functor of strict polynomial degree p^s, evaluation on $E_s \otimes V^{(s)}$ multiplies the last degree by p^s. We gather these conventions in the following statement.

Convention 7.4. We denote by $\mathcal{P}_\Bbbk\mathcal{A}^{*,*}$ (resp. $\mathcal{P}_\Bbbk\mathcal{A}^{*,*,*}$) denote the category of bigraded (resp. trigraded) strict polynomial algebras, whose last degree coincides with the strict polynomial degree. For all $A^{*,*} \in \mathcal{P}_\Bbbk\mathcal{A}^{*,*}$, the tridegree on the strict polynomial algebra $A^{*,*}(E_s \otimes V^{(s)})$ is defined as follows. For all pair of nonnegative integers (i,d), the evaluation of the strict polynomial functor $A^{i,d}$ on $E_s \otimes V^{(s)}$ has tridegrees $(i, *, p^s d)$. This yields a functor

$$\mathrm{ev}_s : \mathcal{P}_\Bbbk\mathcal{A}^{*,*} \to \mathcal{P}_\Bbbk\mathcal{A}^{*,*,*}$$

which commutes with tensor products of algebras (in the naive sense, as in notation 5.10). Moreover, the objects in the image of ev_s are concentrated in even second partial degree.

In practice, the following concrete rule (together with the compatibility with tensor products) will be sufficient to describe the trigrading on $A^{*,*}(E_s\otimes V^{(s)})$ for our cases of interest. Let X^* be a classical exponential functor, and consider the bigraded strict polynomial algebra $A^{*,*}(V) = X^*(V^{(k)}[\ell])$, where the elements of $V^{(k)}[\ell]$ have bidegree (ℓ, p^k). Then

$$A^{*,*}(E_s\otimes V^{(s)}) = X^*\left(E_s^{(k)}\otimes V^{(s+k)}[\ell]\right) = X^*\left(\bigoplus\nolimits_{0\le i<p^s} V^{(s+k)}_{2ip^k}[\ell]\right)$$

is trigraded by considering the $V^{(s+k)}_{2ip^k}[\ell]$ above are copies of $V^{(s+k)}$ with tridegree $(\ell, 2ip^k, p^{s+k})$.

Example 7.5. **(1)** If $A^{*,*}(V) = \Lambda^*(V^{(t)}[p^t-1])$, then

$$A^{*,*}(E_s\otimes V^{(s)}) = \Lambda^*\left(\bigoplus\nolimits_{0\le i<p^s} V^{(s+t)}_{2ip^t}[p^t-1]\right)\ ,$$

where each $V^{(s+t)}_{2ip^t}[p^t-1]$ equals $V^{(s+t)}$ with tridegree $(p^t-1, 2ip^t, p^{s+t})$.
(2) If $A^{*,*}(V) = \Gamma^*(\bigoplus_{k\ge 0} V^{(k+t)}[p^{k+t}-1])$, then

$$A^{*,*}(E_s\otimes V^{(s)}) = \Gamma^*\left(\bigoplus\nolimits_{0\le i<p^s}\bigoplus\nolimits_{k\ge 0} V^{(k+t+s)}_{2ip^{k+t}}[p^{k+t}-1]\right)\ ,$$

where each $V^{(c)}_b[a]$ is a copy of $V^{(c)}$ with tridegree (a,b,p^c).
(3) If $A^{*,*}(V)$ is the algebra from example 7.3(3), then $A_{*,*}(E_s\otimes V^{(s)})$ equals:

$$\Lambda^*\left(\bigoplus_{0\le i<p^s}\bigoplus_{k\ge 0} V^{(k+s+t)}_{2ip^{k+t}}[p^{k+t}-1]\right)\overset{\circ}{\otimes}\Gamma^*\left(\bigoplus_{0\le i<p^s}\bigoplus_{k\ge 0} V^{(k+s+t+1)}_{2ip^{k+t+1}}[p^{k+t+1}-2]\right)$$

where each $V^{(c)}_b[a]$ is a copy of $V^{(c)}$ with tridegree (a,b,p^c).

7.2.2. *The twisting spectral sequence.* Let X^*, Y^* be classical exponential functors. The twisting spectral sequence is a spectral sequence of trigraded strict polynomial algebras,

$$E_2^{i,j}(X^{*\,(t)}, Y^{*p^t}, s; V) \Longrightarrow \mathbb{E}^{i+j}(X^{*\,(s+t)}, Y^{*p^t\,(s)}; V)$$

explicitly described as follows.

(a) The second page equals the evaluation of the bigraded strict polynomial algebra $\mathbb{E}^*(X^{*\,(t)}, Y^*; V)$ on $E_s\otimes V^{(s)}$. The elements of tridegree $(i, *, p^{s+t}d)$ arise from the evaluation of $V\mapsto \mathbb{E}^i(X^{d\,(t)}, Y^{dp^t}; V)$ on $E_s\otimes V^{(s)}$ as specified in convention 7.4.
(b) The spectral sequence converges to the bigraded strict polynomial algebra $\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$. To be more specific, an element with tridegree (i,j,d) in the E_∞ page corresponds to an element of bidegree $(i+j, d)$ of the abutment.

The precise meaning of the convergence of the twisting spectral sequence $E_r^{i,j}(X^{*\,(t)}, Y^*, s; V)$ is the following.

(b1) The bigraded strict polynomial algebra $\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ is filtered (and the product preserves the filtration).
(b2) The associated graded object $\mathrm{Gr}(\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V))$ is isomorphic, as a strict polynomial algebra, to the E_∞ page.

7.2.3. *Collapsing.* We proved in [T3, Thm 8.11] (with the help of Troesch complexes) that the twisting spectral sequence $E_r^{i,j}(X^{*\,(t)}, Y^*, s; V)$ collapses at the second page if with X^* and Y^* are classical exponential functors.

So the bigraded algebra $\mathrm{Gr}(\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V))$ equals the evaluation of the bigraded algebra $\mathbb{E}^*(X^{*\,(t)}, Y^*; V)$ on $E_s \otimes V^{(s)}$. We want an analogous statement for signed algebras. For this, we first check that the operation of taking signed algebras commutes with the operation of taking graded objects, and with the operation of evaluation on $E_s \otimes V^{(s)}$. The first lemma is straightforward.

Lemma 7.6. *Let $A^{*,*}$ be a filtered bigraded strict polynomial algebra. If $A^{*,*}$ is $(1,\epsilon)$-commutative, then $(\mathrm{Gr}A)^{*,*}$ is also $(1,\epsilon)$-commutative and furthermore $(\mathrm{Gr}\overline{A})^{*,*}$ equals $\overline{(\mathrm{Gr}A)}^{*,*}$.*

Lemma 7.7. *Let us denote by 'tot' the partial totalization functor*

$$\mathrm{tot} : \mathcal{P}_{\Bbbk}\mathcal{A}^{*,*,*} \to \mathcal{P}_{\Bbbk}\mathcal{A}^{*,*}\ , \quad A^{*,*,*} \mapsto (\mathrm{tot}A)^{*,*}$$

defined by $(\mathrm{tot}A)^{k,\ell} = \bigoplus_{i+j=k} A^{i,j,\ell}$. If $A^{*,*}$ is $(1,\epsilon)$-commutative, then for all positive integer s, the bigraded strict polynomial algebra $(\mathrm{tot}(\mathrm{ev}_s A))^{*,*}$ is also $(1,\epsilon)$-commutative and furthermore $\overline{(\mathrm{tot}(\mathrm{ev}_s A))}^{*,*}$ equals $(\mathrm{tot}(\mathrm{ev}_s \overline{A}))^{*,*}$.

