

**PENTAGONAL RELATIONS AND THE EXCHANGE MODULE  
OF THE TYPE- $A_n$  CLUSTER ALGEBRA**

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ABSTRACT. In this paper we study relations between the exchange relations in the cluster algebra of type- $A_n$ , which has the 1-skeleton of the associahedron as its exchange graph, and the complex of triangulations of a regular  $(n+3)$ -gon,  $\mathcal{T}_n$ , as its cluster complex. We define the exchange module of the type- $A_n$  cluster algebra to be the  $\mathbb{Z}$ -module generated by all differences of exchangeable cluster variables. Using a notion of discrete homotopy theory, we describe all of the relations in the exchange module and show that these relations are generated by  $\binom{n+2}{4}$  five term relations which correspond to equivalence classes in the abelianization of the discrete fundamental group of  $\mathcal{T}_n$ . Finally, we show that the exchange module is a free  $\mathbb{Z}$ -module and give a minimal generating set of size  $\binom{n+2}{3}$ .

1. INTRODUCTION

In the cluster algebra of type- $A_n$  there is a recurrence relation of the following form:

$$f_1 = x_1, f_2 = x_2 \text{ and } f_{n+1} = \frac{f_n + 1}{f_{n-1}}.$$

It is easy to check that this recurrence has period five, that is,  $f_6 = x_1$  and  $f_7 = x_2$ . In their excellent Park City lecture notes [9], Fomin and Reading present this recurrence and use it as an entry point to motivate the study of reflection groups and cluster algebras. Inspired by this recurrence phenomenon, we study the module generated by all exchanges in the type- $A_n$  cluster algebra and show that the non-commutative relations between exchanges are generated by a special type of relation, which specializes to the recurrence above, that we call *pentagonal relations*.

A cluster algebra, introduced by Fomin and Zelevinsky in [11], is an axiomatically defined commutative ring equipped with a set of distinguished generators, called cluster variables and frozen variables, that are grouped into overlapping collections called clusters. For each cluster, there is an associated matrix which tells us how to

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generate a new cluster via a process called mutation. A pair consisting of a cluster and mutation matrix is called a seed.

Though developed in the context of representation theory, it is noted by Fomin and Zelevinsky that cluster algebras have found applications in fields such as discrete dynamical systems based on rational recurrences [12],  $Y$ -systems in thermodynamic Bethe Ansatz [15], grassmannians and their tropical analogues [18], Poisson geometry and Teichmüller theory [8]. For more information about cluster algebras we refer the reader to the Park City notes by Fomin and Reading [9], the CDM '03 notes by Fomin and Zelevinsky [14], the original series of cluster algebra papers by Fomin and Zelevinsky [11, 13, 15] and of course the comprehensive Cluster Algebras Portal maintained by Fomin at <http://www.math.lsa.umich.edu/~fomin/cluster.html>.

A cluster algebra with a finite number of seeds is said to be of finite type, and Fomin and Zelevinsky showed in [13] that the types of finite cluster algebras correspond exactly to the Cartan-Killing types. The cluster algebra of type- $A_n$ , which is the object we are concerned with, has a combinatorial description using triangulations of an  $(n + 3)$ -gon.

We review the construction of the type  $A_n$  cluster algebra. First, let  $P_n$  be a polygon on  $n + 3$  vertices. Fix a triangulation  $T$  of  $P$ , and label the diagonals of  $T$  with the numbers  $1, \dots, n$  in some order. Also, label the boundary edges of  $P$  with  $n + 1, \dots, 2n + 3$ . Now consider indeterminates  $x_1, \dots, x_{2n+3}$ , where  $x_1, \dots, x_n$  are referred to as *cluster variables*, and  $x_{n+1}, \dots, x_{2n+3}$  are referred to as *frozen variables*. The cluster algebra of type  $A$  is a subring of  $\mathbb{Q}(x_1, \dots, x_{2n+3})$ , which is generated by certain Laurent polynomials in the variables  $x_1, \dots, x_{2n+3}$ . These Laurent polynomials are obtained by exchange relations, which correspond to diagonal flips. The exchange relations are of the form

$$x_k x_{k'} = x_a x_c + x_b x_d,$$

where  $a, b, c, d$  are the diagonals (or edges) that bound the quadrilateral where the flip from  $k$  to  $k'$  occurs. Thus, we obtain a generator  $x_{k'}$ , that is in fact a Laurent polynomial of  $x_a, x_b, x_c, x_d$  and  $x_k$ . Note that  $x_{k'}$  (viewed as a Laurent polynomial) is also referred to as a cluster variable. This is illustrated in Figure 1. The cluster algebra of type  $A$  is generated by all cluster variables coming from diagonal flips. With some work, one sees that there is exactly one such cluster variable for each diagonal, and these cluster variables are all Laurent polynomials in the indeterminates  $x_1, \dots, x_{2n+3}$ . For general cluster algebras, the notion of diagonal flip is replaced with the notion of mutation, the concept of triangulation is replaced with the concept of cluster, and the initial triangulation  $T$  is replaced with an initial *seed*. However, we choose to only give the definitions for type  $A$ , since this cluster algebra is the focus of the current paper.

Associated to each cluster algebra is a *cluster complex* and an *exchange graph*. The cluster complex is the simplicial complex on the set of all cluster variables and whose maximal simplices are clusters; in the type- $A_n$  case this means that the ground set consists of all diagonals of a regular  $(n + 3)$ -gon and the maximal simplices are triangulations of the same  $(n + 3)$ -gon. We denote this complex  $\mathcal{J}_n$ . The exchange graph is simply the dual graph of the cluster complex. That is, the nodes of this graph are in bijection with the clusters, and there is an edge between two nodes if the corresponding clusters differ by a mutation. In the type- $A_n$  cluster algebra, the nodes of the exchange graph are triangulations, and there is an edge between two triangulations if they differ by a single diagonal flip. Readers familiar

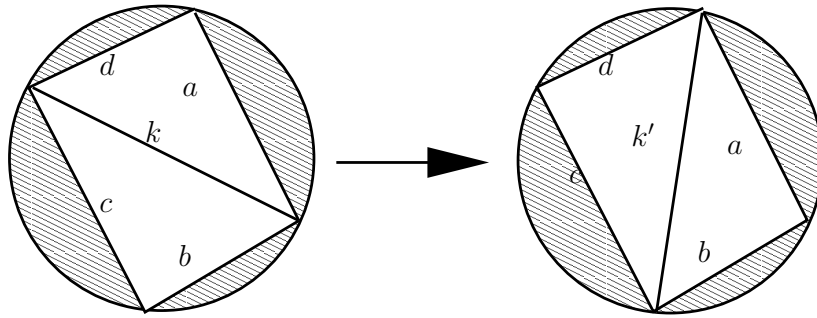


FIGURE 1. A diagonal flip.

with the classical polytope called the *associahedron* will recognize that  $\mathcal{T}_n$ , ordered by reverse inclusion, is the face poset of the associahedron and that the exchange graph is the 1-skeleton of the associahedron. In our concluding remarks we discuss this connection in greater detail. We should note here that in some papers in the literature, e.g., in the work of Simion on the type  $B$  associahedron [19], the associahedron is defined to be the polytope with face poset  $\mathcal{T}_n$  ordered by inclusion rather than reverse inclusion.

