Oscillator Variations of the Classical Theorem on Harmonic Polynomials [1]

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Abstract

We study two-parameter oscillator variations of the classical theorem on harmonic polynomials, associated with noncanonical oscillator representations of $sl(n,\mathbb{F})$ and $o(n,\mathbb{F})$. We find the condition when the homogeneous solution spaces of the variated Laplace equation are irreducible modules of the concerned algebras and the homogeneous subspaces are direct sums of the images of these solution subspaces under the powers of the dual differential operator. This establishes a local $(sl(2,\mathbb{F}),sl(n,\mathbb{F}))$ and $(sl(2,\mathbb{F}),o(n,\mathbb{F}))$ Howe duality, respectively. In generic case, the obtained irreducible $o(n,\mathbb{F})$-modules are infinite-dimensional non-unitary modules without highest-weight vectors. As an application, we determine the structure of noncanonical oscillator representations of $sp(2n,\mathbb{C})$. When both parameters are equal to the maximal allowed value, we obtain an infinite family of explicit irreducible $(\mathcal{G},\mathcal{K})$-modules for $o(n,\mathbb{F})$ and $sp(2n,\mathbb{C})$. Methodologically we have extensively used partial differential equations to solve representation problems.

1 Introduction

Harmonic polynomials are important objects in analysis, differential geometry and physics. A fundamental theorem in classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Bases of these irreducible modules can be obtained easily (e.g., cf. [X]). The algebraic beauty of the above theorem is that Laplace equation characterizes the irreducible submodules of the polynomial algebra and the corresponding quadratic invariant gives a decomposition of the polynomial algebra into a direct sum of irreducible submodules. This actually forms an $(sl(2,\mathbb{F}),o(n,\mathbb{F}))$ Howe duality.

Lie algebras (Lie groups) serve as the symmetries in quantum physics (e.g., cf. [FC, L, LF, G]). Their various representations provide distinct concrete practical physical models. Many important physical phenomena have been interpreted as the consequences of symmetry breaking (e.g., cf. [LF]). Harmonic oscillators are basic objects in quantum mechanics (e.g., cf. [FC, G]). Oscillator representations of finite-dimensional simple Lie algebras are the most fundamental ones in quantum physics. Their infinite-dimensional analogues are free field representations of affine Kac-Moody algebras.

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The aim of this work is to establish certain two-parameter oscillator variations of the classical theorem on harmonic polynomials, associated with noncanonical oscillator representations of special linear Lie algebras and orthogonal Lie algebras, which are obtained by swapping differential operators and multiplication operators in the canonical oscillator representations induced from the natural representations. The Howe duality does not hold on the whole polynomial algebras. But we find the condition when the homogeneous solution spaces of the variated Laplace equation are irreducible modules of the concerned algebras and the homogeneous subspaces are direct sums of the images of these solution subspaces under the powers of the dual differential operator. We may call this a *local* $(sl(2,\mathbb{F}), sl(n,\mathbb{F}))$ *and* $(sl(2,\mathbb{F}), o(n,\mathbb{F}))$ *Howe duality,* respectively. In particular, we obtain explicit infinite-dimensional non-unitary modules of orthogonal Lie algebras that are not of highest-weight type. As an application of our results on special linear Lie algebras, we prove that the homogeneous subspaces in noncanonical oscillator representations of symplectic Lie algebras are irreducible except some singular cases, in which the homogeneous subspaces are direct sums of exactly two explicitly given irreducible submodules. Explicit bases of all the above irreducible modules in generic case are obtained.

Let $\mathcal{G}$ be a semisimple Lie algebra and let $\mathcal{K}$ be a maximal proper reductive Lie subalgebra of $\mathcal{G}$. An infinite-dimensional irreducible $\mathcal{G}$-module is said of $(\mathcal{G},\mathcal{K})$-*type* if it is a direct sum of finite-dimensional irreducible $\mathcal{K}$-submodules. When both parameters are equal to the maximal allowed value, we obtain an infinite family of explicit irreducible $(\mathcal{G},\mathcal{K})$-modules for orthogonal Lie algebras and symplectic Lie algebras. Since our representations are not unitary, the concerned modules are infinite-dimensional and we are dealing with pairs of dual invariant differential operators, traditional methods fail to solve our problems. In fact, we have extensively used the method of solving flag partial differential equations developed in [X] by the second author. Below we give a technical introduction.

For convenience, we will use the notion $\overline{i,i+j}=\{i,i+1,i+2,...,i+j\}$ for integers i and j with $i\leq j$. Denote by $\mathbb{N}$ the additive semigroup of nonnegative integers.

Let $E_{r,s}$ be the square matrix with 1 as its (r,s)-entry and 0 as the others. The compact orthogonal Lie algebra $o(n,\mathbb{R})=\sum_{1\leq r<s\leq n}\mathbb{R}(E_{r,s}-E_{s,r})$, whose representation on the polynomial algebra $\mathcal{A}=\mathbb{R}[x_1,...,x_n]$ is determined by $(E_{r,s}-E_{s,r})|_{\mathcal{A}}=x_r\partial_{x_s}-x_s\partial_{x_r}$, which we call the *canonical oscillator representation of* $o(n,\mathbb{R})$ (e.g., cf. [FSS]). Denote by $\mathcal{A}_k$ the subspace of homogeneous polynomials in $\mathcal{A}$ with degree k. Recall that the Laplace operator $\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$ and its corresponding invariant $\eta=x_1^2+x_2^2+\cdots+x_n^2$. When $n\geq 3$, it is well known that the subspace of harmonic polynomials

$$\mathcal{H}_k=\{f\in\mathcal{A}_k\mid\Delta(f)=0\}\tag{1.1}$$

forms an irreducible $o(n,\mathbb{R})$-module and $\mathcal{A}_k=\mathcal{H}_k\oplus\eta\mathcal{A}_{k-2}$, which is equivalent to that $\mathcal{A}_k=\bigoplus_{i=1}^{[\![k/2]\!]}\eta^i\mathcal{H}_{k-2i}$ is a direct sum of irreducible submodules. Since the space $\mathbb{F}\Delta+\mathbb{F}[\Delta,\eta]+\mathbb{F}\eta$ forms an operator Lie algebra isomorphic to $sl(2,\mathbb{R})$, the above conclusion gives an $(sl(2,\mathbb{R}),o(n,\mathbb{R}))$ Howe duality.

Below all the vector spaces are assumed over a field $\mathbb{F}$ with characteristic 0. Moreover, we always assume that $n\geq 2$ is an integer. Let us reconsider the canonical oscillator representation of $sl(n,\mathbb{F})$:

$$E_{i,j}|_{\mathcal{A}} = x_i\partial_j \qquad \text{for } \ i,j\in\overline{1,n}. \tag{1.2}$$

Fix $1\leq n_1<n$. Note

$$[\partial_{x_r},x_r]=1=[-x_r,\partial_{x_r}]. \tag{1.3}$$

Changing operators $\partial_{x_r}\mapsto -x_r$ and $x_r\mapsto\partial_{x_r}$ in (1.2) for $r\in\overline{1,n_1}$, we obtain the following noncanonical oscillator representation of $sl(n,\mathbb{F})$ determined by:

$$E_{i,j}|_{\mathcal{A}}=\begin{cases}-x_j\partial_{x_i}-\delta_{i,j} & \text{if } i,j\in\overline{1,n_1};\\ \partial_{x_i}\partial_{x_j} & \text{if } i\in\overline{1,n_1},\ j\in\overline{n_1+1,n};\\ -x_ix_j & \text{if } i\in\overline{n_1+1,n},\ j\in\overline{1,n_1};\\ x_i\partial_{x_j} & \text{if } i,j\in\overline{n_1+1,n}.\end{cases} \tag{1.4}$$

For any $k\in\mathbb{Z}$, we denote

$$\mathcal{A}_{\langle k\rangle}=\text{Span}\,\{x^\alpha\mid\alpha\in\mathbb{N}^{\,n};\sum_{r=n_1+1}^{n}\alpha_r-\sum_{i=1}^{n_1}\alpha_i=k\}. \tag{1.5}$$

It was presented by Howe in his work [Ho] that for $m_1,m_2\in\mathbb{N}$ with $m_1>0$, $\mathcal{A}_{\langle -m_1\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-submodule with highest weight $m_1\lambda_{n_1-1}-(m_1+1)\lambda_{n_1}$ and $\mathcal{A}_{\langle m_2\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-submodule with highest weight $-(m_2+1)\lambda_{n_1}+m_2(1-\delta_{n_1,n-1})\lambda_{n_1+1}$.