Proof. In characteristic $p = 2$ there is nothing to prove. So let us assume that the characteristic p is odd. Let $A^{*,*}$ be a bigraded strict polynomial algebra. Then for all i, d the elements of $A^{i,d}(E_s \otimes V^{(s)})$ have degree $(i+*, p^s d)$, with $*$ an even integer, in $(\mathrm{tot}(\mathrm{ev}_s A))^{*,*}$. Hence, the parity of i, resp. d, is the same as the parity of $i + *$, resp. $p^s d$. The result follows. □

We are now ready to prove the main result of section 7.2.

Proposition 7.8. *Let $\Bbbk$ be a field of positive characteristic p, let s, t be nonnegative integers, and let X^*, Y^* be classical exponential functors. By evaluating the bigraded strict polynomial algebras $\mathbb{E}^*(X^{*\,(t)}, Y^*; V)$ and $\overline{\mathbb{E}}^*(X^{*\,(t)}, Y^*; V)$ on the graded strict polynomial functor $E_s \otimes V^{(s)}$, we get trigraded strict polynomial algebras, with tridegree (i, j, d) as in convention 7.4.*

There are filtrations on the bigraded strict polynomial algebras $\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ and $\overline{\mathbb{E}}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ such that we have isomorphisms of strict polynomial algebras:

$$\mathrm{Gr}\left(\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)\right) \simeq \mathbb{E}^*(X^{*\,(t)}, Y^*; E_s \otimes V^{(s)})\ ,$$
$$\mathrm{Gr}\left(\overline{\mathbb{E}}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)\right) \simeq \overline{\mathbb{E}}^*(X^{*\,(t)}, Y^*; E_s \otimes V^{(s)})\ .$$

Elements with tridegree (i, j, d) on the right handside correspond through this isomorphism to elements of bidegree $(i + j, d)$ on the left handside.

Proof. The case of the unsigned algebras $\mathbb{E}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ follows from the discussion in the beginning of section 7.2.3. To get the signed case, we take the associated signed algebra on both sides of the first isomorphism and we apply lemmas 7.6 and 7.7. □

Example 7.9. **(1)** There is, up to a filtration, an isomorphism of bigraded strict polynomial algebras (where the $V^{(s+t)}[2ip^t+p^t-1]$ on the right handside are copies of $V^{(s+t)}$ with bidegree $((2i+1)p^t-1, p^{t+s})$):

$$\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V) \simeq \Lambda^* \left(\bigoplus_{0\le i<p^s} V^{(t+s)}[(2i+1)p^t-1] \right) .$$

Observe that $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$ equals the algebra $\mathbb{E}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$ since everything is concentrated in even Ext-degree if p is odd. As a particular case, if $V=\Bbbk$ then $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\Gamma^{*(s+t)}, \Lambda^{*(s)})$ is an exterior algebra on generators $g_i \in \mathrm{Ext}^{(2i+1)p^t-1}_{\mathcal{P}_\Bbbk}(I^{(s+t)}, \Lambda^{p^t\,(s)})$, $0\le i<p^s$. This result coincides with [FFSS, Thm 5.8(3)].
(2) Assume that $p=2$. There is, up to a filtration, an isomorphism of bigraded strict polynomial algebras (where the $V^{(k+t)}[(2i+1)p^{k+t}-1]$ on the right handside are copies of $V^{(k+t+s)}$ with bidegree $((2i+1)p^{k+t}-1, p^{k+t+s})$):

$$\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Lambda^{*\,(s)}; V) \simeq \Gamma^* \left(\bigoplus_{0\le i<p^s} \bigoplus_{k\ge 0} V^{(k+t+s)}[(2i+1)p^{k+t}-1] \right) .$$

(3) Assume that p is odd. There is, up to a filtration, an isomorphism of bigraded strict polynomial algebras:

$$\begin{aligned}\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Lambda^{*\,(s)}; V) \simeq &\Lambda^* \left(\bigoplus_{0\le i<p^s} \bigoplus_{k\ge 0} V^{(k+s+t)}[(2i+1)p^{k+t}-1] \right) \\ &\overset{\circ}{\otimes} \Gamma^* \left(\bigoplus_{0\le i<p^s} \bigoplus_{k\ge 0} V^{(k+s+t+1)}[(2i+1)p^{k+t+1}-2] \right)\end{aligned}$$

where each $V^{(b)}[a]$ is a copy of $V^{(b)}$ with bidegree (a, p^b), and where $\overset{\circ}{\otimes}$ refers to the naive tensor product from notation 5.10.

Our results hold up to a filtration. But the filtration is actually not a problem: we prove that it must be trivial in section 7.3.

7.3. **Solving filtration problems.** The purpose of this section is to prove that for some families of filtered bigraded strict polynomial algebras with prescribed Gr, the filtration always split. As we will see in section 7.4, all the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ belong to these families, so proposition 7.8 actually computes $\overline{\mathbb{E}}^*(X^{*\,(s+t)}, Y^{*\,(s)}; V)$ (i.e. not up to a filtration). We begin with the case when $\Bbbk$ has odd characteristic.

7.3.1. *Triviality of filtrations for p odd.* Let us begin with general facts about filtered strict polynomial functors.

Lemma 7.10. *Let $\Bbbk$ be a field of positive characteristic p (p even or odd) and let F be a filtered strict polynomial functor.*

(i) *The filtration of F automatically has finite length.*
(ii) *If $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\mathrm{Gr}F, \mathrm{Gr}F) = 0$, there is an isomorphism $F \simeq \mathrm{Gr}F$.*

Proof. To prove (i), observe that a filtration of F is the same as a filtration of the $S(\Bbbk^d, d)$-module $F(\Bbbk^d)$ where d is the strict polynomial degree of F. Hence the filtration is finite for dimension reasons. Let us prove (ii). Let $F_1 \subset F_2 \subset \cdots \subset F_n = F$ be the filtration of F. If $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\mathrm{Gr}F, \mathrm{Gr}F)$ equals zero, then $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F_1, F_2/F_1)$ equals zero. Hence the extension $F_1 \hookrightarrow F_2 \twoheadrightarrow F_2/F_1$ splits, that is $F_2 \simeq F_1 \oplus F_2/F_1$. In this way we build inductively an isomorphism $F \simeq \mathrm{Gr}F$. □

We can find huge families of functors G satisfying $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(G,G)=0$ in odd characteristic.

Lemma 7.11. *Let $\Bbbk$ be a field of odd characteristic. Let m,n be positive integers. If λ and α are m-tuples of nonnegative integers and μ and β are n-tuples of nonnegative integers, we denote by $S^{\lambda\,(\alpha)}\otimes\Lambda^{\mu\,(\beta)}$ the functor:*

$$\left(\bigotimes_{i=1}^m S^{\lambda_i\,(\alpha_i)}\right)\otimes\left(\bigotimes_{i=1}^n \Lambda^{\mu_i\,(\beta_i)}\right)\ .$$

If G is a finite sum of such strict polynomial functors (for various m,n), then $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(G,G)=0$.

Proof. First, by iterated uses of lemma 2.12 (or use [FFSS, Cor 1.8]), the proof reduces to checking that $\mathrm{Ext}^1(X,Y)=0$ if X and Y are of the form $S^{\ell\,(r)}$ or $\Lambda^{m\,(s)}$.

We first check that it is true if the functors are not twisted: $\mathrm{Ext}^1(S^d,S^d)=\mathrm{Ext}^1(\Lambda^d,S^d)=0$ by injectivity of S^d, $\mathrm{Ext}^1(\Lambda^d,\Lambda^d)=0$ by remark 4.6, and $\mathrm{Ext}^1(S^d,\Lambda^d)=0$ by theorem 5.11. By proposition 7.1, Ext^1 also vanish if only one of the two functors X,Y is twisted (indeed, the Ext-degree is shifted by an even integer since p is odd). Finally proposition 7.8 shows the vanishing of $\mathrm{Ext}^1(X,Y)$ in the general case (since the trigraded strict polynomial algebra $\mathbb{E}^*(X^*,Y^*;E_s\otimes V)$ are trivial in odd second partial degree, as explained in convention 7.4). $\square$

We are now ready to prove our first splitting result.

