Uniqueness of normalized homeomorphic solutions to nonlinear Beltrami equations

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Abstract

We settle the problem of the uniqueness of normalized homeomorphic solutions to nonlinear Beltrami equations $\overline{\partial} f(z) = \mathcal{H}(z, \partial f(z))$. It turns out that the uniqueness holds under definite and explicit bounds on the ellipticity at infinity, but not in general.

1 Introduction

Homeomorphic solutions $f \in W^{1,1}_{\text{loc}}(\Omega)$ to the classical Beltrami equation

$$\overline{\partial} f(z) = \mu(z) \partial f(z), \qquad \|\mu\|_\infty \le k < 1, \tag{1.1}$$

are well-known to be unique up to composing with a conformal mapping. Such solutions coincide with the class of the two-dimensional quasiconformal mappings, and hence the equation arises naturally in a great variety of topics. For a modern exposition of the equation and the quasiconformal mappings in the plane, see the recent monograph [1]. We consider global solutions, solutions in the entire plane $\Omega = \mathbb{C}$. In this case the uniqueness of homeomorphic solutions to (1.1) is obtained simply by requiring that $f(0) = 0$ and $f(1) = 1$. We call such homeomorphic solutions f as *normalized* solutions to (1.1).

Enquiring the fundamental properties of the *nonlinear* Beltrami equation $\overline{\partial} f(z) = \mathcal{H}(z, f(z), \partial f(z))$, the existence of homeomorphic solutions can be established in great generality. One merely asks of $\mathcal{H}$ a Lusin type measurability in the first two variables and the k-Lipschitz condition ($k < 1$) in the third; for details, see Theorem 8.2.1 in [1]. The notion of nonlinear Beltrami equations (with more restriction on $\mathcal{H}$ than above) was introduced in [3] and [5].

However, the uniqueness remains more subtle, even for the system

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \text{for almost every } z \in \mathbb{C}. \tag{1.2}$$

In the monograph [1] the uniqueness of normalized homeomorphic solutions to (1.2) was established in the special cases where $\mathcal{H}(z, w)$ has a compact support in z or when it is homogeneous of degree one in w; in particular, we have the uniqueness when $\mathcal{H}(z, w)$ is $\mathbb{R}$-linear in w. For the general equation (1.2) the question remained open.

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In this note we show that the uniqueness of normalized homeomorphic solutions holds if we have small enough bounds on the ellipticity at infinity, but fails in the case of large ellipticity constants. To be more specific, assume $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies

(H1) For every $w \in \mathbb{C}$, the mapping $z \mapsto \mathcal{H}(z, w)$ is measurable on $\mathbb{C}$.

(H2) For $w_1, w_2 \in \mathbb{C}$,

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leq k(z)|w_1 - w_2|, \qquad 0 \leq k(z) \leq k < 1,$$

for almost every $z \in \mathbb{C}$.

(H3) $\mathcal{H}(z, 0) \equiv 0$.

Our main result is the following.

Theorem 1.1. *Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If*

$$\limsup_{|z| \to \infty} k(z) < 3 - 2\sqrt{2} = 0.17157..., \tag{1.3}$$

then the nonlinear Beltrami equation

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \textit{for almost every } z \in \mathbb{C}, \tag{1.4}$$

admits a unique homeomorphic solution $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$.

Furthermore, the bound on k is sharp: for each $k > 3 - 2\sqrt{2}$, there are functions $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ for which (H1)–(H3) hold, such that (1.4) *admits two normalized homeomorphic solutions.*

Note that in terms of the quasiconformal distortion the bound (1.3) reads as

$$\limsup_{|z| \to \infty} K(z) < \sqrt{2}, \qquad K(z) := \frac{1 + k(z)}{1 - k(z)}.$$

Under extra symmetries in $\mathcal{H}$ the equation (1.4) has a unique normalized solution. This holds for instance if $\mathcal{H}(z, tw) \equiv t\mathcal{H}(z, w)$, no matter how large are the ellipticity constants. For another interesting example, note that the above requirement (H3) asks constant functions to be solutions to the nonlinear Beltrami equation in question. If we assume, in addition, that also the identity function satisfies (1.4) or equivalently

(H4) $\mathcal{H}(z, 1) \equiv 0$,

then ellipticity bounds slightly weaker than (1.3) will suffice:

Theorem 1.2. *Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies conditions (H1)–(H4) for some $k < 1$. If*

$$\limsup_{|z| \to \infty} k(z) < \frac{1}{3}, \tag{1.5}$$

then the function $f(z) = z$ is the unique homeomorphic solution $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ to the nonlinear Beltrami equation

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \textit{for almost every } z \in \mathbb{C}, \tag{1.6}$$

normalized by the conditions $f(0) = 0$ *and* $f(1) = 1$.