The exchange graphs for the type- $A_2$  and  $A_3$  cluster algebras are shown in Figures 2 and 3 respectively.

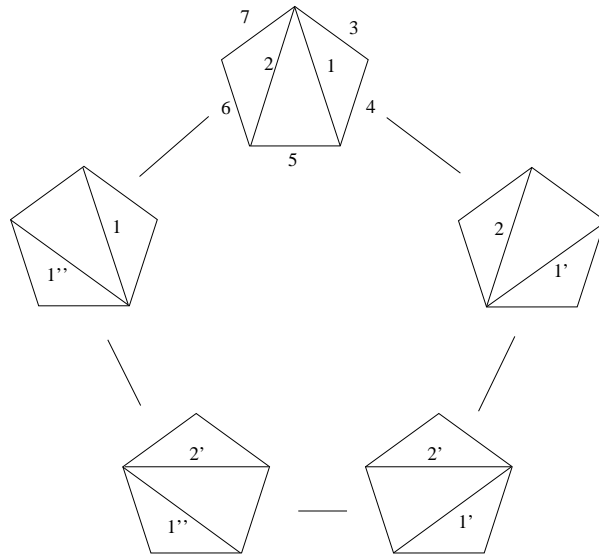


FIGURE 2. The exchange graph of the type- $A_2$  cluster algebra. Vertices are represented by triangulations.

Given a cluster algebra  $\mathcal{A}$ , of type- $A_n$ , we define the exchange module,  $E(\mathcal{A})$  to be the  $\mathbb{Z}$ -module generated by all differences of cluster variables  $x_\alpha - x_\beta$ , where  $\alpha$  and  $\beta$  are crossing diagonals of the  $(n + 3)$ -gon. We let  $F(\mathcal{A})$  be the free  $\mathbb{Z}$ -module generated by all pairs of crossing diagonals of an  $(n + 3)$ -gon, and construct a map

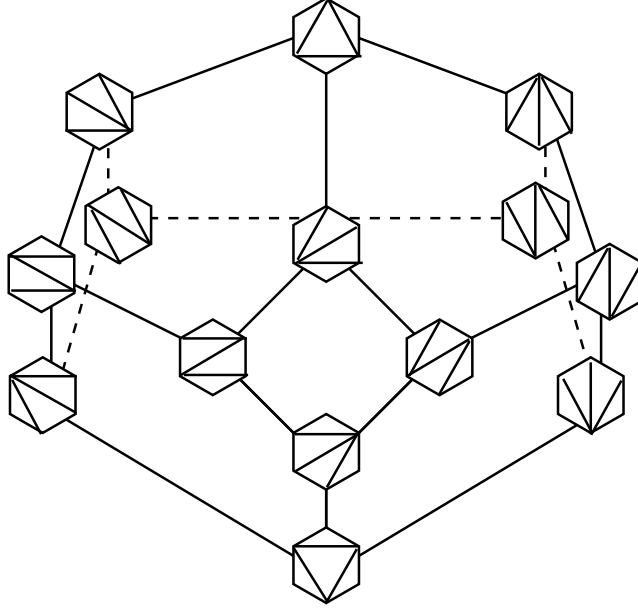


FIGURE 3. The exchange graph of the type- $A_3$  cluster algebra. Vertices are again represented by triangulations.

$\theta : F(\mathcal{A}) \rightarrow E(\mathcal{A})$  where  $X_{\alpha\beta} \mapsto x_\alpha - x_\beta$ . We show that the kernel of  $\theta$  is generated by relations which correspond to inscribed pentagons in an  $(n+3)$ -gon and call any such relation a pentagonal relation. To write down the actual relation we sum over all exchanges which occur entirely inside that pentagon. An example is given below.

*Example 1.1* (The pentagonal relation inside the type- $A_2$  cluster algebra). Fix a triangulation of a regular pentagon, with the diagonals labeled 1, 2 and the edges labeled 3, 4, 5, 6, 7 as shown in Figure 2. The diagonal 1' is obtained by flipping diagonal 1 in the only possible way. The new cluster variable is

$$x_{1'} = \frac{x_3x_5 + x_2x_4}{x_1}.$$

We then flip diagonal 2 to a new diagonal, 2' and obtain the cluster variable

$$x_{2'} = \frac{x_3x_6 + x_{1'}x_7}{x_2} = \frac{x_1x_3x_6 + x_3x_5x_7 + x_2x_4x_7}{x_1x_2}.$$

Next, we flip 1' to 1'' and we have the new cluster variable

$$x_{1''} = \frac{x_{2'}x_5 + x_4x_6}{x_{1'}} = \frac{x_1x_6 + x_5x_7}{x_2}$$

Finally, we flip diagonal 2' to 2'' = 1 and see that

$$x_{2''} = \frac{x_3x_{1''} + x_4x_7}{x_{2'}} = x_1.$$

The cluster algebra of type- $A_2$  is the commutative subring of  $\mathbb{Q}(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$  generated by the cluster variables and frozen variables above. As expected, if we

set  $x_3 = \dots = x_7 = 1$ , then we recover the recurrence from the beginning of the introduction.

Moreover, by definition,  $E(\mathcal{A})$  is generated by  $x_1 - x_{1'}$ ,  $x_2 - x_{2'}$ ,  $x_{1'} - x_{1''}$ ,  $x_{2'} - x_1$ , and  $x_{1''} - x_2$ . However, there is a relation among these generators, which is given by

$$(x_1 - x_{1'}) + (x_2 - x_{2'}) + (x_{1'} - x_{1''}) + (x_{2'} - x_1) + (x_{1''} - x_2) = 0.$$

Given a set of noncrossing diagonals, we always obtain a polygonal dissection of  $P$ . Suppose we have a set  $S$  of noncrossing diagonals for which the corresponding polygonal dissection consists of triangles and exactly one pentagon. Suppose we have labeled the vertices of  $P$  clockwise as  $1, \dots, n+3$ . Let  $a, b, c, d, e$  be the vertices of the pentagon, with  $a < b < c < d < e$ . Given all ways to complete  $S$  to get a triangulation  $T$ , we see that we obtain cluster variables  $x_{a,c}$ ,  $x_{b,d}$ ,  $x_{c,e}$ ,  $x_{a,d}$  and  $x_{b,e}$ , one corresponding to each diagonal in the interior of the pentagon, where we have chosen  $x_{a,c}$  to represent the cluster variable for the diagonal with endpoints  $a$  and  $c$ . From considering diagonal flips involving diagonals in the interior of the pentagon, we see that

$$(x_{a,c} - x_{b,c}) + (x_{a,d} - x_{b,e}) + (x_{b,d} - x_{c,e}) + (x_{b,e} - x_{a,c}) + (x_{c,e} - x_{a,d}) = 0$$

is a relation in the exchange module. Given any such polygonal dissection, we obtain such a relation, and for this reason we shall refer to such relations as *pentagonal relations*.

One of the main results of this paper is that the exchange module is a free module, generated by all exchanges  $x_\alpha - x_\beta$  where one of the endpoints of  $\alpha$  is the vertex 1. This set turns out to be minimal, of size  $\binom{n+3}{2}$ . We prove this result by studying  $\ker \varphi$ . In particular, we prove the following theorem.