Proposition 7.12 (Splitting result I)**.** *Let $\Bbbk$ be a field of odd characteristic, and let $A^{*,*}$ be a $(1,\epsilon)$-commutative filtered bigraded strict polynomial algebra over $\Bbbk$. Assume that $F^{*,*}$ and $G^{*,*}$ are bigraded additive strict polynomial functors such that the bigraded strict polynomial algebra $(\mathrm{Gr}\overline{A})^{*,*}$ equals $S^*(F^{*,*})\overset{\circ}{\otimes}\Lambda^*(G^{*,*})$. Then $A^{*,*}\simeq(\mathrm{Gr}A)^{*,*}$ and $\overline{A}^{*,*}\simeq(\mathrm{Gr}\overline{A})^{*,*}$ as bigraded strict polynomial algebras.*

Proof. If we have an isomorphism $\overline{A}^{*,*}\simeq(\mathrm{Gr}\overline{A})^{*,*}$, then the same isomorphism is compatible with unsigned products. Thus, it suffices to build an isomorphism $\overline{A}^{*,*}\simeq(\mathrm{Gr}\overline{A})^{*,*}$.

Step 1: splitting without products. By lemma 7.11, for all integers k,ℓ, $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}((\mathrm{Gr}\overline{A})^{k,\ell},(\mathrm{Gr}\overline{A})^{k,\ell})$ equals zero. So lemma 7.10 yields an isomorphism of bigraded strict polynomial functors $(\mathrm{Gr}\overline{A})^{*,*}\simeq\overline{A}^{*,*}$. In particular, we have a bigraded injection $\phi^{*,*}:F^{*,*}\oplus G^{*,*}\hookrightarrow\overline{A}^{*,*}$, compatible with filtrations (take the trivial filtration on $F^{*,*}\oplus G^{*,*}$), which equals the injection $F^{*,*}\oplus G^{*,*}\hookrightarrow(\mathrm{Gr}\overline{A})^{*,*}$ after taking graded objects.

Step 2: universal property. We know that $A^{*,*}$ is $(1,\epsilon)$-commutative, so by lemma 3.3, $\mathrm{Tot}^{2+\epsilon}\overline{A}^{*,*}$ is graded commutative. Since $\mathrm{Tot}^{2+\epsilon}(\mathrm{Gr}\overline{A})^{*,*}$ is the free graded commutative algebra on $\mathrm{Tot}^{2+\epsilon}(F^{*,*}\oplus G^{*,*})$, products in $A^{*,*}$ define a morphism of algebras $\psi:\mathrm{Tot}^{2+\epsilon}(\mathrm{Gr}\overline{A})^{*,*}\to\mathrm{Tot}^{2+\epsilon}\overline{A}^{*,*}$ whose restriction to $\mathrm{Tot}^{2+\epsilon}(F^{*,*}\oplus G^{*,*})$ coincides with $\mathrm{Tot}^{2+\epsilon}(\phi^{*,*})$.

Conclusion. Now, $\phi^{*,*}$ preserves the bigrading, and so do products in $\overline{A}^{*,*}$. So ψ actually preserves the bigrading. Thus, $\psi:(\mathrm{Gr}\overline{A})^{*,*}\to\overline{A}^{*,*}$ is a morphism of strict polynomial functors, compatible with filtrations (take the trivial filtration on $(\mathrm{Gr}\overline{A})^{*,*}$), and $\mathrm{Gr}\psi$ is the identity map. So ψ is an isomorphism. $\square$

Now we turn to the case of filtered algebras with divided power algebras as graded objects. To get back to the situation of proposition 7.12, we shall use the following lemma.

Lemma 7.13. *The following statements are equivalent.*

(i) $A^{*,*}$ *is a filtered bigraded strict polynomial algebra, such that* $(\mathrm{Gr}A)^{*,*}$ *is an exponential functor.*
(ii) $A^{*,*}$ *is a filtered bigraded strict polynomial exponential functor, such that* $(\mathrm{Gr}A)^{*,*}$ *is an exponential functor.*
(iii) $A^{*,*}$ *is a filtered bigraded strict polynomial coalgebra, such that* $(\mathrm{Gr}A)^{*,*}$ *is an exponential functor.*

Proof. It is trivial that (ii)⇒(i), and (ii)⇒(iii). Let us prove (i)⇒(ii) (the proof for (iii)⇒(ii) is similar). We know that $(\mathrm{Gr}A)^{*,*}$ is an exponential functor. That is, if we consider the $A^{*,*}(V)\otimes A^{*,*}(W)$ as a bigraded object (with total bidegree), with the tensor product filtration

$$F^i\big(A^{*,*}(V)\otimes A^{*,*}(W)\big)=\textstyle\sum_{k+\ell=i}F^kA^{*,*}(V)\otimes F_\ell A^{*,*}(W)\;,$$

then the multiplication induces a bigraded map $A^{*,*}(V)\otimes A^{*,*}(W)\xrightarrow{\mathrm{mult}^{*,*}} A^{*,*}(V\oplus W)$ compatible with the filtrations, and whose associated graded map $\mathrm{Gr}(\mathrm{mult}^{k,\ell})$ is an isomorphism in each bidegree (k,ℓ). Now since the filtrations have finite length in each bidegree (k,ℓ), by iterated uses of the five lemma, the map $\mathrm{mult}^{k,\ell}$ is also an isomorphism. Whence (ii). □

We are now ready to prove our second splitting result.

Proposition 7.14 (Splitting result II)**.** *Let $\Bbbk$ be a field of odd characteristic, and let $A^{*,*}$ be a $(1,\epsilon)$-commutative filtered bigraded strict polynomial algebra over $\Bbbk$. Assume that $F^{*,*}$ and $G^{*,*}$ are bigraded additive strict polynomial functors such that the bigraded strict polynomial algebra $(\mathrm{Gr}\overline{A})^{*,*}$ equals $\Gamma^*(F^{*,*})\overset{\circ}{\otimes}\Lambda^*(G^{*,*})$. Then $A^{*,*}\simeq(\mathrm{Gr}A)^{*,*}$ and $\overline{A}^{*,*}\simeq(\mathrm{Gr}\overline{A})^{*,*}$ as bigraded strict polynomial algebras.*

Proof. First, by lemma 7.13, $A^{*,*}$ is a $(1,\epsilon)$-commutative filtered bigraded strict polynomial exponential functor, and $(\mathrm{Gr}\overline{A})^{*,*}$ equals $\Gamma^*(F^{*,*})\overset{\circ}{\otimes}\Lambda^*(G^{*,*})$ as a bigraded exponential functor. So by lemma 2.9, it is sufficient to find a coalgebra isomorphism $(\mathrm{Gr}A)^{*,*}\simeq A^{*,*}$.

Step 1: splitting without coproducts. For all k,ℓ, the extension group $\mathrm{Ext}^1((\mathrm{Gr}A)^{k,\ell},(\mathrm{Gr}A)^{k,\ell})$ equals $\mathrm{Ext}^1((\mathrm{Gr}A)^{k,\ell\,\sharp},(\mathrm{Gr}A)^{k,\ell\,\sharp})$, which equals zero by lemma 7.11. So lemma 7.10 yields an isomorphism of bigraded strict polynomial functors $\overline{A}^{*,*}\simeq(\mathrm{Gr}\overline{A})^{*,*}$.