This is complemented with counterexamples: for any $k > 1/3$ *there exists* $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *satisfying (H1)–(H4) such that* (1.6) *admits a normalized solution* $f \not\equiv z$.

As it turns out, the knowledge of the existence of enough solutions gives the uniqueness of normalized solutions. We formulate this as an abstract theorem and then deduce some corollaries from it.

Theorem 1.3. *Assume* $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *satisfies (H1)–(H3) for some* $k < 1$. *Let* $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *be a normalized homeomorphic solution to the equation*

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \text{for almost every } z \in \mathbb{C}. \tag{1.7}$$

Then f *is the unique normalized solution, if there exists a continous flow of solutions* $\{\psi_t : 0 \le t \le 1\} \subset W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *of* (1.7) *such that*

(F1) $\psi_0 \equiv 0$, $\psi_1 = f$,

(F2) $f - \psi_t$ *is quasiconformal,* $0 \le t < 1$,

(F3) for fixed $\epsilon > 0$, *there exist* R *and* δ *such that* $\left|\frac{\psi_t(z) - \psi_s(z)}{\psi_t(z) - f(z)}\right| < \epsilon$, *when* $|z| \ge R$ *and* $|t - s| < \delta$,

(F4) $\psi_t(0) = 0$.

Theorem 1.3 yields new proofs of the uniqueness of normalized solutions in some important particular cases; for instance, when $\mathcal{H}$ is compactly supported in z, the case of the $\mathbb{R}$-linear Beltrami equation or even when $\mathcal{H}$ is 1-homogeneous in w, as discussed above. We point out a couple of further interesting applications. Without the z-dependence in $\mathcal{H}(z, w)$, every homeomorphic solution is affine.

Theorem 1.4. *Suppose* $\mathcal{H} : \mathbb{C} \to \mathbb{C}$ *is* k*-Lipschitz,* $k < 1$, *and* $\mathcal{H}(0) = 0$. *Then homeomorphic solutions* $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *to the nonlinear Beltrami equation*

$$\overline{\partial} f(z) = \mathcal{H}\big(\partial f(z)\big), \qquad \text{for almost every } z \in \mathbb{C}, \tag{1.8}$$

are affine; that is, $f(z) = az + \mathcal{H}(a)\bar{z} + f(0)$, *for some constant* $a \in \mathbb{C}$.

In the case that the identity is a solution, we have the following theorem.

Theorem 1.5. *Suppose that* $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *satisfies (H1)–(H4) for some* $k < 1$. *If there is a continuous path* $\gamma(t) : [0,1] \to \mathbb{C}$ *such that* $\gamma(0) = 0, \gamma(1) = 1$, *and uniformly in* $t \in [0,1]$

$$\mathcal{H}\big(z, \gamma(t)\big) \in L^{p_0}(\mathbb{C}), \qquad \text{for some } p_0 < 2,$$

then $f(z) = z$ *is the unique normalized* $W^{1,2}_{\mathrm{loc}}$*-solution to the nonlinear Beltrami equation*

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \text{for almost every } z \in \mathbb{C}.$$

In particular, if there exists a continuous path of linear solutions connecting 0 and the identity, then Theorem 1.5 applies. Nonlinear equations with a rich set of exact solutions enjoy further properties which will be studied in a forthcoming paper.

Finally, we point out an interesting open problem regarding what happens in the borderline case of Theorems 1.1 and 1.2. We expect that in this case (i.e., when $\limsup_{|z|\to\infty} k(z) = 3-2\sqrt{2}$ or $1/3$, respectively) there is a unique homeomorphic solution $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ to the nonlinear Beltrami equation normalized by the conditions $f(0)=0$ and $f(1)=1$.

2 General case, Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let us assume there exist two normalized and homeomorphic solutions $f, g \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ to the nonlinear Beltrami equation (1.4). Then conditions (H2) and (H3) imply $|\overline{\partial} f(z)| \le k(z)|\partial f(z)|$ and similarly for g. Thus f, g are quasiconformal. Let

$$K_\infty := \limsup_{|z|\to\infty} K(z) < \sqrt{2}, \qquad K(z) := \frac{1+k(z)}{1-k(z)}. \tag{2.1}$$

Then, for any $K > K_\infty$,

$$|f(z)|, |g(z)| \le C(1+|z|)^K. \tag{2.2}$$

Indeed, we can decompose $f = H \circ F$, where H and F are normalized quasiconformal homeomorphisms with the Beltrami coefficient of F given by $\chi_{\mathbb{C}\setminus\mathbb{D}(0,R)}\, \mu_f$; above $\mu_f = \overline{\partial} f / \partial f$ is the Beltrami coefficient of f. Moreover, we may choose R so large that F is K-quasiconformal in $\mathbb{C}$. Then

$$\frac{1}{C_K}|z|^{1/K} \le |F(z)| \le C_K |z|^K, \qquad |z| \ge 1.$$

Since H is conformal near ∞, $H(z) = cz + \mathcal{O}(1/z)$, and the bounds (2.2) follow.

