

TREE HOMOLOGY AND A CONJECTURE OF LEVINE

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ABSTRACT. In his study of the group of homology cylinders, J. Levine [19] made the conjecture that a certain homomorphism $\eta' : \mathcal{T} \rightarrow \mathcal{D}'$ is an isomorphism. Here \mathcal{T} is an abelian group on labeled oriented trees, and \mathcal{D}' is the kernel of a bracketing map on a quasi-Lie algebra. Both \mathcal{T} and \mathcal{D}' have strong connections to a variety of topological settings, including the mapping class group, homology cylinders, finite type invariants, Whitney tower intersection theory, and the homology of $\text{Out}(F_n)$. In this paper, we confirm Levine's conjecture. This is a central step in classifying the structure of links up to grope and Whitney tower concordance, as explained in other papers of this series [1, 2, 3, 4]. We also confirm and improve upon Levine's conjectured relation between two filtrations of the group of homology cylinders.

1. INTRODUCTION

There is an interesting Lie algebra $\mathcal{D}(H)$ that lies at the heart of many areas of topology. It can be defined as the kernel of the bracketing map $H \otimes \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ on the free Lie algebra $\mathcal{L}(H)$ over a symplectic vector space H . An equivalent definition is that $\mathcal{D}(H) = \text{Der}_\omega(\mathcal{L}(H))$ is the set of derivations of the free Lie algebra that kill the symplectic element $\omega = \sum_i [p_i, q_i]$ where $\{p_i, q_i\}$ is a symplectic basis of H .

When $H = H_1(\Sigma_g; \mathbb{Z})$ is the first homology of a closed oriented surface Σ_g of genus g , Dennis Johnson used $\mathcal{D}(H)$ to study the relative weight filtration of the mapping class group of the surface Σ [15]. Specifically, he showed that the associated graded group is a Lie algebra which embeds in $\mathcal{D}(H)$. In a different direction, letting H be the direct limit of finite dimensional symplectic vector spaces, Kontsevich first noticed that the homology of the Lie algebra $\mathcal{D}(H)$ can be used to study the rational homology of outer automorphism groups of free groups, a beautiful connection that was exploited by Morita and then by Conant-Vogtmann [6, 7, 16, 22].

When H is an abelian group with no symplectic structure, one can still define an abelian group $\mathcal{D}(H)$ as the kernel of the bracketing map. Letting H be the first homology of a link complement, $\mathcal{D}(H)$ becomes the natural home for Milnor invariants of the link [3, 14, 24].

In [18], Levine clarified Johnson's construction by enlarging the mapping class group to a group of homology cylinders, proving that the associated graded group becomes all of $\mathcal{D}(H)$. In the context of homology cylinders there is another natural filtration, introduced

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by Habiro and Goussarov [9, 11, 12], which is related to finite type invariants, called the Y -filtration. In order to relate the Y -filtration to the relative weight filtration, Levine worked with a map η_n from an abelian group of trivalent trees with n trivalent vertices, \mathcal{T}_n (defined below), to \mathcal{D}_n . (Close relations of the group \mathcal{T}_n have previously appeared in the theory of Goussarov-Vassiliev invariants [14, 23]. See also [13] for more connections to homology cylinders.) Rationally, η_n was known to be an isomorphism, and initially Levine thought that it was an isomorphism over the integers as well. However, in [19] he published a correction, noting that certain symmetric trees were in the kernel of η_n , but nevertheless establishing that η_n is onto. In order to promote η_n to an isomorphism, Levine realized that in the definition of \mathcal{D}_n as the kernel of a bracketing map, one needs to replace free Lie algebras by free *quasi*-Lie algebras. These are defined like Lie algebras, but with the self-annihilation relation $[Z, Z] = 0$ replaced by antisymmetry $[Y, Z] = -[Z, Y]$. This leads to a new map $\eta'_n: \mathcal{T}_n \rightarrow \mathcal{D}'_n$, and Levine made what we have been calling the *Levine Conjecture*, that this is an isomorphism, proving it in many special cases.

This paper is third in a series of papers on Whitney tower intersection theory, in which the group \mathcal{T}_n is of central importance. The question of whether η' is an isomorphism plays a crucial role in our arguments. For example, establishing that η' is an isomorphism leads to the classification theorems for geometric filtrations of link concordance as described in [1, 2, 3, 4]. Thus we were motivated to prove Levine's conjecture:

Theorem 1. $\eta'_n: \mathcal{T}_n \rightarrow \mathcal{D}'_n$ is an isomorphism for all n .

As discussed in Section 6, this also allows us to prove (Theorem 40) and improve upon (Theorem 41, which we will prove in [5]) Levine's conjectured relationship between the associated graded groups of homology cylinders with respect to the relative weight filtration and the Goussarov-Habiro Y -filtration [19].

Fix an index set $\{1, \dots, m\}$ once and for all. Let $\mathcal{T}_n = \mathcal{T}_n(m)$ be the abelian group of formal linear combinations of oriented univalent trees with n internal vertices (with no distinguished root), where the univalent vertices are labeled by elements of the index set, modulo IHX relations and antisymmetry relations. (In Levine's notation, $\mathcal{T}_n = \mathcal{A}_n^t(H)$, where $H \cong \mathbb{Z}^m$ is the free \mathbb{Z} -module spanned by the index set.)

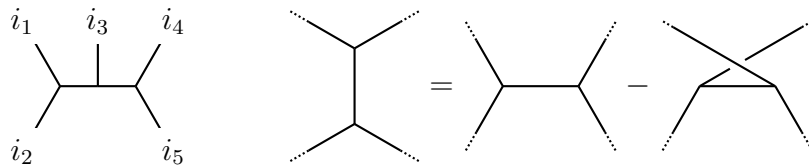


FIGURE 1. A tree in \mathcal{T}_3 , and an IHX relation.

Let $\mathcal{L}_n = \mathcal{L}_n(m)$ be the degree n part of the free \mathbb{Z} -Lie algebra with degree 1 basis X_1, \dots, X_m . It is spanned by formal non-associative bracketing expressions of n basis elements, modulo the Jacobi identity and self-annihilation relation $[Y, Y] = 0$. Replacing this self-annihilation relation with the antisymmetry relation $[Y, Z] = -[Z, Y]$, one obtains $\mathcal{L}'_n = \mathcal{L}'_n(m)$, the degree n part of the free \mathbb{Z} *quasi-Lie algebra* on these same generators.

Let D_n be the kernel of the bracketing map $\mathbb{Z}^m \otimes \mathbf{L}_{n+1} \rightarrow \mathbf{L}_{n+2}$, defined by $X_i \otimes Y \mapsto [X_i, Y]$, where \mathbf{L}_1 is identified with \mathbb{Z}^m ; and let D'_n be the kernel of the corresponding bracketing map, $\mathbb{Z}^m \otimes \mathbf{L}'_{n+1} \rightarrow \mathbf{L}'_{n+2}$, of quasi-Lie algebras.

Levine's map $\eta': \mathcal{T}_n \rightarrow D'_n$ is defined by the formula

$$\eta'(t) = \sum_v X_{\ell(v)} \otimes B'_v(t)$$

where the sum is over all univalent vertices v of t , $\ell(v)$ is the index labeling v , and $B'_v(t)$ is defined to be the iterated bracket in \mathbf{L}'_{n+1} corresponding to the *rooted* oriented tree obtained from t by removing the label of v and letting v be the root. Here is an example.

$$\begin{array}{c} i \qquad j \\ \diagdown \quad / \\ \text{---} \\ / \quad \diagdown \\ l \qquad k \end{array} \mapsto X_l \otimes \left(\begin{array}{c} j \quad k \\ \diagdown \quad / \\ i \\ \bullet \end{array} \right) + X_i \otimes \left(\begin{array}{c} j \quad k \\ \diagdown \quad / \\ l \\ \bullet \end{array} \right) \\ + X_j \otimes \left(\begin{array}{c} l \quad i \\ \diagdown \quad / \\ k \\ \bullet \end{array} \right) + X_k \otimes \left(\begin{array}{c} l \quad i \\ \diagdown \quad / \\ j \\ \bullet \end{array} \right)$$

In [19] Levine conjectured that, for every n , the map $\eta'_n: \mathcal{T}_n \rightarrow D'_n$ is an isomorphism, which has implications concerning the precise relationship between two filtrations of the group of homology cylinders. He obtained partial progress toward the conjecture in the form of the following theorem, and even obtained more progress in [20].

Theorem 2 (Levine). $\eta'_n: \mathcal{T}_n \rightarrow D'_n$ is a split surjection. $\text{Ker } \eta'_n$ is the torsion subgroup of \mathcal{T}_n if n is even. It is the odd torsion subgroup if n is odd. In either case

$$(n+2) \text{Ker } \eta'_n = 0.$$

By [20], D'_n is torsion-free when n is even, and only has 2-torsion of the form $X_i \otimes [Z, Z]$ when n is odd. Hence the Levine conjecture boils down to saying that the \mathcal{T}_n groups have no torsion except the 2-torsion coming from symmetric trees $i \prec_J^j$, which are 2-torsion by the antisymmetry relation. In the theory of Whitney tower intersections, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Q} \otimes \mathcal{T}_n & \twoheadrightarrow & \mathbb{Q} \otimes \mathbf{W}_n \\ & \searrow \eta & \downarrow \mu_n \\ & & \mathbb{Q} \otimes D'_n \end{array}$$

where the group \mathbf{W}_n is defined as the set of links bounding order n Whitney towers, modulo order $(n+1)$ Whitney concordance, and μ_n is the total Milnor invariant of order n (length $n+2$). Thus the fact that η is a rational isomorphism implies that the total Milnor invariant μ_n completely classifies the associated graded group $\mathbb{Q} \otimes \mathbf{W}_n$. So a precise understanding of the torsion in \mathcal{T}_n is necessary for us to be able to classify the groups \mathbf{W}_n over the integers. It turns out that the Levine conjecture is crucial in establishing that \mathbf{W}_n is completely classified by

Milnor invariants, higher-order Sato-Levine invariants and higher-order Arf invariants. See [1, 2, 3, 4] for details.

In our proof of the Levine conjecture, the first step is to reinterpret \mathbb{L}'_n as the zeroth homology of a chain complex $\mathbb{L}_{\bullet,n}$ of rooted labeled trees, where internal vertices may have valence higher than three, with the homological degree given by excess valence. The bracketing operation lifts to an injective map on the chain complex level, giving us a short exact sequence

$$0 \rightarrow \mathbb{Z}^m \otimes \mathbb{L}_{\bullet,n+1} \rightarrow \mathbb{L}_{\bullet,n+2} \rightarrow \overline{\mathbb{L}}_{\bullet,n+2} \rightarrow 0$$

where the complex on the right is by definition the cokernel. Thus we obtain the long exact sequence

$$H_1(\mathbb{L}_{\bullet,n+2}) \rightarrow H_1(\overline{\mathbb{L}}_{\bullet,n+2}) \rightarrow \mathbb{Z}^m \otimes \mathbb{L}'_{n+1} \rightarrow \mathbb{L}'_{n+2} \rightarrow H_0(\overline{\mathbb{L}}_{\bullet,n+2}) \rightarrow 0.$$

Because the bracketing map is onto, $H_0(\overline{\mathbb{L}}_{\bullet,n+2}) = 0$. Thus, if $H_1(\mathbb{L}_{\bullet,n+2})$ were equal to zero, then we would have an alternate characterization of \mathbb{D}'_n as $H_1(\overline{\mathbb{L}}_{\bullet,n+2})$. Indeed, in Section 4 we show that this homology does vanish:

Theorem 3. $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}) = 0$.

With $\mathbb{D}'_n \cong H_1(\overline{\mathbb{L}}_{\bullet,n+2})$, the map η' turns into a map $\bar{\eta}: \mathcal{T}_n \rightarrow H_1(\overline{\mathbb{L}}_{\bullet,n+2})$ which sums over adding a rooted edge to every internal vertex of tree $t \in \mathcal{T}_n$.

To complete the proof of Theorem 1, in section 5 we show that $H_1(\overline{\mathbb{L}}_{\bullet,n+2})$ is isomorphic to \mathcal{T}_n , via a chain map η_{\bullet} which is a lift of $\bar{\eta}$.

Theorem 4. η_{\bullet} induces an isomorphism $\bar{\eta}: \mathcal{T}_n \xrightarrow{\cong} H_1(\overline{\mathbb{L}}_{\bullet,n+2})$.

The proofs of Theorems 3 and 4 use the powerful technique of discrete Morse theory for chain complexes, which we discuss in section 3. Roughly, a discrete gradient vector field on a chain complex is a list of pairs of generators each giving combinatorial data parameterizing an atomic acyclic subcomplex. These acyclic subcomplexes can then be modded out to obtain a simpler quasi-isomorphic complex, called the Morse complex. In practice, it is often possible to find a discrete vector field that drastically reduces the size of the complex being studied. Indeed, in order to show $H_1(\mathbb{L}_{\bullet,n+2}) = 0$, we construct a discrete vector field for which the Morse complex is 0 in degree 1! This vector field is inspired by the Hall basis algorithm for the free Lie algebra. Its lowest degree vectors are defined directly from this algorithm, with a suitably nice choice of a Hall order on trees. One of the conditions on a gradient vector field is that there are no “gradient loops,” which in practice is often the trickiest thing to verify. In this case, the fact that the Hall basis algorithm “works” allows us to rule out loops involving these lowest degree vectors. The complete vector field is a natural extension of the lowest degree case, and ruling out gradient loops in general involves an exhaustive case analysis.