Step 2: duality. Thus, $A^{*,*}$ is a $(1,\epsilon)$-commutative bigraded coalgebra with trivial filtration. That is, each $A^{k,\ell}$ splits as a direct sum of subfunctors $G^mA^{k,\ell}$ and the filtration $\cdots\subset F^iA^{k,\ell}\subset F^{i-1}A^{k,\ell}\subset\dots$ of $A^{k,\ell}$ is defined by $F^iA^{k,\ell}=\bigoplus_{m\ge i}G^mA^{k,\ell}$. And moreover the comultiplication respects the filtration, i.e. for all integer i, it takes $\bigoplus_{m\ge i}G^mA^{k,\ell}$ to

$$\bigoplus_{m+n\ge i\,,\,k_1+k_2=k\,,\,\ell_1+\ell_2=\ell}G^mA^{k_1,\ell_1}\otimes G^nA^{k_2,\ell_2}\;.$$

If we apply duality to this situation, we obtain that $A^{*,*\sharp}$ is a filtered bigraded strict polynomial algebra (with trivial dual filtration), and

$$\mathrm{Gr}(\overline{A}^{*,*\sharp}) = (\mathrm{Gr}\overline{A})^{*,*,\sharp} = S^*(F^{*,*}) \overset{\circ}{\otimes} \Lambda^*(G^{*,*})$$

as bigraded strict polynomial algebras. So by proposition 7.12, we get an isomorphism of bigraded strict polynomial algebras $(\mathrm{Gr}A)^{*,*,\sharp} = \mathrm{Gr}(A^{*,*\sharp}) \simeq A^{*,*\sharp}$. Using once again duality, we recover an isomorphism of bigraded strict polynomial coalgebras $(\mathrm{Gr}A)^{*,*} \simeq A^{*,*}$ and we are done. □

7.3.2. *Triviality of filtrations in characteristic* 2. The following proposition is proved exactly in the same fashion as proposition 7.12 and 7.14. The proof is actually even simpler, since there are no signs to handle.

Proposition 7.15 (Splitting result III)**.** *Let* $\Bbbk$ *be a field of characteristic* 2*, and let* $A^{*,*}$ *be a commutative filtered bigraded strict polynomial algebra over* $\Bbbk$*. Assume that* $F^{*,*}$ *is a bigraded additive strict polynomial functor, and*

$$(\mathrm{Gr}A)^{*,*} = S^*(F_{*,*}) \quad or \quad (\mathrm{Gr}A)^{*,*} = \Gamma^*(F^{*,*}) \;.$$

Then $A^{*,*} \simeq (\mathrm{Gr}A)^{*,*}$ *as bigraded strict polynomial algebras.*

Proposition 7.15 is sufficient to prove that the filtrations on the abutment of the twisting spectral sequence split in characteristic 2, except for the cases of $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, S^{*(s)}; V)$ and $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$. In these cases, the algebras are, up to a filtration, exterior algebras of the form

$$\Lambda^*\left(\bigoplus_{0\le i<p^s} V^{(s+t)}[d_i]\right)$$

where the $V^{(s+t)}[d_i]$ are some copies of $V^{(s+t)}$ placed in some degree d_i (the computation is made in example 7.9(1) for the case of $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$, the case of $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, S^{*(s)}; V)$ is similar).

So we need a analogue of proposition 7.15 when $(\mathrm{Gr}A)^{*,*} = \Lambda^*(F^{*,*})$. Two difficulties arise when we want to adapt the proof of proposition 7.12 to this case. First, there might be non-trivial extensions between certain twisted exterior powers in characteristic 2 (e.g. the extension $\Lambda^2 \hookrightarrow \Gamma^2 \twoheadrightarrow \Lambda^{1\,(1)}$). Second, exterior algebras are universal algebras for strictly anticommutative algebras (i.e. algebras in which squares are trivial). So the best we can easily prove is the following statement.

Proposition 7.16 (Splitting result IV)**.** *Let* $\Bbbk$ *be a field of characteristic* 2*, and let* $A^{*,*}$ *be a strictly anticommutative filtered bigraded strict polynomial algebra over* $\Bbbk$*. Let* r *be an integer, and let* $F^{*,*}$ *be a bigraded strict polynomial functor, such that all* $F^{k,\ell}$ *are finite direct sums of copies of* $I^{(r)}$*. If the bigraded strict polynomial algebra* $(\mathrm{Gr}A)^{*,*}$ *equals* $\Lambda^*(F^{*,*})$*, then* $A^{*,*} \simeq (\mathrm{Gr}A)^{*,*}$ *as bigraded strict polynomial algebras.*

Proof. The proof that $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}((\mathrm{Gr}A)^{k,\ell}, (\mathrm{Gr}A)^{k,\ell})$ equals zero reduces, by iterated uses of lemma 2.12, to checking that for all $d \ge 0$, $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Lambda^{d\,(r)}, \Lambda^{d\,(r)})$. But the latter fact follows from proposition 7.8 and the vanishing of $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(\Lambda^d, \Lambda^d)$. So lemma 7.10 yields an isomorphism of bigraded strict polynomial functors $A^{*,*} \simeq (\mathrm{Gr}A)^{*,*}$. To get an isomorphism of algebras, we use the universal property of exterior algebras, exactly as in the second step of the proof of proposition 7.12. □

In order to use proposition 7.16, we need to show that $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, S^{*(s)}; V)$ and $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$ are strictly anticommutative.

Lemma 7.17. *Let $\Bbbk$ be a field of characteristic 2. For all nonnegative integers s, t, the bigraded strict polynomial algebras*

$$\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, S^{*(s)}; V) \quad \textit{and} \quad \overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$$

are strictly anticommutative (that is, for all x, $x \cdot x = 0$ in these algebras).

Proof. Let us prove that $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$ is strictly anticommutative. Since $\Bbbk$ has characteristic 2, we have an injective morphism of algebras $\alpha : \Lambda^* \hookrightarrow \Gamma^*$. It induces a morphism of algebras:

$$\beta : \overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V) \to \overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, \Lambda^{*(s)}; V)\ .$$

To prove that $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Lambda^{*(s)}; V)$ is strictly anticommutative, it suffices to prove that β is injective and that $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, \Lambda^{*(s)}; V)$ is strictly anticommutative.

Let us prove that β is injective. For all $d \geq 0$, we have a commutative square, where the vertical arrows are injections induced by the canonical map $\Lambda^d \hookrightarrow \otimes^d$:

$$\begin{array}{ccc}
\mathbb{H}(\Gamma^d, \otimes^d; V) & \xrightarrow{\mathbb{H}(\alpha, \otimes^d; V)} & \mathbb{H}(\Lambda^d, \otimes^d; V)\ . \\
\uparrow & & \uparrow \\
\mathbb{H}(\Gamma^d, \Lambda^d; V) & \xrightarrow{\mathbb{H}(\alpha, \Lambda^d; V)} & \mathbb{H}(\Lambda^d, \Lambda^d; V)
\end{array}$$

Now the map $\mathbb{H}(\alpha, \otimes^d; V)$ is an isomorphism (by lemma 2.10), so $\mathbb{H}(\alpha, \Lambda^d; V)$ is injective. Using proposition 7.1, we get an injection $\mathbb{H}(\Gamma^{d\,(t)}, \Lambda^{dp^t}; V) \hookrightarrow \mathbb{H}(\Lambda^{d\,(t)}, \Lambda^{dp^t}; V)$. By proposition 7.8, $\mathrm{Gr}(\beta)$ equals the evaluation of this map on $E_s \otimes V^{(s)}$, hence it is injective. We conclude the injectivity of β.