Denote $\mathcal{B}=\mathbb{F}[x_1,...,x_n,y_1,...,y_n]$. Fix $n_1,n_2\in\overline{1,n}$ with $n_1\leq n_2$. Changing operators $\partial_{x_r}\mapsto -x_r,\ x_r\mapsto\partial_{x_r}$ for $r\in\overline{1,n_1}$ and $\partial_{y_s}\mapsto -y_s,\ y_s\mapsto\partial_{y_s}$ for $s\in\overline{n_2+1,n}$, we get another noncanonical oscillator representation of $sl(n,\mathbb{F})$ on $\mathcal{B}$ determined by

$$E_{i,j}|_{\mathcal{B}}=E^x_{i,j}-E^y_{j,i} \qquad \text{for } \ i,j\in\overline{1,n} \tag{1.6}$$

with

$$E^x_{i,j}|_{\mathcal{B}}=\begin{cases}-x_j\partial_{x_i}-\delta_{i,j} & \text{if } i,j\in\overline{1,n_1};\\ \partial_{x_i}\partial_{x_j} & \text{if } i\in\overline{1,n_1},\ j\in\overline{n_1+1,n};\\ -x_ix_j & \text{if } i\in\overline{n_1+1,n},\ j\in\overline{1,n_1};\\ x_i\partial_{x_j} & \text{if } i,j\in\overline{n_1+1,n}\end{cases} \tag{1.7}$$

and

$$E^y_{i,j}|_{\mathcal{B}}=\begin{cases}y_i\partial_{y_j} & \text{if } i,j\in\overline{1,n_2};\\ -y_iy_j & \text{if } i\in\overline{1,n_2},\ j\in\overline{n_2+1,n};\\ \partial_{y_i}\partial_{y_j} & \text{if } i\in\overline{n_2+1,n},\ j\in\overline{1,n_2};\\ -y_j\partial_{y_i}-\delta_{i,j} & \text{if } i,j\in\overline{n_2+1,n}.\end{cases} \tag{1.8}$$

The related variated Laplace operator becomes

$$\mathcal{D}=-\sum_{i=1}^{n_1}x_i\partial_{y_i}+\sum_{r=n_1+1}^{n_2}\partial_{x_r}\partial_{y_r}-\sum_{s=n_2+1}^{n}y_s\partial_{x_s} \tag{1.9}$$

and its dual

$$\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+\sum_{r=n_1+1}^{n_2}x_ry_r+\sum_{s=n_2+1}^{n}x_s\partial_{y_s}. \tag{1.10}$$

Set

$$\mathcal{B}_{\langle\ell_1,\ell_2\rangle}=\text{Span}\{x^\alpha y^\beta\mid\alpha,\beta\in\mathbb{N}^{\,n};\sum_{r=n_1+1}^{n}\alpha_r-\sum_{i=1}^{n_1}\alpha_i=\ell_1;\sum_{i=1}^{n_2}\beta_i-\sum_{r=n_2+1}^{n}\beta_r=\ell_2\} \tag{1.11}$$

for $\ell_1, \ell_2 \in \mathbb{Z}$. Define

$$\mathcal{H}_{\langle \ell_1,\ell_2\rangle} = \{f \in \mathcal{B}_{\langle \ell_1,\ell_2\rangle} \mid \mathcal{D}(f) = 0\}. \tag{1.12}$$

The following is our first result:

Theorem 1. *For any $\ell_1, \ell_2 \in \mathbb{Z}$ such that $\ell_1 + \ell_2 \leq n_1 - n_2 + 1 - \delta_{n_1,n_2}$, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-module and $\mathcal{B}_{\langle \ell_1,\ell_2\rangle} = \bigoplus_{m=0}^{\infty} \eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle})$ is a decomposition of irreducible submodules. In particular, $\mathcal{B}_{\langle \ell_1,\ell_2\rangle} = \mathcal{H}_{\langle \ell_1,\ell_2\rangle} \oplus \eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})$.*

When $n_1 + 1 < n_2 < n$ and $\ell_1 + \ell_2 > n_1 - n_2 + 1$, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ is not irreducible and contains nonzero elements in $\eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})$. Although the space $\mathbb{F}\mathcal{D} + \mathbb{F}[\mathcal{D},\eta] + \mathbb{F}\eta$ forms an operator Lie algebra isomorphic to $sl(2,\mathbb{R})$, we do not have an $(sl(2,\mathbb{F}), sl(n,\mathbb{F}))$ Howe duality. We may call Theorem 1 an *local* $(sl(2,\mathbb{F}), sl(n,\mathbb{F}))$ *Howe duality.*

Consider the split

$$o(2n,\mathbb{F}) = \sum_{i,j=1}^{n} \mathbb{F}(E_{i,j} - E_{n+j,n+i}) + \sum_{1\leq i<j\leq n} [\mathbb{F}(E_{i,n+j} - E_{j,n+i}) + \mathbb{F}(E_{n+j,i} - E_{n+i,j})] \tag{1.13}$$

and define a noncanonical oscillator representation of $o(2n,\mathbb{F})$ on $\mathcal{B}$ by

$$(E_{i,j} - E_{n+j,n+i})|_{\mathcal{B}} = E^x_{i,j}|_{\mathcal{B}} - E^y_{j,i}|_{\mathcal{B}}, \tag{1.14}$$

$$E_{i,n+j}|_{\mathcal{B}} = \begin{cases} \partial_{x_i}\partial_{y_j} & \text{if } i \in \overline{1,n_1},\ j \in \overline{1,n_2}, \\ -y_j\partial_{x_i} & \text{if } i \in \overline{1,n_1},\ j \in \overline{n_2+1,n}, \\ x_i\partial_{y_j} & \text{if } i \in \overline{n_1+1,n},\ j \in \overline{1,n_2}, \\ -x_iy_j & \text{if } i \in \overline{n_1+1,n},\ j \in \overline{n_2+1,n} \end{cases} \tag{1.15}$$

and

$$E_{n+i,j}|_{\mathcal{B}} = \begin{cases} -x_jy_i & \text{if } j \in \overline{1,n_1},\ i \in \overline{1,n_2}, \\ -x_j\partial_{y_i} & \text{if } j \in \overline{1,n_1},\ i \in \overline{n_2+1,n}, \\ y_i\partial_{x_j} & \text{if } j \in \overline{n_1+1,n},\ i \in \overline{1,n_2}, \\ \partial_{x_j}\partial_{y_i} & \text{if } j \in \overline{n_1+1,n},\ i \in \overline{n_2+1,n}. \end{cases} \tag{1.16}$$

Set

$$\mathcal{B}_{\langle k\rangle} = \bigoplus_{\ell_1,\ell_2\in\mathbb{Z};\ell_1+\ell_2=k} \mathcal{B}_{\langle \ell_1,\ell_2\rangle}, \qquad \mathcal{H}_{\langle k\rangle} = \{f \in \mathcal{B}_{\langle k\rangle} \mid \mathcal{D}(f) = 0\}. \tag{1.17}$$

Below we always take $\mathcal{K} = \sum_{i,j=1}^{n} \mathbb{F}(E_{i,j} - E_{n+j,n+i})$. Our second results is:

Theorem 2. *For any $n_1 - n_2 + 1 - \delta_{n_1,n_2} \geq k \in \mathbb{Z}$, $\mathcal{H}_{\langle k\rangle}$ is an irreducible $o(2n,\mathbb{F})$-submodule and $\mathcal{B}_{\langle k\rangle} = \bigoplus_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle k-2i\rangle})$ is a decomposition of irreducible submodules. In particular, $\mathcal{B}_{\langle k\rangle} = \mathcal{H}_{\langle k\rangle} \oplus \eta(\mathcal{B}_{\langle k-2\rangle})$. The module $\mathcal{H}_{\langle k\rangle}$ under the assumption is of highest-weight type only if $n_2 = n$. When $n_1 = n_2 = n$, all the irreducible modules $\mathcal{H}_{\langle k\rangle}$ with $0 \geq k \in \mathbb{Z}$ are of $(\mathcal{G},\mathcal{K})$-type.*

We may view Theorem 2 as an *local* $(sl(2,\mathbb{F}), o(2n,\mathbb{F}))$ *Howe duality.*

Note the split

$$o(2n+1,\mathbb{F}) = o(2n,\mathbb{F}) \oplus \bigoplus_{i=1}^{n} [\mathbb{F}(E_{0,i} - E_{n+i,0}) + \mathbb{F}(E_{0,n+i} - E_{i,0})]. \tag{1.18}$$

Let $\mathcal{B}'=\mathbb{F}[x_0,x_1,...,x_n,y_1,...,y_n]$. We define a noncanonical oscillator representation of $o(2n+1,\mathbb{F})$ on $\mathcal{B}'$ by the differential operators in (1.14)-(1.16) and

$$E_{0,i}|_{\mathcal{B}'}=\begin{cases}-x_0x_i & \text{if } i\in\overline{1,n_1},\\ x_0\partial_{x_i} & \text{if } i\in\overline{n_1+1,n},\\ x_0\partial_{y_i} & \text{if } i\in\overline{n+1,n+n_2},\\ -x_0y_i & \text{if } i\in\overline{n+n_2+1,2n}\end{cases} \tag{1.19}$$

and

$$E_{i,0}|_{\mathcal{B}'}=\begin{cases}\partial_{x_0}\partial_{x_i} & \text{if } i\in\overline{1,n_1},\\ x_i\partial_{x_0} & \text{if } i\in\overline{n_1+1,n},\\ y_i\partial_{x_0} & \text{if } i\in\overline{n+1,n+n_2},\\ \partial_{x_0}\partial_{y_i} & \text{if } i\in\overline{n+n_2+1,2n}.\end{cases} \tag{1.20}$$

Now the variated Laplace operator becomes

$$\mathcal{D}'=\partial_{x_0}^2-2\sum_{i=1}^{n_1}x_i\partial_{y_i}+2\sum_{r=n_1+1}^{n_2}\partial_{x_r}\partial_{y_r}-2\sum_{s=n_2+1}^{n}y_s\partial_{x_s} \tag{1.21}$$

and its dual operator

$$\eta'=x_0^2+2\sum_{i=1}^{n_1}y_i\partial_{x_i}+2\sum_{r=n_1+1}^{n_2}x_ry_r+2\sum_{s=n_2+1}^{n}x_s\partial_{y_s}. \tag{1.22}$$

Set

$$\mathcal{B}'_{\langle k\rangle}=\sum_{i=0}^{\infty}\mathcal{B}_{\langle k-i\rangle}x_0^i,\qquad \mathcal{H}'_{\langle k\rangle}=\{f\in\mathcal{B}'_{\langle k\rangle}\mid \mathcal{D}'(f)=0\}. \tag{1.23}$$

The following is our third result.