Next, as f, g both satisfy (1.4), we have

$$|\overline{\partial} f(z) - \overline{\partial} g(z)| = |\mathcal{H}\big(z, \partial f(z)\big) - \mathcal{H}\big(z, \partial g(z)\big)| \le k(z)|\partial f(z) - \partial g(z)|, \tag{2.3}$$

for almost every $z \in \mathbb{C}$. Thus the difference is quasiregular, but of course not necessarily injective. By the Stoïlow factorization theorem, $f - g = P \circ h$, where P is a holomorphic mapping and h is a normalized $K(z)$-quasiconformal homeomorphism. By (2.2) and $\|K\|$-quasiconformality of h^{-1}, where $\|K\| = \|K\|_\infty$, for $|z| \ge 1$,

$$|P(h(z))| = |f(z) - g(z)| \le C|z|^K = C|h^{-1}(h(z))|^K \le C|h(z)|^{K\|K\|}.$$

Hence P is a polynomial. Since it has at least two zeroes, points 0 and 1, $\deg(P) \ge 2$.

As above, we can decompose $h = H_1 \circ F_1$. Similarly as before: H_1 is a normalized quasiconformal mapping and conformal near ∞. The mapping F_1 is normalized and K-quasiconformal in $\mathbb{C}$. This gives us a lower bound for h.

Combining upper and lower bounds with the fact that $\deg(P) \geq 2$, we achieve, for $|z|$ large enough,

$$\frac{1}{C}|z|^{2/K} \leq |P(h(z))| = |f(z) - g(z)| \leq C|z|^K.$$

This implies $K \geq \sqrt{2}$ leading to a contradiction with (2.1) when $K > K_\infty$ are sufficiently close.

Our section "Counterexamples" below will prove the sharpness of (1.3). $\square$

Proof of Theorem 1.2. We recall the following topological fact without proof.

Lemma 2.1. *Let γ be a Jordan curve and $f : \mathbb{C} \to \mathbb{C}$ a homeomorphism. Suppose that one of the curves γ or $f(\gamma)$ lies inside the other (that is, is separated from ∞). Then the increment of the argument*

$$\underset{0\leq t\leq 2\pi}{\Delta} \arg\left[f(\xi(t)) - \xi(t)\right] = \pm 2\pi$$

with the sign depending on the orientation of f. Above ξ is any parametrization of γ.

One way to prove the above lemma is to deform the inner curve to a point via a homotopy within the component bounded by the outer curve.

Assume now that there exists a normalized solution $\Phi \neq \mathrm{id}$. Conditions (H2) and (H3), and a similar calculation as in (2.3) imply that Φ and $\Phi - \mathrm{id}$ are $K(z)$-quasiregular, $K(z) = \frac{1+k(z)}{1-k(z)}$.

We have that $\Phi - \mathrm{id}$ is $K(z)$-quasiregular with at least two zeros, points 0 and 1. By the Stoïlow factorization, $\Phi - \mathrm{id} = P \circ h$, where P is a holomorphic mapping and h is a normalized $K(z)$-quasiconformal homeomorphism. Thus, by the argument principle, for all sufficiently large $R > 0$, the increment of the argument

$$\underset{|z|=R}{\Delta} \arg\left[\Phi(z) - z\right] \geq 2 \cdot 2\pi.$$

On the other hand, by Lemma 2.1, if the curve $\partial\mathbb{D}(0, R)$ does not intersect the image $\Phi(\partial\mathbb{D}(0, R))$, the increment can be at most 2π. Therefore, for every R large enough, there is a point z_R such that

$$|\Phi(z_R)| = |z_R| = R. \tag{2.4}$$

The mapping Φ is a $K(z)$-quasiconformal homeomorphism of the plane and thus (2.4) forces linear growth at ∞. That is, by quasisymmetry,

$$\frac{1}{\lambda(\|K\|)}|z| \leq |\Phi(z)| \leq \lambda(\|K\|)|z|, \qquad \text{for } |z| \text{ large enough},$$

where $\|K\| = \|K\|_\infty$. Hence,

$$|\Phi(z) - z| \leq C_{\|K\|}|z|, \qquad \text{for } |z| \text{ large enough}. \tag{2.5}$$

Similarly as in the proof of Theorem 1.1, by (2.5) and $\|K\|$-quasiconformality of h^{-1}, P is a polynomial. Since it has at least two zeroes, points 0 and 1, $\deg(P) \geq 2$.