**Theorem 1.2.**  *$\ker \theta$  is isomorphic to  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ , the abelianization of the discrete fundamental group of  $\mathcal{T}_n$ .*

Discrete homotopy theory is a combinatorial analogue of classical homotopy theory, introduced by Barcelo, et al [2, 3]. We review the basic definitions of discrete homotopy theory in Section 2. As a result of this theorem, we can study the abelianization of the discrete fundamental group instead of  $\ker \varphi$ . An important fact is the following:

**Theorem** (Barcelo, et al.). *Two cycles of length  $> 4$ ,  $C$  and  $C'$  are homotopic (in the sense of discrete homotopy theory) if and only if they differ by 3- and 4-cycles.*

We show that we only need to study nets of 4-cycles between geodesic 5-cycles. This is a useful property, as we can associate a pentagonal relation to each 5-cycle, and two 5-cycles  $C$  and  $C'$  differ by a net of 4-cycles if and only if they have the same associated pentagonal relation. To see this, consider the 5-cycles appearing on the right hand side of Figure 4. To each 5-cycle, we have the corresponding polygonal dissection appearing on the left hand side of Figure 4. We show that the corresponding dissection consists of triangles and exactly one pentagonal region. We already know that each such dissection gives rise to a pentagonal relation, and hence each 5-cycle gives rise to a pentagonal relation. Also note that the polygonal dissections appearing in Figure 4 differ by diagonal flips. Moreover, the diagonal flips involve diagonals that do not intersect the interior of the pentagonal region. As a result, we do find that the corresponding 5-cycles differ by a net of 4-cycles.

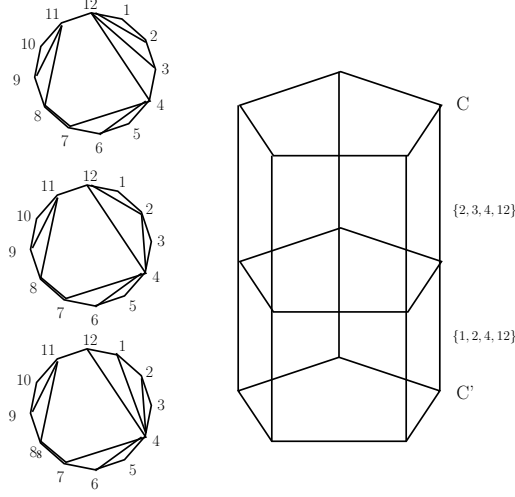


FIGURE 4. A net of 4-cycles between geodesic 5-cycles  $C$  and  $C'$ . The labeled edges describe the quadrilateral each flip takes place in.

In addition, the pentagonal regions have the same vertices on their boundaries, and hence correspond to the same pentagonal relation. This example should serve as motivation that 5-cycles differ by a net of 4-cycles if and only if they correspond to the same pentagonal relation. In the case of this example, the reader can check that the pentagonal relation is:

$$(x_{1,8} - x_{5,9}) + (x_{1,9} - x_{5,12}) + (x_{5,9} - x_{8,12}) + (x_{5,12} - x_{1,8}) + (x_{8,12} - x_{1,9}) = 0$$

In particular, we have that  $\ker \varphi$  is generated by pentagonal relations. In fact, more is true: it is a free  $\mathbb{Z}$ -module with a minimal generating set given by pentagonal relations where one of the cluster variables appearing in the relation corresponds to a diagonal with 1 as one of its endpoints. Note that the pentagonal relation corresponding to the polygonal dissections in Figure 4 is not in this generating. We actually prove the following equivalent result, stated in terms of equivalence classes of 5-cycles and  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ .

**Theorem 1.3.**  $A_1^{n-2}(\mathcal{T}_n)^{ab}$  is a free abelian group with a minimal generating set of size  $\binom{n+2}{4}$ .

These two results are fundamental in our proof of the main result of our paper:

**Theorem 1.4.**  $E(\mathcal{A})$  is a free  $\mathbb{Z}$ -module, with generating set  $\{x_\alpha - x_\beta : \alpha \text{ has } 1 \text{ as an endpoint}\}$ . Moreover, this generating set has size  $\binom{n+2}{3}$ .

We start with a review of discrete homotopy theory. Next, we study the discrete homotopy group of  $\mathcal{T}_n$  in Sections 3 and 4. In Section 4 we prove Theorem 1.3. We study the exchange module in Section 5, and prove Theorems 1.2 and 1.4. We conclude with some remarks about generalizations of the associahedron that we can apply our same method to, as well as a brief discussion of extending our results to non-finite cluster algebras.

2. DISCRETE HOMOTOPY THEORY

All of the definitions and theorems in this section are from [2].

Discrete homotopy theory is a tool to describe the combinatorial connectedness of a simplicial complex  $\Delta$ . Informally, we fix an integer parameter  $0 \leq q \leq \dim(\Delta) - 1$  and say that two simplices  $\sigma$  and  $\tau$  are  $q$ -near if they share a  $q$ -face. With this notion of  $q$ -near we may construct  $q$ -chains

$$\sigma_0 - \sigma_1 - \cdots - \sigma_k$$

such that  $\sigma_i$  and  $\sigma_{i+1}$  are  $q$ -near for  $i \leq k - 1$ . If we fix a simplex,  $\sigma_0$ , we call a  $q$ -chain that starts and ends with  $\sigma_0$  a  $q$ -loop based at  $\sigma_0$ . We say that two simplices are  $q$ -connected if there is a  $q$ -chain between them. Just as in classical homotopy theory, there is an equivalence relation on the set of all  $q$ -loops with a common base simplex  $\sigma_0$ .

**Definition 2.1.** We define an equivalence relation  $\simeq_A$  on the set of all  $q$ -loops with a common base element  $\sigma_0$  in the following way:

(1)

$$\begin{aligned} &\sigma_0 - \sigma_1 - \cdots - \sigma_i - \cdots - \sigma_0 \\ &\simeq_A \sigma_0 - \sigma_1 - \cdots - \sigma_i - \sigma_i - \cdots - \sigma_0 \end{aligned}$$

(2)  $(\sigma) \simeq_A (\tau)$  if they are of the same length and there is a grid between them as in Figure 5.

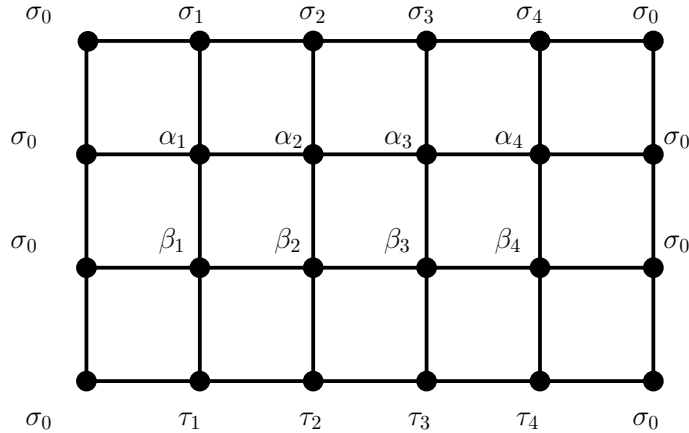


FIGURE 5. A grid between  $(\sigma)$  and  $(\tau)$ . Edges indicate simplices are  $q$ -near.

