ISING CORRELATIONS AND ELLIPTIC DETERMINANTS

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Abstract. Correlation functions of the two-dimensional Ising model on the periodic lattice can be expressed in terms of form factors — matrix elements of the spin operator in the basis of common eigenstates of the transfer matrix and translation operator. Free-fermion structure of the model implies that any multiparticle form factor is given by the pfaffian of a matrix constructed from the two-particle ones. Crossed two-particle form factors can be obtained by inverting a block of the matrix of linear transformation induced on fermions by the spin conjugation. We show that the corresponding matrix is of elliptic Cauchy type and use this observation to solve the inversion problem explicitly. Non-crossed two-particle form factors are then obtained using theta functional interpolation formulas. This gives a new simple proof of the factorized formulas for periodic Ising form factors, conjectured by A. Bugrij and one of the authors.

1. Introduction

The two-dimensional Ising model is the ground of our understanding of critical phenomena and quantum field theory. Starting from Onsager's calculation of the partition function [20], its study has led to remarkable developments in many areas of mathematical physics, including the theory of Toeplitz matrices, quantum integrable systems, Painlevé equations, and conformal field theory. In an effort to keep the bibliography of reasonable size, we refer the reader to [18, 19, 22] for an account of some of these developments and further references.

Correlation functions of the Ising model on an $M\times N$ square lattice with periodic or antiperiodic boundary conditions in each direction can be expressed in terms of elementary blocks, called form factors. These blocks are matrix elements of the spin operator in the basis of common eigenstates of the $2^N\times 2^N$ transfer matrix and the operator of translations. Compact factorized formulas for the form factors of Ising spin have been conjectured in [5, 6] on the basis of (i) long-distance expansions of the two-point correlation function on the cylinder ($M\to\infty$) and (ii) direct transfer matrix calculations for small values of N.

One can try to prove the conjectures of [5, 6] in several ways, reflecting different facets of the integrable structure of the Ising model. First, it can be seen as a special case of the eight-vertex model. This is not very helpful, as the methods of computation of finite-lattice form factors for models with elliptic R-matrix have not yet been developed. Another option is to consider the Ising model as a particular case of the Baxter-Bazhanov-Stroganov $\tau^{(2)}$-model [1, 3, 16]. The corresponding R-matrix is trigonometric, but the L-operator intertwines spin-$\frac{1}{2}$ and cyclic type evaluation representations of the quantum affine algebra $U_q(\widehat{sl_2})$. For this reason, the transfer matrix can not be diagonalized by the standard algebraic Bethe ansatz technique, the corresponding quantum inverse scattering problem does not admit a simple solution, and the Lyon method of computation of form factors for XXZ-type models [15] can not be applied.

These difficulties can be partially overcome in the framework of Sklyanin's method of separation of variables [27]. Local spin operators are then expressed in terms of the quantum monodromy matrix by means of an iterative procedure. The transfer matrix eigenstates are represented by linear combinations of eigenvectors of an auxiliary problem, and the computation of form factors involves summations over this auxiliary basis. Even in the Ising case, performing the sums explicitly and showing that they do reduce to factorized expressions of [5, 6] is a highly nontrivial problem, which was solved in [8, 9]. In spite of this success, the whole procedure looks overwhelmingly cumbersome as compared to final answer, and there remains a strong feeling that the result can (and has to) be obtained in a much simpler way.

The most natural approach would be to take advantage of the free-fermion structure of the Ising model. This structure was discovered in [14], where it was used to give an alternative derivation of the partition function.

Recall that the periodic Ising transfer matrix V commutes with the global spin flip operator U. Since $U^2 = \mathbf{1}$, the eigenstates of V split into two sets, corresponding to $+1$ and -1 eigenvalues of U. The transfer matrix can be naturally written as

$$V = \text{const} \cdot \left(\frac{\mathbf{1} + U}{2} V_a + \frac{\mathbf{1} - U}{2} V_p \right), \tag{1.1}$$

where $V_{a,p}$ commute with the projectors $\dfrac{1 \pm U}{2}$ and are expressed in terms of the generators of a Clifford algebra. They actually belong to the Clifford group, i.e. the generators transform linearly under conjugation by $V_{a,p}$. We emphasize that the same is *not* true for the full transfer matrix V.

Clifford group elements can, in principle, be reconstructed from the induced linear transformations, see e.g. Chapter II in [4]. The case of $V_{a,p}$ is especially simple, as these matrices act on the generators as products of commuting two-dimensional rotations; this becomes manifest after discrete Fourier transforms (antiperiodic for V_a and periodic for V_p). Diagonalization of these rotations gives the spectrum of $V_{a,p}$ and hence, by (1.1), that of V. The eigenvectors of V_a and V_p are given by multiparticle fermionic Fock states, constructed using two different sets of the creation-annihilation operators [14].

Ising spin operator σ also belongs to the Clifford group. However, it changes the value of $\mathbb{Z}_2$-charge U from $+1$ to -1 and vice versa. According to (1.1), these values are associated with fermionic eigenstates of V of different types, and it is not clear how one can compute form factors between them. The problem can be circumvented on the infinite lattice ($N \to \infty$), where the distinction between two types of fermions effectively disappears. In that case, two-particle Ising spin form factors have been found in [26], Theorems 5.2.1–5.2.3, and multiparticle ones in [24], Theorem 5.0.

Further progress for finite N was made by Hystad and Palmer in two recent papers [10, 23]. It was noticed that if we combine the creation-annihilation operators of a- and p-fermions into N-dimensional column vectors $\vec{\psi}^{\,\dagger}$, $\vec{\psi}$, $\vec{\varphi}^{\,\dagger}$, $\vec{\varphi}$, and express the result of conjugation of p-fermions by σ in the a-basis,

$$\sigma \begin{pmatrix} \vec{\varphi}^{\,\dagger} \\ \vec{\varphi} \end{pmatrix} \sigma^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \vec{\psi}^{\,\dagger} \\ \vec{\psi} \end{pmatrix},$$

then form factors of σ between Fock states of different types can be expressed in terms of induced rotations in essentially the same way as if we had an ordinary Bogoliubov transformation involving fermions of only one type. For example, for $T < T_c$

- an arbitrary n-particle form factor is given by the pfaffian of an $n \times n$ matrix, constructed from the two-particle ones;
- normalized two-particle form factors coincide with the elements of $N \times N$ matrices D^{-1}, BD^{-1}, $D^{-1}C$.

Explicit form of the blocks A, B, C, D can be fixed in a rather straightforward way. The computation of finite-lattice form factors therefore reduces to the inversion of D and calculation of the matrix products BD^{-1} and $D^{-1}C$. The present work is devoted to the solution of these two problems.

The key point of our analysis is an elliptic parametrization of the Ising spectral curve, equivalent to the elliptic substitutions used in Yang's derivation of the spontaneous magnetization [30]. We will show that in this parametrization D is given by an elliptic Cauchy matrix. Its determinant and inverse can be computed using the so-called Frobenius determinant formula [7]. (Let us mention that the Frobenius determinant and its generalizations have recently appeared in the computation of the partition function of the eight-vertex SOS model with domain wall boundary conditions [21, 25]). In fact taking the $N \to \infty$ limit of $\det D$ provides a useful alternative to Yang's derivation. Further, the products BD^{-1}, $D^{-1}C$ can be calculated using theta functional analogs of the Lagrange interpolation identities. The matrix of two-particle form factors also turns out to be of elliptic Cauchy type. Computing its pfaffian and going back to the usual trigonometric parametrization, we obtain the general formula for finite-lattice form factors of Ising spin, conjectured in [5, 6].

It is worth noticing that in a recent paper [11] form factors of the quantum Ising chain in a transverse field were derived. They represent a limiting case of the form factors obtained in the present work. Although the derivation in [11] also uses the fermion algebra, it is motivated by the calculation of form factors for superintegrable chiral Potts quantum chain [12] using Baxter's extension [2] of Onsager algebra and it does not use linear transformations induced by the spin operator.

This paper is organized as follows. In Section 2, we introduce basic notation and recall Kaufman's fermionic approach [14] to the diagonalization of the Ising transfer matrix. Section 3 translates the results of [10, 23], expressing spin form factors in terms of induced linear transformations, into the language of creation-annihilation operators. In Section 4, we derive a number of identities relating trigonometric and elliptic parametrization of the Ising spectral curve. The main result is Lemma 4.2, which gives elliptic representations for the matrix elements of A, B, C and D. Section 5 is devoted to the computation of two-particle form factors, i.e. the matrices D^{-1}, BD^{-1} and $D^{-1}C$. The principal results of this section are summarized in Theorem 5.6. Finally, the general factorized formula for the multiparticle form factors is established in Section 6. We conclude with a brief discussion of results and open questions.

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2. Transfer matrix diagonalization

The two-dimensional Ising model on an $M \times N$ lattice is described by the following Hamiltonian:

$$-\beta H[\sigma] = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \Big(\mathcal{K}_x \sigma_{j,k}\sigma_{j+1,k} + \mathcal{K}_y \sigma_{j,k}\sigma_{j,k+1} \Big).$$

Spin variables $\{\sigma_{j,k}\}_{\substack{j=0,\ldots,M-1\\ k=0,\ldots,N-1}}$ take on the values ± 1, and the boundary conditions in each direction are either periodic or antiperiodic. This means that for $j = 0, \ldots, M-1$ and $k = 0, \ldots, N-1$ we have $\sigma_{M,k} = \varepsilon_x \sigma_{0,k}$ and $\sigma_{j,N} = \varepsilon_y \sigma_{j,0}$ with $\varepsilon_{x,y} = \pm 1$. To simplify some of the arguments below, it will be assumed that $\mathcal{K}_{x,y} \in \mathbb{R}_{\geq 0}$. From physical point of view, this involves no loss of generality, as any real values of coupling constants can be made non-negative by a suitable change of spin variables and boundary conditions.

Typically one is interested in calculating n-point correlation functions

$$\langle \sigma_{j_1,k_1} \ldots \sigma_{j_n,k_n} \rangle = \frac{\sum_{[\sigma]} \sigma_{j_1,k_1} \ldots \sigma_{j_n,k_n} e^{-\beta H[\sigma]}}{\sum_{[\sigma]} e^{-\beta H[\sigma]}}. \tag{2.1}$$

The $2^N \times 2^N$ transfer matrix V can be thought of as an operator of discrete evolution in the x-direction. It acts in the space of states, which admits two equivalent descriptions: as a space of maps $f : (\mathbb{Z}_2)^{\times N} \to \mathbb{C}$ or as an N-fold tensor product $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$. To write V explicitly, it is common to define the operators

$$s_j = \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j \text{ times}} \otimes \sigma_z \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1},$$
$$C_j = \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j \text{ times}} \otimes \sigma_x \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1},$$

where $j = 0, \ldots, N-1$ and $\sigma_{x,y,z}$ denote the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The action of $\{s_j\}$, $\{C_j\}$ on the space of maps is given by

$$(s_j f)(\sigma_0, \ldots, \sigma_{N-1}) = \sigma_j f(\sigma_0, \ldots, \sigma_{N-1}),$$
$$(C_j f)(\sigma_0, \ldots, \sigma_{N-1}) = f(\sigma_0, \ldots, \sigma_{j-1}, -\sigma_j, \sigma_{j+1}, \ldots, \sigma_{N-1}),$$

where $\sigma_0, \ldots, \sigma_{N-1} \in \{1, -1\}$. The transfer matrix can now be written as

$$V = (2\sinh 2\mathcal{K}_x)^{\frac{N}{2}}\, V_y^{\frac12} V_x V_y^{\frac12}, \tag{2.2}$$

with

$$V_x = \exp\Big\{\mathcal{K}_x^* \sum_{j=0}^{N-1} C_j\Big\}, \qquad V_y = \exp\Big\{\mathcal{K}_y \sum_{j=0}^{N-1} s_j s_{j+1}\Big\}. \tag{2.3}$$

Here $s_N = \varepsilon_y s_0$ and $\mathcal{K}_x^* = \operatorname{arctanh} e^{-2\mathcal{K}_x} \in \mathbb{R}_{>0}$ denotes the dual coupling.

As already mentioned, V commutes with the global spin flip $U = C_0 C_1 \ldots C_{N-1}$. It also commutes with the operator T_{ε_y} of y-translations, defined by

$$\left(T_{\varepsilon_y} f\right)(\sigma_0, \ldots, \sigma_{N-1}) = f(\sigma_1, \ldots, \sigma_{N-1}, \varepsilon_y \sigma_0),$$

as it may be easily checked that

$$T_{\varepsilon_y} s_j T_{\varepsilon_y}^{-1} = s_{j+1}, \qquad T_{\varepsilon_y} C_j T_{\varepsilon_y}^{-1} = C_{j+1}, \qquad j = 0, \ldots, N-1,$$

with $C_N = C_0$. It is clear that $[T_{\varepsilon_y}, U] = 0$, and therefore the operators T_{ε_y}, U and V can be diagonalized simultaneously. Correlation function (2.1) with $j_1 \le j_2 \le \ldots \le j_n$ is equal to the trace ratio

$$\langle \sigma_{j_1,k_1} \ldots \sigma_{j_n,k_n} \rangle = \frac{\operatorname{Tr}\Big[\Big(V^{j_1} T_{\varepsilon_y}^{k_1} s_0 T_{\varepsilon_y}^{-k_1} V^{-j_1}\Big) \ldots \Big(V^{j_n} T_{\varepsilon_y}^{k_n} s_0 T_{\varepsilon_y}^{-k_n} V^{-j_n}\Big) V^M U^{\frac{1-\varepsilon_x}{2}}\Big]}{\operatorname{Tr}\Big[V^M U^{\frac{1-\varepsilon_x}{2}}\Big]},$$

which can be straightforwardly computed in terms of matrix elements of s_0 in the corresponding basis of normalized common eigenstates.