Theorem 3. *For any $n_1-n_2+1-\delta_{n_1,n_2}\geq k\in\mathbb{Z}$, $\mathcal{H}'_{\langle k\rangle}$ is an irreducible $o(2n+1,\mathbb{F})$-submodule and $\mathcal{B}'_{\langle k\rangle}=\bigoplus_{i=0}^{\infty}(\eta')^i(\mathcal{H}'_{\langle k-2i\rangle})$ is a decomposition of irreducible submodules. In particular, $\mathcal{B}'_{\langle k\rangle}=\mathcal{H}'_{\langle k\rangle}\oplus\eta'(\mathcal{B}'_{\langle k-2\rangle})$. The module $\mathcal{H}'_{\langle k\rangle}$ under the assumption is of highest-weight type only if $n_2=n$. When $n_1=n_2=n$, all the irreducible modules $\mathcal{H}'_{\langle k\rangle}$ with $0\geq k\in\mathbb{Z}$ are of $(\mathcal{G},\mathcal{K})$-type.*

Again Theorem 2 can be viewed as an *local* $(sl(2,\mathbb{F}),o(2n+1,\mathbb{F}))$ *Howe duality.*

Define a noncanonical oscillator representation of $sp(2n,\mathbb{F})$ on $\mathcal{B}$ by (1.14)-(1.16). Using some results in the proof of Theorem 1, we prove:

Theorem 4. *Let $k\in\mathbb{Z}$. If $n_1<n_2$ or $k\neq 0$, the subspace $\mathcal{B}_{\langle k\rangle}$ (cf. (1.17)) is an irreducible $sp(2n,\mathbb{F})$-submodule. When $n_1=n_2$, the subspace $\mathcal{B}_{\langle 0\rangle}$ is a direct sum of two irreducible $sp(2n,\mathbb{F})$-submodules. Moreover, each irreducible submodule is of highest-weight module only if $n_2=n$. When $n_1=n_2=n$, all the irreducible submodules are of $(\mathcal{G},\mathcal{K})$-type.*

In addition, the explicit expressions for all the above irreducible modules are given. In the case of highest-weight type, the highest-weight vector and its weight of the corresponding irreducible modules are also presented. Since the representations with parameters (n_1,n_2) are contragredient to those with parameters $(n-n_2,n-n_1)$, the case $n_2<n_1$ has virtually been handled.

In Section 2, we present some preparatory works, in particular, the method of solving flag partial differential equations found in [X] by the second author. In Section 3, we prove Theorem 1 when $n_1 < n_2$. Section 4 is devoted to the proof of Theorem 1 with $n_1 = n_2$. In Sections 5, 6 and 7, we prove Theorems 2, 3 and 4, respectively.

2 Preparation

It is very often that Lie group theorists characterize certain irreducible modules as kernels of a set of differential operators. But how to solve the corresponding systems of partial differential equations is in general unknown. It was realized by the second author that these equations are of "flag type" when the modules are of highest-weight type. A linear transformation (operator) T on a vector space V is called *locally nilpotent* if for any $v \in V$, there exists a positive integer k such that $T^k(v) = 0$. A *partial differential equation of flag type* is the linear differential equation of the form:

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0, \tag{2.1}$$

where $d_1, d_2, ..., d_n$ are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{F}[x_1, x_2, ..., x_n]$ and $f_1, ..., f_{n-1}$ are polynomials satisfying $d_i(f_j) = 0$ if $i > j$. Many variable-coefficient (generalized) Laplace equations, wave equations, Klein-Gordon equations, Helmholtz equations are of this type. Solving such equations is also important in finding invariant solutions of nonlinear partial differential equations (e.g., cf. [I1, I2]). In representation theory, we are more interested in polynomial solutions of flag partial differential equations. The second author [X] found an effective way of solving for them. The following lemma is a slightly generalized form of Lemma 2.1 in [X].

Lemma 2.1 (Xu [X]). *Let $\mathcal{B}$ be a commutative associative algebra and let $\mathcal{A}$ be a free $\mathcal{B}$-module generated by a filtrated subspace $V = \bigcup_{r=0}^{\infty} V_r$ (i.e., $V_r \subset V_{r+1}$). Let T_1 be a linear operator on $\mathcal{B} \oplus \mathcal{A}$ with a right inverse T_1^- such that*

$$T_1(\mathcal{B}, \mathcal{A}),\ T_1^-(\mathcal{B}, \mathcal{A}) \subset (\mathcal{B}, \mathcal{A}),\ \ T_1(\eta_1\eta_2) = T_1(\eta_1)\eta_2,\ \ T_1^-(\eta_1\eta_2) = T_1^-(\eta_1)\eta_2 \tag{2.2}$$

for $\eta_1 \in \mathcal{B},\ \eta_2 \in V$, and let T_2 be a linear operator on $\mathcal{A}$ such that

$$T_2(V_{r+1}) \subset \mathcal{B}V_r,\ \ T_2(f\zeta) = fT_2(\zeta) \qquad \text{for } r \in \mathbb{N},\ \ f \in \mathcal{B},\ \zeta \in \mathcal{A}. \tag{2.3}$$

Then we have

$$\begin{aligned} &\{g \in \mathcal{A} \mid (T_1 + T_2)(g) = 0\} \\ = \ & Span\{\sum_{i=0}^{\infty} (-T_1^- T_2)^i (hg) \mid g \in V,\ h \in \mathcal{B};\ T_1(h) = 0\}. \end{aligned} \tag{2.4}$$

Set

$$\epsilon_i = (0, ..., 0, \overset{i}{1}, 0, ..., 0) \in \mathbb{N}^n. \tag{2.5}$$

For each $i \in \overline{1, n}$, we define the linear operator $\int_{(x_i)}$ on $\mathcal{A}$ by:

$$\int_{(x_i)} (x^\alpha) = \frac{x^{\alpha+\epsilon_i}}{\alpha_i + 1} \ \text{ for } \alpha \in \mathbb{N}^n. \tag{2.6}$$

Furthermore, we let

$$\int_{(x_i)}^{(0)} = 1, \qquad \int_{(x_i)}^{(m)} = \overbrace{\int_{(x_i)} \cdots \int_{(x_i)}}^{m} \qquad \text{for } \ 0 < m \in \mathbb{Z} \tag{2.7}$$

and denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \quad \int^{(\alpha)} = \int_{(x_1)}^{(\alpha_1)}\int_{(x_2)}^{(\alpha_2)}\cdots\int_{(x_n)}^{(\alpha_n)} \qquad \text{for } \ \alpha \in \mathbb{N}^n. \tag{2.8}$$

Obviously, $\int^{(\alpha)}$ is a right inverse of ∂^α for $\alpha \in \mathbb{N}^n$. We remark that $\int^{(\alpha)}\partial^\alpha \neq 1$ if $\alpha \neq 0$ due to $\partial^\alpha(1) = 0$. In this paper, our T_1's are of the type ∂^α and the right inverse $T_1^- = \int^{(\alpha)}$.

Let $m_1, m_2, ..., m_n$ be positive integers. Taking $T_1 = \partial_{x_1}^{m_1}$, $T_2 = \partial_{x_2}^{m_2} + \cdots + \partial_{x_n}^{m_n}$ and $T_1^- = \int_{(x_1)}^{(m_1)}$, we find that the set

$$\begin{aligned}&\{ \sum_{k_2,...,k_n=0}^{\infty} (-1)^{k_2+\cdots+k_n} \binom{k_2+\cdots+k_k}{k_2,...,k_n} \int_{(x_1)}^{((k_2+\cdots+k_n)m_1)} (x_1^{\ell_1}) \\ &\times \partial_{x_2}^{k_2m_2}(x_2^{\ell_2})\cdots\partial_{x_n}^{k_nm_n}(x_n^{\ell_n}) \mid \ell_1 \in \overline{0, m_1-1},\ \ell_2, ..., \ell_n \in \mathbb{N}\}\end{aligned} \tag{2.9}$$

forms a basis of the space of polynomial solutions for the equation

$$(\partial_{x_1}^{m_1} + \partial_{x_2}^{m_2} + \cdots + \partial_{x_n}^{m_n})(u) = 0. \tag{2.10}$$

When all $m_i = 2$, we get a basis of the space of harmonic polynomials.

Cao [C] used Lemma 2.1 to prove that the subspaces of homogeneous polynomial vector solutions of the n-dimensional Navier equations in elasticity are exactly direct sums of three explicitly given irreducible submodules when $n \neq 4$ and direct sums of four explicitly given irreducible submodules if $n = 4$ of the corresponding orthogonal Lie group (algebra), and the whole polynomial vector space is also a free module over the invariant polynomials generated these solutions. The result can be viewed as a vector generalization of the classical theorem on harmonic polynomials. Moreover, Cao solved the initial value problem for the Navier equations based on the ideas in [X].

The idea of solving flag partial differential equation in [X] leads the second author to find a family of special functions functions

$$\mathcal{Y}_r(y_1, ..., y_m) = \sum_{i_1,...,i_m=0}^{\infty} \binom{i_1+\cdots+i_m}{i_1,...,i_m} \frac{y_1^{i_1}y_2^{i_2}\cdots y_m^{i_m}}{(r+\sum_{s=1}^m si_s)!}, \tag{2.11}$$

by which we can solve the initial value problem of the equation:

$$(\partial_{x_1}^m - \sum_{r=1}^m \partial_{x_1}^{m-i} f_i(\partial_{x_2}, ..., \partial_{x_n}))(u) = 0, \tag{2.12}$$

where $f_i(\partial_{x_2}, ..., \partial_{x_n}) \in \mathbb{R}[\partial_{x_2}, ..., \partial_{x_n}]$.