The grid is a sequence of  $q$ -loops that provide a discrete deformation of  $(\sigma)$  to  $(\tau)$ . Each row of the grid is a  $q$ -loop itself, and each element of the row is  $q$ -near to an element in the loop above and an element in the loop below.

Analogous to classical homotopy theory, there is a group structure on the set of equivalence classes of  $q$ -loops based at  $\sigma_0$  with the operation of concatenation of loops.

**Theorem 2.2.** *The set of all equivalence classes of  $q$ -loops of simplices in  $\Delta$  based at  $\sigma_0$  forms a group, denoted  $A_1^q(\Delta, \sigma_0)$ . The group operation is concatenation of loops. The identity is the equivalence class of constant loop  $\sigma_0$ . The inverse of  $[\sigma]$  is the class  $[\sigma^{-1}]$ , where  $\sigma^{-1}$  is the loop  $\sigma$  read in reverse order.*

The group  $A_1^q(\Delta, \sigma_0)$  is called the  $q$ -discrete fundamental group of  $\Delta$ . When we are working with a fixed value for  $q$ , we will omit the  $q$ - from the notation and refer to  $A_1^q(\Delta, \sigma_0)$  as simply the discrete fundamental group. There are higher discrete homotopy groups as well, and although we will not consider them in this paper we refer the interested reader to [2] for more information about them.

In practice we will also drop the base point from the  $q$ -discrete fundamental group notation. Proposition 2.4 of [2] states that if a complex  $\Delta$  is  $q$ -connected and  $\sigma_0$  and  $\tau_0$  are maximal simplices, then  $A_1^q(\Delta, \sigma_0)$  is isomorphic to  $A_1^q(\Delta, \tau_0)$ .

In practice, the easiest way to understand  $A_1^q(\Delta, \sigma_0)$  is to consider an associated object called the connectivity graph.

**Definition 2.3.** The connectivity graph  $\Gamma_{max}^q(\Delta)$  is the graph whose nodes are in bijection with maximal simplices of  $\Delta$ . There is an edge between node  $\sigma$  and node  $\tau$  if the simplices  $\sigma$  and  $\tau$  are  $q$ -near.

It is easy to see that closed walks in  $\Gamma_{max}^q(\Delta)$  based at  $\sigma_0$  are in bijection with  $q$ -loops in  $\Delta$  based at  $\sigma_0$ . We will use loops and closed walks interchangeably because of this correspondence.

The graph  $\Gamma_{max}^q(\Delta)$  provides a link between discrete homotopy theory and classical homotopy theory. This link is shown in the following theorem.

**Theorem 2.4** ([2], Theorem 5.16).  *$A_1^q(\Delta, \sigma_0) \simeq \pi_1(X_{\Gamma_{max}^q(\Delta)}, \sigma_0)$ , where  $X_{\Gamma_{max}^q(\Delta)}$  is the regular 2-cell complex obtained by attaching a 2-cell along the boundary of every 3- and 4-cycle of the graph  $\Gamma_{max}^q(\Delta)$ , and we use  $\sigma_0$  to denote both a simplex in  $\Delta$  and the corresponding node in  $\Gamma$ .*

From this we can see that two loops ( $\sigma$ ) and ( $\tau$ ) are homotopic if and only if their associated walks in  $\Gamma_{max}^q(\Delta)$  differ by 3- and 4-cycles only. In fact, we can construct a grid by stretching the walk ( $\sigma$ ) around individual 3- and 4-cycles to obtain the sequence of loops that make up the rows of the grid.

In this paper we will be concerned with the abelianization of the discrete fundamental group. It is clear from the isomorphism in Theorem 2.4 and the fact that  $X_{\Gamma_{max}^q(\Delta)}$  is connected that the abelianization  $A_1^q(\mathcal{T}_n)^{ab}$  is isomorphic to the first homology group of  $X_{\Gamma_{max}^q(\Delta)}$ .

While this short review contains the essential information about discrete homotopy theory that we will need, it is by no means a comprehensive look at the topic. Here we refer the interested reader to [1] for more details and a category theoretic treatment of this material.

### 3. DISCRETE HOMOTOPY CLASSES OF $\mathcal{T}_n$

The object we apply discrete homotopy theory to is the cluster complex  $\mathcal{T}_n$ . Maximal simplices in this complex correspond to collections of  $n$  non-crossing diagonals which triangulate a regular  $(n+3)$ -gon  $P_{n+3}$ . If we fix  $q = n-2$  then it is easy to see that  $\Gamma = \Gamma_{max}^{n-2}(\mathcal{T}_n)$  is the graph whose nodes are complete triangulations of  $P_{n+3}$  and has edges corresponding to diagonal flips. Clearly, this graph is the exchange graph of the type- $A_n$  cluster algebra. Note that in this section, and for



the remainder of the paper, we use the term homotopy to mean *discrete homotopy*. Also, to avoid confusion we use the term *node* to refer to the 0-faces in a graph, and *vertex* to refer to the points on the boundary of a labeled regular  $(n + 3)$ -gon.

We observe that every diagonal flip occurs inside a quadrilateral. We will use this quadrilateral to label the corresponding edge of  $\Gamma$ . First, label the vertices of  $P_{n+3}$  in clockwise increasing order with  $1, \dots, n + 3$ .

**Definition 3.1.** Let  $E$  be an edge in  $\Gamma$  with corresponding diagonal flip bounded by the vertices (of  $P_{n+3}$ )  $a, b, c, d$ . We define the label of  $E$ ,  $L(E)$  to be the set  $\{a, b, c, d\}$ .

We note that the same label will be applied to many edges in  $\Gamma$ . A partial triangulation that differs outside of the quadrilateral region a flip takes place in will give us a different edge corresponding to the same flip and hence having the same label.

Next, we consider the special cycles in  $\Gamma$  that correspond to co-dimension 2 simplices in  $\mathcal{T}_n$ . We use the convention of Fomin, Shapiro and Thurston in [10] and refer to these cycles as geodesic cycles. The co-dimension 2 simplices give a partial triangulation of  $P_{n+3}$  which is missing two diagonals which results in two untriangulated quadrilaterals or one untriangulated pentagon.

**Proposition 3.2.** *Let  $T \in \mathcal{T}_n$  be of codimension 2. If  $T$  leaves two quadrilaterals inside  $P_{n+3}$  untriangulated then  $T$  corresponds to a geodesic 4-cycle. Otherwise,  $T$  corresponds to a geodesic 5-cycle.*

*Proof.* Suppose that two quadrilaterals are left untriangulated. There are four ways to triangulate these regions, resulting in the four nodes of  $\Gamma$  that bound a geodesic 4-cycle.

If there are not two untriangulated quadrilaterals, then there must be one untriangulated pentagon. There are exactly five ways to triangulate this region, resulting in five nodes of  $\Gamma$  bounding a geodesic 5-cycle.  $\square$

An important fact about the edge labels of geodesic 4-cycles is that opposite edges have the same label. This can be seen by noting that the diagonal flips occur in two non-overlapping quadrilaterals and so we repeat each flip twice.

As with edge labels, we use the vertices bounding the untriangulated pentagon to label geodesic 5-cycles.

**Definition 3.3.** Let  $C$  be a geodesic 5-cycle in  $\Gamma$  with an untriangulated region (of  $P_{n+3}$ ) bounded by  $a, b, c, d, e$ . We define the label of  $C$ ,  $L(C)$  to be the set  $\{a, b, c, d, e\}$ .