Remark 2.1. The transfer matrix considered in [10, 23] is given, in our notation, by

$$V' = (2\sinh 2\mathcal{K}_x)^{\frac{N}{2}}\, V_x^{\frac12} V_y V_x^{\frac12}.$$

Since V' is related to V in (2.2) by a similarity transformation, both definitions work equally well if one is interested in the partition function. However, V_x does not commute with the spin operators $\{s_j\}$, and therefore V' can not be used in the calculation of Ising correlation functions. This explains discrepancies reported in Section 9 of [10].

On the other hand, the transfer matrix V' possesses the same set of eigenvectors as the hamiltonian of quantum XY-chain in a transverse field. Ising spin matrix elements between the eigenvectors of V' should therefore coincide with the form factors of σ^x between the eigenstates of XY-hamiltonian, which were found in [13] by the method of separation of variables.

For $j=0,\ldots,N-1$, introduce the operators

$$p_j=\underbrace{\sigma_x\otimes\ldots\otimes\sigma_x}_{j\text{ times}}\otimes\sigma_z\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}=C_0\ldots C_{j-1}s_j,$$

$$q_j=\underbrace{\sigma_x\otimes\ldots\otimes\sigma_x}_{j\text{ times}}\otimes\sigma_y\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}=iC_0\ldots C_j s_j,$$

satisfying anticommutation relations for the generators of the Clifford algebra:

$$\{p_j,p_k\}=\{q_j,q_k\}=2\delta_{jk},\qquad \{p_j,q_k\}=0.$$

It can be shown that

$$V=(2\sinh 2\mathcal{K}_x)^{\frac{N}{2}}\left(\frac{\mathbf{1}+\varepsilon_y U}{2}V_a+\frac{\mathbf{1}-\varepsilon_y U}{2}V_p\right),\tag{2.4}$$

where

$$V_\nu=V_{y,\nu}^{\frac12}V_xV_{y,\nu}^{\frac12},\qquad \nu=a,p,$$

$$V_x=\exp\left\{i\mathcal{K}_x^*\sum_{j=0}^{N-1}p_jq_j\right\},\qquad V_{y,\nu}=\exp\left\{-i\mathcal{K}_y\sum_{j=0}^{N-1}p_{j+1}q_j\right\},$$

and $p_N=-p_0$ ($p_N=p_0$) for $\nu=a$ (resp. $\nu=p$). Note that U anticommutes with all $\{p_j\}$, $\{q_j\}$ and hence commutes with $V_{a,p}$.

Consider discrete Fourier transforms

$$p_\theta=\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}e^{-ij\theta}p_j,\qquad q_\theta=\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}e^{-ij\theta}q_j.\tag{2.5}$$

Two sets of quasimomenta will be important for us: $\boldsymbol{\theta}_a=\left\{\frac{\pi}{N},\frac{3\pi}{N},\ldots,2\pi-\frac{\pi}{N}\right\}$ and $\boldsymbol{\theta}_p=\left\{0,\frac{2\pi}{N},\ldots,2\pi-\frac{2\pi}{N}\right\}$. In both cases one has inversion formulas

$$p_j=\frac{1}{\sqrt{N}}\sum_{\theta\in\boldsymbol{\theta}_\nu}e^{ij\theta}p_\theta,\qquad q_j=\frac{1}{\sqrt{N}}\sum_{\theta\in\boldsymbol{\theta}_\nu}e^{ij\theta}q_\theta,\tag{2.6}$$

and anticommutation relations

$$\{p_\theta,p_{\theta'}\}=\{q_\theta,q_{\theta'}\}=2\delta_{\theta+\theta',0\bmod 2\pi},\qquad \{p_\theta,q_{\theta'}\}=0.$$

The conjugation by $V_{a,p}$ acts linearly on the generators $\{p_j\}$ and $\{q_j\}$. In particular, Fourier components with $\theta\in\boldsymbol{\theta}_\nu$ ($\nu=a,p$) transform as

$$V_\nu\begin{pmatrix}p_\theta\\ q_\theta\end{pmatrix}V_\nu^{-1}=\begin{pmatrix}\cosh\gamma_\theta & ie^{-i\theta}w_\theta\\ -ie^{i\theta}w_{-\theta} & \cosh\gamma_\theta\end{pmatrix}\begin{pmatrix}p_\theta\\ q_\theta\end{pmatrix},$$

where the functions w_θ and γ_θ are defined by

$$w_\theta=\cosh^2\mathcal{K}_x^*\sinh 2\mathcal{K}_y\left(1-\alpha e^{i\theta}\right)\left(1-\beta e^{-i\theta}\right),\tag{2.7}$$

$$\cosh\gamma_\theta=\cosh 2\mathcal{K}_x^*\cosh 2\mathcal{K}_y-\sinh 2\mathcal{K}_x^*\sinh 2\mathcal{K}_y\cos\theta,\qquad \gamma_\theta\geq 0,\tag{2.8}$$

with $\alpha=\tanh\mathcal{K}_x^*\coth\mathcal{K}_y$ and $\beta=\tanh\mathcal{K}_x^*\tanh\mathcal{K}_y$. It can be easily checked that

$$\sinh^2\gamma_\theta=\cosh^4\mathcal{K}_x^*\sinh^2 2\mathcal{K}_y\left(1-\alpha e^{i\theta}\right)\left(1-\alpha e^{-i\theta}\right)\left(1-\beta e^{i\theta}\right)\left(1-\beta e^{-i\theta}\right).$$

From now on, unless otherwise is stated explicitly, we will work in the ferromagnetic region of parameters. The latter is defined by $\mathcal{K}_x^*<\mathcal{K}_y$ or, equivalently, $\alpha<1$. In this case, it is convenient to write $w_{\pm\theta}$ in (2.7) as

$$w_{\pm\theta}=b_{\pm\theta}\sinh\gamma_\theta,$$

with

$$b_\theta=\left[b_{-\theta}\right]^{-1}=\sqrt{\frac{\left(1-\alpha e^{i\theta}\right)\left(1-\beta e^{-i\theta}\right)}{\left(1-\beta e^{i\theta}\right)\left(1-\alpha e^{-i\theta}\right)}}.\tag{2.9}$$

The branches of square roots in (2.9) and below are defined so that they have a positive real part.

Finally, introduce creation-annihilation operators of a- and p-fermions, which correspond to $\theta \in \boldsymbol{\theta}_a$ and $\theta \in \boldsymbol{\theta}_p$, respectively:

$$\begin{cases} 2\psi_\theta^\dagger = e^{-i\theta}\sqrt{b_\theta}\, p_{-\theta} - i\sqrt{b_{-\theta}}\, q_{-\theta}, \\ 2\psi_\theta = \ e^{i\theta}\sqrt{b_{-\theta}}\, p_\theta \ + i\sqrt{b_\theta}\, q_\theta. \end{cases} \tag{2.10}$$

Here $-\theta$ is identified with $2\pi - \theta$ for $\theta \neq 0$. The operators in each of the two sets satisfy standard anticommutation relations

$$\{\psi_\theta^\dagger, \psi_{\theta'}^\dagger\} = \{\psi_\theta, \psi_{\theta'}\} = 0, \qquad \{\psi_\theta^\dagger, \psi_{\theta'}\} = \delta_{\theta,\theta'}.$$

Inversion formulas

$$p_\theta = e^{-i\theta}\sqrt{b_\theta}\left(\psi_{-\theta}^\dagger + \psi_\theta\right), \qquad q_\theta = i\sqrt{b_{-\theta}}\left(\psi_{-\theta}^\dagger - \psi_\theta\right), \tag{2.11}$$

combined with Fourier transforms (2.5)–(2.6), allow to express "antiperiodic" creation-annihilation operators as linear combinations of "periodic" ones and vice versa.

These operators transform diagonally under conjugation by $V_{a,p}$,

$$V_\nu \begin{pmatrix} \psi_\theta^\dagger \\ \psi_\theta \end{pmatrix} V_\nu^{-1} = \begin{pmatrix} e^{-\gamma_\theta} & 0 \\ 0 & e^{\gamma_\theta} \end{pmatrix} \begin{pmatrix} \psi_\theta^\dagger \\ \psi_\theta \end{pmatrix}, \qquad \theta \in \boldsymbol{\theta}_\nu.$$

Since $\{s_j\}$, $\{C_j\}$ (as well as $\{p_j\}$, $\{q_j\}$) generate full $2^N \times 2^N$ matrix algebra, it follows from Schur's lemma that induced linear transformations fix $V_{a,p}$ up to a scalar multiple. Now taking into account that $\det V_{a,p} = 1$ and $\operatorname{Tr} V_{a,p} > 0$, we find that

$$V_\nu = \exp\Big\{ -\sum_{\theta\in\boldsymbol{\theta}_\nu} \gamma_\theta \left(\psi_\theta^\dagger\psi_\theta - \frac{1}{2}\right)\Big\}, \qquad \nu = a, p.$$

Fock states

$$|\theta_1, \ldots, \theta_k\rangle_\nu = \psi_{\theta_1}^\dagger \ldots \psi_{\theta_k}^\dagger |vac\rangle_\nu, \qquad \theta_1, \ldots, \theta_k \in \boldsymbol{\theta}_\nu, \tag{2.12}$$

are therefore right eigenvectors of V_ν, with eigenvalues equal to $\exp\left\{\frac{1}{2}\sum_{\theta\in\boldsymbol{\theta}_\nu}\gamma_\theta - \sum_{i=1}^k \gamma_{\theta_i}\right\}$. . Here $|vac\rangle_\nu$ is a vector annihilated by all ψ_θ with $\theta \in \boldsymbol{\theta}_\nu$.

The operators $\{s_j\}$, $\{C_j\}$, $\{p_j\}$, $\{q_j\}$ are hermitian, hence $p_\theta^\dagger = p_{-\theta}$, $q_\theta^\dagger = q_{-\theta}$ and the operator $\psi_\theta^\dagger$ is hermitian conjugate of ψ_θ, just as the notation suggests. Introduce the dual vacua ${}_\nu\langle vac| = |vac\rangle_\nu^\dagger$ annihilated by all $\psi_\theta^\dagger$ with $\theta \in \boldsymbol{\theta}_\nu$. Left eigenvectors of V_ν can then be written as

$${}_\nu\langle\theta_k, \ldots, \theta_1| = |\theta_1, \ldots, \theta_k\rangle_\nu^\dagger = {}_\nu\langle vac|\psi_{\theta_k} \ldots \psi_{\theta_1}, \qquad \theta_1, \ldots, \theta_k \in \boldsymbol{\theta}_\nu.$$

We will impose the normalization condition ${}_\nu\langle vac|vac\rangle_\nu = 1$, which automatically implies orthonormality of the states (2.12).

Recall that only a part of eigenvectors of $V_{a,p}$ are actual eigenstates of V, cf. (2.4). For $\varepsilon_y = 1$ (i.e. periodic boundary conditions on spin variables) only the states with even number of particles are eigenvectors of the full transfer matrix. For $\varepsilon_y = -1$, the number of particles in the surviving states of both types is odd. This follows e.g. from an explicit diagonalization of $V_{a,p}$ in the limit $\mathcal{K}_x^* \to 0$, and observation that the eigenvectors for $\mathcal{K}_x^* > 0$ are related to those with $\mathcal{K}_x^* = 0$ by products of two-mode Bogoliubov transformations, which commute with U. In fact since U also anticommutes with $\{\psi_\theta^\dagger\}$, $\{\psi_\theta\}$, it is sufficient to show that $\mathbb{Z}_2$-charges of the vacuum vectors $|vac\rangle_a$ and $|vac\rangle_p$ are equal to $+1$ and -1, respectively.

To avoid confusion, in what follows we reserve the letters $\psi_\theta^\dagger$, ψ_θ for the creation-annihilation operators with $\theta \in \boldsymbol{\theta}_a$ and denote the corresponding operators with $\theta \in \boldsymbol{\theta}_p$ by $\varphi_\theta^\dagger$, φ_θ. Our aim is to compute spin form factors

$$\mathcal{F}_{m,n}^{(l)}\left(\boldsymbol{\theta}, \boldsymbol{\theta}'\right) = {}_a\langle\theta_1, \ldots, \theta_m|s_l|\theta_1', \ldots, \theta_n'\rangle_p = \left[{}_p\langle\theta_n', \ldots, \theta_1'|s_l|\theta_m, \ldots, \theta_1\rangle_a\right]^*, \tag{2.13}$$

where $\theta_1, \ldots, \theta_m \in \boldsymbol{\theta}_a$, $\theta_1', \ldots, \theta_n' \in \boldsymbol{\theta}_p$ and $*$ in the last expression denotes complex conjugation. As explained above, matrix elements (2.13) are non-zero only if m and n are simultaneously even or odd. These are the only nontrivial form factors: since s_l anticommutes with U, its matrix elements between the eigenstates of V of the same type vanish.