As before, we can decompose $h = H_1 \circ F_1$, where H_1 and F_1 are normalized quasiconformal homeomorphisms. Further, H_1 is conformal near ∞ and F_1 is K-quasiconformal in $\mathbb{C}$ with $K < 2$. The choice of K can be made by assumption (1.5). We get a lower bound for h. Combining the lower bound with the fact that $\deg(P) \geq 2$ and the upper bound (2.5), we achieve, for $|z|$ large enough,

$$c|z|^{2/K} \leq |P(h(z))| = |\Phi(z) - z| \leq C_{\|K\|}|z|.$$

This is a contradiction, since $K < 2$.

The sharpness is obtained in the next section. □

3 Counterexamples

We show that for every $3 - 2\sqrt{2} < k < 1$ there is a function $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, measurable in the first variable and satisfying

$$|\mathcal{H}(z, w_1) - \mathcal{H}(z, w_2)| \leq k|w_1 - w_2| \qquad \text{and} \qquad \mathcal{H}(z, 0) \equiv 0$$

in the second variable, such that the nonlinear Beltrami equation

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \text{for almost every } z \in \mathbb{C}, \tag{3.1}$$

has at least two different homeomorphic solutions $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$, normalized by $f(0) = 0$ and $f(1) = 1$.

We start the construction by setting, for any $0 < t < 1$,

$$F_t(z) = \begin{cases} (1+t)\, z|z| - tz^2, & \text{for } |z| > 1, \\ (1+t)\, z - tz^2, & \text{for } |z| \leq 1, \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)\, z|z| - tz, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leq 1. \end{cases}$$

Both functions are normalized at 0 and 1, and they should be considered as modifications of the radial stretching $\psi(z) = z|z|^{K-1}$, such that their difference is a polynomial vanishing at 0 and 1. Hence one may look for a field $\mathcal{H}(z, w)$ so that F_t, G_t satisfy (3.1). However, composing with an extra quasiconformal factor we will be able to further reduce the distortion constants. For this purpose take

$$\varphi(z) = z|z|^{\sqrt{2}-1}, \quad |z| > 1 \quad \text{with} \quad \varphi(z) = z, \quad |z| \leq 1.$$

and consider the maps $f_t = F_t \circ \varphi^{-1}$ and $g_t = G_t \circ \varphi^{-1}$, or explicitly

$$f_t(z) = \begin{cases} (1+t)\, z|z|^{\sqrt{2}-1} - t(z|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)\, z - tz^2, & \text{for } |z| \leq 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z|z|^{\sqrt{2}-1} - tz|z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leq 1. \end{cases}$$

Both mappings $f = f_t$ and $g = g_t$ are injective by direct argumentation, and normalized. It is immediate that $f - g$ is K-quasiregular with $0 < k = \frac{\sqrt{2}-1}{\sqrt{2}+1} =$

$3-2\sqrt{2}$, $K=\frac{1+k}{1-k}$. Directly estimating $|\overline{\partial} f(z)|$, $|\overline{\partial} g(z)|$ from above and $|\partial f(z)|$, $|\partial g(z)|$ from below gives that f is K_f-quasiregular and g is K_g-quasiregular, where

$$0<k_f=\frac{\sqrt{2}-1+t}{\sqrt{2}+1-t}<1 \quad \text{and} \quad 0<k_g=\frac{2-\sqrt{2}+t}{2+\sqrt{2}+t}<1.$$

Next, define for each fixed $z\notin \partial\mathbb{D}$ the mapping $w\mapsto \mathcal{H}(z,w)$ as follows. First, fix

$$\mathcal{H}(z,0)=0, \qquad \mathcal{H}\big(z,\partial f(z)\big)=\overline{\partial} f(z), \qquad \mathcal{H}\big(z,\partial g(z)\big)=\overline{\partial} g(z). \tag{3.2}$$

The computations above show that the map $\mathcal{H}(z,\cdot):\{0,\partial f(z),\partial g(z)\}\to\mathbb{C}$ is k_0-Lipschitz, where $k_0=\max\{k,k_f,k_g\}$. Using the Kirszbraun extension theorem (for example, Theorem 2.10.43 in [4]) the mapping can be extended to a k_0-Lipschitz map $\mathcal{H}(z,\cdot):\mathbb{C}\to\mathbb{C}$. From an abstract use of the Kirszbraun extension theorem, however, it is not entirely clear that the map $\mathcal{H}$ obtained is measurable in z, i.e., that (H1) is satisfied. To show this, one can proceed as follows.