Actually the previous paragraph simplifies the real story somewhat, in that discrete Morse theory works well for *free* chain complexes, but the chain groups $\mathbb{L}_{\bullet,n+2}$ have both \mathbb{Z} and \mathbb{Z}_2 direct summands. To get around this, we actually construct two slightly different vector fields, one for $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_{\bullet,n+2}$ and one for $\mathbb{Z}_2 \otimes \mathbb{L}_{\bullet,n+2}$. The former complex kills symmetric trees of degree 0, so that $H_0(\mathbb{L}_{\bullet,n+2}; \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_n$ is the free Lie algebra over $\mathbb{Z}[\frac{1}{2}]$. The

	2	3	4	5	6	7	8	9
2	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}_2
3	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^2
4	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}^2	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^3	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^5	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2$
5	\mathbb{Z}_2	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^3	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^6	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^{11}
6	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^3	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^9	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^3$	\mathbb{Z}^{19}	$\mathbb{Z}^{22} \oplus \mathbb{Z}_2^5$
7	\mathbb{Z}_2	\mathbb{Z}	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^6	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^3$	\mathbb{Z}^{19}	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^5$	\mathbb{Z}^{47}
8	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^5	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^{19}	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^5$	\mathbb{Z}^{58}	
9	\mathbb{Z}_2	\mathbb{Z}^2	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2$	\mathbb{Z}^{11}	$\mathbb{Z}^{22} \oplus \mathbb{Z}_2^5$	\mathbb{Z}^{47}		
10	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}^7	$\mathbb{Z}^{13} \oplus \mathbb{Z}_2^3$	\mathbb{Z}^{36}			
11	\mathbb{Z}_2	\mathbb{Z}^2	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^3$	\mathbb{Z}^{18}				
12	\mathbb{Z}	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	\mathbb{Z}^9					

FIGURE 2. A computer generated table of the groups $\mathcal{T}_{(j,k)}$.

second complex is more closely tied to the free quasi-Lie algebra, and indeed the vector field we construct for the case of \mathbb{Z}_2 -coefficients is closely aligned with Levine's generalization of the Hall basis algorithm to the quasi-Lie case [19].

Now we will discuss the proof of Theorem 4. The first step is to generalize η' to a chain map

$$\eta_\bullet: \mathbb{T}_\bullet \rightarrow \overline{\mathbb{L}}_{\bullet+1},$$

where \mathbb{T}_\bullet is a chain complex of *unrooted* trees whose zeroth homology is \mathcal{T}_n . Recall that the chain complex $\overline{\mathbb{L}}_{\bullet,n+2}$ is defined as a quotient of an abelian group of rooted trees by the image of the bracketing map. We construct a discrete vector field Δ on $\overline{\mathbb{L}}_{\bullet,n+2}$, essentially by picking a basepoint and pushing the root away from it, when possible. (This is subtle because trees have nontrivial automorphisms, so one has to be careful doing this.) In any event, this gives rise to the Morse complex $\overline{\mathbb{L}}_{\bullet}^\Delta$, so that the composition

$$\mathbb{T}_\bullet \rightarrow \overline{\mathbb{L}}_{\bullet+1} \rightarrow \overline{\mathbb{L}}_{\bullet+1}^\Delta,$$

has kernel and cokernel which are easy to analyze. In particular, η_\bullet induces an isomorphism of homologies in degree 0, though not in higher degrees.

The *signature* of a tree is an m -tuple $\sigma = (n_1, \dots, n_m)$ that records the multiplicities of each label $1, \dots, m$. Defining $|\sigma| = n_1 + \dots + n_m$, $\mathcal{T}_n = \bigoplus_{|\sigma|=n+2} \mathcal{T}_\sigma$, where \mathcal{T}_σ is the subgroup of \mathcal{T}_n spanned by trees with signature ω . For the reader's amusement, in Figure 2, we list computer calculations of the \mathcal{T}_σ groups for small values of $\sigma = (j, k)$.

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2. TREE HOMOLOGY

In this section, we interpret Levine's conjecture in a homological setting. It is well-known that the free (quasi)-Lie algebra can be regarded as the zeroth homology of a complex of rooted oriented trees (of arbitrary valence $\neq 2$), with univalent vertices labeled by the generators, since the boundary of a tree with a 4-valent vertex is precisely a Jacobi relator. Over the rationals, all the homology is concentrated in degree zero (Proposition 9), but the integral homology appears to be unknown.

Definition 5. Throughout this paper all trees are allowed to have vertices of any valence other than 2, and are considered up to isomorphism.

An *orientation* of a tree is an equivalence class of orderings of the edges, where two orderings are equivalent if they differ by an even permutation. Each tree has at most two orientations, and one is said to be the negative of the other. Note that although the orientation given here seems different from the standard orientation of univalent trees or graphs given by ordering the half-edges around a vertex, according to [6] Proposition 2, these are equivalent notions for odd-valent trees.

Following the previous notation for univalent trees, labels from the index set $\{1, 2, \dots, m\}$ are used to decorate univalent vertices, and a *rooted tree* has all univalent vertices labeled except for a single un-labeled *root* univalent vertex. All non-root univalent vertices are called *leaves*, and all vertices of valence ≥ 3 all called *internal vertices*.

The *bracket* of two oriented rooted trees is the rooted tree (J_1, J_2) defined by identifying the roots of J_1 and J_2 and attaching an edge to the identified vertex, the other vertex of this edge being the new root. The orientation is given by ordering the root edge first, then the edges of J_1 in the order prescribed by J_1 's orientation, and finally the edges of J_2 in the order prescribed by its orientation.

The *homological degree* of a tree is defined to be $\sum_v (|v| - 3)$ where the sum is over all internal vertices v , and $|v|$ represents the valence of the vertex.

Rooted trees will usually be denoted by capital letters, and unrooted trees by lower case letters.

Definition 6. In the following chain complexes, we divide by the relation $(T, -\text{or}) = -(T, \text{or})$ for every oriented tree (T, or) .

For all $k \geq 2$, let $\mathbb{L}_{\bullet, k}$ be the chain complex spanned by oriented rooted trees with $k + 1$ total univalent vertices, and the k leaves labeled by the cardinality m index set. The trees are graded by homological degree.

Let v be an internal vertex of a tree J of valence ≥ 4 , and let P be an unordered partition of the half-edges incident to v into two sets each having at least two elements. P determines an *expansion* of J , where the vertex v expands into a new edge e , and the half-edges are distributed to the ends of e according to the partition P . The induced orientation of an expansion is defined by numbering the new edge first, and increasing the numbering of the other edges by one.

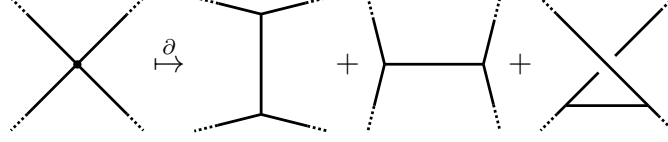


FIGURE 3. An IHX relation appearing as the image of ∂ .

The boundary operator $\partial: \mathbb{L}_{\bullet,k} \rightarrow \mathbb{L}_{\bullet-1,k}$ is defined by setting ∂J equal to the sum of all expansions of J . See Figure 3, which shows the three expansions of a 4-valent vertex. Note that ∂ vanishes on degree zero trees since they have only trivalent internal vertices.

The chain complex $\mathbb{T}_{\bullet,k}$ is spanned by oriented labeled *unrooted* trees with k leaves. The trees are again graded by homological degree. The boundary operator $\partial: \mathbb{T}_{\bullet,k} \rightarrow \mathbb{T}_{\bullet-1,k}$ is defined as before by setting ∂t equal to the sum of all expansions of t .

The chain complex $\bar{\mathbb{L}}_{\bullet,k}$ is the quotient complex of $\mathbb{L}_{\bullet,k}$ by the subcomplex spanned by trees of the form $(i, J) = \leftarrow^i J$. Here the notation i stands for the rooted tree having a single i -labeled leaf.

Some of the homology groups of these chain complexes turn out to be relevant to us.

Proposition 7. *We have the following isomorphisms (\mathbb{Z} -coefficients):*

- (i) $H_0(\mathbb{L}_{\bullet,n}) \cong \mathbb{L}'_n$
- (ii) $H_0(\mathbb{T}_{\bullet,n+2}) \cong \mathcal{T}_n$
- (iii) $H_1(\bar{\mathbb{L}}_{\bullet,n+2}) \cong \mathbb{D}'_n$ for $n > 0$.

Proof. The first isomorphism comes from the fact that in homological degree 0 all trees are trivalent, and hence they are all cycles. The image of the boundary operator is precisely the submodule of IHX relators, since ∂ expands a 4-valent vertex into an IHX relator (Figure 3), where the signs are verified in [6] p.1207. The second isomorphism is similar.

We proceed to explain the third isomorphism, which depends on Theorem 3. Recall that by definition \mathbb{D}'_n is the kernel of the bracketing map

$$\mathbb{Z}^m \otimes \mathbb{L}'_{n+1} \rightarrow \mathbb{L}'_{n+2}$$

via the identification $\mathbb{L}'_1 \cong \mathbb{Z}^m$. This bracketing operation on the quasi-Lie algebra lifts to a chain map

$$\text{Br}_{\bullet}: \mathbb{Z}^m \otimes \mathbb{L}_{\bullet,n+1} \rightarrow \mathbb{L}_{\bullet,n+2}$$

which sends $X_i \otimes J$ to the oriented tree (i, J) . Note that for $n > 0$, Br_{\bullet} is *injective* at the chain level. Thus we get a short exact sequence of chain complexes:

$$0 \rightarrow \mathbb{Z}^m \otimes \mathbb{L}_{\bullet,n+1} \rightarrow \mathbb{L}_{\bullet,n+2} \rightarrow \bar{\mathbb{L}}_{\bullet,n+2} \rightarrow 0$$

where $\bar{\mathbb{L}}_{\bullet,n+2}$ is by definition the cokernel of Br_{\bullet} . Using statement (i) of the proposition, this gives rise to the long exact sequence:

$$H_1(\mathbb{L}_{\bullet,n+2}) \rightarrow H_1(\bar{\mathbb{L}}_{\bullet,n+2}) \rightarrow \mathbb{Z}^m \otimes \mathbb{L}'_{n+1} \rightarrow \mathbb{L}'_{n+2} \rightarrow H_0(\bar{\mathbb{L}}_{\bullet,n+2})$$

We will prove later (Theorem 3) that $H_1(\mathbb{L}_{\bullet, n+2}) = 0$. Since the bracketing map is onto, we get the short exact sequence

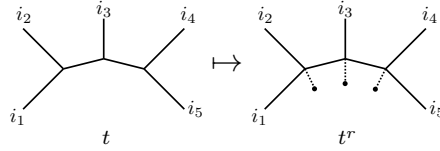
$$0 \rightarrow H_1(\overline{\mathbb{L}}_{\bullet, n+2}) \xrightarrow{\kappa} \mathbb{Z}^m \otimes \mathbb{L}'_{n+1} \rightarrow \mathbb{L}'_{n+2} \rightarrow 0$$

where κ is the connecting homomorphism from the long exact sequence. Hence $D'_n \cong H_1(\overline{\mathbb{L}}_{\bullet, n+2})$. \square

Let us interpret η'_n in this context. Clearly η'_n lifts uniquely to a map $\bar{\eta}_n$ as in the diagram below:

$$\begin{array}{ccc} \mathcal{T}_n & & \\ \downarrow \bar{\eta}_n & \searrow \eta'_n & \\ H_1(\overline{\mathbb{L}}_{\bullet, n+2}) & \xrightarrow{\kappa} & \mathbb{Z}^m \otimes \mathbb{L}'_{n+1} \twoheadrightarrow \mathbb{L}'_{n+2} \end{array}$$

Suppose $t \in \mathbb{T}_{0, n+2}$ is an oriented tree. Define $t^r \in \mathbb{L}_{1, n+2}$ to be the sum of adding a root edge, numbered first in the orientation, to all of the internal vertices of t :



The dotted edges mean that we are summing over putting the root edge in each position. We claim that $\bar{\eta}_n(t) = t^r$. So we must verify that $\kappa(t^r) = \eta'_n(t)$. The map κ is defined via the snake lemma as in the diagram below.

$$\begin{array}{ccccc} \eta'_n(t) & \longmapsto & \sum_v (\ell_v(t), B'_v(t)) & & \\ \mathbb{Z}^m \otimes \mathbb{L}_{0, n+1} & \twoheadrightarrow & \mathbb{L}_{0, n+2} & \twoheadrightarrow & \overline{\mathbb{L}}_{0, n+2} \\ \uparrow 1 \otimes \partial & & \uparrow \partial & & \uparrow \bar{\partial} \\ \mathbb{Z}^m \otimes \mathbb{L}_{1, n+1} & \twoheadrightarrow & \mathbb{L}_{1, n+2} & \twoheadrightarrow & \overline{\mathbb{L}}_{1, n+2} \\ & & \uparrow t^r & & \end{array}$$

Here we use that $\partial t^r = \sum_v (\ell_v(t), B'_v(t))$ because of internal cancellation of the root:

$$\partial(t^r) = \begin{array}{c} \begin{array}{c} i_2 \quad i_3 \quad i_4 \\ \diagdown \quad | \quad / \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad | \quad \diagdown \\ i_1 \quad \bullet \quad i_5 \end{array} \\ = \begin{array}{c} \begin{array}{c} i_2 \quad i_3 \quad i_4 \\ \diagdown \quad | \quad / \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad | \quad \diagdown \\ i_1 \quad \bullet \quad i_5 \end{array} \\ = \sum_v (\ell_v(t), B'_v(t)) \end{array}$$

Tracing through the diagram indeed shows that $t^r \mapsto \eta'_n(t)$.