Now $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, \Lambda^{*(s)}; V)$ is strictly anticommutative since it is a divided power algebra. Indeed, $\overline{\mathbb{E}}^*(\Lambda^*, \Lambda^*; V)$ equals $\mathbb{H}(\Lambda^*, \Lambda^*; V)$ and the latter equals $\Gamma^*(V)$. We apply propositions 7.1, 7.8 and 7.15 to conclude that $\overline{\mathbb{E}}^*(\Gamma^{*(t+s)}, \Gamma^{*(s)}; V)$ is isomorphic to $\Gamma^*(\bigoplus_{0 \leq i < p^s} V^{(t+s)}[(2i+2)p^t - 2])$. Whence the result. The proof for $\overline{\mathbb{E}}^*(\Lambda^{*(t+s)}, S^{*(s)}; V)$ is similar. $\square$

7.4. **Final results.** In this section, we state the computations of the bigraded strict polynomial algebras $\overline{\mathbb{E}}^*(X^{*\,(t+s)}, Y^{*\,(s)}; V)$. Let us recall the conventions used.

Convention 7.18. In the statements of theorems 7.19-7.23, we adopt the following conventions.

(1) On the left handside, the algebras $\overline{\mathbb{E}}^*(X^{*\,(t+s)}, Y^{*\,(s)}; V)$ denote the bigraded strict polynomial functors

$$\textstyle\bigoplus_{h,d \geq 0} \overline{\mathbb{E}}^h(X^{d\,(t+s)}, Y^{dp^t\,(s)}; V)$$

with $\overline{\mathbb{E}}^h(X^{d\,(t+s)}, Y^{d\,(s)}; V) = \mathbb{E}^h(X^{d\,(t+s)}, Y^{d\,(s)}; V)$ placed in bidegree $(h, p^{s+t}d)$, and equipped with the signed product (as in convention 3.6).

(2) On the right handside, the generators $V^{(d+s+t)}[h]$ denote copies of $V^{(d+s+t)}$ with bidegree (h, p^{d+s+t}). Thus, $V^{(d+s+t)}[h]$ is a subfunctor of $\mathbb{E}^h(X^{d\,(t+s)}, Y^{d\,(s)}; V)$.

Let us order the classical exponential functors from the 'more projective' to the 'more injective' one $\Gamma^* < \Lambda^* < S^*$. In theorems 7.19 and 7.20, we get the computations for pairs (X^*, Y^*) with $X^* \le Y^*$. Observe that in these cases $\overline{\mathbb{E}}^*(X^{*\,(t+s)}, Y^{*\,(t)}; V)$ is concentrated in even Ext-degree when p is odd, so this algebra actually equals the unsigned algebra $\mathbb{E}^*(X^{*\,(t+s)}, Y^{*\,(t)}; V)$. These results were computed first in [FFSS, Thm 5.8] (for $V = \Bbbk$) and our results agree with this theorem. Finally, we mention that theorem 7.19 is a particular case of [T3, Cor. 5.8] which computes $\mathbb{E}^*(C^{*\,(s+t)}, S^{*\,(t)}; V)$ for all graded strict polynomial coalgebra C^*.

Theorem 7.19 (Pairs (X^*, S^*)). *Let $\Bbbk$ be a field of positive characteristic p, and let s, t be positive integers. For all classical exponential functor X^*, there is an isomorphism of bigraded strict polynomial algebras:*

$$\overline{\mathbb{E}}^*(X^{*\,(t+s)}, S^{*\,(s)}; V) \simeq X^{*\sharp}\left(\bigoplus_{0\le i<p^s} V^{(t+s)}[2ip^t]\right).$$

Theorem 7.20 (Pairs (X^*, Y^*) with $X^* \le Y^* < S^*$). *Let $\Bbbk$ be a field of positive characteristic p, and let s, t be positive integers. There are isomorphisms of bigraded strict polynomial algebras:*

$$\overline{\mathbb{E}}^*(\Gamma^{*\,(t+s)}, \Lambda^{*\,(s)}; V) \simeq \Lambda^*\left(\bigoplus_{0\le i<p^s} V^{(t+s)}[(2i+1)p^t - 1]\right),$$

$$\overline{\mathbb{E}}^*(\Lambda^{*\,(t+s)}, \Lambda^{*\,(s)}; V) \simeq \Gamma^*\left(\bigoplus_{0\le i<p^s} V^{(t+s)}[(2i+1)p^t - 1]\right),$$

$$\overline{\mathbb{E}}^*(\Gamma^{*\,(t+s)}, \Gamma^{*\,(s)}; V) \simeq \Gamma^*\left(\bigoplus_{0\le i<p^s} V^{(t+s)}[(2i+2)p^t - 2]\right).$$

Now we turn to the pairs (X^*, Y^*), with $X^* > Y^*$. These pairs were not computed in [FFSS], where the authors suspected that there are 'no easy answer' for such pairs. Our approach somehow explains why Ext-groups for these pairs are much more difficult to compute. Indeed, for all pairs (X^*, Y^*), the extension groups $\overline{\mathbb{E}}^*(X^{*\,(t+s)}, Y^{*\,(t)}; V)$ can be deduced from $\overline{\mathbb{E}}^*(X^*, Y^*; V)$. The latter are very easy to compute if $X^* \le Y^*$ (they reduce to Hom-groups), but far from being trivial if $X^* > Y^*$ since they amount to computing the homology of some Eilenberg Mac Lane spaces.

In theorems 7.21 and 7.23, the signed algebras $\overline{\mathbb{E}}^*(X^{*\,(t+s)}, Y^{*\,(t)}; V)$ are actually equal to the unsigned algebras $\mathbb{E}^*(X^{*\,(t+s)}, Y^{*\,(t)}; V)$. But it is not the case in theorem 7.22. Our results are quite different from the results computed in [C2], but as already observed in section 5.3, one can find counterexamples to the latter.

Theorem 7.21 (Pairs (X^*, Y^*) with $X^* > Y^*$ for $p = 2$)**.** *Let $\Bbbk$ be a field of characteristic $p = 2$, and let s, t be positive integers. There are isomorphisms of bigraded strict polynomial algebras:*

$$\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Lambda^{*\,(s)}; V) \simeq \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+t+s)}[(2i+1)p^{k+t}-1]\right)\ ,$$

$$\overline{\mathbb{E}}^*(\Lambda^{*\,(t+s)}, \Gamma^{*\,(s)}; V) \simeq \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+t+s)}[(2i+2)p^{k+t}-p^k-1]\right)\ ,$$

$$\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Gamma^{*\,(s)}; V)$$
$$\simeq \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k\,,\,0\le \ell} V^{(k+\ell+t+s)}[(2i+2)p^{k+\ell+t}-p^k-1]\right)\ .$$

Recall from notation 5.10 that $\overset{\circ}{\otimes}$ denotes the naive tensor product of bigraded algebras, that is $A^{*,*} \overset{\circ}{\otimes} B^{*,*}$ equals $A^{*,*} \otimes B^{*,*}$ as a bigraded object, and the product of $a \otimes b$ and $a' \otimes b'$ equals $aa' \otimes bb'$ with no Koszul sign.