Fix a countable dense set $\mathcal{D}\subset\mathbb{C}$, enumerated as $\mathcal{D}=\{w_4,w_5,w_6,\dots\}$, set $w_1=0$, $w_2=\partial f(z)$, $w_3=\partial g(z)$, and define $\mathcal{H}(z,w_k)$ recursively, starting with (3.2). Assuming $\mathcal{H}(z,w_k)$ is defined for $k\le N$ with $N\ge 3$, following [4], we set

$$Y_s(z)=\bigcap_{j=1}^{N}\overline{\mathbb{D}}\big(a_j(z),s\,r_j\big),$$

where

$$a_j(z)=\mathcal{H}(z,w_j), \quad r_j=k_0|w_j-w_{N+1}|.$$

Let

$$s_0(z):=\inf\{s>0:\ Y_s(z)\neq\emptyset\}.$$

It is shown in [4, Lemma 2.10.40] that $Y_{s_0}(z)$ consists of a single point, say $b(z)$, and in the proof of [4, Theorem 2.10.43] that $s_0(z)\le 1$. Furthermore, an additional elementary argument shows that

$$(a_1,\dots,a_N)\mapsto b$$

is a continuous map. Therefore, we set

$$\mathcal{H}(z,w_{N+1})=b(z).$$

Since $a_1(z),a_2(z),a_3(z)$ defined in (3.2) are measurable, it follows recursively that each $a_i(z)$ is measurable in z. We obtain a k_0-Lipschitz map $\mathcal{H}(z,\cdot):\mathcal{D}\to\mathbb{C}$ such that for each fixed $w\in\mathcal{D}$ the mapping $z\mapsto\mathcal{H}(z,w)$ is measurable. Since $\mathcal{D}$ is dense, for each fixed z we can (uniquely) extend $\mathcal{H}(z,w)$ to a k_0-Lipschitz map $\mathbb{C}\to\mathbb{C}$, which is then measurable in z.

We have now found $\mathcal{H}(z,w)$, satisfying (H1)–(H3) with $k=k_0$, such that (3.1) has two different normalized solutions. Letting $t\to 0$ makes $k_0\to 3-2\sqrt{2}$. The proof of Theorem 1.1 is thus complete.

To prove the sharpness of Theorem 1.2 one may modify the counterexample above. However, a more convenient approach is to simply note that given the functions f_t and g_t, one may change the variables so that both the identity and the composition $\Phi=g_t\circ f_t^{-1}$ satisfy the same nonlinear Beltrami equation. We may thus use the following general factorization result to conclude Theorem 1.2.

Lemma 3.1. *Let $\mathcal{H} : \Omega \times \mathbb{C} \to \mathbb{C}$ be measurable in the first variable, $k(z)$-Lipschitz in the second, $0 \le k(z) \le k < 1$, and $\mathcal{H}(z, 0) = 0$. If $f : \Omega \to \Omega'$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, is a homeomorphic solution to the nonlinear Beltrami equation*

$$\overline{\partial} f(z) = \mathcal{H}\big(z, \partial f(z)\big), \qquad \text{for almost every } z \in \Omega, \tag{3.3}$$

then any other solution $g \in W^{1,2}_{\mathrm{loc}}(\Omega)$ takes the form $g = \Phi \circ f$, where Φ solves

$$\overline{\partial}\Phi(u) = \tilde{\mathcal{H}}\big(u, \partial\Phi(u)\big) \tag{3.4}$$

with $\tilde{\mathcal{H}} : f(\Omega) \times \mathbb{C} \to \mathbb{C}$ measurable in the first variable, $\tilde{k}(u)$-Lipschitz in the second, where $\tilde{k}(u) = \frac{2k(z)}{1+k(z)^2}$, $u = f(z)$, and $\tilde{\mathcal{H}}(u, 0) = 0 = \tilde{\mathcal{H}}(u, 1)$. Furthermore, the function $\tilde{\mathcal{H}}$ depends only on $\mathcal{H}$ and the coordinate change f, but is independent of Φ.

Proof. We use the chain rule and substitute $g(z) = \Phi(f(z))$ to the equation (3.3). We get the nonlinear relation between Φ_u and $\Phi_{\bar{u}}$

$$f_{\bar{z}}\Phi_u + \overline{f_z}\Phi_{\bar{u}} = \mathcal{H}\big(z, f_z\Phi_u + \overline{f_{\bar{z}}}\Phi_{\bar{u}}\big), \tag{3.5}$$

where $z = f^{-1}(u)$ and $f_{\bar{z}} = \mathcal{H}(z, f_z)$. Solving this for $\Phi_{\bar{u}}$ in terms of Φ_u using the contraction mapping principle, see Chapter 9.1 in [1], gives the equation (3.4), where $\tilde{\mathcal{H}} : f(\Omega) \times \mathbb{C} \to \mathbb{C}$ is measurable in the first variable and $\tilde{k}(u)$-Lipschitz in the second.