In fact, $\bar{\eta}$ extends to a map of chain complexes, a fact which we will need in section 5. Define $\eta_{\bullet}: \mathbb{T}_{\bullet, n} \rightarrow \overline{\mathbb{L}}_{\bullet+1, n}$ by letting $\eta_{\bullet}(t)$ be the sum over attaching a root edge to every internal vertex of t , multiplied by the sign $(-1)^{\deg(t)}$.

Lemma 8. $\eta_{\bullet}: \mathbb{T}_{\bullet, n} \rightarrow \overline{\mathbb{L}}_{\bullet+1, n}$ is a chain map.

Proof. For any internal vertex v of an unrooted tree t , let $\alpha_v(t)$ denote the rooted tree gotten by attaching a root edge to v , so that $\eta_{\bullet}(t) = \sum_v \alpha_v(t)$. By definition $\partial\eta_{\bullet}(t)$ is a sum of expansions of $\eta_{\bullet}(t)$, and since those expansions which push the root edge onto an interior edge of t all cancel in pairs, the only terms that are relevant are those where the underlying tree t is expanded. Fix such an expansion, where the vertex v_0 of t gets expanded into two vertices v_1 and v_2 , connected by an edge. Call this expanded tree t^e . Then if v is a vertex of t that is not v_0 , $\partial\alpha_v(t)$ contains one term, $\alpha_v(t^e)$, corresponding to the fixed expansion t^e . On the other hand $\partial\alpha_{v_0}(t)$ contains two such terms: $\alpha_{v_1}(t^e)$ and $\alpha_{v_2}(t^e)$. So for every internal vertex of the expanded tree t^e , there is exactly one summand where the root attaches to it. Thus $\partial\eta_{\bullet}(t) = \sum \eta_{\bullet}(t^e)$ where the sum is over all expansions of t , and so by definition $\partial\eta_{\bullet}(t) = \eta_{\bullet}\partial(t)$. The extra factor of $(-1)^{\deg(t)}$ in the definition of η is designed to make the orientations in this equation agree. \square

2.1. On the rational homology. Although not necessary for the main results of this paper, the following proposition confirms that rationally all the homology is concentrated in degree 0, and gives a crude estimate for the torsion.

Proposition 9.

- (i) $H_k(\mathbb{L}_{\bullet,n}; \mathbb{Z})$ is $n!$ -torsion, and so $H_k(\mathbb{L}_{\bullet,n}; \mathbb{Q}) = 0$, for all $k \geq 1$.
- (ii) $H_k(\mathbb{T}_{\bullet,n+2}; \mathbb{Z})$ is $(n+2)!$ -torsion, and so $H_k(\mathbb{T}_{\bullet,n+2}; \mathbb{Q}) = 0$, for all $k \geq 1$.

Proof. Consider the tree complex $\mathbb{L}_{\bullet,n}^{\dagger}$ defined analogously to $\mathbb{L}_{\bullet,n}$ except that the leaves are always labeled by $1, \dots, n$ without repeats. This actually corresponds to an augmented cochain complex for a simplicial complex K_n defined in the following way. Every tree, except the unique one with only one internal vertex, corresponds to a simplex, given by putting nonnegative lengths summing to 1 on all of its internal edges. (When an edge has length 0, it contracts to a point.) So we have the isomorphism $\tilde{H}^i(K_n) \cong H_{n-i-3}(\mathbb{L}_{\bullet,n}^{\dagger})$, where the index shift comes from the fact that the dimension, i , of a simplex is one less than the number of internal edges of the corresponding tree, and the reader may verify that this corresponds to homological degree $n - i - 3$. This simplicial complex K_n , also known as the Whitehouse complex, is well known to be homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $n - 3$ (which is the top dimension). See [25] for an elementary proof. In particular, this implies that $H_k(\mathbb{L}_{\bullet,n}^{\dagger}) = 0$ for all $k \geq 1$. Let $V = \mathbb{Z}^m$ be the abelian group spanned by the label set. The symmetric group Σ_n acts on $\mathbb{L}_{\bullet,n}^{\dagger}$ by permuting the labels, and we have an isomorphism with the space of coinvariants

$$\mathbb{L}_{\bullet,n} = [V^{\otimes n} \otimes \mathbb{L}_{\bullet,n}^{\dagger}]_{\Sigma_n}$$

where Σ_n acts simultaneously on $L_{\bullet,n}^{\dagger}$ and $V^{\otimes n}$. Now, if $k > 0$, $H_i(V^{\otimes n} \otimes \mathbb{L}_{\bullet,n}^{\dagger}) \cong V^{\otimes n} \otimes H_i(\mathbb{L}_{\bullet,n}^{\dagger}) = 0$. The proof is finished by noting that if a finite group G acts on a chain complex C_{\bullet} , where $H_i(C_{\bullet}) = 0$, then $H_i([C_{\bullet}]_G)$ is $|G|$ -torsion. To see this note that we have a sequence

$$[C_{\bullet}]_G \rightarrow C_{\bullet} \rightarrow [C_{\bullet}]_G$$

where the first map is the map $\sigma \mapsto \sum_{g \in G} g \cdot \sigma$, and the second map is the natural quotient. Their composition is $|G| \cdot \text{Id}$. Applying the functor $H_i(\cdot)$, we have $|G| \cdot \text{Id}$ factoring through 0, implying that $H_i([C_{\bullet}]_G)$ is $|G|$ -torsion.

The proof for $\mathbb{T}_{\bullet, n+2}$ is similar, but one needs to mod out by the action of Σ_{n+2} , and thereby include the root, instead. \square

3. DISCRETE MORSE THEORY FOR CHAIN COMPLEXES

In order to prove Theorems 1 and 3, we adopt a convenient general framework for constructing quasi-isomorphisms based on Forman's theory of discrete vector fields on simplicial complexes [8]. In the setting we require, this has been studied by Kozlov [17], who proves that the Morse complex, defined below, yields isomorphic homology, but does not construct a map to the Morse complex. We give an elaboration of his proof which has the added benefit of constructing a map to the Morse complex, but we claim no originality.

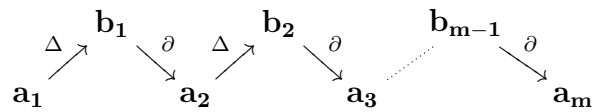
We start by considering, like Kozlov, chain groups which are free modules over a commutative ring. We then analyze a specific non-free case: when the chain groups are direct sums of copies of \mathbb{Z} and \mathbb{Z}_2 . This analysis can be generalized to other non-free cases, but we limit ourselves to what we need in this paper.

Definition 10 (Homological Vector Field). Fix a commutative ground ring R , with unit, and suppose that (C_\bullet, ∂) is a chain complex where each C_k is a free R -module, with a fixed basis $\{\mathbf{b}_k^i\}$.

- (i) A *vector* is a pair of basis elements $(\mathbf{b}_{k-1}^i, \mathbf{b}_k)$ in degrees $k-1$ and k respectively, such that $\partial(\mathbf{b}_k) = r_i \mathbf{b}_{k-1}^i + \sum_{i \neq j} c_j \mathbf{b}_{k-1}^j$, where $r_i \in R$ is invertible, and the coefficients $c_j \in R$ are arbitrary.
- (ii) A (*homological*) *vector field*, Δ , is a collection of vectors (\mathbf{a}, \mathbf{b}) such that every basis element appears in at most one vector of Δ .
- (iii) A basis element is said to be *critical* if doesn't appear in any vector of the vector field Δ . The set of all critical basis elements for Δ will be denoted \mathfrak{X}^Δ .
- (iv) Given a vector field, Δ , a *gradient path* is a sequence of basis elements

$$\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \dots, \mathbf{a}_m$$

where each $(\mathbf{a}_i, \mathbf{b}_i) \in \Delta$, and \mathbf{a}_i has nonzero coefficient in $\partial \mathbf{b}_{i-1}$ and $\mathbf{a}_i \neq \mathbf{a}_{i-1}$. It is often useful to visualize gradient paths using a "zigzag" diagram like the one below.

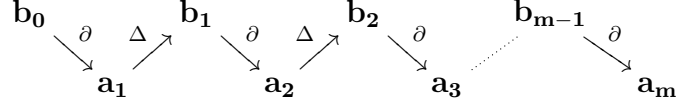


The set of all gradient paths from \mathbf{a} to \mathbf{a}' (that is with $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_m = \mathbf{a}'$) is denoted $\Gamma(\mathbf{a}, \mathbf{a}') = \Gamma^\Delta(\mathbf{a}, \mathbf{a}')$.

- (v) A ∂ -gradient path is a sequence of basis elements

$$\mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_m$$

where $\mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_m$ is a gradient path and \mathbf{a}_1 has nonzero coefficient in $\partial\mathbf{b}_0$. The appropriate zigzag here is



The set of all ∂ -gradient paths from \mathbf{b} to \mathbf{a} will be denoted $\Gamma_{\partial}(\mathbf{b}, \mathbf{a}) = \Gamma_{\Delta}^{\partial}(\mathbf{b}, \mathbf{a})$.

(vi) A vector field is said to be a *gradient field* if there are no closed gradient paths.

Given a vector field, one can construct a degree 1 homomorphism of the same name $\Delta: C_{\bullet} \rightarrow C_{\bullet+1}$ as follows. If \mathbf{a} is a basis element appearing in a vector (\mathbf{a}, \mathbf{b}) , define $\Delta(\mathbf{a}) = \mathbf{b}$, and define Δ to be zero on all other basis elements.

There are functionals on gradient paths and ∂ -gradient paths

$$w: \Gamma(\mathbf{a}, \mathbf{a}') \rightarrow R \text{ and } w: \Gamma_{\partial}(\mathbf{b}, \mathbf{a}) \rightarrow R$$

called the *weight*. The weight of a gradient path $\gamma = (\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \dots, \mathbf{a}_m)$ is defined as follows. For each \mathbf{a}_i , where $i > 1$, suppose that $\partial\mathbf{b}_{i-1} = r_{i-1}\mathbf{a}_{i-1} + c_{i-1}\mathbf{a}_i + \dots$, for $r_{i-1}, c_{i-1} \in R$, with r_{i-1} invertible. Define the weight to be

$$w_{\gamma} = (-1)^{m-1} \frac{c_1 \cdots c_{m-1}}{r_1 \cdots r_{m-1}} \in R.$$

The weight of a ∂ -gradient path μ is also multiplied by the coefficient c_0 of \mathbf{a}_1 in $\partial\mathbf{b}_0$:

$$w_{\mu} = (-1)^{m-1} \frac{c_0 c_1 \cdots c_{m-1}}{r_1 \cdots r_{m-1}} \in R.$$

We now define the Morse complex C_{\bullet}^{Δ} for a gradient vector field Δ . The chain groups of C_{\bullet}^{Δ} are the submodules of C_{\bullet} spanned by critical basis elements. The boundary operator ∂^{Δ} is defined as follows. Suppose $\mathbf{b} \in \mathfrak{X}^{\Delta}$.

$$\partial^{\Delta}(\mathbf{b}) = \sum_{\mathbf{a} \in \mathfrak{X}^{\Delta}} d_{ba} \mathbf{a}$$

where

$$d_{ba} = \sum_{\gamma \in \Gamma_{\partial}(\mathbf{b}, \mathbf{a})} w_{\gamma}.$$

There is a map $\phi = \phi^{\Delta}: (C_{\bullet}, \partial) \rightarrow (C_{\bullet}^{\Delta}, \partial^{\Delta})$ defined as follows. If \mathbf{a} is critical, then $\phi(\mathbf{a}) = \mathbf{a}$. Otherwise

$$\phi(\mathbf{a}) = \sum_{\mathbf{a}' \in \mathfrak{X}^{\Delta}} c_{aa'} \mathbf{a}',$$

where

$$c_{aa'} = \sum_{\gamma \in \Gamma(\mathbf{a}, \mathbf{a}')} w_{\gamma}.$$

In particular, it follows from these definitions that for any \mathbf{a} , $\phi(\Delta(\mathbf{a})) = 0$.

The map ϕ is in some sense defined to be the flow along a vector field. Clearly $\partial^{\Delta} = \phi^{\Delta} \partial$.

Theorem 11. *The Morse complex is a chain complex: $(\partial^\Delta)^2 = 0$, and ϕ^Δ is a chain map which induces an isomorphism $H_*(C_\bullet) \xrightarrow{\cong} H_*(C_\bullet^\Delta)$.*

Proof. We prove all statements simultaneously by induction on the number of vectors in the vector field. The base case of our analysis will be one vector, say $\Delta(\mathbf{a}) = \mathbf{b}$, and $\partial\mathbf{b} = r\mathbf{a} + \dots$. Consider the acyclic subcomplex $A = (0 \rightarrow R(\mathbf{b}) \rightarrow R(\partial\mathbf{b}) \rightarrow 0)$. We get a quasi-isomorphism $C_\bullet \rightarrow C_\bullet/A$. Now as a free R -module, we have an isomorphism $C_\bullet/A \xrightarrow{\cong} C_\bullet^\Delta$ under the map defined on generators $[\mathbf{b}] \mapsto 0$, $[\mathbf{a}] \mapsto \mathbf{a} - \frac{1}{r}\partial\mathbf{b}$, and $[\mathbf{c}] \mapsto \mathbf{c}$ otherwise. The gradient paths in C_\bullet are all of the form $\mathbf{a}, \mathbf{b}, \mathbf{c}$ where \mathbf{c} is a nonzero term in $\partial\mathbf{b}$ not equal to \mathbf{a} . The weight of this path is the negative of the coefficient of \mathbf{c} in $\partial\mathbf{b}$, divided by r . Hence $\phi(\mathbf{a}) = -\frac{1}{r}\partial\mathbf{b} + \mathbf{a}$. On the other hand $\phi(\mathbf{b}) = 0$ and $\phi(\mathbf{c}) = \mathbf{c}$ for critical generators. So the isomorphism is given by ϕ as claimed.