Let $\mathcal{A} = \mathbb{F}[x_1, ..., x_n]$ and let $gl(n, \mathbb{F})$ act on $\mathcal{A}$ by (1.4). With the notion in (1.5), $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_{\langle k\rangle}$ is a $\mathbb{Z}$ graded algebra and each homogeneous subspace $\mathcal{A}_{\langle k\rangle}$ is infinite-dimensional. Set

$$\flat = \sum_{r=n_1+1}^n x_r\partial_{x_r} - \sum_{i=1}^{n_1} x_i\partial_{x_i}. \tag{2.13}$$

Then

$$\mathcal{A}_{\langle k\rangle}=\{f\in\mathcal{A}\mid \flat(f)=kf\}. \tag{2.14}$$

Moreover, we have

$$\flat E_{i,j}=E_{j,i}\flat \;\;\text{on}\;\; \mathcal{A}\qquad \text{for } i,j\in\overline{1,n}. \tag{2.15}$$

Thus $\mathcal{A}_{\langle k\rangle}$ forms a $\mathcal{G}$-module for any subalgebra $\mathcal{G}$ of $gl(n,\mathbb{F})$.

For $\alpha\in\mathbb{N}^n$, we denote $\alpha!=\prod_{i=1}^n\alpha_i!$ and define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathcal{A}$ by

$$(x^\alpha|x^\beta)=\delta_{\alpha,\beta}(-1)^{\sum_{i=1}^{n_1}\alpha_i}\alpha!\qquad \text{for }\;\alpha,\beta\in\mathbb{N}^n. \tag{2.16}$$

Then we have:

Lemma 2.2. *For any $A\in gl(n,\mathbb{F})$ and $f,g\in\mathcal{A}$, we have $(A(f)|g)=(f|A^t(g))$, where A^t denote the transpose of the matrix A.*

Proof. Let $\alpha,\beta\in\mathbb{N}^n$. For $i,j\in\overline{1,n_1}$,

$$(E_{i,j}(x^\alpha)|x^\beta)=-\alpha_i(x^{\alpha+\epsilon_j-\epsilon_i}|x^\beta)-\delta_{i,j}(x^\alpha|x^\beta) \tag{2.17}$$

and

$$(x^\alpha|E_{j,i}(x^\beta))=-\beta_j(x^\alpha|x^{\beta+\epsilon_i-\epsilon_j})-\delta_{i,j}(x^\alpha|x^\beta) \tag{2.18}$$

by (1.4). Note

$$\begin{aligned}\alpha_i(x^{\alpha+\epsilon_j-\epsilon_i}|x^\beta) &= \delta_{\alpha+\epsilon_j-\epsilon_i,\beta}(-1)^{\sum_{i=1}^{n_1}\alpha_i}(\alpha_j+1)\alpha!\\ &= \beta_j\delta_{\alpha,\beta+\epsilon_i-\epsilon_j}(-1)^{\sum_{i=1}^{n_1}\alpha_i}\alpha!=\beta_j(x^\alpha|x^{\beta+\epsilon_i-\epsilon_j})\end{aligned} \tag{2.19}$$

by (2.16). Hence

$$(E_{i,j}(x^\alpha)|x^\beta)=(x^\alpha|E_{j,i}(x^\beta)). \tag{2.20}$$

If $i,j\in\overline{n_1+1,n}$, then (2.19) holds and so does (2.20).

Consider $i\in\overline{1,n_1}$ and $j\in\overline{n_1+1,n}$.

$$(E_{i,j}(x^\alpha)|x^\beta)=\alpha_i\alpha_j(x^{\alpha-\epsilon_i-\epsilon_j}|x^\beta)=-\delta_{\alpha-\epsilon_i-\epsilon_j,\beta}(-1)^{\sum_{i=1}^{n_1}\alpha_i}\alpha! \tag{2.21}$$

and

$$(x^\alpha|E_{j,i}(x^\beta))=-(x^\alpha|x^{\beta+\epsilon_i+\epsilon_j})=-\delta_{\alpha,\beta+\epsilon_i+\epsilon_j}(-1)^{\sum_{i=1}^{n_1}\alpha_i}\alpha! \tag{2.22}$$

by (1.4) and (2.16). So (2.20) holds. Therefore, the lemma holds by the symmetry of the form. □

Let $\mathcal{G}$ be simple Lie subalgebra of $gl(n,\mathbb{F})$ such that $A^t\in\mathcal{G}$ if $A\in\mathcal{G}$. Let H be a Cartan subalgebra of $\mathcal{G}$ and assume that $\mathcal{A}$ forms a weighted $\mathcal{G}$-module with respect to H. Fix the positivity of roots and denote by $\mathcal{G}_+$ the sum of positive root subspaces. A *singular vector* is a weight vector annihilated by positive root vectors.

From now on, we count the number of singular vectors up to a scalar multiple. Moreover, an element $g\in\mathcal{A}$ is called *nilpotent with respect to* $\mathcal{G}_+$ if there exist a positive integer m such that

$$\xi_1\cdots\xi_m(g)=0\qquad \text{for any } \xi_1,...,\xi_m\in\mathcal{G}_+. \tag{2.23}$$

A subspace V of $\mathcal{A}$ is called *nilpotent with respect to* $\mathcal{G}_+$ if all its elements are nilpotent with respect to $\mathcal{G}_+$. If the elements of $\mathcal{G}_+|_\mathcal{A}$ are locally nilpotent and $\mathcal{G}_+(\mathcal{A}_i) \subset \sum_{r=0}^{i} \mathcal{A}_r$ for any $i \in \mathbb{N}$, then any element of $\mathcal{A}$ is nilpotent with respect to $\mathcal{G}_+$ by Engel's Theorem.

Lemma 2.3. *If a submodule N of $\mathcal{A}$ is nilpotent with respect to $\mathcal{G}_+$, N contains only one singular vector v and $(v|v) \neq 0$, then N is an irreducible summand of $\mathcal{A}$.*

Proof. Under the nilpotent assumption, any nonzero submodule of N contains a singular vector. In particular, $N_1 = U(\mathcal{G})(v)$ is an irreducible submodule by the uniqueness of singular vector. Set

$$\bar{N}_1^\perp = \{u \in N|(u|w) = 0 \mid w \in N_1\}. \tag{2.24}$$

and

$$\mathcal{R} = \{u \in N|(u|w) = 0 \mid w \in N\}. \tag{2.25}$$

Note that $\bar{N}_1^\perp$ and $\mathcal{R}$ are submodules of N by Lemma 2.2. If $\mathcal{R} \neq 0$, it should contain a nonzero singular vector, which is impossible according to the assumption $(v|v) \neq 0$. Therefore $\mathcal{R} = \{0\}$, and $N = N_1 \bigoplus \bar{N}_1^\perp$. But $\bar{N}_1^\perp = 0$ by the same argument, and so $N = N_1$. The fact $\mathcal{R} = \{0\}$ implies that

$$\mathcal{A} = N \oplus \{f \in \mathcal{A} \mid (f|g) = 0 \text{ for } g \in N\} \tag{2.26}$$

is a direct sum of $\mathcal{G}$-submodules. □

Let $\mathcal{Q} = \mathbb{F}(x_1, ..., x_n, y_1, ..., y_n)$ be the space of rational functions in $x_1, ..., x_n, y_1, ..., y_n$. Define a representation of $sl(n, \mathbb{F})$ on $\mathcal{Q}$ via

$$E_{i,j}|_\mathcal{Q} = x_i\partial_{x_j} - y_j\partial_{y_i} \qquad \text{for } i, j \in \overline{1, n}. \tag{2.27}$$

Set $\zeta = \sum_{i=1}^{n} x_i y_i$. Then

$$\xi(\zeta) = 0 \qquad \text{for } \xi \in sl(n, \mathbb{F}). \tag{2.28}$$

Take

$$H = \sum_{i=1}^{n-1} \mathbb{F}(E_{i,i} - E_{i+1,i+1}) \tag{2.29}$$

as a Cartan subalgebra of $sl(n, \mathbb{F})$ and the subspace spanned by positive root vectors:

$$sl(n, \mathbb{F})_+ = \sum_{1 \leq i < j \leq} \mathbb{F}E_{i,j}. \tag{2.30}$$

The following lemma was proved in [X], which will be used in next section.