Theorem 7.22. *Let $\Bbbk$ be a field of odd characteristic p, and let s, t be positive integers. The bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Lambda^{*\,(s)}; V)$ is isomorphic to the naive tensor product:*

$$\Lambda^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+t+s)}[(2i+1)p^{k+t}-1]\right)$$
$$\overset{\circ}{\otimes}\, \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+1+t+s)}[(2i+1)p^{k+1+t}-2]\right)\ .$$

The bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(\Lambda^{*\,(t+s)}, \Gamma^{*\,(s)}; V)$ is isomorphic to the naive tensor product

$$\Lambda^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+t+s)}[(2i+2)p^{k+t}-p^k-1]\right)$$
$$\overset{\circ}{\otimes}\, \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+1+t+s)}[(2i+2)p^{k+1+t}-p^k-2]\right)\ .$$

Theorem 7.23. *Let $\Bbbk$ be a field of odd characteristic p, and let s, t be positive integers. The bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(S^{*\,(t+s)}, \Gamma^{*\,(s)}; V)$*

is isomorphic to the naive tensor product:

$$\Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k} V^{(k+t+s)}[(2i+2)p^{k+t}-2]\right)$$

$$\overset{\circ}{\otimes}\ \Lambda^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k\,,\,0\le \ell} V^{(k+\ell+1+t+s)}[(2i+2)p^{k+\ell+1+t}-2p^k-1]\right)$$

$$\overset{\circ}{\otimes}\ \Gamma^*\left(\bigoplus_{0\le i<p^s\,,\,0\le k\,,\,0\le \ell} V^{(k+\ell+2+t+s)}[(2i+2)p^{k+\ell+2+t}-2p^{k+1}-2]\right)$$

proof of theorems 7.19-7.23. **Step 1.** We first compute the bigraded strict polynomial algebra $\overline{\mathbb{E}}^*(X^*,Y^*;V)$. There are two cases. If $X^*\le Y^*$, then $\overline{\mathbb{E}}^*(X^*,Y^*;V)$ reduces to $\mathbb{H}(X^*,Y^*;V)$ and is very easy to compute (see section 1.3). If $X^*>Y^*$, then $\overline{\mathbb{E}}^*(X^*,Y^*;V)$ is rather complicated, and computed in theorems 5.11 and 5.14. In all cases, $\overline{\mathbb{E}}^*(X^*,Y^*;V)$ is a symmetric, an exterior or a divided power algebra (or a tensor product of these) on some generators $V^{(b)}[a]$ which are copies of the functor $V^{(b)}$ placed in bidegree (a,p^b).

Step 2. By proposition 7.1, $\overline{\mathbb{E}}^*(X^{*\,(t)},Y^*;V)$ is isomorphic, up to a regrading, to $\overline{\mathbb{E}}^*(X^*,Y^*;V^{(t)})$. To be more specific, each generator $V^{(b)}[a]$ of $\overline{\mathbb{E}}^*(X^*,Y^*;V)$ corresponds to a generator $V^{(b+t)}[a+(p^t-1)p^b\alpha(Y^*)]$ of $\overline{\mathbb{E}}^*(X^{*\,(t)},Y^*;V)$, where $\alpha(S*)=0$, $\alpha(\Lambda^*)=1$ and $\alpha(\Gamma^*)=2$.

Step 3. By proposition 7.8, $\overline{\mathbb{E}}^*(X^{*\,(t+s)},Y^{*\,(s)};V)$ is, up to a filtration, an algebra of the same kind as $\overline{\mathbb{E}}^*(X^{*\,(t)},Y^*;V)$, but with more generators. To be more specific, each generator $V^{(b+t)}[a+(p^t-1)p^b\alpha(Y^*)]$ of $\overline{\mathbb{E}}^*(X^{*\,(t)},Y^*;V)$ gives birth to a family of generators (indexed by an integer i, with $0\le i<p^s$):

$$V^{(b+t+s)}[a+(p^t-1)p^b\alpha(Y^*)+2ip^{b+t}]=V^{(b+t+s)}[(2i+\alpha(Y^*))p^{b+t}-\alpha(Y^*)p^b+a].$$

Step 4. Finally, all the filtrations involved are trivial by section 7.3. Whence the results. □

8. Appendix: the basic theory of strict polynomial functors

In this appendix, we recall (without proofs) the main features of strict polynomial functors. The basic references are [FS, Sections 2 and 3] or [F, P] for strict polynomial functors over fields, and [SFB, Section 3] for the variants. One can also consult [T2, Section 2].

8.1. **Strict polynomial functors over a field.** Let $\Bbbk$ be a field. We denote by $\mathcal{V}_\Bbbk$ the category of finite dimensional $\Bbbk$-vector spaces. For $X,Y\in\mathcal{V}_\Bbbk$ we denote by $\mathrm{Hom}_{\mathrm{Pol}}(X,Y)$ the polynomials with source X and target Y, that is $\mathrm{Hom}_{\mathrm{Pol}}(X,Y)=S^*(X^\vee)\otimes Y$, where '$\vee$' stands for $\Bbbk$-linear duality.

8.1.1. *Definition.* We let $\mathcal{P}_\Bbbk$ be the abelian category of strict polynomial functors over field $\Bbbk$.

The objects of $\mathcal{P}_{\Bbbk}$ are functors $F : \mathcal{V}_{\Bbbk} \to \mathcal{V}_{\Bbbk}$ equipped with an extra 'strict polynomial structure'. Such a strict polynomial structure is a collection of polynomials $F_{V,W} \in \mathrm{Hom}_{\mathrm{Pol}}(\mathrm{Hom}_{\Bbbk}(V,W), \mathrm{Hom}_{\Bbbk}(F(V), F(W))$ for all $V, W \in \mathcal{V}_{\Bbbk}$, which satisfy the following conditions.

(i) For all $\Bbbk$-linear map $f : V \to W$ the $\Bbbk$-linear map $F(f) : F(V) \to F(W)$ equals $F_{V,W}(f)$.
(ii) For all $U, V, W \in \mathcal{V}_{\Bbbk}$, the *polynomials* $F_{V,W}(g) \circ F_{U,V}(f)$ and $F_{U,W}(g \circ f)$ are equal.
(iii) The set of integers $\{\deg(F_{V,W})\ ,\ V, W \in \mathcal{V}_{\Bbbk}\}$ is bounded.

The integer $\sup_{V,W \in \mathcal{V}_{\Bbbk}}\{\deg F_{V,W}\}$ is the strict polynomial degree (or simply the degree when no confusion is possible) of F. If all the polynomial $F_{V,W}$ are homogeneous of degree d, then F is said to be homogeneous of degree d.

The morphisms of $\mathcal{P}_{\Bbbk}$ are the natural transformations $\theta : F \to G$, satisfying the following condition.

(iv) For all $V, W \in \mathcal{V}_{\Bbbk}$ the polynomials $\theta_W \circ F_{V,W}(f)$ and $G_{V,W}(f) \circ \theta_W$ are equal (both are polynomials in the variable $f \in \mathrm{Hom}_{\Bbbk}(V,W)$, with values in $\mathrm{Hom}_{\Bbbk}(F(V), G(W))$).

8.1.2. *Examples.* The tensor powers $\otimes^d : V \mapsto V^{\otimes d}$ are homogeneous strict polynomial functors of degree d. The strict polynomial structure can be explicitly described as follows. Let (e_i) be a basis of $\mathrm{Hom}_{\Bbbk}(V,W)$ and let $(e_i^\vee)$ be the dual basis. The polynomial $\otimes^d_{V,W} \in S^d(\mathrm{Hom}_{\Bbbk}(V,W)^\vee) \otimes \mathrm{Hom}_{\Bbbk}(V,W)^{\otimes d}$ equals:

$$\sum_{i_1 \le \cdots \le i_d} e_{i_1}^\vee \cdots e_{i_d}^\vee \otimes \left(\sum_{\sigma \in \mathfrak{S}(i_1,\dots,i_d)} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(d)}} \right) ,$$

where $\mathfrak{S}(i_1,\dots,i_d)$ is the subset of the symmetric group $\mathfrak{S}_d$ formed by the permutations σ satisfying $\sigma(k) < \sigma(\ell)$ for all $1 \le k < \ell \le d$ such that $i_k = i_\ell$ (that is, the parenthesis contains one copy of each elementary tensor which can be obtained from $e_{i_1} \otimes \cdots \otimes e_{i_d}$ by changing the order of the e_{i_k}). An elementary check shows that this definition of $\otimes^d_{V,W}$ does not depend on the choice of the basis (e_i), and that conditions (i), (ii) and (iii) are satisfied.