The boundary ∂ on C_\bullet/A induces the boundary operator $\phi\partial\phi^{-1}$ on C_\bullet^Δ , and we must now determine its form. Consider a critical generator \mathbf{c} . $\phi^{-1}(\mathbf{c}) = [\mathbf{c}]$. So $\phi\partial\phi^{-1}(\mathbf{c}) = \phi\partial\mathbf{c} = \partial^\Delta\mathbf{c}$ as desired.

Now suppose the theorem is true for gradient vector fields with k vectors, and assume we now have one with $k + 1$ vectors. Choose a vector, $\mathbf{b} = \Delta\mathbf{a}$.

Let Δ_k be the vector field consisting of the k vectors aside from (\mathbf{a}, \mathbf{b}) . Inductively we have a quasi-isomorphism

$$\phi^k := \phi^{\Delta_k}: (C_\bullet, \partial) \rightarrow (C_\bullet^{\Delta_k}, \partial^{\Delta_k})$$

Now, we claim the pair (\mathbf{a}, \mathbf{b}) still represents a vector in this Morse complex. We need only verify that $\partial^{\Delta_k}\mathbf{b} = r\mathbf{a} + \dots$. This follows from the nonexistence of closed gradient paths in the original vector field, since a ∂ -gradient path from \mathbf{b} to \mathbf{a} aside from the path (\mathbf{b}, \mathbf{a}) would combine with the vector (\mathbf{a}, \mathbf{b}) to form a closed gradient path. Let Δ_1 be the vector field with the single vector (\mathbf{a}, \mathbf{b}) on the Morse complex $C_\bullet^{\Delta_k}$. Then $(C_\bullet^{\Delta_k})^{\Delta_1} = C_\bullet^\Delta$. So we also have a quasi-isomorphism

$$\phi^1 = \phi^{\Delta_1}: (C_\bullet^{\Delta_k}, \partial^{\Delta_k}) \rightarrow (C_\bullet^\Delta, \partial^\Delta)$$

where we emphasize that ∂^{Δ_1} is defined to be the weighted sum of ∂ -gradient paths alternating between ∂^{Δ_k} and Δ_1 . We now need to check that $\phi^\Delta = \phi^1\phi^k$, and that $\partial^{\Delta_1} = \partial^\Delta$. In the following calculations let $\mathfrak{X}^i = \mathfrak{X}^{\Delta_i}$, and let $\mathfrak{X} = \mathfrak{X}^\Delta$.

$$\begin{aligned} \phi^1\phi^k(\mathbf{u}) &= \phi_1 \sum_{\mathbf{v} \in \mathfrak{X}^k} c_{uv}^k \mathbf{v} \\ &= \sum_{\mathbf{v} \in \mathfrak{X}^k} \sum_{\mathbf{w} \in \mathfrak{X}} c_{uv}^k c_{vw}^1 \mathbf{w} \end{aligned}$$

where c_{uv}^i measures gradient paths with respect to Δ_i . So we need to check that

$$\forall \mathbf{w} \in \mathfrak{X}, \quad \sum_{\mathbf{v} \in \mathfrak{X}^k} c_{uv}^k c_{vw}^1 = c_{uw}.$$

Note that $\mathfrak{X} = \mathfrak{X}^k \setminus \{\mathbf{a}, \mathbf{b}\}$, so we can write

$$\begin{aligned} \sum_{\mathbf{v} \in \mathfrak{X}^k} c_{uv}^k c_{vw}^1 &= \left(\sum_{\mathbf{v} \in \mathfrak{X}} c_{uv}^k c_{vw}^1 \right) + c_{ub}^k c_{bw}^1 + c_{ua}^k c_{aw}^1 \\ &= c_{uw}^k + c_{ua}^k c_{aw}^1 \end{aligned}$$

The first term simplifies as indicated, because when $\mathbf{v}, \mathbf{w} \in \mathfrak{X}$, c_{vw}^1 is only nonzero if $\mathbf{v} = \mathbf{w}$. The term c_{bw}^1 is zero because \mathbf{b} being $\Delta_1(\mathbf{a})$ does not begin any gradient paths. The term c_{uw}^k measures gradient paths from \mathbf{u} to \mathbf{w} which do not involve the vector (\mathbf{a}, \mathbf{b}) . c_{ua}^k measures all gradient paths that end in \mathbf{a} , and c_{aw}^1 represents all gradient paths (alternating between Δ_1 and ∂^{Δ_k}) from \mathbf{a} to \mathbf{w} . There is only one vector in Δ_1 , and ∂^{Δ_k} measures alternating paths between ∂ and Δ_k , so c_{aw}^1 represents all gradient paths with respect to Δ that start with the basis element \mathbf{a} and end at \mathbf{w} . So the product $c_{ua}^k c_{aw}^1$ measures all gradient paths that pass through the basis element \mathbf{a} . Thus the sum measures all gradient paths from \mathbf{u} to \mathbf{w} : $c_{uw}^k + c_{ua}^k c_{aw}^1 = c_{uw}$.

Note that by induction $\partial^{\Delta_k} = \phi^k \partial$ and $\partial^{\Delta_1} = \phi^1 \partial^{\Delta_k}$. Hence $\partial^{\Delta_1} = \phi^1 \phi^k \partial = \phi \partial = \partial^{\Delta}$. \square

3.1. Non-free chain complexes. We will need to adapt the above construction to chain complexes which are not free. Indeed our tree complexes all have 2-torsion, so we adapt the notion of a gradient vector field to the case where the chain groups consist of both \mathbb{Z} - and \mathbb{Z}_2 -summands. We replace the notion of “basis of a free R -module” with the notion of “minimal generating set,” where some generators span copies of \mathbb{Z} and some span copies of \mathbb{Z}_2 . We define a vector field as above except that a vector (\mathbf{a}, \mathbf{b}) cannot mix a \mathbb{Z}_2 -generator and a \mathbb{Z} -generator. Then C_{\bullet}^{Δ} is defined to be the subgroup of the chain group C_{\bullet} spanned by critical generators. (In particular, \mathbb{Z}_2 -generators remain 2-torsion in the Morse complex.) Gradient paths are defined as above, and come in two types. A gradient path ending in \mathbf{a} is called a \mathbb{Z}_2 -path if \mathbf{a} is a \mathbb{Z}_2 -generator and is called a \mathbb{Z} -path if \mathbf{a} is a \mathbb{Z} -generator. Notice that if a gradient path involves a \mathbb{Z}_2 generator at some stage, then every subsequent generator in the path will be a \mathbb{Z}_2 -generator. In particular, a \mathbb{Z} -path will only consist of \mathbb{Z} -generators. The weight of a gradient path is defined as follows. If is defined by the same formula as in the free case for \mathbb{Z} -paths: $w_{\gamma} = (-1)^{m-1} \frac{c_1 \cdots c_{m-1}}{r_1 \cdots r_{m-1}} \in \mathbb{Z}$. For \mathbb{Z}_2 -paths, $w_{\gamma} = c_1 \cdots c_{m-1} \in \mathbb{Z}_2$, where some of these coefficients may be in \mathbb{Z} , but are interpreted mod 2. The r_i 's are omitted as they are all ± 1 , so are trivial mod 2. The weights of ∂ -gradient paths are defined similarly.

The flow ϕ^{Δ} is defined as a sum of weights of gradient paths, as in the free case. With these definitions, the proof of Theorem 11 then goes through with little modification in this particular non-free setting:

Theorem 12. *Let C_{\bullet} be a chain complex where each C_k is a finitely generated abelian group whose only torsion is 2-torsion, and suppose that Δ is a gradient vector field in the above sense on C_{\bullet} . Then the Morse complex is a chain complex: $(\partial^{\Delta})^2 = 0$, and ϕ^{Δ} is a chain map which induces an isomorphism $H_*(C_{\bullet}) \xrightarrow{\cong} H_*(C_{\bullet}^{\Delta})$.*

4. PROOF OF THEOREM 3

Recall the statement of Theorem 3: $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}) = 0$. The proof is done in two stages by Propositions 13 and 14, whose proofs occupy most of this section.

Let $\mathbb{L}_{\bullet,n}^{(2)} \subset \mathbb{L}_{\bullet,n}$ be the subcomplex spanned by $2\mathbb{L}_{\bullet,n}$ and symmetric degree 2 trees as on the right side of Figure 5 which have no orientation-reversing symmetry. (The obvious symmetry turns out to be orientation preserving, because the trees being swapped have an even number of edges. See the discussion in the proof of Proposition 13.) This is evidently a subcomplex since the boundary of such a tree is a multiple of 2.

Proposition 13. *For all n , $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}[\frac{1}{2}]) = 0$, and $H_1(\mathbb{L}_{\bullet,n}^{(2)}; \mathbb{Z}) = 0$.*

Proposition 14. *For all n , $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}_2) = 0$.*

Before proving these propositions, we check that they imply Theorem 3. We would like to use the universal coefficient theorem to conclude that $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}) = 0$, but our chain groups are not free modules, so we pause to state a lemma that holds in this context. For any chain complex C_\bullet define

$$H_k^{(2)}(C_\bullet) = \frac{Z_k \cap 2C_k}{2Z_k \cup (B_k \cap 2C_k)},$$

where Z_k and B_k are the submodules of cycles and boundaries, respectively.

Lemma 15. *For all k , there is an exact sequence*

$$0 \rightarrow H_k^{(2)}(C_\bullet) \rightarrow \mathbb{Z}_2 \otimes H_k(C_\bullet) \rightarrow H_k(C_\bullet; \mathbb{Z}_2).$$

Proof. The right-hand map is defined by $1 \otimes [z] \mapsto [1 \otimes z]$ for cycles z . Clearly the $[1 \otimes z]$ is still a cycle. This is well-defined because boundaries $1 \otimes \partial w$ map to boundaries $\partial(1 \otimes w)$. We claim that the left-hand map is an injection. Suppose a cycle $2u$ maps to 0 in $\mathbb{Z}_2 \otimes H_k(C_\bullet)$. So $1 \otimes 2u = \partial(1 \otimes w)$, implying $1 \otimes 2u = 1 \otimes \partial w$ in $\mathbb{Z}_2 \otimes C_\bullet$. Therefore, $\partial w = 2u + 2z$, where z is a cycle. Therefore $2u \in 2Z_k \cup (B_k \cap 2C_k)$, and hence equals 0 in the domain. Finally, to see exactness at the middle, suppose that $1 \otimes [z] \mapsto 0 \in H_k(C_\bullet; \mathbb{Z}_2)$. Then $1 \otimes z = \partial(1 \otimes w)$. Therefore $\partial w = z + 2x$. Hence $z - \partial w \in 2C_\bullet$, and the homology class $[z] = [z - \partial w]$ is represented by a cycle in $2C_\bullet$. \square

Proof of Theorem 3. We apply Lemma 15, using propositions 13 and 14: Notice first that $H_1(\mathbb{L}_{\bullet,n}^{(2)}; \mathbb{Z})$ surjects onto $H_1^{(2)}(\mathbb{L}_{\bullet,n}; \mathbb{Z})$, because $\mathbb{L}_{1,n}^{(2)} = 2\mathbb{L}_{1,n}$. Thus $\mathbb{Z}_2 \otimes H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z})$ is trapped between two zero groups and is therefore zero. And tensoring with $\mathbb{Z}[\frac{1}{2}]$ things are even easier, since $\mathbb{Z}[\frac{1}{2}] \otimes H_1(\mathbb{L}_{\bullet,n})$ injects into $H_1(\mathbb{L}_{\bullet,n}; \mathbb{Z}[\frac{1}{2}])$. \square

4.1. Plan of the proofs of Propositions 13 and 14. Because the proofs of these two propositions are somewhat technical, we would like to give an overview of the structure of these proofs. Fix n , and to simplify notation let $\mathbb{L}_\bullet = \mathbb{L}_{\bullet,n}$. The idea will be, with two different sets of coefficients, to construct a vector field $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_i: \mathbb{L}_i \rightarrow \mathbb{L}_{i+1}$, which has no critical vectors in degree 1 and then appeal to Theorem 11 to conclude the degree 1 homology vanishes.

First consider $\mathbb{Z}[\frac{1}{2}]$ coefficients. Since $H_0(\mathbb{L}_\bullet; \mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_n$ has a well-known basis, called the Hall basis, our strategy in the $\mathbb{Z}[\frac{1}{2}]$ case will be to define $\Delta_0(J)$ to be some nontrivial contraction of J for every non-Hall tree J . In particular, we will define a “Hall problem” of a tree J to be a place in J where two Hall trees meet at a vertex, but their bracket is not itself Hall. $\Delta_0(J)$ then contracts an edge at the base of one of the two Hall trees. The resulting vector field has no closed gradient paths, in some sense because the Hall basis algorithm “works.” The way we rigorously prove it is to show that a natural “Hall order” defined on trees always increases as one moves along a gradient path. This is the most one can hope to do for Δ_0 , because Hall trees need to survive as a basis for $H_0(\mathbb{L}_\bullet; \mathbb{Z}[\frac{1}{2}])$.