Lemma 4. *Any singular function in $\mathcal{Q}$ is a rational function in x_1, y_n, ζ.*

3 The $sl(n, \mathbb{F})$-Variation I: $n_1 < n_2$

Fix $n_1, n_2 \in \overline{1, n}$ such that $n_1 \leq n_2$. Recall that $\mathcal{Q}$ is the space of rational functions in $x_1, ..., x_n, y_1, ..., y_n$. Define a representation of $sl(n, \mathbb{F})$ on $\mathcal{Q}$ determined by

$$E_{i,j}|_\mathcal{Q} = E_{i,j}^x - E_{j,i}^y \qquad \text{for } i, j \in \overline{1, n} \tag{3.1}$$

with

$$E^x_{i,j}|_{\mathcal{Q}} = \begin{cases} -x_j\partial_{x_i} - \delta_{i,j} & \text{if } i,j \in \overline{1,n_1}; \\ \partial_{x_i}\partial_{x_j} & \text{if } i \in \overline{1,n_1},\ j \in \overline{n_1+1,n}; \\ -x_ix_j & \text{if } i \in \overline{n_1+1,n},\ j \in \overline{1,n_1}; \\ x_i\partial_{x_j} & \text{if } i,j \in \overline{n_1+1,n} \end{cases} \tag{3.2}$$

and

$$E^y_{i,j}|_{\mathcal{Q}} = \begin{cases} y_i\partial_{y_j} & \text{if } i,j \in \overline{1,n_2}; \\ -y_iy_j & \text{if } i \in \overline{1,n_2},\ j \in \overline{n_2+1,n}; \\ \partial_{y_i}\partial_{y_j} & \text{if } i \in \overline{n_2+1,n},\ j \in \overline{1,n_2}; \\ -y_j\partial_{y_i} - \delta_{i,j} & \text{if } i,j \in \overline{n_2+1,n}. \end{cases} \tag{3.3}$$

Recall $\flat$ in (2.13) and define

$$\flat' = \sum_{i=1}^{n_2} y_i\partial y_i - \sum_{r=n_2+1}^{n} y_r\partial y_r. \tag{3.4}$$

Moreover, the deformed Laplace operator $\mathcal{D}$ in (1.9) and its dual η in (1.10). Then

$$TE_{i,j}|_{\mathcal{Q}} = E_{i,j}|_{\mathcal{Q}}T \qquad \text{for } T = \flat, \flat', \mathcal{D}, \eta;\ i,j \in \overline{1,n}. \tag{3.5}$$

In addition,

$$[\flat, \mathcal{D}] = [\flat', \mathcal{D}] = -\mathcal{D}, \qquad [\flat, \eta] = [\flat', \eta] = \eta. \tag{3.6}$$

By (3.1)-(3.3), we find

$$E_{i,r}|_{\mathcal{Q}} = -x_r\partial_{x_i} - y_r\partial_{y_i} \qquad \text{for } 1 \le i < r \le n_1, \tag{3.7}$$

$$E_{i,n_1+s}|_{\mathcal{Q}} = \partial_{x_i}\partial_{x_{n_1+s}} - y_{n_1+s}\partial_{y_i} \qquad \text{for } i \in \overline{1,n_1},\ s \in \overline{1,n_2-n_1}, \tag{3.8}$$

$$E_{r,s}|_{\mathcal{Q}} = x_r\partial_{x_s} - y_s\partial_{y_r} \qquad \text{for } n_1 < r < s \le n_2, \tag{3.9}$$

$$E_{n_2,n_2+1} = x_{n_2}\partial_{x_{n_2+1}} - \partial_{y_{n_2}}\partial_{y_{n_2+1}}, \tag{3.10}$$

$$E_{i,r}|_{\mathcal{Q}} = x_i\partial_{x_r} + y_i\partial_{y_r} \qquad \text{for } n_2+1 \le i < r \le n. \tag{3.11}$$

The subalgebra $sl(n,\mathbb{F})_+$ in (2.30) is generated by the above $E_{i,j}$.

Denote

$$\zeta_1 = x_{n_1-1}y_{n_1} - x_{n_1}y_{n_1-1},\ \ \zeta = \sum_{r=n_1+1}^{n_2} x_ry_r,\ \ \zeta_2 = x_{n_2+1}y_{n_2+2} - x_{n_2+2}y_{n_2+1}. \tag{3.12}$$

We will process according to three cases.

Case 1. $n_1 + 1 < n_2$

Assume $n_1 + 1 < n_2 < n$. Suppose that $f \in \mathcal{Q}$ is a singular vector. By Lemma 2.4, f can be written as a rational function in

$$\{x_i, y_r, \zeta_1, \zeta, \zeta_2 \mid n_2+2 \neq i \in \overline{1,n_1+1}\bigcup\overline{n_2+1,n},\ n_1-1 \neq r \in \overline{1,n_1}\bigcup\overline{n_2,n}\}. \tag{3.13}$$

Note

$$E_{n_1-1,n_1}(f) = -x_{n_1}\partial_{x_{n_1-1}}(f) = 0 \tag{3.14}$$

by (3.7) and

$$E_{n_2+1,n_2+2}(f) = y_{n_2+1}\partial_{y_{n_2+2}}(f) = 0 \tag{3.15}$$

by (3.11). So f is independent of x_{n_1-1} and y_{n_2+2}. For $i\in\overline{1,n_1-2}$, we have

$$\begin{aligned}E_{i,n_1-1}(f) &= -x_{n_1-1}\partial_{x_i}(f)-y_{n_1-1}\partial_{y_i}(f)\\ &= -x_{n_1-1}(\partial_{x_i}(f)+x_{n_1}^{-1}y_{n_1}\partial_{y_i}(f))+x_{n_1}^{-1}\zeta_1\partial_{y_i}(f)=0\end{aligned}\tag{3.16}$$

by (3.7). Since both $\partial_{x_i}(f)+x_{n_1}^{-1}y_{n_1}\partial_{y_i}(f)$ and $x_{n_1}^{-1}\zeta_1\partial_{y_i}(f)$ are independent of x_{n_1-1}, we have $\partial_{y_i}(f)=0$, which implies $\partial_{x_i}(f)$=0 by (3.16). Thus f is independent of $\{x_i,y_i\mid i\in\overline{1,n_1-1}$. Similarly, we can prove that f is independent of $\{x_i,y_i\mid i\in\overline{n_2+1,n}$. Therefore, f only depends on

$$\{x_{n_1},x_{n_1+1},x_{n_2+1},y_{n_1},y_{n_2},y_{n_2+1},\zeta_1,\zeta,\zeta_2\}.\tag{3.17}$$

According to (3.8) and (3.12), $E_{n_1,n_1+1}|_{\mathcal{Q}}=\partial_{x_{n_1}}\partial_{x_{n_1+1}}-y_{n_1+1}\partial_{y_{n_1}}$ and

$$E_{n_1,n_1+1}(f)=f_{x_{n_1}x_{n_1+1}}+y_{n_1+1}(f_{x_{n_1}\zeta}-y_{n_1-1}f_{\zeta_1\zeta}-f_{y_{n_1}}-x_{n_1-1}f_{\zeta_1})=0.\tag{3.18}$$

Applying $E_{n_1+1,n_2}|_{\mathcal{Q}}=x_{n_1+1}\partial_{x_{n_2}}-y_{n_2}\partial_{y_{n_1+1}}$ to the above equation, we get

$$-f_{x_{n_1}\zeta}+y_{n_1-1}f_{\zeta_1\zeta}+f_{y_{n_1}}+x_{n_1-1}f_{\zeta_1}=0\tag{3.19}$$

by (3.9). According to (3.12),

$$x_{n_1-1}=y_{n_1}^{-1}\zeta_1+x_{n_1}y_{n_1}^{-1}y_{n_1-1}.\tag{3.20}$$

Substituting it into (3.19), we get

$$y_{n_1-1}(f_{\zeta_1\zeta}+y_{n_1}^{-1}x_{n_1}f_{\zeta_1})+f_{y_{n_1}}+y_{n_1}^{-1}\zeta_1f_{\zeta_1}-f_{x_{n_1}\zeta}=0.\tag{3.21}$$

Since f is independent of y_{n_1-1}, we have

$$f_{\zeta_1\zeta}+y_{n_1}^{-1}x_{n_1}f_{\zeta_1}=0.\tag{3.22}$$

Thus

$$f_{\zeta_1}=e^{-y_{n_1}^{-1}x_{n_1}\zeta}g\tag{3.23}$$

for some function g in the variables of (3.17) except ζ, i.e., $g_\zeta=0$. But f is a rational function in the variables of (3.17) and so is f_{ζ_1}. Hence (3.23) forces $f_{\zeta_1}=0$, that is, f is independent of ζ_1. Similarly, we can prove that f is independent of ζ_2. Now f only depends on

$$\{x_{n_1},x_{n_1+1},x_{n_2+1},y_{n_1},y_{n_2},y_{n_2+1},\zeta\}.\tag{3.24}$$

Since $\zeta=\sum_{i=n_1+1}^{n_2}x_iy_i$, $f\in\mathcal{B}=\mathbb{F}[x_1,...,x_n,y_1,...,y_n]$ if and only if f is a polynomial in the variables (3.24). Now (3.18) and (3.19) are equivalent to

$$f_{x_{n_1}x_{n_1+1}}=0,\qquad f_{x_{n_1}\zeta}-f_{y_{n_1}}=0.\tag{3.25}$$

Similarly, we can prove

$$f_{y_{n_2}y_{n_2+1}}=0,\qquad f_{y_{n_2+1}\zeta}-f_{x_{n_2+1}}=0.\tag{3.26}$$

Set

$$\phi(m_1,m_2)=\sum_{i=0}^{\infty}\frac{y_{n_1}^i(\partial_{x_{n_1}}\partial_\zeta)^i(x_{n_1}^{m_1}\zeta^{m_2})}{i!}\qquad\text{for }\ m_1,m_2\in\mathbb{N}.\tag{3.27}$$

By Lemma 2.1 with $T_1=\partial_{y_{n_1}},\ T_1^-=\int_{(y_{n_1})}$ (cf. (2.6)) and $T_2=-\partial_{x_{n_1}}\partial_\zeta$, the polynomial solution space of (3.25) is

$$[\mathbb{F}[x_{n_1+1},\zeta]+\sum_{m_1=1}^\infty\sum_{m_2=0}^\infty\mathbb{F}\phi(m_1,m_2)][\mathbb{F}[x_{n_2+1},y_{n_2},y_{n_2+1}]]. \tag{3.28}$$