We are left to check $\tilde{\mathcal{H}}(u, 0) = 0 = \tilde{\mathcal{H}}(u, 1)$. For this we let $\Phi_u = 0$ and $\Phi_u = 1$ in (3.5) and solve it for $\Phi_{\bar{u}} = \tilde{\mathcal{H}}(u, 0)$ and $\Phi_{\bar{u}} = \tilde{\mathcal{H}}(u, 1)$, respectively. This is equivalent to the equations

$$\overline{f_z}\Phi_{\bar{u}} = \mathcal{H}\big(z, \overline{f_{\bar{z}}}\Phi_{\bar{u}}\big),$$
$$f_{\bar{z}} + \overline{f_z}\Phi_{\bar{u}} = \mathcal{H}\big(z, f_z + \overline{f_{\bar{z}}}\Phi_{\bar{u}}\big).$$

In both cases we find that

$$|\overline{f_z}||\Phi_{\bar{u}}| \le k^2|\overline{f_z}||\Phi_{\bar{u}}|,$$

and thus $\Phi_{\bar{u}} = 0$ almost everywhere as wanted. Above we use the k-Lipschitz property of $\mathcal{H}$ and K-quasiconformality of f, $K = \frac{1+k}{1-k}$, which is a straightforward calculation as in the beginning of the proof of Theorem 1.1. □

Note that

$$k = 3 - 2\sqrt{2} \;\Leftrightarrow\; \frac{2k}{1+k^2} = \frac{1}{3},$$

thus examples proving sharpness of the bound (1.3) yield, via factorization and Lemma 3.1, also examples showing the sharpness of Theorem 1.2. A similar reasoning shows that the uniqueness part of Theorem 1.1 could be deduced from Theorem 1.2.

4 Flow of solutions, Theorems 1.3, 1.4, and 1.5

Proof of Theorem 1.3. We use similar methods as in the proof of Theorem 6.2.2 in [1]. Let f be as in the statement of the theorem and g be another normalized solution. We construct two different flows of maps. The flow $L_t = f - \psi_t$ is a family of quasiconformal mappings joining f and 0. The flow $g_t = g - \psi_t$ is a family of quasiregular mappings joining the homeomorphism g with the noninjective map $g - f$.

Let $T \subset [0, 1)$ denote the set of parameters t for which g_t is a homeomorphism. One such parameter is $t = 0$. By the Hurwitz-type theorem, Theorem 3.9.4 in [1], we find that T is a relatively closed subset of $[0, 1)$. Thus we need to show that T is open.

Now, fix a parameter $t \in T$. The mapping

$$g - f = g_t - L_t$$

is, by assumption, a nonconstant K-quasiregular mapping with at least two zeros, points 0 and 1. Therefore, the composition

$$(g - f) \circ L_t^{-1} = g_t \circ L_t^{-1} - \mathrm{id}$$

is K^2-quasiregular and has also two zeros.

We use the same ideas as in the proof of Theorem 1.2. First, applying the argument principle and Lemma 2.1 to the difference $g_t \circ L_t^{-1} - \mathrm{id}$, we get that for every R large enough there is a point z_R such that

$$|g_t \circ L_t^{-1}(z_R)| = |z_R| = R.$$

As $t \in T$, $g_t \circ L_t^{-1}$ is quasisymmetric, and since by (F4) it fixes the origin, we obtain

$$\frac{1}{\lambda(K)}|z| \le |g_t \circ L_t^{-1}(z)| \le \lambda(K)|z| \qquad \text{whenever } |z| \ge R, \tag{4.1}$$

if $R > 0$ is large enough.

Secondly, the continuity assumption (F3) and the equation (4.1) allow us to compare $g_t \circ L_t^{-1}$ with $g_s \circ L_t^{-1}$ when $|t - s|$ is small enough. Indeed, there is $\delta > 0$ and $R_0 > 0$ such that if $|t - s| < \delta$, one has

$$|g_s(w) - g_t(w)| = |\psi_t(w) - \psi_s(w)| \le \frac{1}{2\lambda(K)}|L_t(w)|,$$

for $|w| \ge R_0$. Writing $L_t(w) = z$, we obtain that

$$|g_s \circ L_t^{-1}(z) - g_t \circ L_t^{-1}(z)| \le \frac{1}{2\lambda(K)}|z|, \tag{4.2}$$

for every z outside the set $L_t(\mathbb{D}(0, R_0))$.