*Remark 3.4.* Let  $C$  be a geodesic 5-cycle with edges  $E_1, \dots, E_5$ . Then we have the following:

- (1)  $L(C) = \bigcup_{i=1}^5 L(E_i)$ .
- (2)  $L(E_i) \neq L(E_j)$  for  $i \neq j$ . That is, every edge of a geodesic 5-cycle has a distinct label.

Just as in the case of edge labels, there are many distinct cycles which share the same label. In fact, the set of 5-cycle labels is in correspondence with the set of discrete homotopy classes in  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ . This is the main result of Section 4, but before we can prove it we will introduce a simplified description of the relation  $\simeq_A$  due to fact that  $\Gamma$  is triangle free. This triangle free version of the relation  $\simeq_A$  has

appeared previously in the work of Barcelo and Smith [5] and Barcelo, Severs and White [4].

Fix a base node  $T_0$  in  $\Gamma$ . Let  $\ell = T_0 - T_1 - \cdots - T_0$  be an  $(n-2)$ -loop (hereafter just a loop). If we read off the labels of the edges of  $\ell$  in order we obtain a word  $w_\ell$  on the alphabet of all  $\binom{n+3}{4}$  possible edge labels. Use of these words will make the proof of our main results more clear.

We now show three possible changes that we can make to  $\ell$  (and  $w_\ell$ ) which preserve its homotopy class and from which we can construct any homotopy.

The first change is **stretch**. This change is from the first part of Definition 2.1. In the loop  $\ell$ , we repeat a simplex  $T_i$ . In  $w_\ell$  there is no change because no new edge was added.

The second change is **insert**. In this change we insert a new simplex,  $T_j$  to the loop  $\ell$ , between two existing simplices  $T_i$  and  $T_{i+1}$ . The simplex  $T_j$  must be  $(n-2)$ -near (hereafter just near) to both  $T_i$  and  $T_{i+1}$  in order to have a valid loop after insertion. However, since we already know that  $T_i$  and  $T_{i+1}$  are near, it must be the case that  $T_i = T_{i+1}$  for otherwise we would have a triangle in  $\Gamma$ . So, this change always inserts a new simplex in the loop between two identical simplices. Now consider the effect on  $w_\ell$ . The insertion of the simplex  $T_j$  creates two new edges,  $(T_i, T_j)$  and  $(T_j, T_{i+1})$ . However, since  $T_i = T_{i+1}$  These edges are the same. In the word  $w_\ell$  then, we have inserted the square of a letter. Note also that no edge was removed due to the equality of  $T_i$  and  $T_{i+1}$ . An example of this change is shown in Figure 6.

The last change is **switch**. In this change we replace one simplex,  $T_i$  with a new simplex  $T_j$  which is near to both  $T_{i-1}$  and  $T_{i+1}$ . All of  $T_{i-1}, T_i, T_{i+1}$  and  $T_j$  are distinct and bound a geodesic 4-cycle in  $\Gamma$ . The change corresponds to using the path  $T_{i-1}, T_j, T_{i+1}$  in place of the path  $T_{i-1}, T_i, T_{i+1}$ . Due to the fact that the opposite edges in a geodesic 4-cycle share the same label, in the word  $w_\ell$  this change commutes the letters corresponding to the edges  $(T_{i-1}, T_i)$  and  $(T_i, T_{i+1})$ . Figure 7 shows an example of this change.

**Proposition 3.5.** *If two loops  $(\sigma)$  and  $(\tau)$  are homotopic then we may deform  $(\sigma)$  into  $(\tau)$  using only changes stretch, insert and switch.*

*Proof.* By Theorem 2.4 and because  $\Gamma$  is triangle free, we know that two loops are homotopic if and only if they differ by 4-cycles only. It is also easy to see that if  $(\sigma)$  and  $(\tau)$  differ by multiple 4-cycles that there is a series of intermediate loops such that each loop differs from the next by only a single 4-cycle. If we consider the loops in  $\Gamma$ , we may start with  $(\sigma)$  and stretch it around the first 4-cycle where  $(\sigma)$  and  $(\tau)$  differ. This produces a new loop,  $(\sigma_1)$  that is homotopic to  $(\sigma)$ . We repeat this process now with  $(\sigma_1)$ , stretching it around the first 4-cycle where it differs from  $(\tau)$ . In this way we construct a series of loops, all homotopic to one another and differing from one another in sequence by a single 4-cycle only. Thus it will suffice to consider cases where  $(\sigma)$  and  $(\tau)$  differ by only a single 4-cycle.

- Case 1.  $(\sigma)$  and  $(\tau)$  differ by a degenerate 4-cycle. That is, the loop  $(\tau)$  traverses an edge to a new simplex and then immediately follows the same edge back. See the top case in Figure 8. This corresponds to the change insert.
- Case 2.  $(\sigma)$  and  $(\tau)$  differ by a non-degenerate 4-cycle as in the second case of Figure 8. This is clearly the same as the change switch since it involves replacing two edges in a 4-cycle with the opposite two edges.

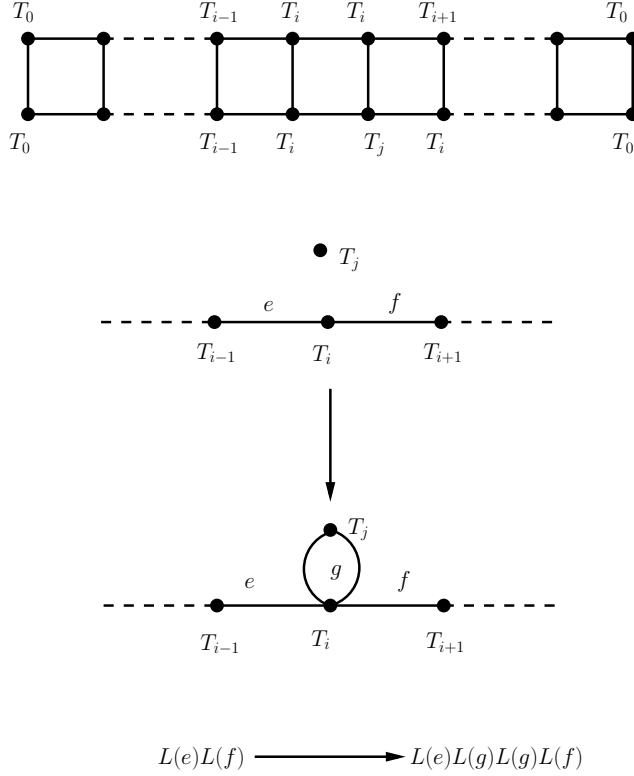


FIGURE 6. The operation insert

- Case 3. The loops differ as in the third case of Figure 8. We insert the simplex  $\tau$  and then switch to the new path using  $\gamma$ .
- Case 4. This case may also be accounted for using only stretch, insert and switch. We first insert both  $\tau$  and  $\gamma$ . Then we change the path  $\gamma, \tau, \sigma_2$  to the path  $\gamma, \delta, \sigma_2$  using switch.

□

*Remark 3.6.* The changes stretch, insert and switch preserve the parity of the letters in a word.