All the trees in the image of Δ_0 are not critical, so it suffices now to define Δ_1 to be nonzero on all the other degree 1 trees. In analogy with degree 0, we say a tree is Hall₁ if it is in the image of Δ_0 , because these are the trees on which Δ_1 needs to vanish. We combinatorially characterize what it means to be Hall₁, and define a “Hall₁ problem” to be one of an exhaustive list of ways that a tree can fail to be Hall₁. Finally we define Δ_1 for each different type of Hall₁ problem as a certain contraction of an edge within the Hall₁ problem. The resulting vector field is again shown to be gradient by arguing that the Hall order increases as one moves along gradient paths. This then proves the $\mathbb{Z}[\frac{1}{2}]$ case, and in fact proves Proposition 13 since the chain complexes $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet$ and $\mathbb{Z} \otimes \mathbb{L}_\bullet^{(2)}$ are isomorphic.

For \mathbb{Z}_2 coefficients the argument is similar, except now we use the fact proven by Levine [20], that $H_0(\mathbb{L}_\bullet; \mathbb{Z}_2) \cong \mathbb{Z}_2 \otimes \mathbb{L}'_n$ has a basis given by Hall trees plus trees (H, H) where H is Hall. We call the trees in this basis Hall' trees, and proceeding as before, we define Δ'_0 by contracting Hall' problems. Defining a Hall'₁ tree to be a tree in the image of Δ'_0 , a Hall'₁ problem is one of an exhaustive list of ways that a tree can fail to be Hall'₁. Then Δ'_1 is defined for these different Hall'₁ problems by contracting certain edges within the problems. We argue that the Hall order increases along gradient paths, except in one case, and using special arguments to take care of this case, this shows that there are no closed gradient paths.

4.2. Proof of Proposition 13. We prove both cases simultaneously. Recall that the chain group \mathbb{L}_\bullet is defined as the quotient of a free \mathbb{Z} module of oriented trees by relations $(J, -\text{or}) = -(J, \text{or})$. In particular, a tree J will either generate a \mathbb{Z} - or a \mathbb{Z}_2 -summand depending on whether it has an orientation reversing automorphism, so that \mathbb{L}_\bullet is a direct sum of copies of \mathbb{Z} and \mathbb{Z}_2 . We claim that the nonzero trees in $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet$ and $\mathbb{L}_\bullet^{(2)}$ are the same. Multiplying by 2 and adjoining 1/2 both have the effect of killing the 2-torsion, and the additional symmetric trees in $\mathbb{L}_2^{(2)}$ are not 2-torsion, so that this is indeed true.

To define a gradient vector field, first we need to specify a basis. Choose orientations for each tree. Define the basis of the free $\mathbb{Z}[\frac{1}{2}]$ -module $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet$ to be the trees with specified orientations, except for nonzero symmetric degree 2 trees as on the right of figure 5. In that case the basis element is defined to be 1/2 the given oriented tree. For the free \mathbb{Z} -module $\mathbb{L}_\bullet^{(2)}$, define the basis to be exactly twice the oriented trees just mentioned. In fact $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet \cong \mathbb{L}_\bullet^{(2)}$ as chain complexes with these specified generators. Thus it makes sense to construct a gradient vector field on both $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet$ and $\mathbb{L}_\bullet^{(2)}$ simultaneously. We will construct such a vector field with no critical generators in degree 1. We will work with $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_\bullet$ but

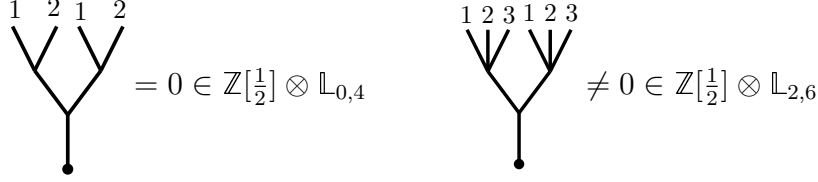


FIGURE 4. The tree $((1, 2), (1, 2))$ on the left has an orientation reversing symmetry, whereas the tree $((1, 2, 3), (1, 2, 3))$ on the right has an orientation preserving symmetry.

since the correspondence of bases respects the boundary operator, this will simultaneously prove the $\mathbb{L}_{\bullet}^{(2)}$ case.

The vector field will consist of two pieces $\Delta_0: \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_0 \rightarrow \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1$, and $\Delta_1: \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1 \rightarrow \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_2$. Δ_0 is constructed via the Hall Basis algorithm for the free Lie algebra. So we need to set up some machinery to explain this. Our presentation follows and expands upon [26].

Recall that for rooted trees J_1 and J_2 , the rooted tree (J_1, J_2) is defined by identifying the roots together to a single vertex, and attaching a new rooted edge to this vertex. Extend this notation to (J_1, \dots, J_k) , which is defined by identifying the roots of all the trees J_i to a single vertex, and attaching a new rooted edge to this vertex. Thus the two trees in Figure 4 are notated $((1, 2), (1, 2))$ and $((1, 2, 3), (1, 2, 3))$, respectively.

If J_1, \dots, J_k are oriented, the tree (J_1, \dots, J_k) will be oriented by numbering the root edge first, then the edges of J_1 , followed by the edges of J_2 , etc.

Given a rooted tree, suppose one deletes an internal vertex. Then there are multiple connected components, including the one containing the root. Any of the connected components that does not contain the root can itself be regarded as a rooted tree, by filling in the deleted vertex as a root. Such a tree will be called a *full subtree*.

With this notation and terminology in hand, we can characterize those rooted trees which have orientation-reversing automorphisms, and are therefore zero when tensoring with $\mathbb{Z}[\frac{1}{2}]$. Indeed, any rooted tree is zero which contains a full subtree of the form (J_1, \dots, J_m) where $J_i = J_k$ for some $i \neq k$ with $J_i = J_k$ having an odd number of edges. For example, this will be true when J_i is univalent. Hence, over $\mathbb{Z}[\frac{1}{2}]$ we may assume that for trees in \mathbb{L}_0 and in \mathbb{L}_1 the emanating subtrees at every vertex are *distinct*. Exemplar trees with orientation preserving and reversing automorphisms are pictured in Figure 4.

Definition 16. Define the *weight* of a tree, denoted by $|J|$, to be the number of leaves (the number of univalent vertices not counting the root). We recursively define an order relation, called the *Hall order*, on (unoriented) labeled trees in the following way.

- (i) Trees of weight 1 are ordered by an ordering on the index set.
- (ii) If $|J| < |K|$, then $J \succ K$.
- (iii) If $|J| = |K|$ and $J = (J_1, \dots, J_\ell)$, $K = (K_1, \dots, K_m)$ with $\ell < m$ then $J \succ K$.

- (iv) If $|J| = |K|$ and $J = (J_1, \dots, J_m)$, $K = (K_1, \dots, K_m)$, assume that $J_1 \preceq J_2 \preceq \dots \preceq J_m$ and $K_1 \preceq K_2 \preceq \dots \preceq K_m$. Then compare the two trees lexicographically: $J \prec K$ if and only if there is some index ℓ , $1 \leq \ell < m$, such that $J_i = K_i$ for all $i \leq \ell$ and $J_{\ell+1} \prec K_{\ell+1}$.

Lemma 17. *Suppose one replaces a full subtree of a tree by a tree with increased Hall order. Then the new tree has larger Hall order than the original.*

Proof. If the new subtree tree has lower weight, then this is clear. If the new tree has the same weight, then recursively it will suffice to consider the replacement $(J_1, \dots, J_\ell, \dots, J_m) \mapsto (J_1, \dots, J'_\ell, \dots, J_m)$ where the full subtree J_ℓ has been replaced by J'_ℓ , with $J_\ell \prec J'_\ell$. By the lexicographic definition (case (iv) of Definition 16), the whole tree is also larger with respect to \prec . \square

- Definition 18.**
- (i) The set of *Hall trees* is the subset of (unoriented) labeled rooted univalent trees, defined recursively as follows. All weight 1 trees are Hall. The bracket of two Hall trees (H_1, H_2) with $H_1 \prec H_2$ is Hall if and only if either H_1 is of weight 1, or $H_1 = (H', H'')$ with $H' \prec H''$ and $H'' \succeq H_2$.
 - (ii) Given a tree J , a *Hall problem* is a full subtree of J of the form (H_1, H_2) where $H_1 \prec H_2$ are both Hall, but $H_1 = (H', H'')$ with $H' \prec H'' \prec H_2$.
 - (iii) The *contraction of a Hall problem* is the tree obtained by replacing the full subtree (H_1, H_2) by the full subtree (H', H'', H_2) .

The above definitions are made for unoriented trees, but we will often not distinguish between oriented and unoriented trees. For example, when we say that an oriented tree is Hall, we mean that the underlying unoriented tree is Hall.

We now define a vector field $\Delta_0: \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_0 \rightarrow \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1$ as follows. If H is a Hall tree, then $\Delta_0(H) = 0$. Otherwise, suppose J is a tree where there is at least one Hall problem. Define

$$\Delta_0(J) = \max\{J^c \mid J^c \text{ is the contraction of a Hall problem in } J\}.$$

Lemma 19. Δ_0 is a gradient vector field.

Proof. We need to check the following three conditions.

- (i) $\Delta_0(J_1) \neq \Delta_0(J_2)$ for distinct J_1, J_2 which are non-Hall.
- (ii) $\partial\Delta_0(J) = \pm J + \text{other trees}$, for J non-Hall.
- (iii) All gradient paths terminate.

All three of these will follow from the fact that the two expansions of $\Delta_0(J)$, other than J itself, have larger Hall order, which we now check. Indeed, by Lemma 17, we just need to show

$$((H', H''), H_2) \prec ((H', H_2), H'') \text{ and } ((H', H''), H_2) \prec ((H'', H_2), H')$$

Note we are assuming that $H' \prec H'' \prec H_2$, corresponding to the Hall problem $((H', H''), H_2)$ in J . Then $(H', H_2) \prec H' \prec H''$, so the term $((H', H_2), H'')$ is in non-decreasing order. So

applying the lexicographic definition, it suffices to show that $(H', H'') \prec (H', H_2)$ which follows from Lemma 17. Now for the second inequality above, the two component trees of $((H'', H_2), H')$ may occur in either order, or even be equal. If they are equal, the tree is zero, and we don't need to consider it, but because we want to re-use this analysis in the case of \mathbb{Z}_2 coefficients, we prefer not to use this fact. In any event, no matter what the order, it suffices to observe that $(H', H'') \prec (H'', H_2)$ and $(H', H'') \prec H'$. Thus the two expansions of $\Delta_0(J)$, other than J itself, have larger Hall order, as claimed.

Condition (i) follows because $\Delta_0(J_1) = \Delta_0(J_2)$ implies that J_2 is another expansion of $\Delta_0(J_1)$ so that $J_1 \prec J_2$. Symmetrically $J_2 \prec J_1$, which is a contradiction. Condition (ii) follows since $\partial\Delta_0(J) = J + J' + J''$ where J' and J'' have increased Hall order. Condition (iii) now follows since the Hall order increases as one flows along a gradient path, implying that all gradient paths must terminate. \square

So far we have adapted the well-known Hall algorithm to the context of homological vector fields. Since the critical generators in degree zero are Hall trees, we have reproduced the standard fact that the free Lie algebra $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_n \cong H_0(\mathbb{L}_\bullet; \mathbb{Z}[\frac{1}{2}])$ is generated by Hall trees. Since these trees actually form a basis, to make further progress in killing degree 1 generators we will need to construct a vector field $\Delta_1: \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1 \rightarrow \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_2$, extending Δ_0 .

Definition 20. A nonzero tree, $H \in \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1$, is said to be *Hall₁* if $H = \Delta_0(J)$ for some tree $J \in \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_0$.

We now characterize Hall₁ trees.

Lemma 21. A tree $H \in \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1$ is Hall₁ if and only if both of the following two conditions hold.

- (i) It contains a full subtree (A, B, C) , where $A \prec B \prec C$ are all Hall, such that either $|A| = 1$, or $A = (A', A'')$ with $A' \prec A''$ and $A'' \succeq B$.
- (ii) The tree obtained by replacing (A, B, C) with $((A, B), C)$ is nonzero, and H is the largest among all contractions of Hall problems of this expanded tree.

Proof. The first condition says exactly that (A, B, C) is the contraction of a Hall problem $((A, B), C)$, using the fact that (A, B) is Hall. Moreover the other two expansions $((A, C), B)$ and $((B, C), A)$ are not Hall problems, so to check whether H is in the image of Δ_0 we need only check that it is the maximal contraction of this expanded tree, as per the second statement. \square

A *Hall₁ problem* of a nonzero tree J , will be defined by negating the characterization of the previous lemma. Cases (i) and (ii) of the following Definition 22 are the two ways that condition (i) of the previous Lemma can fail, and cases (iii) and (iv) are the two ways that condition (ii) of the previous Lemma can fail.

Definition 22. A *Hall₁ problem* in a nonzero tree $J \in \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_1$ is defined to be any one of the following situations:

- (i) J contains the full subtree (A, B, C) where at least one of A, B, C fails to be Hall. So there is a Hall problem in one of these trees.
- (ii) J contains the full subtree (A, B, C) where $A \prec B \prec C$ are all Hall, but $A = (A', A'')$ with $A' \prec A'' \prec B$.
- (iii) J contains a full subtree (A, B, C) , where $A \prec B \prec C$ are all Hall, such that either $|A| = 1$, or $A = (A', A'')$ with $A' \prec A''$ and $A'' \succeq B$; and the expanded tree containing $((A, B), C)$ is nonzero, but J is not the largest of all contractions of the expanded tree.
- (iv) J contains a full subtree (A, B, C) , where $A \prec B \prec C$ are all Hall, such that either $|A| = 1$, or $A = (A', A'')$ with $A' \prec A''$ and $A'' \succeq B$; but the expanded tree containing $((A, B), C)$ is zero. This means that $((A, B), C)$ occurs in the tree J , and there is a symmetry in the expanded tree exchanging this for the $((A, B), C)$ expansion of (A, B, C) . Such a J is pictured in the middle of Figure 5, where the expanded tree on the left is 0.