We now fix $w_0 \in \mathbb{C}$. Since $g_t \circ L_t^{-1}$ is a homeomorphism by assumption, the winding number of $(g_t \circ L_t^{-1})(\partial\mathbb{D}(0, R))$ around w_0 is 1, for R large enough. Therefore conditions (4.1) and (4.2) show that for $|t-s| < \delta$ the winding number of $g_s \circ L_t^{-1}$ is 1 as soon as $R \ge 2\lambda(K)|w_0|$. It follows that the mappings g_s for $|t - s| < \delta$ are homeomorphisms and $s \in T$. Thus T is open.

We have proven that, for all $t \in [0, 1)$, the mappings g_t are quasiconformal homeomorphisms of the plane. Hence by Hurwitz-type arguments, e.g., Theorem 3.9.4 in [1], their locally uniform limit $g - f$ is either a homeomorphism or a constant. Having at least two zeroes, it must be the constant map 0. □

Proof of Theorem 1.4. First we find a homeomorphic, linear, and normalized solution to the nonlinear Beltrami equation (1.8): By the Banach fixed point theorem, the contraction $w \mapsto 1 - \mathcal{H}(w)$ has a unique fixed point $a \in \mathbb{C}$. Then the linear mapping $f(z) = az + \mathcal{H}(a)\bar{z}$ is solution to (1.8), fixes 0 and 1, and is injective by the inequality $|\mathcal{H}(a)| \le k|a|$.

Now, we can apply Theorem 1.3 with the linear maps $\psi_t(z) = taz + \mathcal{H}(ta)\overline{z}$ for $t \in [0,1]$ to see that f is the only normalized solution.

To show that any homeomorphic solution g to (1.8) is affine, we may assume $g(0) = 0$. Given $g(1) = b$, then $h(z) = g(z)/b$ is the normalized solution to $\overline{\partial} h = \widetilde{\mathcal{H}}(\partial h)$, where $\widetilde{\mathcal{H}}(w) = \mathcal{H}(b\,w)/b$. By the above h, and hence g, is linear. ☐

Proof of Theorem 1.5. Since $\mathcal{H}(z,1) = 0$, we already know that $f(z) = z$ is a normalized solution. We will get the uniqueness by applying Theorem 1.3. We can assume $\gamma(t) \notin \{0,1\}$, when $t \in (0,1)$.

We will construct a concrete flow of solutions. The crucial point is to solve the following nonlinear and inhomogeneous Beltrami equation

$$\overline{\partial}\eta_t(z) = \mathcal{H}\big(z, \partial\eta_t(z) + \gamma(t)\big) \qquad \text{for almost every } z \in \mathbb{C}. \tag{4.3}$$

By [2], there exists exactly one solution η_t to the above equation (4.3) such that $D\eta_t \in L^p(\mathbb{C})$. Namely, this can be established via the invertibility of the nonlinear Beltrami operator $B_t = \mathbf{I} - \mathcal{H}_t(z, \mathcal{S}) := \mathbf{I} - [\mathcal{H}(z, \mathcal{S} + \gamma(t)) - \mathcal{H}(z, \gamma(t))]$. Here $\mathcal{S}$ stands for the Beurling transform. With this notation, (4.3) gets the form $\overline{\partial}\eta_t = \mathcal{H}_t(z, \partial\eta_t) + \mathcal{H}(z, \gamma(t))$. The L^p-invertibility, $1 + k < p < 1 + 1/k$, of the above operator $B_t = \mathbf{I} - \mathcal{H}_t(z, \mathcal{S})$ is proven in [2]. We have assumed that $\mathcal{H}(z, \gamma(t)) \in L^{p_0}(\mathbb{C})$ for some $p_0 < 2$. The ellipticity of $\mathcal{H}$ gives an L^∞-bound, $|\mathcal{H}(z, \gamma(t))| \le k|\gamma(t)|$. Thus have the solution η_t with

$$\|\overline{\partial}\eta_t\|_{L^p(\mathbb{C})} \le C_p, \qquad \max\{p_0, 1+k\} < p < 1 + \frac{1}{k}.$$

In particular, by the mapping properties of the Cauchy transform, see, for instance, Theorem 4.3.11 in [1], we obtain a uniform L^∞-estimate

$$\|\eta_t\|_{L^\infty(\mathbb{C})} \le C_\infty \qquad \text{with} \quad \eta_t \in C_0(\hat{\mathbb{C}}).$$

We now claim that $\psi_t(z) := \gamma(t)z + \eta_t(z) - \eta_t(0)$ defines a flow with all the properties required in Theorem 1.3. By definition, ψ_t solves the original equation

$$\overline{\partial}\psi_t(z) = \mathcal{H}\big(z, \partial\psi_t(z)\big) \qquad \text{for almost every } z \in \mathbb{C}.$$

The condition $\mathcal{H}(z,1) = 0$ implies $\eta_1(z) \equiv 0$, and similarly $\eta_0(z) \equiv 0$. Thus $\psi_0(z) = 0$, $\psi_1(z) = z = f(z)$. Hence we have (F1) and (F4) in Theorem 1.3.