One can also compute that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\otimes^d, \otimes^d)$ is the free $\Bbbk$-module with basis the natural transformations induced by the permutations $\sigma \in \mathfrak{S}_d$

$$\begin{array}{ccc} V^{\otimes d} & \to & V^{\otimes d} \\ \otimes_{i=1}^d v_i & \mapsto & \otimes_{i=1}^d v_{\sigma^{-1}(i)} \end{array} .$$

The symmetric powers S^d, the exterior powers Λ^d or the divided powers $\Gamma^d(V) = (V^{\otimes d})^{\mathfrak{S}_d} = (S^d(V^\vee))^\vee$ are also homogeneous strict polynomial functors of degree d. And if $X = S, \Lambda$ or Γ, the multiplications $X^k \otimes X^\ell \to X^{k+\ell}$ and the comultiplications $X^{k+\ell} \to X^k \otimes X^\ell$ are morphisms in $\mathcal{P}_{\Bbbk}$. If $\Bbbk$ has prime characteristic $p > 0$, another example is the Frobenius twist $I^{(r)}$ which sends V to the subspace of $S^{p^r}(V)$ generated by the elements of the form v^{p^r}. Also, tensor products of strict polynomial functors $F \otimes G : V \mapsto F(V) \otimes G(V)$, and compositions $F \circ G : V \mapsto F(G(V))$ are once again strict polynomial functors. (the degree is additive with respect to tensor products and multiplicative with respect to compositions of homogeneous strict polynomial functors).

8.1.3. *Structure of the category* $\mathcal{P}_{\Bbbk}$. The main structural results about $\mathcal{P}_{\Bbbk}$ are the following:

(1) The abelian category $\mathcal{P}_{\Bbbk}$ has enough projectives; a projective generator is the family of tensor products of divided powers. It also has enough injectives ; an injective cogenerator is the family of tensor products of symmetric powers. In particular $\mathcal{P}_{\Bbbk}$ is a nice framework for homological algebra and computing extensions groups.
(2) The Kuhn dual $F^\sharp : V \mapsto F(V^\vee)^\vee$ of a strict polynomial functor ('$^\vee$' denotes $\Bbbk$-linear duality) is a strict polynomial functor of the same degree, and it induces an isomorphism, natural in F, G: $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(G^\sharp, F^\sharp)$. We have $\Lambda^{d\,\sharp} = \Lambda^d$, $I^{(r)\,\sharp} = I^{(r)}$ and $S^{d\,\sharp} = \Gamma^d$.
(3) The abelian category $\mathcal{P}_{\Bbbk}$ splits as the direct sum of its full abelian subcategories $\mathcal{P}_{\Bbbk,d}$ of homogeneous strict polynomial functors of degree d. In practice, it means that:
   - Each functor F splits as a finite direct sum of homogeneous functors.
   - If F, G are homogeneous functors, $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) = 0$ if F and G have different degrees, and $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) = \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk,d}}(F,G)$ if F, G are homogeneous of degree d.
(4) For $d \ge 1$, let $\Gamma^d\mathcal{V}_{\Bbbk}$ be the category with objects the finite dimensional $\Bbbk$-vector spaces and with morphism $V \to W$ the $\mathfrak{S}_d$-equivariant $\Bbbk$-linear maps $V^{\otimes d} \to W^{\otimes d}$:
$$\mathrm{Hom}_{\Gamma^d\mathcal{V}_{\Bbbk}}(V,W) = \mathrm{Hom}_{\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}) = \Gamma^d(\mathrm{Hom}_{\Bbbk}(V,W)) \;.$$
Then the category $\mathcal{P}_{\Bbbk,d}$ is isomorphic to the category of $\Bbbk$-linear functors $\Gamma^d\mathcal{V}_{\Bbbk} \to \mathcal{V}_{\Bbbk}$. So, the Yoneda lemma yields an isomorphism
$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk,d}}(\Gamma^d(\mathrm{Hom}_{\Bbbk}(V,-)), F(-)) \simeq F(V) \;.$$
(5) Evaluating a strict polynomial functor F on $V \in \mathcal{V}_{\Bbbk}$ yields a functor from $\mathcal{P}_{\Bbbk,d}$ to the category of finite dimensional modules over the Schur algebra $S(V,d) = \mathrm{End}_{\Gamma^d\mathcal{V}_{\Bbbk}}(V) = \mathrm{End}_{\mathfrak{S}_d}(V^{\otimes d})$. One proves that if $\dim V \ge d$, the evaluation functor induces an equivalence of categories $\mathcal{P}_{\Bbbk,d} \simeq S(V,d)$-mod. Now $S(V,d)$-modules are rational $GL(V)$-modules, so one has an evaluation morphism:
$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) \to \mathrm{Ext}^*_{\mathrm{rat}-GL(V)}(F(V), G(V)) \;,$$
which is an isomorphism if $\dim V \ge \deg F\,, \deg G$.

8.1.4. *Strict polynomial vs ordinary functors.* Let $\mathcal{F}ct(\mathcal{V}_{\Bbbk}, \mathcal{V}_{\Bbbk})$ denote the category of endofunctors of $\mathcal{V}_{\Bbbk}$. Then a strict polynomial functor is an element of $\mathcal{F}ct(\mathcal{V}_{\Bbbk}, \mathcal{V}_{\Bbbk})$ together with a strict polynomial structure. Thus, forgetting the strict polynomial structure yields a forgetful functor
$$\mathcal{U} : \mathcal{P}_{\Bbbk} \to \mathcal{F}ct(\mathcal{V}_{\Bbbk}, \mathcal{V}_{\Bbbk}) \;.$$
Many subtle properties of the forgetful functor are proved in [FFSS, Section 2 and 3], we only recall very basic properties here. First, $\mathcal{U}$ is faithful (indeed,

a morphism of strict polynomial functors is a natural transformation of functors satisfying an extra condition). Moreover, $\mathcal{U}$ reflects isomorphisms, that is $f : F \to G$ is an isomorphism if and only if $\mathcal{U}f$ is an isomorphism. Indeed we can assume that F, G are homogeneous of degree d. If $\mathcal{U}f$ is an isomorphism, $f_V : F(V) \to G(V)$, for $\dim V \geq d$ is a $\Bbbk$-linear isomorphism. Thus, f_V is an isomorphism of $S(V, d)$-modules, which is equivalent to say that $f : F \to G$ is an isomorphism.

Assume that $\Bbbk$ is an infinite field. So, evaluation induces an *injective* map $\mathrm{Hom}_{\mathrm{Pol}}(X, Y) \hookrightarrow \mathrm{Map}(X, Y)$. Thus, for a given functor $F \in \mathcal{F}ct(\mathcal{V}_\Bbbk, \mathcal{V}_\Bbbk)$, there can be at most one collection of polynomials $F_{V,W}$ satisfying conditions (i) and (iii) from the definition (and condition (ii) is automatically satisfied). Moreover, all natural transformations between strict polynomial functors automatically satisfy condition (iv). Thus, $\mathcal{U}$ is an embedding.

If $\Bbbk$ is a finite field, the situation is quite different. First $\mathcal{U}$ is not full. For example, if $\Bbbk = \mathbb{F}_p$, then $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(S^1, S^p) = 0$ since these two functors are homogeneous of different degrees. But the natural transformation $v \mapsto v^p$ is a non trivial element of $\mathrm{Hom}_{\mathcal{F}ct(\mathcal{V}_\Bbbk, \mathcal{V}_\Bbbk)}(S^1, S^p)$. Also, $\mathcal{U}$ is not injective on objects. For example, if $\Bbbk = \mathbb{F}_p$, the Frobenius twist functors $I^{(r)}$ and $I^{(s)}$, for $r \neq s$ are not ismorphic, but $\mathcal{U}I^{(r)}$ and $\mathcal{U}I^{(s)}$ both equal the identity functor.