Remark 2.2. Let us show that the eigenstates of V constructed above diagonalize the translation operator T_{ε_y} (since $\gamma_\theta = \gamma_{-\theta}$, the eigenvalues of V are degenerate and this is not automatic). Though translation invariance will be manifest in the final answer, which satisfies

$$\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}') = e^{il\left(\sum_{i=1}^n \theta'_i - \sum_{i=1}^m \theta_i\right)} \mathcal{F}^{(0)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}'), \tag{2.14}$$

it is instructive to explain it already at this stage.

Indeed, the vacua $|vac\rangle_{a,p}$ are non-degenerate eigenstates of the periodic Ising transfer matrix, and therefore they are also eigenvectors of T_+. It can be deduced from the limit $\mathcal{K}_x^* \to 0$ by continuity that the corresponding eigenvalues are both equal to 1. It is also straightforward to check that

$$T_\pm \begin{pmatrix} p_j \\ q_j \end{pmatrix} T_\mp^{-1} \frac{\mathbf{1}\pm\epsilon U}{2} = \begin{pmatrix} p_{j+1} \\ q_{j+1} \end{pmatrix} \frac{\mathbf{1}\pm\epsilon U}{2}, \qquad j=0,\ldots,N-1,$$

where $\epsilon = \pm 1$ and $p_N = \epsilon p_0$, $q_N = \epsilon q_0$. Performing discrete Fourier transforms, we obtain

$$T_\pm \begin{pmatrix} \psi_\theta^\dagger \\ \psi_\theta \end{pmatrix} T_\mp^{-1} \frac{\mathbf{1}\mp U}{2} = \begin{pmatrix} e^{-i\theta}\psi_\theta^\dagger \\ e^{i\theta}\psi_\theta \end{pmatrix} \frac{\mathbf{1}\mp U}{2}, \tag{2.15}$$

$$T_\pm \begin{pmatrix} \varphi_\theta^\dagger \\ \varphi_\theta \end{pmatrix} T_\mp^{-1} \frac{\mathbf{1}\pm U}{2} = \begin{pmatrix} e^{-i\theta}\varphi_\theta^\dagger \\ e^{i\theta}\varphi_\theta \end{pmatrix} \frac{\mathbf{1}\pm U}{2}, \tag{2.16}$$

which in turn yields the desired formulas:

$$T_+|\theta_1,\ldots,\theta_{2k}\rangle_{a,p} = e^{-i\sum_{i=1}^{2k}\theta_i}\,|\theta_1,\ldots,\theta_{2k}\rangle_{a,p}, \tag{2.17}$$

$$T_-|\theta_1,\ldots,\theta_{2k+1}\rangle_{a,p} = e^{-i\sum_{i=1}^{2k+1}\theta_i}|\theta_1,\ldots,\theta_{2k+1}\rangle_{a,p}. \tag{2.18}$$

3. Spin form factors and induced rotations

Just as in the above case of $V_{a,p}$, conjugation by the spin operator s_l acts linearly on the Clifford algebra generators $\{p_j\}$, $\{q_j\}$:

$$\begin{cases} s_l p_j s_l^{-1} = \operatorname{sgn}(l-j)p_j, \\ s_l q_j s_l^{-1} = \operatorname{sgn}(l-1-j)q_j, \end{cases} \qquad j=0,\ldots,N-1, \tag{3.1}$$

where $\operatorname{sgn}(x)=1$ if $x\geq 0$ and -1 if $x<0$. Recall that induced linear transformations fix s_l up to a scalar multiple.

Let us combine $\{\psi_\theta^\dagger\}$, $\{\psi_\theta\}$ and $\{\varphi_\theta^\dagger\}$, $\{\varphi_\theta\}$ into N-dimensional column vectors $\vec{\psi}^\dagger$, $\vec{\psi}$, $\vec{\varphi}^\dagger$, $\vec{\varphi}$. Their entries are thus given by $2^N\times 2^N$ matrices. It will be very convenient to write the result of conjugation of $\{\varphi_\theta^\dagger\}$, $\{\varphi_\theta\}$ by s_l as a linear combination of $\{\psi_\theta^\dagger\}$, $\{\psi_\theta\}$:

$$s_l \begin{pmatrix} \vec{\varphi}^\dagger \\ \vec{\varphi} \end{pmatrix} s_l^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \vec{\psi}^\dagger \\ \vec{\psi} \end{pmatrix}, \tag{3.2}$$

where A, B, C, D are $N\times N$ matrices. The entries of the $2N$-column in the l.h.s. satisfy canonical anticommutation relations, from which it can be deduced that

$$AB^T + BA^T = CD^T + DC^T = 0, \qquad AD^T + BC^T = \mathbf{1}. \tag{3.3}$$

Since $s_l = s_l^{-1} = s_l^\dagger$, the operator s_l is unitary. This yields further relations

$$\bar{A} = D, \qquad \bar{B} = C. \tag{3.4}$$

Vacuum vectors $|vac\rangle_\nu$ ($\nu = a,p$) are fixed by the annihilation operators $\{\psi_\theta\}$, $\{\varphi_\theta\}$ and the normalization ${}_\nu\langle vac|vac\rangle_\nu = 1$ up to inessential pure phase factors. This can be used to establish a relation between the two vacua.

Lemma 3.1. *We have*

$$|vac\rangle_p = \lambda \cdot s_l^{-1}\mathcal{O}|vac\rangle_a, \tag{3.5}$$

where

$$|\lambda| = |\det D|^{\frac{1}{2}}, \qquad \mathcal{O} = \exp\Bigl\{-\frac{1}{2}\sum_{\theta,\theta'\in\boldsymbol{\theta}_a}\psi^\dagger_\theta\left(D^{-1}C\right)_{\theta,\theta'}\psi^\dagger_{\theta'}\Bigr\}.$$

Proof. Lemma is formulated and will be proved under assumption that D is invertible. This point will be checked in Section 5, where $\det D$ is shown to be non-zero by explicit computation.

Acting on $s_l^{-1}\mathcal{O}|vac\rangle_a$ with the N-column $\vec{\varphi}$ of annihilation operators, one finds

$$\vec{\varphi}s_l^{-1}\mathcal{O}|vac\rangle_a = s_l^{-1}(C\vec{\psi}^{\,\dagger} + D\vec{\psi})\mathcal{O}|vac\rangle_a = s_l^{-1}\mathcal{O}(C\vec{\psi}^{\,\dagger} + D\mathcal{O}^{-1}\vec{\psi}\mathcal{O})|vac\rangle_a. \tag{3.6}$$

For two matrices X, Y satisfying $[X,[X,Y]]=0$ holds $e^XYe^{-X} = Y + [X,Y]$. From this and the fact that $D^{-1}C$ is skew-symmetric because of (3.3) we get

$$\mathcal{O}^{-1}\vec{\psi}\mathcal{O} = -D^{-1}C\vec{\psi}^{\,\dagger} + \vec{\psi}. \tag{3.7}$$

Since $\vec{\psi}|vac\rangle_a = 0$, it follows from (3.6) and (3.7) that $\vec{\varphi}s_l^{-1}\mathcal{O}|vac\rangle_a = 0$, and thus the vector $s_l^{-1}\mathcal{O}|vac\rangle_a$ is proportional to $|vac\rangle_p$.

Absolute value of the scalar multiple λ in (3.5) can be determined from the normalization. Indeed, the unitarity of s_l implies that

$$1 = {}_p\langle vac|vac\rangle_p = |\lambda|^2\,{}_a\langle vac|\mathcal{O}^\dagger\mathcal{O}|vac\rangle_a. \tag{3.8}$$

To compute the vacuum expectation value on the right, one can e.g. write $D^{-1}C$ as G^TFG, where G is unitary and F has block-diagonal form, with 2×2 skew-symmetric blocks containing nontrivial eigenvalues of $D^{-1}C$ and 1×1 blocks containing zeros. Introducing new creation-annihilation operators $\vec{\xi}^{\,\dagger} = G\vec{\psi}^{\,\dagger}$, $\vec{\xi} = \bar{G}\vec{\psi}$ which satisfy canonical anticommutation relations, and using that $|vac\rangle_a$ is annihilated by $\vec{\xi}$, one obtains

$${}_a\langle vac|\mathcal{O}^\dagger\mathcal{O}|vac\rangle_a = \left[\det\left(\mathbf{1}+FF^\dagger\right)\right]^{\frac{1}{2}} = \left[\det\left(\mathbf{1}+\left(D^{-1}C\right)\left(D^{-1}C\right)^\dagger\right)\right]^{\frac{1}{2}} = |\det D|^{-1}.$$

The last equality follows from the relation $CC^\dagger = \mathbf{1} - DD^\dagger$, which can be derived from (3.3)–(3.4). Now (3.8) implies that $|\lambda| = |\det D|^{\frac{1}{2}}$ and the lemma is proved. □

Since $T_+|vac\rangle_{a,p} = |vac\rangle_{a,p}$, the factor $\lambda = {}_a\langle vac|s_l|vac\rangle_p$ is independent of l and thus its phase can always be absorbed into the definition of vacua. We will therefore assume below that $\lambda \in \mathbb{R}_{>0}$. Under this convention, it follows from Lemma 3.1 that

$${}_a\langle vac|s_l|vac\rangle_p = |\det D|^{\frac{1}{2}}. \tag{3.9}$$

Note that in the limit $N\to\infty$ the l.h.s. of this formula gives Ising spontaneous magnetization.

Let us now compute general form factors $\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}')$. By (2.13) and (3.5), one has

$$\begin{aligned}\lambda^{-1}\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}') &= {}_a\langle vac|\psi_{\theta_1}\ldots\psi_{\theta_m}s_l\varphi^\dagger_{\theta'_1}\ldots\varphi^\dagger_{\theta'_n}s_l^{-1}\mathcal{O}|vac\rangle_a = \\ &= {}_a\langle vac|\psi_{\theta_1}\ldots\psi_{\theta_m}(A\vec{\psi}^{\,\dagger}+B\vec{\psi})_{\theta'_1}\ldots(A\vec{\psi}^{\,\dagger}+B\vec{\psi})_{\theta'_n}\mathcal{O}|vac\rangle_a,\end{aligned}$$

where we used (3.2) to pass from the first to the second line. Now pull $\mathcal{O}$ in the last expression to the left vacuum, on which it acts as the identity operator. This can be done using (3.7) and the fact that $\mathcal{O}$ commutes with all entries of $\vec{\psi}^{\,\dagger}$. Moreover, (3.3) implies that

$$\mathcal{O}^{-1}(A\vec{\psi}^{\,\dagger}+B\vec{\psi})\mathcal{O} = (A-BD^{-1}C)\vec{\psi}^{\,\dagger}+B\vec{\psi} = D^{-T}\vec{\psi}^{\,\dagger}+B\vec{\psi},$$

where $D^{-T} = \left(D^{-1}\right)^T$, so that

$$\begin{aligned}\lambda^{-1}\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}') = {}_a\langle vac|&(-D^{-1}C\vec{\psi}^{\,\dagger}+\vec{\psi})_{\theta_1}\ldots(-D^{-1}C\vec{\psi}^{\,\dagger}+\vec{\psi})_{\theta_m}\times\\ &\times(D^{-T}\vec{\psi}^{\,\dagger}+B\vec{\psi})_{\theta'_1}\ldots(D^{-T}\vec{\psi}^{\,\dagger}+B\vec{\psi})_{\theta'_n}|vac\rangle_a.\end{aligned}$$

Vacuum expectation value of any product of linear combinations of $\{\psi^\dagger_\theta\}$, $\{\psi_\theta\}$ can be easily computed using Wick theorem. The result is given by the pfaffian of the matrix of pairings between different combinations. In our case, there are three types of pairings:

$$\begin{aligned}
{}_a\langle vac|(-D^{-1}C\vec{\psi}^\dagger+\vec{\psi})_{\theta_j}(-D^{-1}C\vec{\psi}^\dagger+\vec{\psi})_{\theta_k}|vac\rangle_a &= \left(D^{-1}C\right)_{\theta_j,\theta_k},\\
{}_a\langle vac|(-D^{-1}C\vec{\psi}^\dagger+\vec{\psi})_{\theta_j}(D^{-T}\vec{\psi}^\dagger+B\vec{\psi})_{\theta'_k}|vac\rangle_a &= D^{-1}_{\theta_j,\theta'_k},\\
{}_a\langle vac|(D^{-T}\vec{\psi}^\dagger+B\vec{\psi})_{\theta'_j}(D^{-T}\vec{\psi}^\dagger+B\vec{\psi})_{\theta'_k}|vac\rangle_a &= \left(BD^{-1}\right)_{\theta'_j,\theta'_k}.
\end{aligned}$$

Accordingly, we obtain

Lemma 3.2. *Form factors $\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}')$ of Ising spin have the following representation in terms of induced rotations (3.2):*

$$\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}') = |\det D|^{\frac12}\cdot \mathrm{Pf}\, R, \qquad R=\begin{pmatrix} R_{\boldsymbol{\theta}\times\boldsymbol{\theta}} & R_{\boldsymbol{\theta}\times\boldsymbol{\theta}'}\\ R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}} & R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'}\end{pmatrix}, \tag{3.10}$$

where matrix elements of the blocks of skew-symmetric $(m+n)\times(m+n)$ matrix R are given by

$$\left(R_{\boldsymbol{\theta}\times\boldsymbol{\theta}}\right)_{jk} = \left(D^{-1}C\right)_{\theta_j,\theta_k}, \qquad j,k=1,\ldots,m, \tag{3.11}$$

$$\left(R_{\boldsymbol{\theta}\times\boldsymbol{\theta}'}\right)_{jk} = -\left(R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}}\right)_{kj} = D^{-1}_{\theta_j,\theta'_k}, \qquad j=1,\ldots,m, \quad k=1,\ldots,n, \tag{3.12}$$

$$\left(R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'}\right)_{jk} = \left(BD^{-1}\right)_{\theta'_j,\theta'_k}, \qquad j,k=1,\ldots,n. \tag{3.13}$$

Observe that matrix elements (3.11)–(3.13) coincide with normalized two-particle form factors:

$$\frac{{}_a\langle\theta,\theta'|s_l|vac\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = \left(D^{-1}C\right)_{\theta,\theta'}, \quad \frac{{}_a\langle\theta|s_l|\theta'\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = D^{-1}_{\theta,\theta'}, \quad \frac{{}_a\langle vac|s_l|\theta,\theta'\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = \left(BD^{-1}\right)_{\theta,\theta'}. \tag{3.14}$$

The reader might have noticed that the only important point for the above derivation is the formula (3.2). In particular, we have not as yet used the fact that s_l belongs to the Clifford group, i.e. that $\vec{\psi}^\dagger$, $\vec{\psi}$ and $\vec{\varphi}^\dagger$, $\vec{\varphi}$ are related by a linear transformation. However, this does play a role in the proof of the following Lemma, which gives the explicit form of the induced rotation matrix.