Denote

$$\psi(m_1,m_2)=\sum_{i=0}^\infty\frac{x_{n_2+1}^i(\partial_{y_{n_2+1}}\partial_\zeta)^i(y_{n_2+1}^{m_1}\zeta^{m_2})}{i!}\qquad\text{for }\ m_1,m_2\in\mathbb{N}, \tag{3.29}$$

$$\begin{aligned}\phi(m_1,m_2,m_3)&=\sum_{r=0}^\infty\frac{x_{n_2+1}^r(\partial_{y_{n_2+1}}\partial_\zeta)^r(\phi(m_1,m_2)y_{n_2+1}^{m_3})}{r!}\\&=\sum_{i,r=0}^\infty\frac{y_{n_i}^i x_{n_2+1}^r\partial_{x_{n_1}}^i\partial_{y_{n_2+1}}^r\partial_\zeta^{i+r}(x_{n_1}^{m_1}\zeta^{m_2}y_{n_2+1}^{m_3})}{i!r!}.\end{aligned} \tag{3.30}$$

Solving (3.26) by Lemma 2.1 with $T_1=\partial_{x_{n_2+1}}\ T_1^-=\int_{(x_{n_2+1})}$ (cf. (2.6)) and $T_2=-\partial_{y_{n_2+1}}\partial_\zeta$, we find the polynomial solution space of the system (3.25) and (3.26) is

$$\begin{aligned}&\mathbb{F}[x_{n_1+1},y_{n_2},\zeta]+\sum_{m_1,m_3=1}^\infty\sum_{m_2=0}^\infty\mathbb{F}\phi(m_1,m_2,m_3)\\&+\sum_{m_1=1}^\infty\sum_{m_2=0}^\infty(\mathbb{F}[y_{n_2}]\phi(m_1,m_2)+\mathbb{F}[x_{n_1+1}]\psi(m_1,m_2)).\end{aligned} \tag{3.31}$$

According to (1.10),

$$x_{n_1+1}^{m_1}y_{n_2}^{m_2}\zeta^{m_3}=\eta^{m_3}(x_{n_1+1}^{m_1}y_{n_2}^{m_2}), \tag{3.32}$$

$$\eta^{m_2}(x_{n_1}^{m_1}y_{n_2}^{m_3})=(\zeta+y_{n_1}\partial_{x_{n_1}})^{m_2}(x_{n_1}^{m_1}y_{n_2}^{m_3})=\phi(m_1,m_2)y_{n_2}^{m_3}, \tag{3.33}$$

$$\eta^{m_2}(y_{n_2+1}^{m_1}x_{n_1+1}^{m_3})=(\zeta+x_{n_2+1}\partial_{y_{n_2+1}})^{m_2}(y_{n_2+1}^{m_1}x_{n_1+1}^{m_3})=\psi(m_1,m_2)x_{n_1+1}^{m_3}, \tag{3.34}$$

$$\eta^{m_2}(x_{n_1}^{m_1}y_{n_2+1}^{m_3})=(\zeta+y_{n_1}\partial_{x_{n_1}}+x_{n_2+1}\partial_{y_{n_2+1}})^{m_2}(x_{n_1}^{m_1}y_{n_2+1}^{m_3})=\phi(m_1,m_2,m_3). \tag{3.35}$$

It can be verified that $\{\eta^{m_1}(x_i^{m_2}y_j^{m_3})\mid m_1,m_2,m_3\in\mathbb{N};i=n_1,n_1+1;j=n_2,n_2+1\}$ are singular vectors. By (3.31)-(3.35), the nonzero vectors in

$$\{\mathbb{F}[\eta](x_i^{m_1}y_j^{m_2})\mid m_1,m_2\in\mathbb{N};i=n_1,n_1+1;j=n_2,n_2+1\} \tag{3.36}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}=\mathbb{F}[x_1,...,x_{n_1},y_1,...,y_{n_2}]$.

Similarly, when $n_2=n$ and $n_1\le n-2$, the nonzero vectors in

$$\{\mathbb{F}[\eta](x_i^{m_1}y_n^{m_2})\mid m_1,m_2\in\mathbb{N};i=n_1,n_1+1\} \tag{3.37}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}$.

Denote

$$\mathcal{H}=\{f\in\mathcal{B}\mid\mathcal{D}(f)=0\}. \tag{3.38}$$

By (3.5), $\mathcal{H}$ forms an $sl(n,\mathbb{F})$-submodule. Recall $\mathcal{B}_{\langle\ell_1,\ell_2\rangle}$ defined in (1.11). Then

$$\mathcal{B}_{\langle\ell_1,\ell_2\rangle}=\{f\in\mathcal{B}\mid\flat(f)=\ell_1f;\flat'(f)=\ell_2f\} \tag{3.39}$$

by (2.13) and (3.4). Moreover, $\mathcal{B}=\bigoplus_{\ell_1,\ell_2\in\mathbb{Z}}\mathcal{B}_{\langle\ell_1,\ell_2\rangle}$ becomes a $\mathbb{Z}^2$-graded algebra. According to (3.5), $\mathcal{B}_{\langle\ell_1,\ell_2\rangle}$ forms an $sl(n,\mathbb{F})$-submodule, and so does

$$\mathcal{H}_{\langle\ell_1,\ell_2\rangle}=\mathcal{B}_{\langle\ell_1,\ell_2\rangle}\bigcap\mathcal{H}. \tag{3.40}$$

Next (1.9) and (1.10) imply

$$[\mathcal{D},\eta]=n_2-n_1+\flat+\flat',\ \ \mathcal{D}(x_i^{m_1}y_j^{m_2})=0 \tag{3.41}$$

for $m_1,m_2\in\mathbb{N},\ i=n_1,n_1+1$ and $j=n_2,n_2+1$. Thus

$$x_{n_1+1}^{m_1}y_{n_2}^{m_2}\in\mathcal{H}_{\langle m_1,m_2\rangle},\qquad x_{n_1+1}^{m_1}y_{n_2+1}^{m_2}\in\mathcal{H}_{\langle m_1,-m_2\rangle}, \tag{3.42}$$

$$x_{n_1}^{m_1}y_{n_2}^{m_2}\in\mathcal{H}_{\langle -m_1,m_2\rangle},\qquad x_{n_1}^{m_1}y_{n_2+1}^{m_2}\in\mathcal{H}_{\langle -m_1,-m_2\rangle}. \tag{3.43}$$

For any $g\in\mathcal{H}_{\langle\ell_1,\ell_2\rangle}$and $0<m\in\mathbb{N}$, we have $\eta^m(g)\in\mathcal{B}_{\ell_1+m,\ell_2+m}$ and

$$\mathcal{D}(\eta^m(g))=m(n_2-n_1+\ell_1+\ell_2+m-1)\eta^{m-1}(g). \tag{3.44}$$

Thus

$$\mathcal{D}(\eta^m(g))=0\ \text{ if and only if }\ \ell_1+\ell_2\leq n_1-n_2\ \text{ and }\ m=n_1-n_2-\ell_1-\ell_2+1. \tag{3.45}$$

If so,

$$\eta^m(g)\in\mathcal{H}_{n_1-n_2-\ell_2+1,n_1-n_2-\ell_1+1}. \tag{3.46}$$

Note

$$(n_1-n_2-\ell_2+1)+(n_1-n_2-\ell_1+1)=n_1-n_2+2+(n_1-n_2-\ell_1-\ell_2)\geq n_1-n_2+2. \tag{3.47}$$

Let $f_{\langle\ell_1,\ell_2\rangle}\in\mathcal{H}_{\langle\ell_1,\ell_2\rangle}$ be a singular vector in (3.42) and (3.43). Then the singular vectors in $\mathcal{H}$ are nonzero weight vectors in

$$\text{Span}\{f_{\langle\ell_1,\ell_2\rangle},\eta^{n_1-n_2+1-r_1-r_2}(f_{\langle r_1,r_2\rangle})\mid\ell_1,\ell_2,r_1,r_2\in\mathbb{Z};r_1+r_2\leq n_1-n_2\} \tag{3.48}$$

by (3.36), where

$$\eta^{n_1-n_2+1-r_1-r_2}(f_{\langle r_1,r_2\rangle})\in\mathcal{H}_{\langle n_1-n_2+1-r_2,n_1-n_2+1-r_1\rangle}. \tag{3.49}$$

Thus when $n_1+1<n_2<n$, we have

$$\mathcal{H}_{\langle\ell_1,\ell_2\rangle}\text{ has a unique singular vector if }\ \ell_1+\ell_2\leq n_1-n_2+1 \tag{3.50}$$

and

$$\mathcal{H}_{\langle\ell_1,\ell_2\rangle}\text{ has exactly two singular vectors if }\ \ell_1+\ell_2>n_1-n_2+1. \tag{3.51}$$

In the case $n_1+1<n_2=n$, $\mathcal{B}_{\langle\ell_1,\ell_2\rangle}=0$ if $\ell_2<0$, and for $\ell_1\in\mathbb{Z}$, $\ell_2\in\mathbb{N}$,

$$\mathcal{H}_{\langle\ell_1,\ell_2\rangle}\text{ has a unique singular vector if }\ \ell_1\geq n_1-n+2\text{ or }\ell_1+\ell_2\leq n_1-n+1. \tag{3.52}$$

$$\mathcal{H}_{\langle\ell_1,\ell_2+1\rangle}\text{ has exactly two singular vector if and }\ n_1-n+1-\ell_2\leq\ell_1\leq n_1-n+1. \tag{3.53}$$

Observe that the symmetric bilinear form $(\cdot|\cdot)$ on $\mathcal{B}$ is determined by

$$(x^\alpha y^\beta|x^{\alpha_1}y^{\beta_1})=\delta_{\alpha,\alpha_1}\delta_{\beta,\beta_1}(-1)^{\sum_{i=1}^{n_1}\alpha_i+\sum_{r=n_2+1}^{n}\beta_r}\alpha!\beta!\qquad\text{for }\ \alpha,\beta,\alpha_1,\beta_1\in\mathbb{N}^{\,n}. \tag{3.54}$$