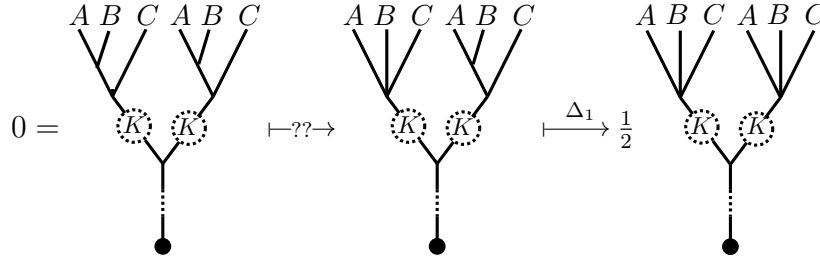


FIGURE 5. The tree, J , in the middle has a type (iv) Hall₁ problem, since the expanded tree on the left is equal to zero in $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_0$. The image of J under Δ_1 is pictured on the right. (The sub-trees indicated by the dotted arcs are all trivalent.)

Now we define the vector field Δ_1 .

Definition 23. If a tree is Hall₁ we define Δ_1 to be zero. (As we need to, in order to make Δ_1 disjoint from Δ_0 .) Otherwise, referring to Definition 22, we consider the four types of Hall₁ problems:

- (i) If J has a type (i) Hall₁ problem, then define $\Delta_1(J)$ to contract some Hall problem in the trees $\{A, B, C\}$.
- (ii) If J has a type (ii) Hall₁ problem, then define $\Delta_1(J)$ to contract the base edge of A .
- (iii) If J has a type (iii) Hall₁ problem, to define $\Delta_1(J)$, consider the expanded tree J^e containing $((A, B), C)$. Then $\Delta_0(J^e)$ contracts a different Hall problem in J^e (which is maximal among all contractions) than the one that contracts to J . Define $\Delta_1(J)$ by contracting the image in J of this other Hall problem in J^e .
- (iv) If J has a type (iv) Hall₁ problem, define $\Delta_1(J)$ to be the contraction of the other copy of $((A, B), C)$ with a coefficient of $1/2$ (Figure 5). This contracted tree has

a symmetry, but it is orientation-preserving because it exchanges two trees which have a single 4-valent vertex and are otherwise univalent. As mentioned in the introductory remarks to the proof, $1/2$ the contracted tree is indeed a basis element.

Lemma 24. Δ_1 is a gradient vector field.

Proof. As we did for Δ_0 , we will show that if J is not Hall_1 , the expansions of $\Delta_1(J)$ distinct from J have increased Hall order. So assume J is not Hall_1 . That means there is a Hall_1 problem in J .

We analyze the cases separately.

Case (i). We have a tree J with a full subtree (A, B, C) which has a Hall problem in one of the A, B, C trees. Contracting this Hall problem and then applying ∂ without backtracking will either expand the Hall problem into one of the two other trees besides J , which we already showed increases the Hall order, or will expand the (A, B, C) vertex, which increases the Hall order because $(A, B, C) \prec (U, V)$ if $|(A, B, C)| = |(U, V)|$.

Case (ii). This case has the most numerous collection of terms. The full subtree $((A', A''), B, C)$ with $A' \prec A'' \prec B \prec C$ gets contracted to (A', A'', B, C) and we must show that all terms of $\partial(A', A'', B, C)$ except $((A', A''), B, C)$ increase in Hall order. If the 4-valent vertex gets pushed up the tree, like in $((A', A'', B), C)$, then the Hall order goes up, so we need only consider the 5 possibilities where this doesn't happen:

$$\begin{aligned} &((A', B), A'', C), ((A', C), A'', B), ((A'', B), A', C), \\ &((A'', C), A', B), ((B, C), A', A'') \end{aligned}$$

The components of these trees are not necessarily in order, although they definitely are in the first two trees, and in fact the smallest tree in each of the first two triples has larger Hall order than (A', A'') . For the third tree, it could be that either of (A'', B) or A' is larger, and they could even be equal, however it suffices to observe that both are greater than (A', A'') . Similar remarks hold for the fourth tree. For the fifth tree, we know $A' \prec A''$, but we have no information on (B, C) . So it will suffice to observe that $(A', A'') \prec (B, C)$ and $(A', A'') \prec A'$.

Case (iii). Let J be a nonzero univalent tree with at least two Hall problems, and suppose that J^1 is the largest contraction of a Hall problem and that J^2 is the contraction of another one, so that $J^1 \succeq J^2$, and by definition $J^1 = \Delta_0(J)$. Case (iii) is the case of the tree J^2 , assuming it is distinct from J^1 . Now by definition, $\Delta_1(J^2) = J^{12}$, the tree with both Hall problems collapsed. Now we wish to analyze the trees appearing in $\partial(J^{12})$ distinct from J^2 . Expanding the first vertex yields the tree J^2 and two other trees with larger Hall order. Expanding the second vertex yields J^1 and two other trees larger than J^1 . Since $J_1 \succ J_2$, this shows the Hall order has increased for all of these trees.

Case (iv). Here when we expand the tree with two copies of (A, B, C) , we will get twice the original tree J , which is why we needed to divide by 2. The other trees are all increased with respect to \prec . \square

Thus we have constructed a gradient vector field $\Delta_0 \cup \Delta_1$ on $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_0$ such that every basis element in degree 1 is either in the range of Δ_0 or the support of Δ_1 . So in the Morse complex, there are no nonzero degree 1 chains, implying the first homology is zero. Incidentally, this

also reproduces the classical result that the Hall trees (in degree 0) are independent and form a basis for $\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{L}_n \cong H_0(\mathbb{L}_\bullet; \mathbb{Z}[\frac{1}{2}])$.

This concludes the proof of Proposition 13.

4.3. Proof of Proposition 14. The proof is very similar to the proof of Proposition 13, but now trees with orientation-reversing automorphisms are no longer equal to zero in $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$. We therefore adapt the vector field to take account of these, in some sense generalizing Levine's quasi-Hall basis algorithm [20]. Define the specified basis for $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$ to consist of unoriented trees. This makes sense because tensoring with \mathbb{Z}_2 erases the orientation data.

- Definition 25.**
- (i) A tree in $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$ is a *Hall'* tree if it is either a Hall tree as defined previously in Definition 18 for the $\mathbb{Z}[\frac{1}{2}]$ -coefficients case, or a tree of the form (H, H) where H is a Hall tree.
 - (ii) A *Hall problem* in a tree J in $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$ is either a Hall problem as defined in Definition 18, or it is a full subtree of J of the form (H, H) , for a Hall tree H . Call this latter type of problem a *symmetric Hall problem*.
 - (iii) A *Hall' problem* in a tree J is a Hall problem which is not of the form (H, H) where $J = (H, H)$ and H is Hall.
 - (iv) The *contraction* of a Hall' problem is defined as in Definition 18, with the contraction of a symmetric Hall problem defined by contracting the root edge of the (H, H) subtree.

These definitions were set up so that

$$J \text{ is not Hall} \Leftrightarrow J \text{ has a Hall problem.}$$

$$J \text{ is not Hall}' \Leftrightarrow J \text{ has a Hall}' \text{ problem.}$$

Now, similarly to the $\mathbb{Z}[\frac{1}{2}]$ -coefficients case, Δ'_0 is defined to vanish on Hall' trees; and for J not Hall', $\Delta'_0(J)$ is defined to be the maximal contraction of a Hall' problem in J .

Lemma 26. $\Delta'_0: \mathbb{Z}_2 \otimes \mathbb{L}_0 \rightarrow \mathbb{Z}_2 \otimes \mathbb{L}_1$ is a gradient vector field.

Proof. As in the proof of Proposition 13, we argue that terms in $\partial\Delta'_0(J)$, other than J itself, have increased Hall order. For Hall' problems which are not symmetric Hall problems, we have already argued this in the proof of Lemma 19. For a symmetric Hall problem we contract $((H, H), B)$ to (H, H, B) . Notice that $\partial((H, H), B) = 2((H, B), H) + ((H, H), B) = (H, H, B)$. Thus the only nonzero term in ∂J , where J is the tree containing (H, H, B) , is the original tree. Thus there are no gradient paths involving the vector $((H, H), B) \mapsto (H, H, B)$. \square

Definition 27. A *Hall'*₁ tree is defined to be a non-zero tree in the image of Δ'_0 .

Lemma 28. A tree H is *Hall'*₁ if and only if both of the following two conditions hold:

- (i) H either
 - (a) contains a full subtree (A, B, C) , where $A \prec B \prec C$ are all Hall, such that either $|A| = 1$ or $A = (A', A'')$ with $A' \prec A''$ and $A'' \succeq B$;

(b) or it contains a full subtree (A, A, B) where A is Hall.

- (ii) Let H^e be the tree where the full subtree (A, B, C) is expanded to $((A, B), C)$ in case (a) above, or where the full subtree (A, A, B) is expanded to $((A, A), B)$ in case (b). Then H is the largest tree among all contractions of Hall' problems in H^e .

Proof. The first condition characterizes being the contraction of a Hall' problem and the second makes sure that it is a maximal contraction. Because all trees are nonzero in the complex $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$, there is no longer the subtlety that the expanded tree could be zero (as was the case in Lemma 21). \square

Definition 29. Negating the characterization from Lemma 28, we define a *Hall'₁ problem* in a tree $J \in \mathbb{Z}_2 \otimes \mathbb{L}_1$ to be any one of the following situations. Assume J contains the full subtree (A, B, C) with $A \preceq B \preceq C$.

- (i) One of the trees A, B, C has a Hall problem:
 - (a) $A = B$ and A is not Hall.
 - (b) $A \prec B = C$ and B is not Hall.
 - (c) $A \prec B \prec C$ and one of A, B, C is not Hall.
- (ii) A, B, C are Hall with $A \prec B \prec C$ but $A = (A', A'')$ with $A' \prec A'' \prec B$.
- (iii) $A = B$ are Hall or $B = C$ are Hall or $A \prec B \prec C$ with $|A| = 1$ or $A'' \succeq B$, but this is not the maximal contraction of a Hall' problem in the expanded tree.

Definition 30. Define $\Delta'_1: \mathbb{Z}_2 \otimes \mathbb{L}_1 \rightarrow \mathbb{Z}_2 \otimes \mathbb{L}_2$ to vanish on Hall'₁ trees. Otherwise, referring to Definition 29, Δ'_1 is defined for each case as follows:

- (i) (a) Define Δ'_1 to contract some Hall problem in A .
- (b) Define Δ'_1 to contract some Hall problem in B .
- (c) Define Δ'_1 to contract some Hall problem in A, B , or C .

Each of these three subcases includes the possibility that the contraction of the Hall problem contracts the root edge of A, B or C . For example A might be a tree of the form (H, H) , and Δ'_1 is then defined by contracting the root edge of A to give a full subtree of the form (H, H, B, C) .

- (ii) As in Definition 23, define Δ'_1 to contract the root edge of A .
- (iii) As in Definition 23, define Δ'_1 to contract the other, maximal, Hall' problem.

Lemma 31. $\Delta'_1: \mathbb{Z}_2 \otimes \mathbb{L}_1 \rightarrow \mathbb{Z}_2 \otimes \mathbb{L}_2$ is a gradient vector field.

This will complete the proof of Proposition 14, as we have constructed a gradient vector field $\Delta'_0 \cup \Delta'_1$ on $\mathbb{Z}_2 \otimes \mathbb{L}_\bullet$ with no critical basis elements in degree 1.

4.4. **Proof of Lemma 31.** We need to verify three things as in the proof of Lemma 19:

- (i) $\Delta'_1(J_1) \neq \Delta'_1(J_2)$ for distinct J_1, J_2 that are not Hall'_1 .
- (ii) $\partial\Delta'_1(J) = J + \text{ other trees, for } J \text{ not } \text{Hall}'_1$.
- (iii) All gradient paths terminate.

We proceed similarly to that proof by arguing that in all but one case, the trees in the sum $\partial\Delta'_1(J) - J$ all have larger Hall order than J . This exceptional case is dealt with by supplemental arguments that verify the three conditions.

For type (i) Hall'_1 problems, we are contracting a Hall' problem “above” the 4-valent vertex. If this problem is a Hall problem, we have already done the required analysis in the $\mathbb{Z}[\frac{1}{2}]$ -coefficients proof to show that the other expansions have increased Hall order. If it is a contraction of an (H, H) the same analysis applies with the exception of the case singled out in item (i) of the definition of Δ'_1 , namely that the full subtree $((H, H), B, C)$ contracts to (H, H, B, C) . For type (ii) and (iii) Hall'_1 problems, the $\mathbb{Z}[\frac{1}{2}]$ -coefficient analysis remains valid, and the other expansions have increased Hall order.