Lemma 3.3. *Matrix elements of A, B, C, D are given by*

$$\bar{A}_{\theta,\theta'} = D_{\theta,\theta'} = \frac{e^{-i(l-\frac12)(\theta-\theta')}}{2iN\sin\frac{\theta'-\theta}{2}}\left(\sqrt{\frac{b_{\theta'}}{b_\theta}}+\sqrt{\frac{b_\theta}{b_{\theta'}}}\right), \tag{3.15}$$

$$\bar{B}_{\theta,\theta'} = C_{\theta,\theta'} = \frac{e^{-i(l-\frac12)(\theta+\theta')}}{2iN\sin\frac{\theta+\theta'}{2}}\left(\sqrt{b_\theta b_{\theta'}}-\frac{1}{\sqrt{b_\theta b_{\theta'}}}\right), \tag{3.16}$$

where $\theta\in\boldsymbol{\theta}_p$, $\theta'\in\boldsymbol{\theta}_a$.

Proof. Express φ_θ in terms of p_j, q_j using (2.10) and the Fourier transform (2.5), compute the result of conjugation by s_l using (3.1), and rewrite it in terms of $\{\psi^\dagger_{\theta'}\}$, $\{\psi_{\theta'}\}$ using the inverse transform (2.6) and the formula (2.11). Explicitly,

$$\begin{aligned}
2s_l\varphi_\theta s_l^{-1} &= s_l\left(e^{i\theta}\sqrt{b_{-\theta}}\,p_\theta + i\sqrt{b_\theta}\,q_\theta\right)s_l^{-1} = \\
&= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-ij\theta}\left(e^{i\theta}\sqrt{b_{-\theta}}\,s_l p_j s_l^{-1} + i\sqrt{b_\theta}\,s_l q_j s_l^{-1}\right) = \\
&= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-ij\theta}\left(e^{i\theta}\sqrt{b_{-\theta}}\,\mathrm{sgn}(l-j)p_j + i\sqrt{b_\theta}\,\mathrm{sgn}(l-1-j)q_j\right) = \\
&= \frac{1}{N}\sum_{\theta'\in\boldsymbol{\theta}_a}\sum_{j=0}^{N-1} e^{ij(\theta'-\theta)}\left(e^{i\theta}\sqrt{b_{-\theta}}\,\mathrm{sgn}(l-j)p_{\theta'} + i\sqrt{b_\theta}\,\mathrm{sgn}(l-1-j)q_{\theta'}\right) = \\
&= -\frac{2}{N}\sum_{\theta'\in\boldsymbol{\theta}_a}\frac{e^{il(\theta'-\theta)}}{1-e^{i(\theta'-\theta)}}\left(e^{i\theta'}\sqrt{b_{-\theta}}\,p_{\theta'} + i\sqrt{b_\theta}\,q_{\theta'}\right) = \\
&= -\frac{2}{N}\sum_{\theta'\in\boldsymbol{\theta}_a}\frac{e^{il(\theta'-\theta)}}{1-e^{i(\theta'-\theta)}}\left\{\left(\sqrt{\frac{b_{\theta'}}{b_\theta}}-\sqrt{\frac{b_\theta}{b_{\theta'}}}\right)\psi^\dagger_{-\theta'} + \left(\sqrt{\frac{b_{\theta'}}{b_\theta}}+\sqrt{\frac{b_\theta}{b_{\theta'}}}\right)\psi_{\theta'}\right\},
\end{aligned}$$

which gives C and D. The formulas for A and B immediately follow from (3.4). □

Remark 3.4. An essentially equivalent result was obtained in [10], Theorem 4.1. However, the function b_θ appearing in the kernels of A, B, C, D differs from the one in [10]. This seems to be related to the inappropriate initial choice of the transfer matrix, see Remark 2.1.

4. Elliptic parametrization of the Ising spectral curve

Let us now go back to the lattice dispersion relation (2.8). Introduce the variables $z = e^{i\theta}$, $\lambda = e^{\gamma_\theta}$ and rewrite it as an algebraic curve

$$\sinh 2\mathcal{K}_x\frac{\lambda+\lambda^{-1}}{2} + \sinh 2\mathcal{K}_y\frac{z+z^{-1}}{2} = \cosh 2\mathcal{K}_x\cosh 2\mathcal{K}_y. \tag{4.1}$$

Topologically this is a torus realized as a two-fold covering of $\mathbb{P}^1$, branched at 4 points

$$\begin{aligned}
z_{1,2} &= (\tanh\mathcal{K}_x^*\coth\mathcal{K}_y)^{\pm1} = \alpha^{\pm1}, \\
z_{3,4} &= (\tanh\mathcal{K}_x^*\tanh\mathcal{K}_y)^{\pm1} = \beta^{\pm1}.
\end{aligned}$$

The points $\beta < \alpha < \alpha^{-1} < \beta^{-1}$ can be mapped to $0 < k < k^{-1} < \infty$ by a Möbius transformation. Such transformations preserve the anharmonic ratio $(z_1, z_2; z_3, z_4) = \dfrac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}$, and therefore

$$k = \frac{\beta-\alpha}{\alpha\beta-1} = \frac{\sinh 2\mathcal{K}_x^*}{\sinh 2\mathcal{K}_y}.$$

The parameter k plays the role of the modulus of the Jacobi elliptic functions uniformizing (4.1). One also needs to introduce elliptic integrals

$$K = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad K' = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

where the complementary modulus is defined by $k'^2 = 1-k^2$. It is known (see Theorem 2.2.1 in [22]) that the functions

$$z(u) = \frac{\mathrm{sn}\,(u+i\eta)}{\mathrm{sn}\,(u-i\eta)}, \tag{4.2}$$

$$\lambda(u) = \left[k\,\mathrm{sn}\,(u+i\eta)\,\mathrm{sn}\,(u-i\eta)\right]^{-1}, \tag{4.3}$$

with $u \in [-K, K) \times i[-K', K')$, provide a uniformization of the Ising spectral curve (4.1). The real parameter $\eta \in \left(-\frac{K'}{2}, 0\right)$ is determined by $\sinh 2\mathcal{K}_x = i \,\mathrm{sn}\, 2i\eta$ (it is related to a of [22] by $a = \eta + \frac{K'}{2}$).

Elliptic parametrization (4.2)–(4.3) establishes a one-to-one correspondence between the "physical" cycle $\mathcal{C}_\theta = \left\{(z, \lambda) = (e^{i\theta}, e^{\gamma_\theta}) \,|\, \theta \in [0, 2\pi)\right\}$ and the set $\mathcal{C}_u = \{u \,|\, \mathrm{Re}\, u \in [-K, K),\, \mathrm{Im}\, u = 0\}$. It is convenient to denote by u_θ the image of the point $(e^{i\theta}, e^{\gamma_\theta}) \in \mathcal{C}_\theta$ in $\mathcal{C}_u$. In particular, one has $u_0 = -K$, $u_\pi = 0$ and, more generally, $u_\theta = -u_{2\pi-\theta}$ for $\theta \in (0, \pi]$.

The rest of this section is devoted to the derivation of a suitable elliptic representation for the elements of the induced rotation matrix. We proceed by establishing a number of identities between the two parametrizations. First note that for $\theta \in [0, 2\pi)$ holds

$$e^{-\frac{\gamma_\theta \pm i\theta}{2}} = -\sqrt{k}\, \mathrm{sn}(u_\theta \mp i\eta). \tag{4.4}$$

Indeed, the squares of both sides of this relation coincide because of (4.2)–(4.3). Since they never vanish, it is sufficient to fix the sign for one value of θ. Set $\theta = \pi$, then the l.h.s. becomes $\mp i e^{-\gamma_\pi/2}$ and $\mathrm{sn}(u_\theta \mp i\eta)$ in the r.h.s. reduces to $\mp \mathrm{sn}\, i\eta$. On the other hand $i\, \mathrm{sn}\, i\eta > 0$, and the result follows.

Setting $u = u_\theta$, $u' = u_{\theta'}$, $v = i\eta$ in the easily verified formula

$$\frac{\mathrm{sn}(u+v)\,\mathrm{sn}(u'-v) - \mathrm{sn}(u-v)\,\mathrm{sn}(u'+v)}{1 - k^2 \mathrm{sn}(u+v)\,\mathrm{sn}(u-v)\,\mathrm{sn}(u'+v)\mathrm{sn}(u'-v)} = -\mathrm{sn}(u-u')\,\mathrm{sn}\, 2v,$$

and using (4.4), one finds that

$$\mathrm{sn}(u_\theta - u_{\theta'}) = \sinh 2\mathcal{K}_y \,\frac{\sin \frac{\theta - \theta'}{2}}{\sinh \frac{\gamma_\theta + \gamma_{\theta'}}{2}}. \tag{4.5}$$

This identity has several important consequences. Notice especially that for $\theta' = 0, \pi, 2\pi - \theta$ it reduces to

$$\frac{\mathrm{sn}(u_\theta + K)}{\sinh 2\mathcal{K}_y} = \frac{\sin \frac{\theta}{2}}{\sinh \frac{\gamma_\theta + \gamma_0}{2}}, \qquad \frac{\mathrm{sn}\, u_\theta}{\sinh 2\mathcal{K}_y} = -\frac{\cos \frac{\theta}{2}}{\sinh \frac{\gamma_\pi + \gamma_\theta}{2}}, \qquad \frac{\mathrm{sn}\, 2u_\theta}{\sinh 2\mathcal{K}_y} = -\frac{\sin\theta}{\sinh \gamma_\theta}. \tag{4.6}$$

Since $\cosh\gamma_\theta - \cosh\gamma_{\theta'} = \sinh 2\mathcal{K}_x^* \sinh 2\mathcal{K}_y (\cos\theta' - \cos\theta)$ by (2.8), it can be inferred from the second relation in (4.6) that

$$k\, \mathrm{sn}^2 u_\theta = \frac{\sinh \frac{\gamma_\pi - \gamma_\theta}{2}}{\sinh \frac{\gamma_\pi + \gamma_\theta}{2}}, \tag{4.7}$$

$$1 - k^2\, \mathrm{sn}^2 u_\theta\, \mathrm{sn}^2 u_{\theta'} = \frac{\sinh\gamma_\pi \sinh \frac{\gamma_\theta + \gamma_{\theta'}}{2}}{\sinh \frac{\gamma_\pi + \gamma_\theta}{2} \sinh \frac{\gamma_\pi + \gamma_{\theta'}}{2}}. \tag{4.8}$$

Next we want to compute $\mathrm{cn}\, u_\theta$ and $\mathrm{dn}\, u_\theta$. For that, set $\theta' = \theta$ in the last identity and combine the result with the doubling formula $\mathrm{sn}\, 2u = \dfrac{2\,\mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1 - k^2 \mathrm{sn}^4 u}$ and the second relation in (4.6) to fix the product $\mathrm{cn}\, u_\theta\, \mathrm{dn}\, u_\theta$. Together with the first identity in (4.6) (since $\mathrm{sn}(u+K) = \dfrac{\mathrm{cn}\, u}{\mathrm{dn}\, u}$ and $\mathrm{cn}\, u_\theta$, $\mathrm{dn}\, u_\theta \geq 0$), this gives

$$\mathrm{cn}\, u_\theta = \sin\frac{\theta}{2} \sqrt{\frac{\sinh 2\mathcal{K}_y \sinh\gamma_\pi}{\sinh \frac{\gamma_\theta + \gamma_0}{2} \sinh \frac{\gamma_\theta + \gamma_\pi}{2}}}, \qquad \mathrm{dn}\, u_\theta = \sqrt{\frac{\sinh\gamma_\pi \sinh \frac{\gamma_\theta + \gamma_0}{2}}{\sinh 2\mathcal{K}_y \sinh \frac{\gamma_\theta + \gamma_\pi}{2}}}. \tag{4.9}$$