When $n_1+1<n_2<n$, Lemma 2.3 tells us that $\mathcal{H}_{\langle\ell_1,\ell_2\rangle}$ for $\ell_1,\ell_2\in\mathbb{Z}$ is an irreducible summand of $\mathcal{B}_{\ell_1,\ell_2}$ if and only if $\ell_1+\ell_2\leq n_1-n_2+1$. It can be verified that

$$(\mathcal{D}(x^\alpha y^\beta)|x^{\alpha_1}y^{\beta_1})=(x^\alpha y^\beta|\eta(x^{\alpha_1}y^{\beta_1})). \tag{3.55}$$

Recall that $f_{\langle\ell_1,\ell_2\rangle}\in\mathcal{H}_{\langle\ell_1,\ell_2\rangle}$ is a singular vector in (3.42) and (3.43). Thus

$$(f_{\langle\ell_1,\ell_2\rangle}|f_{\langle\ell_1,\ell_2\rangle})\neq 0 \tag{3.56}$$

and

$$(f_{\langle\ell_1,\ell_2\rangle}|f_{\langle\ell_1',\ell_2'\rangle})=0\qquad\text{if }\;(\ell_1,\ell_2)\neq(\ell_1',\ell_2'). \tag{3.57}$$

Recall $sl(n,\mathbb{F})_+$ in (2.30) and let $sl(n,\mathbb{F})_-=\sum_{1\leq i<j\leq n}\mathbb{F}E_{j,i}$ be the subalgebra spanned by the negative root vectors. Moreover, $(sl(n,\mathbb{F})_-)^t=sl(n,\mathbb{F})_+$. According to (3.7)-(3.11), $\mathcal{B}$ is nilpotent with respect to $sl(n,\mathbb{F})_+$. Thus all $\mathcal{H}_{\langle\ell_1,\ell_2\rangle}$ with $\ell_1+\ell_2\leq n_1-n_2+1$ are irreducible $sl(n,\mathbb{F})$-submodules by Lemma 2.3 and (3.50), and so are $\eta^m(\mathcal{H}_{\langle\ell_1,\ell_2\rangle})$ for any $m\in\mathbb{N}$ by (3.5).

We extend the transpose to an algebraic anti-isomorphism on $U(sl(n,\mathbb{F}))$ by $1^t=1$ and

$$(A_1A_2\cdots A_r)^t=A_r^t\cdots A_2^tA_1^t\qquad\text{for }\;A_i\in sl(n,\mathbb{F}). \tag{3.58}$$

By the irreducibility,

$$\mathcal{H}_{\langle\ell_1,\ell_2\rangle}=U(sl(n,\mathbb{F})_-)(f_{\langle\ell_1,\ell_2\rangle})\qquad\text{if }\;\ell_1+\ell_2\leq n_1-n_2+1. \tag{3.59}$$

Let $\ell_1,\ell_2,\ell_1',\ell_2'\in\mathbb{Z}$ such that $\ell_1+\ell_2,\ell_1'+\ell_2'\leq n_1-n_2+1$ and $(\ell_1,\ell_2)\neq(\ell_1',\ell_2')$. Then

$$(w(f_{\langle\ell_1,\ell_2\rangle})|f_{\langle\ell_1',\ell_2'\rangle})=(f_{\langle\ell_1,\ell_2\rangle}|w^t(f_{\langle\ell_1',\ell_2'\rangle}))=0\qquad\text{for }\;w\in U(sl(n,\mathbb{F})_-)sl(n,\mathbb{F})_- \tag{3.60}$$

by Lemma 2.2. Since $f_{\langle\ell_1,\ell_2\rangle}$ is a weight vector, we have $U(H)(f_{\langle\ell_1,\ell_2\rangle})\subset\mathbb{F}f_{\langle\ell_1,\ell_2\rangle}$ (cf. (2.29)). Thus for any $w_1,w_2\in U(sl(n,\mathbb{F})_-)$,

$$(w_1(f_{\langle\ell_1,\ell_2\rangle})|w_1(f_{\langle\ell_1',\ell_2'\rangle}))=(w_2^tw_1(f_{\langle\ell_1,\ell_2\rangle})|f_{\langle\ell_1',\ell_2'\rangle})=c(f_{\langle\ell_1,\ell_2\rangle})|f_{\langle\ell_1',\ell_2'\rangle}) \tag{3.61}$$

for some $c\in\mathbb{F}$ by (3.60). Hence (3.59) implies

$$(\mathcal{H}_{\langle\ell_1,\ell_2\rangle}|\mathcal{H}_{\langle\ell_1',\ell_2'\rangle})=\{0\}. \tag{3.62}$$

For $f\in\mathcal{H}_{\langle\ell_1,\ell_2\rangle},\ g\in\mathcal{B}$ and $m,m_1\in\mathbb{N}$ such that $m\leq m_1$, we find

$$(\eta^m(f)|\eta^{m_1}(g))=(\mathcal{D}^{m_1}\eta^m(f)|g)=\delta_{m_1,m}m![\prod_{r=0}^{m-1}(\ell_1+\ell_2+n_2-n_1+r)](f|g) \tag{3.63}$$

by (3.44) and (3.55). In particular, the singular vectors $\eta^{n_1-n_2+1-r_1-r_2}(f_{\langle r_1,r_2\rangle})$ for $r_1,r_2\in\mathbb{Z}$ with $r_1+r_2\leq n_1-n_2$ are isotropic polynomials. Moreover, for $m,m_1\in\mathbb{N}$ and $\ell_1,\ell_2,\ell_1',\ell_2'\in\mathbb{Z}$ such that $\ell_1+\ell_2,\ell_1'+\ell_2'\leq n_1-n_2+1$,

$$(\eta^m(\mathcal{H}_{\langle\ell_1,\ell_2\rangle})|\eta^{m_1}(\mathcal{H}_{\langle\ell_1',\ell_2'\rangle}))=\{0\}\qquad\text{if }\;(m,\ell_1,\ell_1)\neq(m_1,\ell_1',\ell_1') \tag{3.64}$$

by (3.62) and (3.63). On the other hand,

$$(\eta^m(f_{\langle\ell_1,\ell_2\rangle})|\eta^m(f_{\langle\ell_1,\ell_2\rangle}))=m![\prod_{r=0}^{m-1}(\ell_1+\ell_2+n_2-n_1+r)](f_{\langle\ell_1,\ell_2\rangle}|f_{\langle\ell_1,\ell_2\rangle})\neq 0 \tag{3.65}$$

by (3.63). Since the radical of $(\cdot|\cdot)$ on $\eta^m(\mathcal{H}_{\langle \ell_1,\ell_2\rangle})$ is a proper submodule by Lemma 2.2, the irreducibility of $\eta^m(\mathcal{H}_{\langle \ell_1,\ell_2\rangle})$ implies that

$$(\cdot|\cdot) \text{ is nondegenerate rewtricted to } \eta^m(\mathcal{H}_{\langle \ell_1,\ell_2\rangle}). \tag{3.66}$$

Fix $\ell_1,\ell_2\in\mathbb{Z}$ with $\ell_1+\ell_2\leq n_1-n_2+1$. Set

$$\hat{\mathcal{B}}_{\langle \ell_1,\ell_2\rangle}=\sum_{m=0}^{\infty}\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle}). \tag{3.67}$$

By (3.64) and (3.66), the above sum is a direct sum and $(\cdot|\cdot)$ is nondegenerate restricted to $\hat{\mathcal{B}}_{\langle \ell_1,\ell_2\rangle}$. Hence

$$\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\hat{\mathcal{B}}_{\langle \ell_1,\ell_2\rangle}\oplus(\hat{\mathcal{B}}^{\perp}_{\langle \ell_1,\ell_2\rangle}\bigcap\mathcal{B}_{\langle \ell_1,\ell_2\rangle}). \tag{3.68}$$

If $\hat{\mathcal{B}}^{\perp}_{\langle \ell_1,\ell_2\rangle}\bigcap\mathcal{B}_{\langle \ell_1,\ell_2\rangle}\neq\{0\}$, then it contains a singular vector, which must be of the form $\eta^{m_1}(f_{\langle \ell_1-m_1,\ell_2-m_1\rangle})$ for some $m_1\in\mathbb{N}$ by (3.36). This contradicts (3.65). Thus $\hat{\mathcal{B}}^{\perp}_{\langle \ell_1,\ell_2\rangle}\bigcap\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\{0\}$, equivalently

$$\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\bigoplus_{m=0}^{\infty}\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle}) \tag{3.69}$$

is completely reducible. Applying (3.69) to $\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle}$, we have

$$\mathcal{B}_{\ell_1,\ell_2}=\mathcal{H}_{\langle \ell_1,\ell_2\rangle}\oplus\eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})\qquad\text{if }\ \ell_1+\ell_2\leq n_1-n_2+1. \tag{3.70}$$

Assume $n_1+1<n_2=n$. For $\ell_1\in\mathbb{Z}$ and $\ell_2\in\mathbb{N}$ such that $\ell_1\geq n_1-n+2$ or $\ell_1+\ell_2\leq n_1-n+1$, all $\mathcal{H}_{\ell_1,\ell_2}$ are irreducible submodules of $\mathcal{B}_{\ell_1,\ell_2}$ by Lemma 2.3, (3.52) and (3.54). When $\ell_1+\ell_2\leq n_1-n_2+1$, (3.64), (3.66), (3.69) and (3.70) also hold by the same arguments as in the above. In summary, we have:

Theorem 3.1. *Suppose $n_1+1<n_2$. For $\ell_1,\ell_2\in\mathbb{Z}$ with $\ell_1+\ell_2\leq n_1-n_2+1$, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-module and*

$$\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\bigoplus_{m=0}^{\infty}\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle}) \tag{3.71}$$

is an orthogonal decomposition of irreducible submodules. In particular, $\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\mathcal{H}_{\langle \ell_1,\ell_2\rangle}\oplus\eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})$. The symmetric bilinear form $(\cdot|\cdot)$ restricted to $\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle})$ is nondengerate. If $n_2<n$, all $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ for $\ell_1,\ell_2\in\mathbb{Z}$ with $\ell_1+\ell_2>n_1-n_2+1$ have exactly two singular vectors.