Now that we know that the Hall order increases along gradient flows in all but the case where the full subtree $((H, H), B, C)$ contracts to (H, H, B, C) , we proceed to verify conditions (i), (ii) and (iii) for this exceptional case. First we check that condition (i) holds. By the proof of Lemma 19, the only potential difficulty is when at least one of J_1 and J_2 is in the exceptional case. Say J_1 contains the full subtree $((H, H), B, C)$, which contracts to (H, H, B, C) . There are four nonzero terms in $\partial\Delta'_1(J_1)$: J_1 , and three trees containing the full subtrees $(H, H, (B, C))$, $((H, H, B), C)$ and $((H, H, C), B)$. With the exception of J_1 , none of these trees contains a Hall'_1 problem that contracts to (H, H, B, C) . Thus (H, H, B, C) is the image of a unique tree under Δ'_1 .

Condition (ii) also needs only to be checked in the exceptional case, as it automatically follows in the other cases from the fact that the other terms of $\partial\Delta'_1(J)$ have increased Hall order. So we must verify that the nonzero terms of $\partial(H, H, B, C)$ contain only one copy of $((H, H), B, C)$. The only other term that could be equal to $((H, H), B, C)$ is $(H, H, (B, C))$, which would imply that $B = C = H$. In this case $((H, H), H, H)$ is the contraction of a Hall' problem in $((H, H), (H, H))$, and Δ'_1 is either 0, if the tree containing $((H, H), (H, H))$ is in the image of Δ'_0 , or Δ'_1 contracts some other Hall problem elsewhere in the tree. So the hypothesis that Δ'_1 contracts $((H, H), B, C)$ to (H, H, B, C) is not satisfied.

Finally, we tackle (iii). We will show that any gradient path that starts in the exceptional case can never return to its starting point. This implies that a closed gradient path will not contain these exceptional cases, and then the fact that the Hall order increases along gradient paths away from the exceptional cases implies there are no closed paths.

Define a *descendant* of a tree T , to be a tree S such that there is a gradient path starting at T and ending at S .

Lemma 32. *Suppose Δ'_1 contracts the full subtree $((H, H), B, C)$ to (H, H, B, C) . Every descendant of the tree containing $((H, H), B, C)$ is either in the image of Δ'_0 or contains a full subtree of the form (H, H, A) for some tree A .*

Corollary 33. *Any gradient path starting with the tree containing $((H, H), B, C)$ never returns to $((H, H), B, C)$.*

Proof of Corollary 33. This obviously follows from Lemma 32 unless $B = C = H$. However we argued just above in the proof of Lemma 31 that this case violates the hypothesis that Δ'_1 contracts $((H, H), B, C)$ to (H, H, B, C) . \square

Proof of Lemma 32. First, consider descendants which are connected by a length 2 gradient path, i.e., descendants which are terms of $\partial\Delta - \text{id}$ applied to the tree containing $((H, H), B, C)$. There are three such descendants, and they contain trees of the following forms:

$$(H, H, (B, C)), ((H, H, B), C), ((H, H, C), B)$$

All three contain a subtree of the appropriate type. So now assume inductively that J is a k th descendant containing a full subtree of the form (H, H, A) and consider what trees are connected to J by a length 2 gradient path starting at J . Since (H, H, A) is the contraction of a Hall' problem, $\Delta'_1(J)$ will either be 0, if it was the maximal contraction, or it will contract a different Hall' problem which gives the maximal contraction, possibly in the tree A . So $\Delta'_1(J)$ contains one of the following subtrees, depending on the nature of the contracted Hall' problem:

- (i) (H, H, A) if the Hall' problem giving the maximal contraction is not in A .
- (ii) (H, H, A^c) if the Hall' problem giving the maximal contraction is in A but this is not a contraction of the root. Here A^c represents the contraction of this Hall' problem in A .
- (iii) (H, H, K, K) if $A = (K, K)$ and Δ'_1 contracts the root of A .

Applying ∂ without backtracking will always yield a subtree (H, H, B) except for the trees containing, respectively, $((H, H), A)$, $((H, H), A^c)$ and $((H, H), K, K)$. However these trees are all in the image of Δ'_0 as they are each by definition the tree with the maximal contraction of a Hall' problem. \square

5. PROOF OF THEOREM 4

Recall the statement of Theorem 4: η_\bullet induces an isomorphism $\bar{\eta}: \mathcal{T}_n \xrightarrow{\cong} H_1(\bar{\mathbb{L}}_{\bullet, n+2})$. Recalling that \mathcal{T}_n is isomorphic to $H_0(\mathbb{T}_{\bullet, n+2}; \mathbb{Z})$ (Proposition 7), we will construct a gradient vector field Δ on the chain complex $\bar{\mathbb{L}}_{\bullet, n+2}$, and then show that the resulting chain map $\phi^\Delta \circ \eta$ induces an isomorphism

$$H_0(\mathbb{T}_{\bullet, n+2}; \mathbb{Z}) \xrightarrow{\bar{\eta}} H_1(\bar{\mathbb{L}}_{\bullet, n+2}; \mathbb{Z}) \xrightarrow[\cong]{\phi^\Delta} H_1(\bar{\mathbb{L}}_{\bullet, n+2}^\Delta; \mathbb{Z})$$

$\curvearrowright \cong \curvearrowleft$

implying $\bar{\eta}$ is an isomorphism as desired. (Where ϕ^Δ is an isomorphism by Theorem 12.)

To define Δ we begin by noting that via the operation of “removing the rooted edge” each tree $J \in \bar{\mathbb{L}}_{\bullet, n+2}$ defines a unique tree $t \in \mathbb{T}_{\bullet, n+2}$ which we call the *underlying tree* of J . Specifically, if the root of J is adjacent to an internal vertex of valence greater than 3, then t is gotten by deleting the root and (the interior of) its edge but leaving the internal vertex.

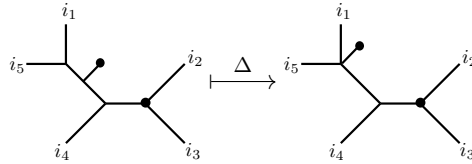
If the root of J is adjacent to a trivalent vertex, then t is gotten by deleting the root and (the interior of) its edge but converting the resulting 2-valent vertex into a non-vertex point of t . This operation in the case where the root is adjacent to a trivalent vertex is succinctly described using the notion of the *inner product* $\langle J_1, J_2 \rangle$ of two rooted trees J_1 and J_2 , which is the unrooted tree defined by identifying the roots of J_1 and J_2 to a single non-vertex point. (The orientation of $\langle J_1, J_2 \rangle$ is given by numbering the middle edge first, followed by the edges of J_1 and then the edges of J_2 in the orderings prescribed by their orientations.) Then $\langle J_1, J_2 \rangle$ is the underlying tree of the bracket (J_1, J_2) .

Next choose a fixed basepoint for every isomorphism class of tree $t \in \mathbb{T}_{\bullet, n+2}$ in the following way. If an internal vertex is fixed by the symmetry group of t , choose the basepoint to be such a vertex. Otherwise, by the following lemma, the tree is of the form $t = \langle T, T \rangle$ for some tree T . In this case choose the basepoint of t to be the midpoint of the middle edge joining the two copies of T .

Lemma 34. *Every $t \in \mathbb{T}_{\bullet, n+2}$ is either of the form $\langle T, T \rangle$ or it has an internal vertex fixed by its symmetry group.*

Proof. The barycenter of a tree t is defined as the midpoint of a maximal geodesic, and is uniquely defined. Note that the symmetry group of the tree fixes this barycenter. If the barycenter is in the middle of an edge, and one endpoint of that edge is not fixed by the symmetry group of the tree, then the tree is of the form $t = \langle T, T \rangle$ for some tree T . \square

Now define Δ as follows: If the root of J is adjacent to an internal vertex of valence greater than 3, then $\Delta(J) = 0$. If $J = \langle T, T \rangle$ then we also define $\Delta(J) = 0$. Otherwise, $\Delta(J)$ is defined by “sliding” the rooted edge of J away from the basepoint of the underlying tree until it attaches to the next internal vertex:



Lemma 35. $\Delta: \bar{\mathbb{L}}_{\bullet, n+2} \rightarrow \bar{\mathbb{L}}_{\bullet+1, n+2}$ is a well-defined gradient vector field.

Proof. Note that Δ is well-defined as a map: Since there are no trees of the form (i, T) in $\bar{\mathbb{L}}_{\bullet, n+2}$, there will always be a neighboring internal vertex to slide to as required. Also $\partial(\Delta(J)) = \pm J$ plus other terms not equal to J since other expansions of $\Delta(J)$ where the root is adjacent to a trivalent vertex have the root further away from the basepoint.

We must check that Δ does not map any \mathbb{Z} -trees to \mathbb{Z}_2 -trees, and does not map any \mathbb{Z}_2 -trees to \mathbb{Z} -trees. We only need to check the cases where $J = (J_1, J_2)$ has the root adjacent to a trivalent vertex. Note that every symmetry of J obviously fixes the root. If J has a symmetry which flips the two outgoing trees J_1 and J_2 , then $J_1 = J_2$ and $\Delta(J)$ was defined to be 0. If $J_1 \neq J_2$, then any symmetry of J restricts to the identity on the edge of the underlying tree to which the root edge attaches, so when we slide the root along this edge to get $\Delta(J)$, the symmetry is still there. If there are no non-trivial symmetries of J , then sliding the root

edge to the neighboring internal vertex can not create a symmetry because such a symmetry would not fix the basepoint on the underlying tree, contradicting our choice of basepoints.

Let us analyze what a gradient path is in this vector field. A non-vanishing application of Δ pushes the root away from the basepoint. Now the only terms of $\partial(\Delta(J)) - J$ on which Δ will evaluate nontrivially are those where the root has been pushed by ∂ onto another edge of the underlying tree that is further away from the basepoint. Repeated applications of Δ moves the rooted edges in such terms further and further away, until they eventually reach univalent edges, where the trees are zero in $\overline{\mathbb{T}}_{\bullet, n+2}$. Thus there are no closed gradient paths and Δ is a gradient field. \square

Next we analyze the critical generators. Given a tree $t \in \mathbb{T}_{\bullet, n+2}$, let t^b denote the tree with a rooted edge attached to the basepoint. Also let $\langle T, T \rangle^s$ denote the tree where a rooted edge is attached to one of the endpoints of the central edge.

Lemma 36. *The Morse complex is generated by the following two types of trees.*

- (i) *Trees of the form t^b , for $t \in \mathbb{T}_{\bullet, n+2}$*
- (ii) *Trees of the form $\langle T, T \rangle^s$.*

Proof. If J is a tree where the root edge is adjacent to a trivalent vertex, then Δ is nonzero, so J is not critical. If the root attaches to a higher-valence vertex v away from the basepoint of the underlying tree t , then consider the tree J' where the root edge attaches to the middle of the edge of t adjacent to v that is closer to the basepoint. Then, in most cases, $\Delta(J') = J$. The one case that is ruled out is the case when $J' = \langle T, T \rangle$, where Δ was defined as 0. This explains why the tree $\langle T, T \rangle^s$ is critical. Finally, if the root edge attaches to the basepoint, t is not in the image of Δ , so it is critical. \square

Recall now the chain map $\eta_{\bullet}: \mathbb{T}_{\bullet} \rightarrow \overline{\mathbb{T}}_{\bullet+1}$ from Lemma 8 defined by summing over attaching a root to all internal vertices. We calculate $\phi^{\Delta} \circ \eta_{\bullet}$ as follows. If t is not of the form $\langle T, T \rangle$, then $\phi^{\Delta}(\eta(t)) = t^b \in \overline{\mathbb{T}}_{\bullet, n+2}^{\Delta}$. This is because Δ vanishes on the tree summands of $\eta(t)$, so $\phi^{\Delta}(\eta(t))$ is the sum of critical generators in $\eta(t)$. On the other hand, suppose $t = \langle T, T \rangle$ where T is of even degree. Then $\phi^{\Delta}(\eta(\langle T, T \rangle)) = 2\langle T, T \rangle^s$. If T has odd degree, then $\eta(\langle T, T \rangle) = 0$, because there is an orientation reversing automorphism exchanging the two endpoints of the central edge.

Now consider the chain map $\phi^{\Delta} \circ \eta$ which will have both a kernel Ker_{\bullet} and a cokernel Cok_{\bullet} :

$$0 \rightarrow \text{Ker}_{\bullet} \rightarrow \mathbb{T}_{\bullet} \rightarrow \overline{\mathbb{T}}_{\bullet+1}^{\Delta} \rightarrow \text{Cok}_{\bullet} \rightarrow 0$$

Let us now analyze the kernel and cokernel. First we set up some convenient notation. A tree denoted by A must have an orientation reversing automorphism. A tree denoted by K must have no orientation reversing automorphism. A tree denoted by J may or may not have one.

Lemma 37. *The cokernel can be written as follows.*

$$\begin{aligned}\text{Cok}_{4i+2} &= \mathbb{Z}\{\langle K, K \rangle^s \mid \deg(K) = 2i + 1\} \oplus \mathbb{Z}_2\{\langle A, A \rangle^s \mid \deg(A) = 2i + 1\} \\ \text{Cok}_{4i+1} &= \mathbb{Z}\{\langle K, K \rangle \mid \deg(K) = 2i + 1\} \oplus \mathbb{Z}_2\{\langle A, A \rangle \mid \deg(A) = 2i + 1\} \\ \text{Cok}_{4i} &= \mathbb{Z}_2\{\langle J, J \rangle^s \mid \deg(J) = 2i\} \\ \text{Cok}_{4i-1} &= \mathbb{Z}_2\{\langle J, J \rangle \mid \deg(J) = 2i\}\end{aligned}$$

Moreover Cok_\bullet is an acyclic complex.