#### 4. A CALCULATION OF $A_1^{n-2}(\mathcal{T}_n)^{ab}$ .

In the previous section we introduced a labeling scheme for the geodesic 5-cycles of  $\Gamma$ . In this section we show that the homotopy classes of  $A_1^{n-2}(\mathcal{T}_n)^{ab}$  correspond exactly to the geodesic 5-cycle label classes. We start by showing that we can build a net of 4-cycles between any two geodesic 5-cycles with the same label by using a series of diagonal flips which correspond to a path between the two cycles.

**Proposition 4.1.** *Let  $C$  and  $C'$  be geodesic 5-cycles in  $\Gamma$  with nodes  $T_1, \dots, T_5$  and  $T'_1, \dots, T'_5$  respectively. Also, assume that  $L(C) = L(C')$  and  $L((T_i, T_{i+1})) = L((T'_i, T'_{i+1}))$ . Then there is a sequence of diagonal flips  $f_1, \dots, f_k$  that give paths from  $T_i$  to  $T'_i$  and such that  $f_j$  commutes with every edge of  $C$  and  $C'$ .*

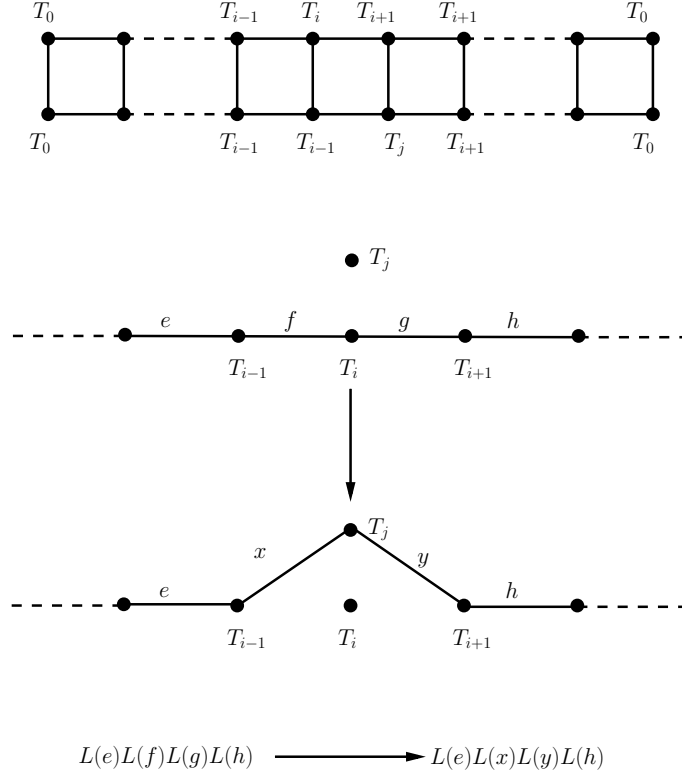


FIGURE 7. The operation switch

*Proof.* The partial triangulation of  $P_{n+3}$  corresponding to the cycle  $C$  leaves a pentagon untriangulated. Outside of this pentagon there are at most five other regions that are triangulated. The partial triangulation corresponding to  $C'$  leaves the same pentagon untriangulated, but differs in the triangulation of at least one of the other regions. In each region where  $C$  and  $C'$  differ, we may perform a series of diagonal flips to change the triangulation of  $C$  to be the same as that of  $C'$ . Each flip corresponds to an edge in  $\Gamma$  and because it occurs outside of the untriangulated pentagon, it commutes with all of the edges of both  $C$  and  $C'$ . Changing each region where  $C$  and  $C'$  differ by performing individual flips produces the desired sequence of edges.  $\square$

We now construct a net of 4-cycles between two 5-cycles with the same label in the following way. First identify pairs of nodes, one from each 5-cycle, such that the triangulations associated to the nodes are identical inside the pentagon described by the cycle label. There are five such pairs. The previous Proposition gives us a path between nodes that are paired together. Each path has the same sequence of edge labels, so they never intersect. Furthermore, at each step in the sequence, the new nodes of paths that originated from adjacent nodes on the 5-cycle differ by exactly a diagonal flip. Thus, the induced graph on the nodes of the cycles and paths is a net of 4-cycles between the starting 5-cycles, and each level of the net is again a 5-cycle in the same homotopy class.

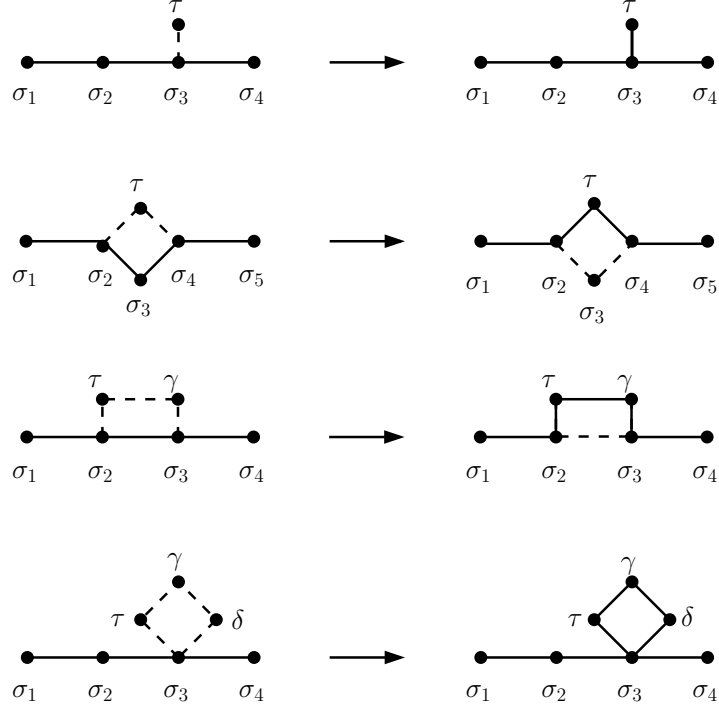


FIGURE 8. The ways in which two loops may differ by a geodesic 4-cycle.

The following theorem establishes that the label classes are in fact the homotopy classes.

**Theorem 4.2.** *Let  $C$  and  $C'$  be geodesic 5-cycles. Then  $C \simeq_A C'$  if and only if  $L(C) = L(C')$ .*

*Proof.* By Proposition 4.1, if  $C$  and  $C'$  have the same label then there is a net of 4-cycles between them and are by definition homotopic. It suffices to show that cycles in the same homotopy class must have the same label.

Suppose that  $L(C) \neq L(C')$  and let  $w_C$  and  $w_{C'}$  be the words associated to  $C$  and  $C'$  respectively. Also let  $c_1, \dots, c_5$  ( $c'_1, \dots, c'_5$ ) be the edges of  $C$  (resp.  $C'$ ). Due to the assumption that  $L(C) \neq L(C')$ , there is an  $i$  such that  $L(c_i) \neq L(c'_j)$  for any  $1 \leq j \leq 5$ . Now recall that any homotopy may be constructed using changes **stretch**, **insert** and **switch**, but also that these changes preserve the parity of letters in a word. Since  $L(c_i)$  occurs exactly once in  $w_C$  and zero times in  $w_{C'}$ , there is no way to change  $w_C$  into  $w_{C'}$  using **stretch**, **insert** and **switch**. Thus, if  $L(C) \neq L(C')$  it must be the case that  $C$  and  $C'$  are in different homotopy classes.  $\square$

The label, and hence homotopy, classes of  $\Gamma$  correspond to pentagons inside  $P_{n+3}$ . There are  $\binom{n+3}{5}$  such classes and these classes certainly generate  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ . However, we can reduce the number of generators and show that the group is free abelian on the smaller set of generators.