To find an elliptic representation for b_θ in (2.9), recall that

$$b_\theta = \cosh^2 \mathcal{K}_x^* \sinh 2\mathcal{K}_y \,\frac{\left(1 - \alpha e^{i\theta}\right)\left(1 - \beta e^{-i\theta}\right)}{\sinh\gamma_\theta}.$$

As a function of θ, the numerator of this formula is a linear combination of 1, $\cos\theta$ and $\sin\theta$. On the other hand, (4.7) and (4.8) imply that

$$\frac{2k\,\mathrm{sn}^2 u_\theta}{1-k^2\mathrm{sn}^4 u_\theta}=\frac{\cosh\gamma_\pi-\cosh\gamma_\theta}{\sinh\gamma_\pi\sinh\gamma_\theta}, \tag{4.10}$$

$$\frac{1+k^2\mathrm{sn}^4 u_\theta}{1-k^2\mathrm{sn}^4 u_\theta}=\frac{\cosh\gamma_\pi\cosh\gamma_\theta-1}{\sinh\gamma_\pi\sinh\gamma_\theta}. \tag{4.11}$$

Hence b_θ is a linear combination of the l.h.s.'s of (4.10), (4.11) and the third relation in (4.6). A little calculation then gives

$$b_\theta=ik\,\mathrm{sn}\,2u_\theta-\frac{2k^2\mathrm{sn}^2 u_\theta}{1-k^2\mathrm{sn}^4 u_\theta}+\frac{1+k^2\mathrm{sn}^4 u_\theta}{1-k^2\mathrm{sn}^4 u_\theta}=\frac{\left(\mathrm{dn}\,u_\theta+ik\,\mathrm{sn}\,u_\theta\,\mathrm{cn}\,u_\theta\right)^2}{1-k^2\mathrm{sn}^4 u_\theta}.$$

Taking the square root and fixing the signs via $\sqrt{b_\pi}=1$, one obtains

$$\left[\sqrt{b_\theta}\right]^{\pm1}=\frac{\mathrm{dn}\,u_\theta\pm ik\,\mathrm{sn}\,u_\theta\,\mathrm{cn}\,u_\theta}{\sqrt{1-k^2\mathrm{sn}^4 u_\theta}}. \tag{4.12}$$

From (4.12) we can now deduce what we really want — namely, a nice elliptic representation of the matrix elements of A, B, C, D in Lemma 3.3:

Lemma 4.1. *Let $\theta,\theta'\in[0,2\pi)$ with $\theta\pm\theta'\neq 0,2\pi$. Then*

$$\frac{1}{2\sin\frac{\theta-\theta'}{2}}\left(\sqrt{\frac{b_\theta}{b_{\theta'}}}+\sqrt{\frac{b_{\theta'}}{b_\theta}}\right)=\frac{\sinh 2\mathcal{K}_y}{\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\mathrm{dn}(u_\theta-u_{\theta'})}{\mathrm{sn}(u_\theta-u_{\theta'})}, \tag{4.13}$$

$$\frac{1}{2\sin\frac{\theta+\theta'}{2}}\left(\sqrt{b_\theta b_{\theta'}}-\frac{1}{\sqrt{b_\theta b_{\theta'}}}\right)=\frac{-i\sinh 2\mathcal{K}_x^*}{\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\,\mathrm{cn}(u_\theta-u_{\theta'}). \tag{4.14}$$

Proof. From (4.12) it follows that e.g.

$$\frac{1}{2\sin\frac{\theta-\theta'}{2}}\left(\sqrt{\frac{b_\theta}{b_{\theta'}}}+\sqrt{\frac{b_{\theta'}}{b_\theta}}\right)=\frac{\mathrm{dn}\,u_\theta\,\mathrm{dn}\,u_{\theta'}+k^2\mathrm{sn}\,u_\theta\,\mathrm{cn}\,u_\theta\,\mathrm{sn}\,u_{\theta'}\,\mathrm{cn}\,u_{\theta'}}{\sin\frac{\theta-\theta'}{2}\,\sqrt{\left(1-k^2\mathrm{sn}^4 u_\theta\right)\left(1-k^2\mathrm{sn}^4 u_{\theta'}\right)}}.$$

Standard addition formula $\mathrm{dn}(u-u')=\dfrac{\mathrm{dn}\,u\,\mathrm{dn}\,u'+k^2\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{sn}\,u'\,\mathrm{cn}\,u'}{1-k^2\mathrm{sn}^2 u\,\mathrm{sn}^2 u'}$ transforms the right side of the last relation into

$$\frac{1-k^2\mathrm{sn}^2 u_\theta\mathrm{sn}^2 u_{\theta'}}{\sqrt{\left(1-k^2\mathrm{sn}^4 u_\theta\right)\left(1-k^2\mathrm{sn}^4 u_{\theta'}\right)}}\,\frac{\mathrm{dn}(u_\theta-u_{\theta'})}{\sin\frac{\theta-\theta'}{2}}.$$

By (4.8), the first factor may be reduced to $\dfrac{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}$, and the desired identity (4.13) becomes an immediate consequence of (4.5).

Similarly, applying (4.12) to the l.h.s. of (4.14) we get

$$\frac{1}{2\sin\frac{\theta+\theta'}{2}}\left(\sqrt{b_\theta b_{\theta'}}-\frac{1}{\sqrt{b_\theta b_{\theta'}}}\right)=ik\frac{\mathrm{sn}\,u_\theta\,\mathrm{cn}\,u_\theta\,\mathrm{dn}\,u_{\theta'}+\mathrm{dn}\,u_\theta\,\mathrm{sn}\,u_{\theta'}\,\mathrm{cn}\,u_{\theta'}}{\sin\frac{\theta+\theta'}{2}\,\sqrt{\left(1-k^2\mathrm{sn}^4 u_\theta\right)\left(1-k^2\mathrm{sn}^4 u_{\theta'}\right)}}.$$

Now (4.14) can be obtained from the addition formula

$$\mathrm{cn}(u-u')\,\mathrm{sn}(u+u')=\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u'+\mathrm{dn}\,u\,\mathrm{sn}\,u'\,\mathrm{cn}\,u'}{1-k^2\mathrm{sn}^2 u\,\mathrm{sn}^2 u'}$$

in a way analogous to the above. □

Finally, let us introduce two diagonal matrices Λ_ν ($\nu=a,p$) with elements

$$\left(\Lambda_\nu\right)_{\theta,\theta'}=\frac{e^{i(l-\frac12)\theta}}{\sqrt{\sinh\gamma_\theta}}\,\delta_{\theta,\theta'},\qquad \theta,\theta'\in\boldsymbol{\theta}_\nu, \tag{4.15}$$

and two non-diagonal matrices Φ, Ψ defined by

$$\Phi_{\theta,\theta'} = \frac{\mathrm{dn}(u_\theta - u_{\theta'})}{\mathrm{sn}(u_\theta - u_{\theta'})}, \qquad \Psi_{\theta,\theta'} = \mathrm{cn}(u_\theta - u_{\theta'}), \qquad \theta \in \boldsymbol{\theta}_p, \theta' \in \boldsymbol{\theta}_a. \tag{4.16}$$

Combining the results of Lemmas 3.3 and 4.1, we arrive at

Lemma 4.2. *Matrices B, C and D defined by (3.15)–(3.16) satisfy the following identities:*

$$D^{-1} = \frac{-iN}{\sinh 2\mathcal{K}_y}\, \Lambda_a^{-1}\Phi^{-1}\bar{\Lambda}_p^{-1}, \tag{4.17}$$

$$BD^{-1} = i\,\frac{\sinh 2\mathcal{K}_x^*}{\sinh 2\mathcal{K}_y}\, \Lambda_p\Psi\Phi^{-1}\bar{\Lambda}_p^{-1}, \tag{4.18}$$

$$D^{-1}C = i\,\frac{\sinh 2\mathcal{K}_x^*}{\sinh 2\mathcal{K}_y}\, \Lambda_a^{-1}\Phi^{-1}\Psi\bar{\Lambda}_a, \tag{4.19}$$

$$|\det D| = \frac{(\sinh 2\mathcal{K}_y)^N\, |\det \Phi|}{N^N \sqrt{\prod_{\theta\in\boldsymbol{\theta}_p} \sinh\gamma_\theta \prod_{\theta\in\boldsymbol{\theta}_a} \sinh\gamma_\theta}}. \tag{4.20}$$

Proof. Check that $B = \bar{C} = -\dfrac{\sinh 2\mathcal{K}_x^*}{N}\,\Lambda_p\Psi\Lambda_a$ and $D = -\dfrac{\sinh 2\mathcal{K}_y}{iN}\,\bar{\Lambda}_p\Phi\Lambda_a$. □

Thus in order to obtain all finite-lattice form factors of Ising spin, it is sufficient to compute the inverse Φ^{-1} and two matrix products $\Psi\Phi^{-1}$, $\Phi^{-1}\Psi$. This will be done in the next section using that Φ, Ψ defined by (4.16) are special cases of elliptic Cauchy matrices.

5. Elliptic Cauchy matrices and two-particle form factors

5.1. **Elliptic determinants.** Consider $2N$ complex variables $x_1,\ldots,x_N$, $y_1,\ldots,y_N$. The determinant of the Cauchy matrix $V_{ij} = \dfrac{1}{x_i - y_j}$ $(i,j = 1,\ldots,N)$ is a rational function of $\{x_i\}$ and $\{y_i\}$, which vanishes whenever $x_i = x_j$ or $y_i = y_j$ for some pair of distinct indices i,j. It is well-known that there are no other zeros and the determinant is given by

$$\det V = \frac{\prod_{i<j}^N (x_i - x_j)(y_j - y_i)}{\prod_{i,j}^N (x_i - y_j)}.$$

There is an elliptic version of this relation due to Frobenius [7]. Namely, if we denote by $\vartheta_{1\ldots4}(z)$ Jacobi theta functions of nome $q = e^{\pi i\tau}$, then for the elliptic Cauchy matrix $\tilde{V}_{ij} = \dfrac{\vartheta_1(x_i - y_j + \alpha)}{\vartheta_1(x_i - y_j)\vartheta_1(\alpha)}$ depending on an arbitrary parameter α holds the identity

$$\det\tilde{V} = \frac{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \alpha\right)}{\vartheta_1(\alpha)} \frac{\prod_{i<j}^N \vartheta_1(x_i - x_j)\vartheta_1(y_j - y_i)}{\prod_{i,j}^N \vartheta_1(x_i - y_j)}. \tag{5.1}$$

Corollary 5.1. *We have*

$$\tilde{V}_{mn}^{-1} = -\frac{\vartheta_1\left(\sum_{i\neq n}^N x_i - \sum_{i\neq m}^N y_i + \alpha\right)}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \alpha\right)\vartheta_1(x_n - y_m)} \frac{\prod_i^N \vartheta_1(x_n - y_i)\prod_i^N \vartheta_1(y_m - x_i)}{\prod_{i\neq n}^N \vartheta_1(x_n - x_i)\prod_{i\neq m}^N \vartheta_1(y_m - y_i)}. \tag{5.2}$$

Proof. Use (5.1) to compute the determinant and cofactors and (5.2) follows. □

Theta functions are related to the Jacobi elliptic functions by

$$\mathrm{sn}\, u = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}, \qquad \mathrm{cn}\, u = \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}, \qquad \mathrm{dn}\, u = \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}, \tag{5.3}$$

where $\vartheta_i = \vartheta_i(0)$ for $i = 2, 3, 4$. Elliptic modulus and half-periods are given by

$$k = \frac{\vartheta_2^2}{\vartheta_3^2}, \qquad 2K = \pi\vartheta_3^2, \qquad 2iK' = \pi\tau\vartheta_3^2.$$

The functions $\vartheta_{2,3,4}(z)$ can be obtained from $\vartheta_1(z)$ by shifting its argument,

$$\vartheta_2(z) = \vartheta_1\left(z + \frac{\pi}{2}\right), \tag{5.4}$$

$$\vartheta_3(z) = e^{-i\left(z-\frac{\pi\tau}{4}\right)}\vartheta_1\left(z + \frac{\pi}{2} - \frac{\pi\tau}{2}\right), \tag{5.5}$$

$$\vartheta_4(z) = ie^{-i\left(z-\frac{\pi\tau}{4}\right)}\vartheta_1\left(z - \frac{\pi\tau}{2}\right). \tag{5.6}$$

Therefore one can deduce from (5.1), (5.2) the determinants and inverses for a number of matrices with entries written in terms of the Jacobi elliptic functions.