Proof. The critical trees that are not hit by $\phi^\Delta \eta_\bullet$ are of two kinds: (T, T) and $\langle T, T \rangle^s$. No multiple of (T, T) is in the image, whereas $2\langle T, T \rangle^s$ is in the image if and only if T has even degree.

If T is of odd degree and itself has an orientation-reversing automorphism, then both (T, T) and $\langle T, T \rangle^s$ will be 2-torsion, accounting for the \mathbb{Z}_2 -summands in degrees $4i + 1$ and $4i + 2$ above. If T has no such automorphism, then neither does $\langle T, T \rangle^s$ nor (T, T) , accounting for the \mathbb{Z} -summands. Finally, when T has even degree, $2\langle J, J \rangle^s$ is in the image of $\phi^\Delta \eta_\bullet$ and (T, T) is 2-torsion, accounting for the remaining terms above,

Notice that $\partial^\Delta \langle J, J \rangle^s = (J, J)$, because one term of ∂ is critical and equal to (J, J) and the other terms lead to gradient flows that push the root away from the basepoint and which eventually terminate in 0. Thus $\text{Cok}_{4i} \rightarrow \text{Cok}_{4i-1}$ is a direct sum of acyclic complexes of the form $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. Similarly $\text{Cok}_{4i+2} \rightarrow \text{Cok}_{4i+1}$ is a direct sum of acyclic complexes either of the form $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ or $\mathbb{Z} \rightarrow \mathbb{Z}$. Therefore Cok_\bullet is acyclic. \square

Now we turn to an analysis of Ker_\bullet . Again a tree called A must have an orientation reversing automorphism.

Lemma 38. *The kernel can be written as follows.*

$$\begin{aligned}\text{Ker}_{4i+2} &= \mathbb{Z}_2\{\langle T, T \rangle \mid \deg(T) = 2i - 1\} \\ \text{Ker}_{4i+1} &= 0 \\ \text{Ker}_{4i} &= \mathbb{Z}_2\{\langle A, A \rangle \mid \deg(A) = 2i\} \\ \text{Ker}_{4i-1} &= 0\end{aligned}$$

Proof. Clearly $\phi^\Delta \circ \eta_\bullet$ is injective away from symmetric trees $\langle T, T \rangle$. If the degree of T is odd, these are all in the kernel. If the degree of T is even, then $\langle T, T \rangle \mapsto 2\langle T, T \rangle^s$, and so is nonzero unless $\langle T, T \rangle$ is 2-torsion, implying that T has an orientation-reversing automorphism. \square

We are interested in establishing that η_0 induces an isomorphism on homology. This will follow because Ker_0 is generated by trees which are zero in \mathcal{T}_n . (In degree 0, A will contain a subtree of the form (J, J) . Apply IHX to the base edge of this copy of (J, J) to see that $\langle A, A \rangle$ is 0 in \mathcal{T}_n .)

Formally, we argue by splitting the exact sequence into two short exact sequences:

$$\begin{array}{ccccc}
 \text{Ker}_i & \xrightarrow{\iota} & \mathbb{T}_i & \xrightarrow{\phi^{\Delta} \eta_i} & \overline{\mathbb{L}}_{i+1}^{\Delta} \twoheadrightarrow \text{Cok}_i \\
 & & \searrow & & \nearrow \\
 & & \mathbb{T}_i / \text{Im } \iota & &
 \end{array}$$

Then because the inclusion $\iota_0: H_0(\text{Ker}_\bullet) \rightarrow H_0(\mathbb{T}_\bullet)$ induces the zero map, we have that $H_0(\mathbb{T}_\bullet) \cong H_0(\mathbb{T}_\bullet / \text{Im } \iota)$. On the other hand, the cokernel is acyclic, so that $H_i(\mathbb{T}_\bullet / \text{Im } \iota) \cong H_{i+1}(\overline{\mathbb{L}}_\bullet^{\Delta})$ for all i . Therefore

$$\mathcal{T}_n \cong H_0(\mathbb{T}_\bullet) \cong H_1(\overline{\mathbb{L}}_\bullet^{\Delta}) \cong H_1(\overline{\mathbb{L}}_\bullet) \cong \mathcal{D}'_n.$$

This completes the proof of Theorem 4.

Remark: It follows from the above discussion that η_\bullet does not induce an isomorphism $H_i(\mathbb{T}_\bullet) \rightarrow H_{i+1}(\overline{\mathbb{L}}_\bullet)$ for arbitrary i . Indeed, at the next degree there is an exact sequence

$$0 \rightarrow H_1(\mathbb{T}_\bullet) \rightarrow H_2(\overline{\mathbb{L}}_\bullet) \rightarrow \mathbb{Z}_2\{\langle A, A \rangle \mid \deg(A) = 0\} \rightarrow 0$$

This demonstrates a failure of surjectivity of η_1 . In general η can also fail to be injective. For example, $\langle (1, 1, 2), (1, 1, 2) \rangle \in \mathbb{T}_2$ represents a nontrivial 2-torsion homology class, which can be easily checked since there is only one degree 3 tree with the same signature. This tree is clearly in the kernel of η_2 .

6. COMPARING FILTRATIONS OF THE GROUP OF HOMOLOGY CYLINDERS

Let $\Sigma_{g,1}$ denote the compact orientable surface of genus g with one boundary component. A *homology cylinder* over $\Sigma_{g,1}$ is a compact 3 manifold M which is homologically equivalent to the cylinder $\Sigma_{g,1} \times [0, 1]$, equipped with standard parameterizations of the two copies of $\Sigma_{g,1}$ at each “end.” Two homology cylinders M_0 and M_1 are said to be *homology cobordant* if there is a compact oriented 4-manifold W with $\partial W = M_0 \cup_{\Sigma_{g,1}} (-M_1)$, such that the inclusions $M_i \hookrightarrow W$ are homology isomorphisms. This defines an equivalence relation on the set of homology cylinders. Let \mathcal{H}_g be the set of homology cylinders up to homology cobordism over $\Sigma_{g,1}$. \mathcal{H}_g is a group via the “stacking” operation.

Adapting the usual string link definition, Garoufalidis and Levine [10] introduced an Artin-type representation $\sigma_n: \mathcal{H}_g \rightarrow A_0(F/F_{n+2})$ where F is the free group on $2g$ generators, and $A_0(F/F_{n+1})$ is the group of automorphisms ϕ of F/F_{n+1} such that ϕ fixes the product $[x_1, y_1] \cdots [x_g, y_g]$ modulo F_{n+1} . Here $\{x_i, y_i\}_{i=1}^g$ is a standard symplectic basis for $\Sigma_{g,1}$. The Johnson (relative weight) filtration of \mathcal{H}_g is defined by $\mathcal{J}_n = \text{Ker } \sigma_n$. Define the associated graded group $\mathcal{J}_n = \mathcal{J}_n / \mathcal{J}_{n+1}$. Levine shows that $\mathcal{J}_n \cong \mathcal{D}_n$.

On the other hand, there is a filtration related to Goussarov-Habiro’s theory of finite type 3-manifold invariants. We define the relation of A_n -equivalence to be generated by the following move: $M \sim_n M'$ if M' is diffeomorphic to M_C , for some connected clasper C with n nodes. Let \mathcal{Y}_n be the subgroup of \mathcal{H}_g of all homology cylinders A_n -equivalent to the trivial one, and let $\mathcal{Y}_n = \mathcal{Y}_n / \sim_{n+1}$. Rationally, Levine showed the associated graded groups for these two filtrations are the same, and are even classified by the tree group \mathcal{T}_n :

Theorem 39 (Levine). *There is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{T}_n & \xrightarrow{\theta_n} & \mathcal{Y}_n & \longrightarrow & \mathcal{J}_n \xrightarrow{\cong} \mathcal{D}_n \\ & & & \searrow \eta_n & \\ & & & & \end{array}$$

All of these maps are rational isomorphisms.

The story is more subtle over the integers. Levine conjectured the statements in the following theorem, which are straightforward consequences of the fact that η' is an isomorphism.

Theorem 40. *There are exact sequences:*

$$\begin{aligned} 0 \rightarrow \mathcal{Y}_{2n} \rightarrow \mathcal{J}_{2n} \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}_{n+1} \rightarrow 0 \quad n \geq 1 \\ \mathbb{Z}_2^m \otimes \mathcal{L}_n \rightarrow \mathcal{Y}_{2n-1} \rightarrow \mathcal{J}_{2n-1} \rightarrow 0 \quad n \geq 2 \end{aligned}$$

Levine did not conjecture that the map $\mathbb{Z}_2^m \otimes \mathcal{L}_n \rightarrow \mathcal{Y}_{2n-1}$ is injective, and in fact it is *not* injective, basically because the framing relations discussed in [2] are also present in this context. As we prove in [5], for odd numbers of the form $4n - 1$ this allows us to get a sharp answer to what the kernel of $\mathcal{Y}_{4n-1} \rightarrow \mathcal{J}_{4n-1}$ is, while for odd numbers of the form $4n + 1$ we determine it up to $\mathbb{Z}_2 \otimes \mathcal{L}_{n+1}$:

Theorem 41 ([5]). *There are exact sequences*

- (i) $0 \rightarrow \mathcal{Y}_{2n} \rightarrow \mathcal{J}_{2n} \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}_{n+1} \rightarrow 0 \quad n \geq 1$
- (ii) $0 \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}_{2n+1} \rightarrow \mathcal{Y}_{4n-1} \rightarrow \mathcal{J}_{4n-1} \rightarrow 0 \quad n \geq 1$
- (iii) $0 \rightarrow \mathcal{K}_{4n+1}^{\mathcal{Y}} \rightarrow \mathcal{Y}_{4n+1} \rightarrow \mathcal{J}_{4n+1} \rightarrow 0 \quad n \geq 0$, where the kernel $\mathcal{K}_{4n+1}^{\mathcal{Y}}$ fits into the exact sequence $\mathbb{Z}_2 \otimes \mathcal{L}_{n+1} \xrightarrow{a_{n+1}} \mathcal{K}_{4n+1}^{\mathcal{Y}} \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}_{2n+2} \rightarrow 0$.

The calculation of the kernel $\mathcal{K}_{4n+1}^{\mathcal{Y}}$ is thus reduced to the calculation of $\text{Ker}(a_{n+1})$. This is the precise analog of the question in the Whitney tower world of whether α_{n+1} is injective and how nontrivial are the higher-order Arf invariants.

Conjecture 42. *The homomorphisms a_{n+1} are injective for all $n \geq 0$, implying that there is an exact sequence $0 \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}'_{2n+2} \rightarrow \mathcal{Y}_{4n+1} \rightarrow \mathcal{J}_{4n+1} \rightarrow 0$.*

To prove Theorem 40 we will use the following exact sequences

$$\begin{aligned} 0 \rightarrow \mathbb{Z}_2^m \otimes \mathcal{L}_n \rightarrow \mathcal{D}'_{2n-1} \rightarrow \mathcal{D}_{2n-1} \rightarrow 0 \\ 0 \rightarrow \mathcal{D}'_{2n} \rightarrow \mathcal{D}_{2n} \rightarrow \mathbb{Z}_2 \otimes \mathcal{L}_{n+1} \rightarrow 0 \end{aligned}$$

proven by Levine in [19, 20], and the commutative diagram of Theorem 39.

Proof of Theorem 40. In the even case, we get the following diagram:

$$\begin{array}{ccccccc} & & & & & & \mathcal{D}'_{2n} \\ & & & & & & \downarrow \\ & & & & & & \mathcal{D}_{2n} \\ & & & & & & \downarrow \\ \mathcal{T}_{2n} & \xrightarrow{\theta_{2n}} & \mathcal{Y}_{2n} & \longrightarrow & \mathcal{J}_{2n} & \xrightarrow{\cong} & \mathcal{D}_{2n} \\ & & & & & & \downarrow \\ & & & & & & \mathbb{Z}_2 \otimes \mathcal{L}_{n+1} \end{array}$$

from which it follows that the map $Y_{2n} \rightarrow J_{2n}$ is injective with cokernel $\mathbb{Z}_2 \otimes L_{n+1}$. Indeed after making various identifications using the isomorphisms in the diagram, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{2n} & \xrightarrow{\quad} & J_{2n} \twoheadrightarrow \mathbb{Z}_2 \otimes L_{n+1} \\ \downarrow \theta_{2n} & \nearrow & \\ Y_{2n} & & \end{array}$$

where the top row is exact. By commutativity of the triangle, θ_{2n} is an isomorphism.

Turning to the odd case, we have a commutative diagram:

$$\begin{array}{ccccccc} & & & & \mathbb{Z}_2^m \otimes L_n & & \\ & & & & \downarrow & & \\ & & & & D'_{2n-1} & & \\ & & \cong & \nearrow & \downarrow & & \\ \mathcal{T}_{2n-1} & \xrightarrow{\theta_{2n}} & Y_{2n-1} & \longrightarrow & J_{2n-1} & \xrightarrow{\cong} & D_{2n-1} \\ & & \eta' & & & & \end{array}$$

which collapses, after identifications, to the following diagram, where the top row is exact.

$$\begin{array}{ccc} \mathbb{Z}_2^m \otimes L_n & \xrightarrow{\quad} & \mathcal{T}_{2n-1} \twoheadrightarrow J_{2n-1} \\ & & \downarrow \theta_{2n-1} \\ & & Y_{2n-1} \nearrow \end{array}$$

Thus the map $Y_{2n-1} \rightarrow J_{2n-1}$ is surjective, and $\mathbb{Z}_2^m \otimes L_n$ maps onto the kernel. \square

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