We first present an example. Consider the graph  $\Gamma^1(\mathcal{T}_3)$ , whose nodes correspond to triangulations of a hexagon. This graph, with edge labels, is shown in Figure 9.

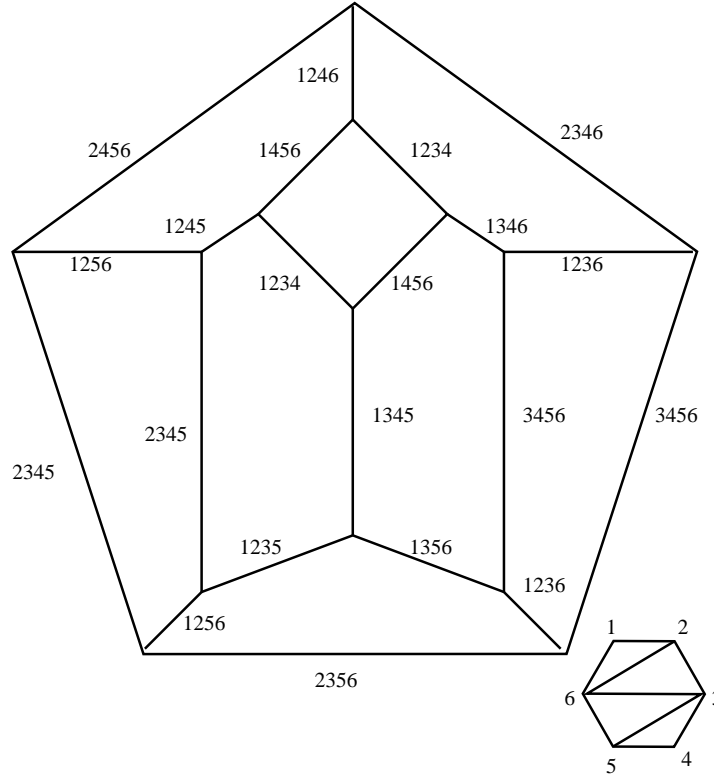


FIGURE 9. The graph  $\Gamma^1(\mathcal{T}_3)$  with edge labels.

It is easy to see that one may write the geodesic 5-cycle with label  $\{2, 3, 4, 5, 6\}$  as a sum of geodesic 4-cycles and those geodesic 5-cycles that have a 1 in their label. Also, we show in the following theorem that no geodesic 5-cycle with a 1 in its label may be written as a sum of other geodesic 5-cycles with a 1 in their label. Thus,  $A_1^1(\mathcal{T}_3)^{ab}$  is generated by geodesic 5-cycles with a 1 in their label. Since there is no relation among these generators,  $A_1^1(\mathcal{T}_3)^{ab}$  is free abelian. There are  $\binom{5}{4} = 5$  such cycles.

We now prove Theorem 1.3 from the introduction.

**Proof of Theorem 1.3.** This proof contains two parts. In the first part, we show that any geodesic 5-cycle without a 1 in its label may be written as a sum of those cycles with a 1 in their label. The idea is that we may find an isomorphic copy of  $\Gamma^1(\mathcal{T}_3)$  and reduce to a case as in the example above. The second part of the proof is showing that there are no relations between geodesic 5-cycles that have distinct labels which contain the element 1. This will follow from a parity argument similar to the proof of Theorem 4.2.

Let  $C$  be a geodesic 5-cycle with  $L(C) = \{a, b, c, d, e\}$ ,  $a < b < c < d < e$  and such that  $1 \notin L(C)$ . Then there exists a geodesic 5-cycle  $C'$  such that  $L(C') = L(C)$

and  $C'$  corresponds to a polygonal dissection of  $P$  into triangles and one pentagon, where the pentagon has vertices  $a, b, c, d, e$  and one of the triangles has vertices  $1, a, e$ . An example appears in Figure 4. We focus on the polygon  $P'$  given by vertices  $1, 2, \dots, a-1, a, e, e+1, \dots, n+3, 1$ . We flip diagonals inside this polygon until we obtain a triangulation of  $P'$  that has the triangle with vertices  $1, a, e$ . The fact that this is possible follows from the fact that such diagonal flips correspond to edges, and triangulations of  $P'$  correspond to vertices in a smaller dimensional associahedron. Since the graph of the associahedron is always connected, and there is at least some triangulation of  $P'$  with a triangle  $1ae$ , there must be some sequence of flips leading to this triangulation. This sequence of flips gives rise to a net of 4-cycles between  $C$  and some 5-cycle  $C'$  with the desired properties.

Now there is a copy of  $\Gamma^1(\mathcal{T}_3)$  inside  $\Gamma$  that corresponds to triangulating the region bounded by  $\{1, a, b, c, d, e\}$  in all possible ways, while keeping the triangulation of the regions outside fixed. This copy of  $\Gamma^1(\mathcal{T}_3)$  looks like Figure 9 but has 2, 3, 4, 5 and 6 replaced by  $a, b, c, d$  and  $e$  respectively. The geodesic 5-cycle on the boundary of the graph is  $C'$  and clearly as in our example it may be written as a sum of those cycles with a 1 in their label. Since  $C'$  is in the same homotopy class as  $C$  we can thus also write  $C'$  as a sum of cycles with 1 in their labels.

Now suppose that  $C$  is a geodesic 5-cycle with  $L(C) = \{1, a, b, c, d\}$ . We claim that  $C$  cannot be written as a sum of geodesic 5-cycles that are from a different homotopy class but also have a 1 in their label. To see this we note that in the word  $w_C$  there is a letter  $\{a, b, c, d\}$ . This letter is from the edge corresponding to the diagonal flip that occurs in the region bounded by  $a, b, c, d$ . This letter appears exactly once in  $w_C$ . If  $C'$  is another geodesic 5-cycle with 1 in its label and such that  $L(C') \neq L(C)$  then the letter  $\{a, b, c, d\}$  cannot appear in  $w_{C'}$  since it would imply  $L(C') = \{1, a, b, c, d\} = L(C)$ . Hence,  $C$  cannot be written as a sum of cycles from different homotopy classes with 1 in their label.

From this we conclude that there are  $\binom{n+2}{4}$  homotopy classes (corresponding to picking a label that contains 1) which generate  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ . Furthermore, there are no relations among these generators. Hence the group is free abelian, and this is a minimum generating set.  $\square$

## 5. THE EXCHANGE MODULE $E(\mathcal{A})$

We now describe the exchange module by its generators and relations. Let  $\mathcal{A}$  be the cluster algebra of type- $A_n$ . We start with a free  $\mathbb{Z}$ -module on the pairs of crossing diagonals. Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be two crossing diagonals of the  $(n+3)$ -gon  $P_{n+3}$ , where the elements  $\alpha_i$  and  $\beta_i$  denote the endpoints of  $\alpha$  and  $\beta$  in  $P_{n+3}$ . Fix a partial order on all diagonals such that  $\alpha < \beta$  if  $\alpha_1 < \beta_1$ . Let  $X_{\alpha\beta}$  be an indeterminate corresponding to the pair of crossing diagonals  $\alpha$  and  $\beta$  and let  $F(\mathcal{A})$  be the free  $\mathbb{Z}$ -module generated by all such  $X_{\alpha\beta}$ .