Lemma 5.2. *For $1 \le i, j \le N$ with even N we have*

$$\det\left(\sqrt{k}\,\mathrm{sn}(u_i - u_j)\right) = \left(\prod_{i<j}^{N}\sqrt{k}\,\mathrm{sn}(u_i - u_j)\right)^2. \tag{5.7}$$

Proof. Set $x_i = \vartheta_3^{-2}u_i$, $y_i = \vartheta_3^{-2}u_i + \frac{\pi\tau}{2}$ and use (5.3)–(5.6) and the formula $\vartheta_1\left(\frac{\pi\tau}{2}\right) = ie^{-\frac{i\pi\tau}{4}}\vartheta_4$ to write

$$\sqrt{k}\,\mathrm{sn}(u_i - u_j) = \vartheta_4 e^{i(x_i - y_j)}\frac{\vartheta_1\left(x_i - y_j + \frac{\pi\tau}{2}\right)}{\vartheta_1\left(x_i - y_j\right)\vartheta_1\left(\frac{\pi\tau}{2}\right)}.$$

The determinant on the left side of (5.7) can therefore be deduced from (5.1) with $\alpha = \frac{\pi\tau}{2}$. Since in our case $y_i = x_i + \frac{\pi\tau}{2}$ and in particular $\sum_i^N x_i - \sum_i^N y_i = -\frac{N\pi\tau}{2}$, it reduces to

$$(-1)^{\frac{N}{2}-1}e^{-\frac{iN\pi\tau}{4}}\frac{\vartheta_1\left(\frac{(N-1)\pi\tau}{2}\right)}{\vartheta_1\left(\frac{\pi\tau}{2}\right)}\prod_{i<j}^{N}\frac{\vartheta_1(x_i - x_j)\vartheta_1(x_j - x_i)}{\vartheta_1\left(x_i - x_j - \frac{\pi\tau}{2}\right)\vartheta_1\left(x_j - x_i - \frac{\pi\tau}{2}\right)}.$$

To transform the theta functions back to sn's, use again (5.3)–(5.6) and the quasiperiodicity relation $\vartheta_1(z + \ell\pi\tau) = (-1)^\ell e^{-i\pi\ell^2\tau - 2i\ell z}\vartheta_1(z)$, which holds for $\ell \in \mathbb{Z}$. □

Remark 5.3. Taking the square root at both sides of (5.7) and fixing the signs via the rational limit, one obtains a pfaffian version of this formula:

$$\mathrm{Pf}\left(\sqrt{k}\,\mathrm{sn}(u_i - u_j)\right) = \prod_{i<j}^{N}\sqrt{k}\,\mathrm{sn}(u_i - u_j). \tag{5.8}$$

We note that this identity has already appeared in the proof of Theorem 5.0 in [24], where it was used to obtain multiparticle Ising form factors on the infinite lattice from the two-particle ones.

Lemma 5.4. *Let Φ denote a matrix with elements $\Phi_{ij} = \dfrac{\mathrm{dn}(u_i - v_j)}{\mathrm{sn}(u_i - v_j)}$ with $i, j = 1, \ldots, N$. Then*

$$\det\Phi = \frac{\vartheta_2^N\vartheta_4^N}{\vartheta_3^{N+1}}e^{-i\left(\sum_i^N x_i - \sum_i^N y_i - \frac{\pi\tau}{4}\right)}\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)\times$$
$$\times\frac{\prod_{i<j}^N\vartheta_1(x_i - x_j)\vartheta_1(y_j - y_i)}{\prod_{i,j}^N\vartheta_1(x_i - y_j)}, \tag{5.9}$$

$$\Phi^{-1}_{mn} = -\frac{\vartheta_3}{\vartheta_2\vartheta_4}\frac{e^{i(x_n - y_m)}\vartheta_1\left(\sum_{i\neq n}^N x_i - \sum_{i\neq m}^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)\vartheta_1\left(x_n - y_m\right)}\times$$
$$\times\frac{\prod_i^N\vartheta_1\left(x_n - y_i\right)\prod_i^N\vartheta_1\left(y_m - x_i\right)}{\prod_{i\neq n}^N\vartheta_1\left(x_n - x_i\right)\prod_{i\neq m}^N\vartheta_1\left(y_m - y_i\right)}, \tag{5.10}$$

where $x_i = \vartheta_3^{-2} u_i$, $y_i = \vartheta_3^{-2} v_i$.

Proof. From (5.3) and the relation $\vartheta_1\left(\frac{\pi}{2} - \frac{\pi\tau}{2}\right) = e^{-\frac{i\pi\tau}{4}}\vartheta_3$ it follows that Φ_{ij} can be written as

$$\Phi_{ij} = e^{-i(x_i - y_j)} \frac{\vartheta_2\vartheta_4}{\vartheta_3} \frac{\vartheta_1\left(x_i - y_j + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)}{\vartheta_1(x_i - y_j)\,\vartheta_1\left(\frac{\pi}{2} - \frac{\pi\tau}{2}\right)}.$$

Now to derive (5.9), (5.10), it suffices to set $\alpha = \frac{\pi}{2} - \frac{\pi\tau}{2}$ in (5.1), (5.2). □

5.2. **Theta functional interpolation.** Let $z_1, \ldots, z_M$, $f_1, \ldots, f_M$ be $2M$ complex variables. The Lagrange polynomial $P(z)$ is the unique polynomial of degree $\leq M-1$, interpolating the set of points (z_i, f_i) (in other words, $f_i = P(z_i)$ for $i = 1, \ldots, M$). As is well-known,

$$P(z) = \sum_i^M f_i \prod_{j \neq i}^M \frac{z - z_j}{z_i - z_j}. \tag{5.11}$$

Introduce M additional parameters $z'_1, \ldots, z'_M$ and the $(M-1)$th degree polynomial

$$f(z) = \prod_i^M (z - z'_i) - \prod_i^M (z - z_i).$$

If we now set $f_j = f(z_j)$ for $j = 1, \ldots, M$, then the evaluation of the coefficient of z^{M-1} at both sides of (5.11) leads to the identity

$$\sum_i^M \frac{\prod_j^M (z_i - z'_j)}{\prod_{j\neq i}^M (z_i - z_j)} = \sum_i^M z_i - \sum_i^M z'_i.$$

The last relation has an elliptic analog, which turns out to be very helpful in the calculation of the products BD^{-1} and $D^{-1}C$. Namely, if the parameters $\{z_i\}$, $\{z'_i\}$ satisfy the balancing condition $\sum_i^M z_i - \sum_i^M z'_i = 0$, then

$$\sum_i^M \frac{\prod_j^M \vartheta_1(z_i - z'_j)}{\prod_{j\neq i}^M \vartheta_1(z_i - z_j)} = 0. \tag{5.12}$$

This identity appears in a somewhat disguised form in [29], Example 3 on p. 451, where it is formulated in terms of the Weierstrass σ-function. It can be proved by induction on M, using the Riemann's addition formula in the base case, see e.g. Theorem 7 in [28].

Lemma 5.5. *Let* Ψ *denote a matrix with elements* $\Psi_{ij} = \operatorname{cn}(u_i - v_j)$ $(i, j = 1, \ldots, N)$ *and* Φ, $\{x_i\}$, $\{y_i\}$ *be as in Lemma 5.4. Then we have*

$$\left(\Psi\Phi^{-1}\right)_{ln} = ie^{i\left(x_n - x_l - \frac{\pi\tau}{2}\right)} \frac{\vartheta_3^2}{\vartheta_2^2} \frac{\vartheta_1\left(x_l - x_n + \sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2}\right)}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)} \times \times \prod_i^N \frac{\vartheta_1(x_n - y_i)}{\vartheta_1\left(x_l - y_i + \frac{\pi\tau}{2}\right)} \prod_{i\neq n}^N \frac{\vartheta_1\left(x_l - x_i + \frac{\pi\tau}{2}\right)}{\vartheta_1(x_n - x_i)}, \tag{5.13}$$

$$\left(\Phi^{-1}\Psi\right)_{ml} = ie^{i\left(y_l - y_m - \frac{\pi\tau}{2}\right)} \frac{\vartheta_3^2}{\vartheta_2^2} \frac{\vartheta_1\left(y_m - y_l + \sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2}\right)}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)} \times \times \prod_i^N \frac{\vartheta_1(y_m - x_i)}{\vartheta_1\left(y_l - x_i - \frac{\pi\tau}{2}\right)} \prod_{i\neq m}^N \frac{\vartheta_1\left(y_l - y_i - \frac{\pi\tau}{2}\right)}{\vartheta_1(y_m - y_i)}. \tag{5.14}$$

Proof. We will prove only the first identity, since for the second the argument is completely analogous. By (5.3), one can write

$$\Psi_{lm} = -ie^{i\left(y_m - x_l - \frac{\pi\tau}{4}\right)} \frac{\vartheta_4}{\vartheta_2} \frac{\vartheta_1\left(y_m - x_l + \frac{\pi}{2}\right)}{\vartheta_1\left(y_m - x_l - \frac{\pi\tau}{2}\right)}.$$

Lemma 5.4 then implies that the l.h.s. of (5.13) is given by

$$\left(\Psi\Phi^{-1}\right)_{ln} = \frac{\vartheta_3}{\vartheta_2^2} \frac{ie^{i\left(x_n - x_l - \frac{\pi\tau}{4}\right)}}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \frac{\pi}{2} - \frac{\pi\tau}{2}\right)} \frac{\prod_i^N \vartheta_1(x_n - y_i)}{\prod_{i\neq n}^N \vartheta_1(x_n - x_i)} \times \tag{5.15}$$

$$\times \sum_m^N \left\{ \vartheta_1\left(\sum_{i\neq n}^N x_i - \sum_{i\neq m}^N y_i - \frac{\pi}{2} - \frac{\pi\tau}{2}\right) \frac{\vartheta_1\left(y_m - x_l + \frac{\pi}{2}\right)}{\vartheta_1\left(y_m - x_l - \frac{\pi\tau}{2}\right)} \frac{\prod_{i\neq n}^N \vartheta_1(y_m - x_i)}{\prod_{i\neq m}^N \vartheta_1(y_m - y_i)} \right\}.$$

If we denote

- $z_i = y_i$ for $i = 1, \ldots, N$, $z_{N+1} = x_l + \frac{\pi\tau}{2}$,
- $z_i' = x_i$ for $i = 1, \ldots, n-1$ and for $i = n+1, \ldots, N$,
- $z_n' = x_n - \sum_i^N x_i + \sum_i^N y_i + \frac{\pi}{2} + \frac{\pi\tau}{2}$ and $z_{N+1}' = x_l - \frac{\pi}{2}$,

then the sum in the second line of (5.15) can be written as $\displaystyle\sum_i^N \frac{\prod_j^{N+1} \vartheta_1(z_i - z_j')}{\prod_{j\neq i}^{N+1} \vartheta_1(z_i - z_j)}$. This almost coincides with the sum in the l.h.s. of (5.12) with $M = N+1$. The missing $(N+1)$th term is given by

$$\frac{\prod_i^{N+1} \vartheta_1(z_{N+1} - z_j')}{\prod_i^N \vartheta_1(z_{N+1} - z_j)} = \vartheta_1\left(\frac{\pi}{2} + \frac{\pi\tau}{2}\right) \vartheta_1\left(x_l - x_n + \sum_i^N x_i - \sum_i^N y_i - \frac{\pi}{2}\right) \frac{\prod_{i\neq n}^N \vartheta_1\left(x_l - x_i + \frac{\pi\tau}{2}\right)}{\prod_i^N \vartheta_1\left(x_l - y_i + \frac{\pi\tau}{2}\right)}.$$

Since the balancing condition is satisfied, our sum is equal to minus this missing term. Together with the first line in (5.15), this leads to (5.13). □

5.3. **Trigonometric reduction.** The matrices Φ and Ψ, corresponding to the Ising case, are indexed by two sets of quasimomenta. Explicitly, $u_j = u_\theta$ with $\theta = \frac{2\pi(j-1)}{N} \in \boldsymbol{\theta}_p$, and $v_j = u_{\theta'}$ with $\theta' = \frac{2\pi}{N}\left(j - \frac{1}{2}\right) \in \boldsymbol{\theta}_a$. Since for $\theta \in (0, 2\pi)$ each set contains both θ and $2\pi - \theta$, there are additional constraints on $\{x_j\}$, $\{y_j\}$. Their form slightly differs for odd and even values of N:

- For odd N, the value $\theta = 0$ belongs to the periodic set of momenta, but $\theta = \pi$ is in the antiperiodic one. Then for all $j = 2, \ldots, \frac{N+1}{2}$ and $k = 1, \ldots, \frac{N-1}{2}$ we have

$$x_j + x_{N+2-j} = y_k + y_{N+1-k} = x_1 + \frac{\pi}{2} = y_{(N+1)/2} = 0. \tag{5.16}$$

- For even N, both exceptional values $\theta = 0, \pi$ belong to the periodic set. Thus for all $j = 2, \ldots, \frac{N}{2}$ and $k = 1, \ldots, \frac{N}{2}$ holds

$$x_j + x_{N+2-j} = y_k + y_{N+1-k} = x_1 + \frac{\pi}{2} = x_{N/2+1} = 0. \tag{5.17}$$