8.2. **Generalizations.**

8.2.1. *Strict polynomial functors over a commutative ring* $\Bbbk$. Let $\Bbbk$ be a commutative ring, and let $\mathcal{V}_\Bbbk$ be the category of finitely generated $\Bbbk$-modules. Then the definition of the category $\mathcal{P}_\Bbbk$ can be transposed without change over $\Bbbk$. (i.e. strict polynomial functors are functors $F : \mathcal{V}_\Bbbk \to \mathcal{V}_\Bbbk$, endowed with a strict polynomial structure, satisfying conditions (i), (ii) and (iii).

Then all that is written in section 8.1 carries word for word in this more general setting, up to the following minor change. The categories $\mathcal{P}_\Bbbk$ and $\mathcal{P}_{\Bbbk,d}$ are no longer abelian: there are not enough kernels or cokernels in general. For example, if $\Bbbk = \mathbb{Z}$, the cokernel of the multiplication by 2: $F \xrightarrow{\times 2} F$ does not take projective values, hence is not an object of $\mathcal{P}_\Bbbk$. However, if we define admissible short exact sequences to be the $F \to G \to H$ which are short exact sequences after evaluation on all $V \in \mathcal{V}_\Bbbk$, then $\mathcal{P}_\Bbbk$ and $\mathcal{P}_{\Bbbk,d}$ become exact categories in the sense of Quillen, with enough admissible projectives (divided powers) and injectives (symmetric powers). Hence, $\mathcal{P}_\Bbbk$ is still a good framework for homological algebra, Ext-computations and so on, see [Bu] for a recent detailed account of exact categories (classical homological algebra works without change in this setting if one replaces the concept of short exact sequences by the concept of admissible ones).

Such strict polynomial functors were used in [SFB], and the reader can find more details in this article.

8.2.2. *Strict polynomial functors with values in arbitrary* $\Bbbk$*-modules.* Let $\Bbbk$ be a commutative ring. One can also define strict polynomial functors with values in arbitrary $\Bbbk$-modules. To be more specific, a strict polynomial functor with values in arbitrary $\Bbbk$-modules is a functor: $F : \mathcal{V}_\Bbbk \to \Bbbk\text{-Mod}$, equipped with a collection of polynomials $F_{V,W} \in$

$\mathrm{Hom}_{\mathrm{Pol}}(\mathrm{Hom}_{\Bbbk}(V,W), \mathrm{Hom}_{\Bbbk}(F(V),F(W)))$ satisfying the same conditions as in the field case.

Let $\widetilde{\mathcal{P}_{\Bbbk}}$ denote the category of strict polynomial functors with values in arbitrary $\Bbbk$-modules. Then it is easy to prove (the proofs are the same as in the field case) that:

- $\widetilde{\mathcal{P}_{\Bbbk}}$ is an abelian category, which splits as a direct sum of its full abelian subcategories $\widetilde{\mathcal{P}_{\Bbbk,d}}$ of homogeneous functors of degree d, exactly as in the field case.
- The abelian categories $\widetilde{\mathcal{P}_{\Bbbk,d}}$ are equivalent to the categories of $\Bbbk$-linear functors $\Gamma^d\mathcal{V}_{\Bbbk} \to \Bbbk$-Mod.
- $\widetilde{\mathcal{P}_{\Bbbk}}$ has enough projectives (a projective generator is the tensor products of divided powers), so $\widetilde{\mathcal{P}_{\Bbbk}}$ is a nice framework for classical homological algebra. (But we warn the reader that unlike in the category $\mathcal{P}_{\Bbbk}$, symmetric powers are no longer injective, that Kuhn duality is no longer a self anti-equivalence of categories, and that tensor products are no longer exact).

Finally, $\mathcal{P}_{\Bbbk}$ equals the full subcategory of $\widetilde{\mathcal{P}_{\Bbbk}}$ whose objects are functors taking values in finitely generated projective modules. Moreover the inclusion $\mathcal{P}_{\Bbbk} \hookrightarrow \widetilde{\mathcal{P}_{\Bbbk}}$ is exact and preserves projectives. So for all $F, G \in \mathcal{P}_{\Bbbk}$ the inclusion induces an isomorphism:

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G) \simeq \mathrm{Ext}^*_{\widetilde{\mathcal{P}_{\Bbbk}}}(F,G)\ .$$

Thus, one can think of the extension groups between F and G, as being computed in the exact category $\mathcal{P}_{\Bbbk}$, or in the abelian category $\widetilde{\mathcal{P}_{\Bbbk}}$ (since the two categories have slightly different properties, the two viewpoints might be interesting).

References

[A] K. Akin, Extensions of symmetric tensors by alternating tensors. J. Algebra 121 (1989), no. 2, 358–363.

[BB] G. Boffi, D. Buchsbaum, Threading homology through algebra: selected patterns. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2006. xii+255 pp. ISBN: 978-0-19-852499-1; 0-19-852499-4

[Br] K.S. Brown, Cohomology of groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982. x+306 pp. ISBN: 0-387-90688-6

[Bu] T. Bühler, Exact Categories, Expositiones Mathematicae 28 (2010) 1–69, doi: 10.1016/j.exmath.2009.04.004

[Car] H. Cartan, Séminaire Henri Cartan de l'École Normale supérieure, 1954/1955. Algèbres d'Eilenberg Mac Lane et homotopie, Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955 (French). Available online on www.numdam.org

[C1] M. Chałupnik, Extensions of strict polynomial functors. Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 5, 773–792.

[C2] M. Chałupnik, Koszul duality and extensions of exponential functors. Adv. Math. 218 (2008), no. 3, 969–982.

[CPSVdK] E. Cline, B. Parshall, L. Scott, W. van der Kallen, Rational and generic cohomology. Invent. Math. 39 (1977), no. 2, 143–163.

[D] G. Decker, The integral homology algebra of an Eilenberg-Mac Lane space, Ph.D. Thesis, Ph. D. Thesis, University of Chicago, 1974, (under the direction of S. Mac Lane).

[EML] S. Eilenberg, S. Mac Lane, On the groups of $H(\Pi, n)$. I. Ann. of Math. (2) 58, (1953). 55–106.

[EML2] S. Eilenberg, S. Mac Lane, On the groups of $H(\Pi, n)$. II. Methods of computation. Ann. of Math. (2) 60, (1954). 49–139.

[FFSS] V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin, General linear and functor cohomology over finite fields, Ann. of Math. (2) 150 (1999), no. 2, 663–728.

[F] E. Friedlander, Lectures on the cohomology of finite group schemes. Rational representations, the Steenrod algebra and functor homology, 27–53, Panor. Synthèses, 16, Soc. Math. France, Paris, 2003.

[FS] E. Friedlander, A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997), 209–270.

[J] J.C. Jantzen, Representations of algebraic groups. Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003. xiv+576 pp. ISBN: 0-8218-3527-0

[ML] S. Mac Lane, Homology. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. x+422 pp. ISBN: 3-540-58662-8.

[P] T. Pirashvili, Introduction to functor homology. Rational representations, the Steenrod algebra and functor homology, 27–53, Panor. Synthèses, 16, Soc. Math. France, Paris, 2003.

[SFB] A. Suslin, E. Friedlander, C. Bendel, Infinitesimal 1-parameter subgroups and cohomology. J. Amer. Math. Soc. 10 (1997), no. 3, 693–728.

[To] B. Totaro, Projective resolutions of representations of $\mathrm{GL}(n)$, J. Reine Angew. Math. 482 (1997), 1–13.

[T1] A. Touzé, Universal classes for algebraic groups, Duke Math. J. 151 (2010), no. 2, 219–250.

[T2] A. Touzé, Cohomology of classical algebraic groups from the functorial viewpoint, Adv. in Math. 225(1), 33–68, 2010.

[T3] A. Touzé, Troesch complexes and the twisting spectral sequence, `arXiv:1005.3133`.