In order to study  $E(\mathcal{A})$ , the exchange module of  $\mathcal{A}$ , we introduce a map  $\theta : F(\mathcal{A}) \rightarrow E(\mathcal{A})$  such that  $X_{\alpha\beta} \mapsto x_\alpha - x_\beta$  (where  $\alpha < \beta$ ). We claim that  $\ker \theta \simeq A_1^{n-2}(\mathcal{T}_n)^{ab}$ .

Recall that  $X_\Gamma$  is the exchange graph of  $\mathcal{A}$  with a 2-cell attached to the boundary of every 4-cycle in a regular way. We orient each edge by considering the exchange it represents. If the edge changes diagonals  $\alpha$  and  $\beta$  then orient the edge from the triangulation containing the smaller of  $\alpha$  and  $\beta$  to the triangulation containing the larger. In this way, edges that are connected by a net of 4-cycles always have a

parallel orientation. We say that two edges are equivalent if they differ by such a net of 4-cycles. Note that we are assuming that the vertices in this net correspond to distinct vertices of  $\Gamma$ . Let  $C_i(X_\Gamma)$  be the  $i$ th chain group of the 2-cell complex  $X_\Gamma$  and let  $\partial$  be the usual boundary operator.

**Proposition 5.1.**

$$C_1(X_\Gamma)/\partial(C_2(X_\Gamma)) \simeq F(\mathcal{A})$$

*Proof.* We claim that the terms  $X_{\alpha\beta}$  are in bijection with the equivalence classes of edges. This follows from the same argument in Proposition 4.1; if we fix a quadrilateral region inside the  $(n+3)$ -gon and then triangulate outside that region we obtain a homotopic edge. Now send  $\sum_i X_{\alpha_i\beta_i}$  to the sum of edge equivalence classes to produce the isomorphism.  $\square$

The map  $\theta$  may be thought of as sending an edge to the difference of its endpoints. In this way,  $\theta$  behaves exactly as  $\partial$  does when applied to  $C_1(X_\Gamma)$ . We use this to establish an isomorphism between  $\ker \theta$  and  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ .

**Proof of Theorem 1.2.** We first show that  $H_1(X_\Gamma) \simeq \ker \theta$ . Let  $\psi : H_1(X_\Gamma) \rightarrow \ker \theta$  be the map that sends a cycle, which is a sum of equivalence classes of edges indexed by the pairs  $(\alpha_i, \beta_i)$ , to the relation  $\sum_i X_{\alpha\beta}$ . We claim  $\psi$  is an isomorphism.

Any cycle in  $H_1(X_\Gamma)$  is a sum of edges such that under the boundary map the endpoints of the edges sum to zero. The image of such a cycle is a sum  $\sum_i X_{\alpha_i\beta_i}$  and the image of this sum when  $\theta$  is applied is  $\sum_i x_{\alpha_i} - x_{\beta_i}$ . We may identify each  $\alpha_i$  and  $\beta_i$  with the endpoints of the edge which changes diagonal  $\alpha_i$  to  $\beta_i$ . Since the sum of the endpoints of the edges is zero it follows that  $\sum_i x_{\alpha_i} - x_{\beta_i}$  is a relation and so  $\sum_i X_{\alpha_i\beta_i}$  is in  $\ker \theta$ . In the proposition above we showed that elements  $X_{\alpha\beta}$  are in bijection with equivalence classes of edges. From this it is easy to see that no two distinct sums of edges map to the same relation, and that any relation has a well defined pre-image. Note that elements of  $\ker \theta$  of the form  $X_{\alpha_i\beta_i} + X_{\alpha_j\beta_j} - X_{\alpha_i\beta_j} - X_{\alpha_j\beta_i}$  are the images of 4-cycles, which have a 2-face attached, in  $X_\Gamma$ .

To finish the proof we appeal to Theorem 2.4 to establish the isomorphism between  $A_1^{n-2}(\mathcal{T}_n)^{ab}$  and  $H_1(X_\Gamma)$ .  $\square$

Our presentation of  $E(\mathcal{A})$  follows from the presentation of  $A_1^{n-2}(\mathcal{T}_n)^{ab}$ .

**Proof of Theorem 1.4.** Let  $\alpha = (\alpha_1, \alpha_2)$  and  $\beta = (\beta_1, \beta_2)$  be two crossing diagonals such that none of the  $\alpha_i$  or  $\beta_i$  are equal to 1. Then there is a 5-cycle with label  $\{1, \alpha_1, \alpha_2, \beta_1, \beta_2\}$  which corresponds to a pentagonal relation in  $\ker \theta$ . This means that we can write  $x_\alpha - x_\beta$  as a sum of the other four exchange relations in the pentagonal relation.  $\square$

## 6. FUTURE DIRECTIONS

It is well known that the exchange graph of the type- $A_n$  cluster algebra is the 1-skeleton of the classical polytope called the *associahedron* [13]. The associahedron is a polytope first discovered by Stasheff in the course of his research on operads [20] and later rediscovered by Haiman (unpublished) and Lee [16]. The 1-skeleton of the associahedron has also appeared under a different name, the Tamari lattice [21]. The associahedron has also been generalized to a type- $B$  associahedron, called the cyclohedron [19, 6] as well as other classes of polytopes such as graph



associahedra [7], generalized associahedra [13] and generalized permutahedra [17]. A natural extension of our work here is to use the same process to study some of these generalizations.

We first look to the generalized associahedra of Fomin and Zelevinsky in [13]. The generalized associahedra of type- $B$  has a description in terms of centrally symmetric triangulations. The type- $D$  associahedron also has a description in terms of centrally symmetric triangulations with some extra restrictions due to the similarities between the Coxeter groups of type- $B$  and  $D$ . In the case of the type- $B$  associahedron, there are 5-cycles appearing in the same way as the type- $A$  associahedron, and 6-cycles appearing when the central diagonal of a triangulation and an adjacent diagonal are removed. Using the same techniques we developed in this paper, it is easy to study both pentagonal and hexagonal relations in the cluster algebra of type- $B_n$ . The type- $D$  associahedron has a more difficult description that we will omit here, however it is easy to show that it contains only geodesic 4- and 5-cycles and that we can use the same technique to study pentagonal relations in the type- $D_n$  cluster algebra.

Still within the realm of cluster algebras, there are combinatorial descriptions of the cluster complexes of any cluster algebra arising from a triangulated surface, as introduced by Fomin, Shapiro and Thurston [10]. A further, but possibly more difficult, extension of our work would be to study the exchange modules for this class of cluster complexes. Many of the finite type cluster algebras have a description in terms of triangulations of surfaces, however the majority of cluster algebras described this way are not of finite type.

Outside of cluster algebras, we might also study the exchange modules of the graph associahedra of Carr and Devadoss [7]. This generalization gives combinatorial descriptions for polytopes in terms of connected components of graphs. Our initial investigations suggest that much of the work done here will also carry over to these objects.

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