In particular, in both cases we have

$$\sum_i^N x_i - \sum_i^N y_i = -\frac{\pi}{2}. \tag{5.18}$$

This allows to rewrite (5.9)–(5.10) and (5.13)–(5.14) as

$$(\det\Phi)^2 = \frac{\vartheta_3^{-2}\vartheta_4^2}{\prod_i^N f_i g_i}, \qquad \Phi_{mn}^{-1} = \frac{f_n g_m}{\mathrm{sn}(u_n - v_m)}, \tag{5.19}$$

$$\left(\Psi\Phi^{-1}\right)_{ln} = f_n h\big(x_l + \frac{\pi\tau}{2}\big)\,\mathrm{sn}(u_l - u_n), \tag{5.20}$$

$$\left(\Phi^{-1}\Psi\right)_{ml} = -\frac{g_m}{h\left(y_l - \frac{\pi\tau}{2}\right)}\,\mathrm{sn}(v_m - v_l), \tag{5.21}$$

where $h(z) = \prod_i^N \frac{\vartheta_1(z - x_i)}{\vartheta_1(z - y_i)}$ and $\{f_n\}$, $\{g_m\}$ are defined by

$$f_n = \frac{i\vartheta_3}{\vartheta_2\vartheta_4}\frac{\prod_i^N \vartheta_1(x_n - y_i)}{\prod_{i\neq n}^N \vartheta_1(x_n - x_i)}, \qquad g_m = \frac{i\vartheta_3}{\vartheta_2\vartheta_4}\frac{\prod_i^N \vartheta_1(y_m - x_i)}{\prod_{i\neq m}^N \vartheta_1(y_m - y_i)}.$$

Notice that (5.18), combined with the quasiperiodicity of $\vartheta_1(z)$, also implies that $h(z+\pi\tau) = -h(z)$. Further, it is easy to check using (5.3), (5.6) that the quantities $\chi_n = -f_n h\big(x_n - \frac{\pi\tau}{2}\big)$ and $\kappa_m = g_m / h\left(y_m - \frac{\pi\tau}{2}\right)$ are given by

$$\chi_n = \frac{\prod_i^N \mathrm{sn}(u_n - v_i)}{\prod_{i\neq n}^N \mathrm{sn}(u_n - u_i)}, \qquad \kappa_m = \frac{\prod_i^N \mathrm{sn}(v_m - u_i)}{\prod_{i\neq m}^N \mathrm{sn}(v_m - v_i)}. \tag{5.22}$$

Thus, to write (5.19)–(5.21) in terms of sn's, it suffices to find a suitable representation for the function $\mu(z,z') = h(z)/h(z')$. For that, one can use the constraints (5.16)–(5.17).

Suppose e.g. that N is even, then

$$\mu(z,z') = \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_1(z')\vartheta_2(z')}\prod_{i=2}^{N/2}\frac{\vartheta_1(z - x_i)\vartheta_1(z + x_i)}{\vartheta_1(z' - x_i)\vartheta_1(z' + x_i)}\prod_{i=1}^{N/2}\frac{\vartheta_1(z' - y_i)\vartheta_1(z' + y_i)}{\vartheta_1(z - y_i)\vartheta_1(z + y_i)}, \tag{5.23}$$

where we have put together the theta functions with arguments coming in pairs according to (5.17). In each product of two theta functions, use the well-known addition formula, written in the form

$$\begin{aligned}\vartheta_1(s+t)\vartheta_1(s-t)\vartheta_4^2 &= \vartheta_1^2(s)\vartheta_4^2(t) - \vartheta_4^2(s)\vartheta_1^2(t) = \\ &= \vartheta_1(s)\vartheta_4(s)\vartheta_1(t)\vartheta_4(t)\frac{\mathrm{sn}^2\vartheta_3^2 s - \mathrm{sn}^2\vartheta_3^2 t}{\mathrm{sn}\,\vartheta_3^2 s\;\mathrm{sn}\,\vartheta_3^2 t}.\end{aligned} \tag{5.24}$$

After substitution into (5.23), almost all of the theta functions and denominators appearing in (5.24) cancel each other, so that $\mu(z,z')$ reduces to

$$\mu(z,z') = \frac{\mathrm{cn}\,\vartheta_3^2 z\left(1 + \mathrm{sn}\,\vartheta_3^2 z'\right)}{\mathrm{cn}\,\vartheta_3^2 z'\left(1 + \mathrm{sn}\,\vartheta_3^2 z\right)}\prod_{i=1}^N \frac{\left(\mathrm{sn}\,\vartheta_3^2 z - \mathrm{sn}\,u_i\right)\left(\mathrm{sn}\,\vartheta_3^2 z' - \mathrm{sn}\,v_i\right)}{\left(\mathrm{sn}\,\vartheta_3^2 z - \mathrm{sn}\,v_i\right)\left(\mathrm{sn}\,\vartheta_3^2 z' - \mathrm{sn}\,u_i\right)}. \tag{5.25}$$

Similar manipulations for odd N lead to the same formula.

If we now denote $\lambda(u,v) = \mu\left(\vartheta_3^{-2}u - \frac{\pi\tau}{2}, \vartheta_3^{-2}v - \frac{\pi\tau}{2}\right)$, then by (5.25)

$$\lambda(u,v) = \frac{\mathrm{dn}\,u\,(1 + k\,\mathrm{sn}\,v)}{\mathrm{dn}\,v\,(1 + k\,\mathrm{sn}\,u)}\prod_{i=1}^N \frac{(1 - k\,\mathrm{sn}\,u_i\,\mathrm{sn}\,u)(1 - k\,\mathrm{sn}\,v_i\,\mathrm{sn}\,v)}{(1 - k\,\mathrm{sn}\,v_i\,\mathrm{sn}\,u)(1 - k\,\mathrm{sn}\,u_i\,\mathrm{sn}\,v)}. \tag{5.26}$$

Relations (5.22) and (5.26) then allow to rewrite (5.19)–(5.21) in terms of the Jacobi elliptic functions:

$$(\det\Phi)^2 = \frac{(-1)^N\sqrt{1-k^2}}{\prod_i^N \kappa_i\chi_i\,\lambda(v_i,u_i)}, \qquad \Phi^{-1}_{mn} = \frac{\kappa_m\chi_n\lambda(v_m,u_n)}{\mathrm{sn}(v_m - u_n)}, \tag{5.27}$$

$$\left(\Psi\Phi^{-1}\right)_{ln} = \chi_n\lambda(u_l,u_n)\,\mathrm{sn}(u_l - u_n), \tag{5.28}$$

$$\left(\Phi^{-1}\Psi\right)_{ml} = \kappa_m\lambda(v_m,v_l)\,\mathrm{sn}(v_l - v_m). \tag{5.29}$$

So far we have used only a reflection symmetry of the periodic and antiperiodic spectrum of momenta. That these momenta take rather special values (recall that $e^{i\theta}$ is an Nth root of ± 1)

leads to further simplifications. It will be convenient to switch the notation and write all matrix indices in terms of momenta, as suggested in the beginning of this subsection. Thus, e.g.

$$\chi_\theta=\frac{\prod_{\theta'\in\boldsymbol{\theta}_a}\operatorname{sn}(u_\theta-u_{\theta'})}{\prod_{\theta'\in\boldsymbol{\theta}_p,\theta'\neq\theta}\operatorname{sn}(u_\theta-u_{\theta'})},\qquad \kappa_\theta=\frac{\prod_{\theta'\in\boldsymbol{\theta}_p}\operatorname{sn}(u_\theta-u_{\theta'})}{\prod_{\theta'\in\boldsymbol{\theta}_a,\theta'\neq\theta}\operatorname{sn}(u_\theta-u_{\theta'})},\tag{5.30}$$

where in the first formula $\theta\in\boldsymbol{\theta}_p$ and in the second one $\theta\in\boldsymbol{\theta}_a$. Likewise, for $\theta,\theta'\in[0,2\pi)$ denote

$$\lambda_{\theta,\theta'}=\lambda(u_\theta,u_{\theta'})=\frac{\operatorname{dn}u_\theta}{\operatorname{dn}u_{\theta'}}\prod_{\substack{\theta''\in\boldsymbol{\theta}_p\\ \theta''\neq 0}}\frac{1-k\operatorname{sn}u_{\theta''}\operatorname{sn}u_\theta}{1-k\operatorname{sn}u_{\theta''}\operatorname{sn}u_{\theta'}}\prod_{\theta''\in\boldsymbol{\theta}_a}\frac{1-k\operatorname{sn}u_{\theta''}\operatorname{sn}u_{\theta'}}{1-k\operatorname{sn}u_{\theta''}\operatorname{sn}u_\theta}.\tag{5.31}$$

Our aim is to express χ_θ, κ_θ and $\lambda_{\theta,\theta'}$ in the initial trigonometric parametrization (i.e. in terms of θ and γ_θ). Introduce the function

$$\nu_\theta=\ln\frac{\prod_{\theta'\in\boldsymbol{\theta}_a}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\prod_{\theta'\in\boldsymbol{\theta}_p}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}},\qquad \theta\in[0,2\pi).\tag{5.32}$$

Use (4.9) to rewrite the first factor in (5.31) and transform the remaining two products with the help of (4.8). The calculation is slightly different for odd and even N, but the final result is given by the same simple formula:

$$\lambda_{\theta,\theta'}=e^{(\nu_{\theta'}-\nu_\theta)/2}.\tag{5.33}$$

Similarly, the identity (4.5) allows to write (5.30) as

$$\chi_\theta=e^{-\nu_\theta}\frac{\sinh 2\mathcal{K}_y}{\sinh\gamma_\theta}\frac{\prod_{\theta'\in\boldsymbol{\theta}_a}\sin\frac{\theta-\theta'}{2}}{\prod_{\theta'\in\boldsymbol{\theta}_p,\theta'\neq\theta}\sin\frac{\theta-\theta'}{2}}\quad\text{for }\theta\in\boldsymbol{\theta}_p,\tag{5.34}$$

$$\kappa_\theta=e^{\nu_\theta}\frac{\sinh 2\mathcal{K}_y}{\sinh\gamma_\theta}\frac{\prod_{\theta'\in\boldsymbol{\theta}_p}\sin\frac{\theta-\theta'}{2}}{\prod_{\theta'\in\boldsymbol{\theta}_a,\theta'\neq\theta}\sin\frac{\theta-\theta'}{2}}\quad\text{for }\theta\in\boldsymbol{\theta}_a.\tag{5.35}$$

The products in the r.h.s. can be easily computed explicitly. One has

$$2^{N-1}\prod_{\theta'\in\boldsymbol{\theta}_p}\sin\frac{\theta-\theta'}{2}=(-1)^{N-1}\sin\frac{N\theta}{2},\tag{5.36}$$

$$2^{N-1}\prod_{\theta'\in\boldsymbol{\theta}_a}\sin\frac{\theta-\theta'}{2}=\ (-1)^{N}\ \cos\frac{N\theta}{2},\tag{5.37}$$

$$2^{N-1}\prod_{\theta'\in\boldsymbol{\theta}_p,\theta\neq\theta'}\sin\frac{\theta-\theta'}{2}=(-1)^{N-1}N\cos\frac{N\theta}{2}\quad\text{for }\theta\in\boldsymbol{\theta}_p,\tag{5.38}$$

$$2^{N-1}\prod_{\theta'\in\boldsymbol{\theta}_a,\theta\neq\theta'}\sin\frac{\theta-\theta'}{2}=(-1)^{N-1}N\sin\frac{N\theta}{2}\quad\text{for }\theta\in\boldsymbol{\theta}_a.\tag{5.39}$$

The first two relations are valid for any $\theta\in\mathbb{C}$ and follow from the obvious product formulas $\prod_{\theta'\in\boldsymbol{\theta}_p}(z-e^{i\theta'})=z^N-1$ and $\prod_{\theta'\in\boldsymbol{\theta}_a}(z-e^{i\theta'})=z^N+1$. The third and fourth formula may be deduced from (5.36)–(5.37) by taking appropriate limits.