The quasiconformality of $f - \psi_t$ for $0 \le t < 1$, condition (F2), follows because

$$f - \psi_t = \big(1 - \gamma(t)\big)z - \eta_t(z) + \eta_t(0).$$

Indeed, since $\eta_t(z) \in C_0(\hat{\mathbb{C}})$, the map $h := f - \psi_t$ can be shown to be homotopic to the homeomorphism $(1 - \gamma(t))z + \eta_t(0)$ in $\partial\mathbb{D}(0,R)$ with respect to 0, for R large enough. Thus, for example, by Theorem 2.8.1 in [1] (for the proof, cf. Theorem 2.2.4 in [6]), it follows that $\deg(h,0) = 1$. Since $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the degree is constant and hence equal to 1 for each $w \in \hat{\mathbb{C}}$. By quasiregularity of h, this forces h to be a homeomorphism.

It remains to address the continuity assumption (F3) in Theorem 1.3. Notice that

$$\overline{\partial}(\eta_t - \eta_s) = \mathcal{H}\big(z, \partial\eta_t - \gamma(t)\big) - \mathcal{H}\big(z, \partial\eta_s - \gamma(s)\big).$$

By applying the Lipschitz condition on $\mathcal{H}$, we get the pointwise inequality

$$|\overline{\partial}(\eta_t - \eta_s)| \le k(|\partial\eta_t - \partial\eta_s| + |\gamma(t) - \gamma(s)|). \tag{4.4}$$

For a compactly supported $\mathcal{H}$ the continuity estimate follows directly from the invertibility of the Beltrami operators [2], but in the general case an additional argument is required.

Let $\varphi \in C_0^\infty(\mathbb{D}(0,2R))$ with $\varphi(z) = 1$, if $|z| \le R$, and $|\nabla\varphi(z)| \le \frac{1}{R}$. Then multiply (4.4) by φ and apply the Caccioppoli type estimate, Theorem 5.4.3 in [1], with the exponent $2 < r < 1 + \frac{1}{k}$ to get

$$\begin{aligned}\left(\int_{\mathbb{D}(0,R)} |D(\eta_t - \eta_s)|^r\right)^{\frac{1}{r}} &\le \left(\int_{\mathbb{D}(0,2R)} |\varphi\, D(\eta_t - \eta_s)|^r\right)^{\frac{1}{r}} \\ &\le CR^{\frac{2}{r}-1}(R|\gamma(t) - \gamma(s)| + \|\eta_t - \eta_s\|_\infty).\end{aligned}$$

Next, we combine this estimate with the Hölder estimates for Sobolev functions. More precisely, since by construction $\eta_t - \eta_s \in W^{1,r}_{\mathrm{loc}}(\mathbb{C})$ for every $r \in [2, 1 + \frac{1}{k})$, at points z with $|z| = R$ one has

$$\begin{aligned}|\eta_t(z) - \eta_s(z) - \eta_t(0) - \eta_s(0)| &\le C\left(\int_{\mathbb{D}(0,R)} |D(\eta_t - \eta_s)|^r\right)^{\frac{1}{r}} R^{1-\frac{2}{r}} \\ &\le C\left(|\gamma(t) - \gamma(s)|\, |z| + C_\infty\right).\end{aligned} \tag{4.5}$$

Furthermore, the definition of ψ_t and (4.5) yield that there exists a constant C such that if $|z| = R$, then

$$|\psi_t(z) - \psi_s(z)| \le C(|\gamma(t) - \gamma(s)||z| + 1). \tag{4.6}$$

On the other hand, clearly $|f(z) - \psi_t(z)| \ge |1 - \gamma(t)||z| - 2C_\infty$, which for $|z| \ge \frac{4C_\infty}{|1-\gamma(t)|}$ implies that

$$|f(z) - \psi_t(z)| \ge \frac{|1 - \gamma(t)|}{2}|z|. \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$\left|\frac{\psi_t(z) - \psi_s(z)}{f(z) - \psi_t(z)}\right| \le \frac{C}{|1 - \gamma(t)|}\left(|\gamma(t) - \gamma(s)| + \frac{1}{|z|}\right)$$

and the continuity estimate (F3) follows by letting $s \to t$ and $R \to \infty$. □

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