Substituting (5.36)–(5.39) into (5.34)–(5.35), and combining the result with (4.5) and (5.33), we can rewrite (5.27)–(5.29) as follows:

$$(\det\Phi)^2 = \frac{N^{2N}\sqrt{1-k^2}}{(\sinh 2\mathcal{K}_y)^{2N}} \prod_{\theta\in\boldsymbol{\theta}_p} e^{\nu_\theta/2}\sinh\gamma_\theta \prod_{\theta\in\boldsymbol{\theta}_a} e^{-\nu_\theta/2}\sinh\gamma_\theta, \tag{5.40}$$

$$\Phi^{-1}_{\theta,\theta'} = -\frac{\sinh 2\mathcal{K}_y e^{(\nu_\theta-\nu_{\theta'})/2}}{N^2\sinh\gamma_\theta\sinh\gamma_{\theta'}}\frac{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\sin\frac{\theta-\theta'}{2}}, \qquad \theta\in\boldsymbol{\theta}_a, \theta'\in\boldsymbol{\theta}_p, \tag{5.41}$$

$$\left(\Psi\Phi^{-1}\right)_{\theta,\theta'} = -\frac{\sinh^2 2\mathcal{K}_y e^{-(\nu_\theta+\nu_{\theta'})/2}}{N\sinh\gamma_{\theta'}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, \qquad \theta,\theta'\in\boldsymbol{\theta}_p, \tag{5.42}$$

$$\left(\Phi^{-1}\Psi\right)_{\theta,\theta'} = -\frac{\sinh^2 2\mathcal{K}_y e^{(\nu_\theta+\nu_{\theta'})/2}}{N\sinh\gamma_{\theta}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, \qquad \theta,\theta'\in\boldsymbol{\theta}_a. \tag{5.43}$$

Now from (5.40)–(5.43) and Lemma 4.2 one easily derives the main result of this paper:

Theorem 5.6. *Let B, C and D be the matrices defined by (3.15)–(3.16). Then*

$$D^{-1}_{\theta,\theta'} = ie^{-i(l-\frac12)(\theta-\theta')}\frac{e^{(\nu_\theta-\nu_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\sin\frac{\theta-\theta'}{2}}, \tag{5.44}$$

$$\left(BD^{-1}\right)_{\theta,\theta'} = -ie^{i(l-\frac12)(\theta+\theta')}\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\frac{e^{-(\nu_\theta+\nu_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, \tag{5.45}$$

$$\left(D^{-1}C\right)_{\theta,\theta'} = -ie^{-i(l-\frac12)(\theta+\theta')}\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\frac{e^{(\nu_\theta+\nu_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, \tag{5.46}$$

$$|\det D| = \left[\left(1-k^2\right)\prod_{\theta\in\boldsymbol{\theta}_p} e^{\nu_\theta}\prod_{\theta\in\boldsymbol{\theta}_a} e^{-\nu_\theta}\right]^{\frac14}, \tag{5.47}$$

where $\theta\in\boldsymbol{\theta}_a$, $\theta'\in\boldsymbol{\theta}_p$ in (5.44), $\theta,\theta'\in\boldsymbol{\theta}_p$ in (5.45), $\theta,\theta'\in\boldsymbol{\theta}_a$ in (5.46) and the function ν_θ is given by (5.32).

By (3.9) and (3.14), Theorem 5.6 gives the vacuum expectation value ${}_a\langle vac|s_l|vac\rangle_p$ and two-particle form factors of Ising spin. Multiparticle form factors are determined by Lemma 3.2.

6. Multiparticle form factors

We now compute general matrix elements $\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}')$ using a trick learnt from [10]. There it was shown that, roughly speaking, if the conjectures of [5, 6] are true for two-particle form factors, then they also hold for multiparticle ones. Theorem 5.6 implies that the last statement is no longer conditional.

The argument of [10] may be sketched as follows. Introduce an $(m+n)\times(m+n)$ diagonal matrix Ω with non-zero elements defined by

$$\Omega_{jj} = \begin{cases} -\dfrac{\exp\left\{-i(l-\frac12)\theta_j+\nu_{\theta_j}/2\right\}}{\sqrt{N\sinh\gamma_{\theta_j}}} & \text{for } j=1,\ldots,m, \\[2ex] \dfrac{\exp\left\{i(l-\frac12)\theta'_{j-m}-\nu_{\theta'_{j-m}}/2\right\}}{\sqrt{N\sinh\gamma_{\theta'_{j-m}}}} & \text{for } j=m+1,\ldots,m+n. \end{cases}$$

Taking into account Theorem 5.6, the skew-symmetric matrix R in Lemma 3.2 can be written as

$$R = -i\rho\,\Omega\tilde{R}\Omega, \qquad \tilde{R} = \begin{pmatrix} \tilde{R}_{\boldsymbol{\theta}\times\boldsymbol{\theta}} & \tilde{R}_{\boldsymbol{\theta}\times\boldsymbol{\theta}'} \\ \tilde{R}_{\boldsymbol{\theta}'\times\boldsymbol{\theta}} & \tilde{R}_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'} \end{pmatrix}$$

where $\rho = \sqrt{\dfrac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}}$ and matrix elements of the blocks of $\tilde{R}$ are given by

$$\left(\tilde{R}_{\boldsymbol{\theta}\times\boldsymbol{\theta}}\right)_{jk} = \frac{\rho \sin\frac{\theta_j-\theta_k}{2}}{\sinh\frac{\gamma_{\theta_j}+\gamma_{\theta_k}}{2}}, \qquad j,k=1,\ldots,m, \tag{6.1}$$

$$\left(\tilde{R}_{\boldsymbol{\theta}\times\boldsymbol{\theta}'}\right)_{jk} = -\left(\tilde{R}_{\boldsymbol{\theta}'\times\boldsymbol{\theta}}\right)_{kj} = \frac{\sinh\frac{\gamma_{\theta_j}+\gamma_{\theta'_k}}{2}}{\rho\sin\frac{\theta_j-\theta'_k}{2}}, \qquad j=1,\ldots,m, \quad k=1,\ldots,n, \tag{6.2}$$

$$\left(\tilde{R}_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'}\right)_{jk} = \frac{\rho \sin\frac{\theta'_j-\theta'_k}{2}}{\sinh\frac{\gamma_{\theta'_j}+\gamma_{\theta'_k}}{2}}, \qquad j,k=1,\ldots,n. \tag{6.3}$$

Let us recall the identity (4.5) from Section 4, which we rewrite in the form

$$\sqrt{k}\,\mathrm{sn}(u_\theta - u_{\theta'}) = \left[\sqrt{k}\,\mathrm{sn}(u_\theta - u_{\theta'} \pm iK')\right]^{-1} = \frac{\rho\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, \qquad \theta,\theta'\in[0,2\pi). \tag{6.4}$$

Introducing $(m+n)$ variables

$$\tilde{u}_j = \begin{cases} u_{\theta_j} + iK' & \text{for } j=1,\ldots,m, \\ u_{\theta'_{j-m}} & \text{for } j=m+1,\ldots,m+n, \end{cases} \tag{6.5}$$

matrix elements of $\tilde{R}$ can be written as $\tilde{R}_{ij} = \sqrt{k}\,\mathrm{sn}(\tilde{u}_i - \tilde{u}_j)$ with $i,j=1,\ldots,m+n$. The pfaffian (3.10) can then be easily computed using the elliptic pfaffian identity (5.8):

$$\mathrm{Pf}\,R = (-i\rho)^{\frac{m+n}{2}}\cdot\det\Omega\cdot\mathrm{Pf}\,\tilde{R} = (-i\rho)^{\frac{m+n}{2}}\prod_{i=1}^{m+n}\Omega_{ii}\prod_{i<j}^{m+n}\sqrt{k}\,\mathrm{sn}(\tilde{u}_i-\tilde{u}_j).$$

Transforming sn's back to trigonometric functions by (6.4)–(6.5) and taking into account (3.10) and (5.47), we finally obtain the desired general formula for finite-lattice form factors of Ising spin:

Theorem 6.1. *Spin matrix elements $\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}')$, defined by (2.13), for even $m+n$ can be explicitly written as*

$$\mathcal{F}^{(l)}_{m,n}(\boldsymbol{\theta},\boldsymbol{\theta}') = i^{2mn-\frac{m+n}{2}}\sqrt{\xi\xi_T}\prod_{j=1}^{m}\frac{e^{-i(l-\frac12)\theta_j+\nu_{\theta_j}/2}}{\sqrt{N\sinh\gamma_{\theta_j}}}\prod_{j=1}^{n}\frac{e^{i(l-\frac12)\theta'_j-\nu_{\theta'_j}/2}}{\sqrt{N\sinh\gamma_{\theta'_j}}}\times \tag{6.6}$$

$$\times\left(\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\right)^{\frac{(m-n)^2}{4}}\prod_{1\leq i<j\leq m}\frac{\sin\frac{\theta_i-\theta_j}{2}}{\sinh\frac{\gamma_{\theta_i}+\gamma_{\theta_j}}{2}}\prod_{1\leq i<j\leq n}\frac{\sin\frac{\theta'_i-\theta'_j}{2}}{\sinh\frac{\gamma_{\theta'_i}+\gamma_{\theta'_j}}{2}}\prod_{\substack{1\leq i\leq m\\1\leq j\leq n}}\frac{\sinh\frac{\gamma_{\theta_i}+\gamma_{\theta'_j}}{2}}{\sin\frac{\theta_i-\theta'_j}{2}},$$

where $\xi = \left|1-(\sinh 2\mathcal{K}_x\sinh 2\mathcal{K}_y)^{-2}\right|^{\frac14}$, the function ν_θ is defined by (5.32) and

$$\xi_T = \prod_{\theta\in\boldsymbol{\theta}_p}e^{\nu_\theta/4}\prod_{\theta\in\boldsymbol{\theta}_a}e^{-\nu_\theta/4} = \left[\frac{\prod_{\theta\in\boldsymbol{\theta}_p}\prod_{\theta'\in\boldsymbol{\theta}_a}\sinh^2\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\prod_{\theta,\theta'\in\boldsymbol{\theta}_p}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}\prod_{\theta,\theta'\in\boldsymbol{\theta}_a}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}\right]^{\frac14}. \tag{6.7}$$

This result coincides with the conjectures of [5, 6]. It clearly satisfies translation invariance constraint (2.14). In the thermodynamic limit $\xi_T\to 1$, $\nu_\theta\to 0$, the spectra of quasimomenta become continuous and (6.6) reproduces infinite-lattice form factors found in [24, 26]. In particular, considering the $N\to\infty$ limit of (5.47), it is straightforward to recover Yang's formula $\langle\sigma\rangle = \sqrt{\xi}$ for the spontaneous magnetization [30].

Remark 6.2. We finally comment on the paramagnetic region of parameters ($\mathcal{K}_x^* > \mathcal{K}_y$). The above results remain valid if one performs analytic continuation in e.g. $\mathcal{K}_x^*$. Nontrivial dependence of (6.6) on $\mathcal{K}_x^*$ is hidden in the functions γ_θ with $\theta\in\boldsymbol{\theta}_{a,p}$. The result of analytic continuation of almost all such γ_θ from the region $\mathcal{K}_x^* < \mathcal{K}_y$ to $\mathcal{K}_x^* > \mathcal{K}_y$ coincides with the definition (2.8). The only exception is γ_0, which is continued to $-\gamma_0$. Then one finds that under continuation

$$\frac{\sin\frac{\theta}{2}}{\sinh\frac{\gamma_\theta+\gamma_0}{2}} \to \frac{\sinh 2\mathcal{K}_x}{\sinh 2\mathcal{K}_y}\frac{\sinh\frac{\gamma_\theta+\gamma_0}{2}}{\sin\frac{\theta}{2}}, \qquad e^{\nu_\theta/2} \to e^{\nu_\theta/2}\sqrt{\frac{\sinh 2\mathcal{K}_x}{\sinh 2\mathcal{K}_y}\frac{\sinh\frac{\gamma_\theta+\gamma_0}{2}}{\sin\frac{\theta}{2}}}, \tag{6.8}$$

for $\theta \in (0, 2\pi)$ and

$$\frac{e^{-\nu_0/2}}{\sqrt{N}\sinh\gamma_0} \to \sqrt{\frac{\sinh 2\mathcal{K}_x}{\sinh 2\mathcal{K}_y}}\left(\frac{e^{-\nu_0/2}}{\sqrt{N}\sinh\gamma_0}\right)^{-1}, \qquad \xi\xi_T \to \xi\xi_T\sqrt{\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}}\frac{e^{-\nu_0}}{N\sinh\gamma_0}. \tag{6.9}$$

Quasiparticle interpretation of eigenvectors also changes. If the particle with $\theta = 0$ was initially present or absent in a p-eigenstate, then it is annihilated (resp. created) after analytic continuation. This means e.g. that the number of particles in p-eigenstates of the periodic Ising transfer matrix becomes odd instead of even. Relations (6.8)–(6.9) then show that spin form factors are given by the same formula (6.6), with odd $m + n$ and appropriately adjusted first numerical factor.

It is also possible to derive this result along the above lines. However, for $\mathcal{K}_x^* > \mathcal{K}_y$ the definition of the creation-annihilation operators (2.10) should be modified and the corresponding matrix D is degenerate, which leads to additional subtleties.

7. Discussion

In this paper, exact expressions for finite-lattice form factors of the spin operator in the two-dimensional Ising model were obtained. Starting point of our derivation was the idea of expressing induced linear transformations of fermions in a particular basis, which was put forward by Hystad and Palmer in [10, 23]. New crucial ingredient is the use of elliptic determinants and theta functional interpolation. The presented approach seems to be quite natural and straightforward. It is likely to be applicable to other free-fermion lattice models, such as those considered in [13, 17, 26].

A more ambitious challenge is to complete the program of form factor derivation for $\mathbb{Z}_N$-symmetric superintegrable chiral Potts quantum chain, which presumably possesses a hidden fermion structure. Two parts of the hamiltonian of this model generate Onsager algebra and the space of states decomposes into a set of invariant irreducible subspaces (Onsager sectors) with Ising-like hamiltonian spectrum in each of them.

Using Baxter's idea of extending Onsager algebra by the spin operator [2], its form factors between the hamiltonian eigenstates were found in [12] up to unknown scalar factors for each pair of Onsager sectors. In the case $N = 2$ (quantum Ising chain in a transverse field), Onsager algebra can be embedded into a fermion algebra, so that all irreducible representations of the former are combined into one irreducible representation of the latter. This allows to fix all unknown scalar factors and obtain form factors of the quantum Ising chain [11]. It is therefore natural to look for a fermion-like algebra extending Onsager algebra for general N.

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