Assume $n_2=n$. Then $\mathcal{B}_{\langle \ell,0\rangle}=\mathcal{H}_{\langle \ell,0\rangle}$ with $\ell\in\mathbb{Z}$ are irreducible highest-weight $sl(n,\mathbb{F})$-modules. All $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ for $\ell_1\in\mathbb{Z}$ and $\ell_2\in\mathbb{N}$ such that $\ell_1\geq n_1-n+2$ are also irreducible highest-weight $sl(n,\mathbb{F})$-modules. Moreover, for $\ell_2\in 1+\mathbb{N}$ and $n_1-n_2+1+\ell_2\leq\ell_1\in\mathbb{Z}$, the orthogonal decompositions in (3.71) also holds. Furthermore, $\mathcal{H}_{\langle \ell_1,\ell_2+1\rangle}$ for $\ell_1\in\mathbb{Z}$ and $\ell_2\in\mathbb{N}$ such that $n_1-n+1-\ell_2\leq\ell_1\leq n_1-n+1$ have exactly two singular vectors.

Indeed, we have more detailed information. Suppose $n_1+1<n_2<n$. For $m_1,m_2\in\mathbb{N}$ with $m_1+m_2\geq n_2-n_1-1$, $\mathcal{H}_{\langle -m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}y_{n_2+1}^{m_2}$ of weight $m_1\lambda_{n_1-1}-(m_1+1)\lambda_{n_1}-(m_2+1)\lambda_{n_2}+m_2(1-\delta_{n_2,n-1})\lambda_{n_2+1}$. When $m_1,m_2\in\mathbb{N}$ with

$m_2 - m_1 \geq n_2 - n_1 - 1$, $\mathcal{H}_{\langle m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1}y_{n_2+1}^{m_2}$ of weight $-(m_1+1)\lambda_{n_1} + m_1\lambda_{n_1+1} - (m_2+1)\lambda_{n_2} + m_2(1-\delta_{n_2,n-1})\lambda_{n_2+1}$. If $m_1, m_2 \in \mathbb{N}$ with $m_1 - m_2 \geq n_2 - n_1 - 1$, $\mathcal{H}_{\langle -m_1,m_2\rangle}$ is has a highest-weight vector $x_{n_1}^{m_1}y_{n_2}^{m_2}$ of weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1} + m_2\lambda_{n_2-1} - (m_2+1)\lambda_{n_2}$.

Assume $n_1 + 1 < n_2 = n$. When $m_1, m_2 \in \mathbb{N}$, $\mathcal{H}_{\langle m_1,m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1}y_n^{m_2}$ of weight $-(m_1+1)\lambda_{n_1} + m_1\lambda_{n_1+1} + m_2\lambda_{n-1}$. If $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq n - n_1 - 2$ or $m_2 - m_1 \leq n_1 - n + 1$, $\mathcal{H}_{\langle -m_1,m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}y_n^{m_2}$ of weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1} + m_2\lambda_{n-1}$.

By Lemma 2.1 with $T_1 = \partial_{x_{n_1+1}}\partial_{y_{n_1+1}}$, $T_1^- = \int_{(x_{n_1+1})}\int_{(y_{n_1+1})}$ and $T_2 = \mathcal{D} - \partial_{x_{n_1+1}}\partial_{y_{n_1+1}}$, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ has a basis

$$\{\sum_{i=0}^{\infty}\frac{(x_{n_1+1}y_{n_1+1})^i(\mathcal{D}-\partial_{x_{n_1+1}}\partial_{y_{n_1+1}})^i(x^\alpha y^\beta)}{\prod_{r=1}^i(\alpha_{n_1+1}+r)(\beta_{n_1+1}+r)} \mid \alpha,\beta \in \mathbb{N}^n;$$
$$\alpha_{n_1+1}\beta_{n_1+1} = 0; \sum_{r=n_1+1}^{n}\alpha_r - \sum_{i=1}^{n_1}\alpha_i = \ell_1; \sum_{i=1}^{n_2}\beta_i - \sum_{r=n_2+1}^{n}\beta_r = \ell_2\}. \tag{3.72}$$

Case 2. $n_1 + 1 = n_2$

In this case, $\zeta = x_{n_1+1}y_{n_1+1}$. First we consider the subcase $n_2 < n$. Suppose that $f \in \mathcal{Q}$ is a singular vector. According to the arguments in (3.13)-(3.17), f is a rational function in

$$\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, y_{n_1}, y_{n_1+2}, \zeta, \zeta_1, \zeta_2\}. \tag{3.73}$$

Moreover, (3.18) holds. Substituting (3.20) and $y_{n_1+1} = x_{n_1+1}^{-1}\zeta$ into (3.18), we still get (3.22), which implies $f_{\zeta_1} = 0$. Symmetrically, $f_{\zeta_2} = 0$. Hence we can rewrite f as a rational function in

$$\{x_{n_1}, x_{n_1+1}, x_{n_1+2}, y_{n_1}, y_{n_1+1}, y_{n_1+2}\}. \tag{3.74}$$

Now f is a singular vector if and only if it is a weight vector satisfying the following system of differential equations

$$(\partial_{x_{n_1}}\partial_{x_{n_1+1}} - y_{n_1+1}\partial_{y_{n_1}})(f) = 0, \tag{3.75}$$

$$(x_{n_1+1}\partial_{x_{n_1+2}} - \partial_{y_{n_1+1}}\partial_{y_{n_1+2}})(f) = 0 \tag{3.76}$$

by (3.8) and (3.10) with $n_2 = n_1 + 1$. Note

$$E_{n_1,n_1+2}|_{\mathcal{Q}} = [E_{n_1,n_1+1}|_{\mathcal{Q}}, E_{n_1+1,n_1+2}|_{\mathcal{Q}}] = \partial_{x_{n_1}}\partial_{x_{n_1+2}} - \partial_{y_{n_1}}\partial_{y_{n_1+2}} \tag{3.77}$$

by (3.8) and (3.10) with $n_2 = n_1 + 1$. So

$$(\partial_{x_{n_1}}\partial_{x_{n_1+2}} - \partial_{y_{n_1}}\partial_{y_{n_1+2}})(f) = 0. \tag{3.78}$$

For our purpose of representation, we only consider f is a polynomial in $\{x_i, y_i \mid i = n_1, n_1+1, n_1+2\}$. Set

$$\begin{aligned}\phi_{m_1,m_2,m_3} &= [\prod_{s=1}^{m_2}(m_1+s)]\sum_{i=0}^{\infty}\frac{x_{n_1}^{m_1+i}x_{n_1+2}^i(\partial_{y_{n_1}}\partial_{y_{n_1+2}})^i(y_{n_1}^{m_2}y_{n_1+2}^{m_3})}{i!\prod_{r=1}^i(m_1+r)} \\ &= (y_{n_1}\partial_{x_{n_1}} + x_{n_1+2}\partial_{y_{n_1+2}})^{m_2}(x_{n_1}^{m_1+m_2}y_{n_1+2}^{m_3}) \end{aligned}\tag{3.79}$$

and

$$
\begin{aligned}
\psi_{m_1,m_2,m_3} &= \frac{(m_1+m_2)!\prod_{s=1}^{m_1}(m_3+s)}{m_1!}\sum_{i=0}^{\infty}\frac{x_{n_1}^i x_{n_1+2}^{m_1+i}(\partial_{y_{n_1}}\partial_{y_{n_1+2}})^i(y_{n_1}^{m_2}y_{n_1+2}^{m_3})}{i!\prod_{r=1}^{i}(m_1+r)}\\
&= \sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(m_1+m_2)!x_{n_1}^i x_{n_1+2}^{m_1+i}y_{n_1}^{m_2-i}\partial_{y_{n_1+2}}^{m_1+i}(y_{n_1+2}^{m_1+m_3})}{(m_1+i)!}\\
&= \sum_{i=0}^{m_2}\binom{m_2}{m_2-i}\frac{(m_1+m_2)!x_{n_1}^i y_{n_1}^{m_2-i}(x_{n_1+2}\partial y_{n_1+2})^{m_1+i}(y_{n_1+2}^{m_1+m_3})}{(m_1+i)!}\\
&= \sum_{i=0}^{m_2}\frac{(m_1+m_2)!(y_{n_1}\partial_{x_{n_1}})^{m_2-i}(x_{n_1}^{m_2})(x_{n_1+2}\partial y_{n_1+2})^{m_1+i}(y_{n_1+2}^{m_1+m_3})}{(m_2-i)!(m_1+i)!}\\
&= \sum_{r=0}^{\infty}\frac{(m_1+m_2)!(y_{n_1}\partial_{x_{n_1}})^{r}(x_{n_1}^{m_2})(x_{n_1+2}\partial y_{n_1+2})^{m_1+m_2-r}(y_{n_1+2}^{m_1+m_3})}{r!(m_1+m_2-r)!}\\
&= (y_{n_1}\partial_{x_{n_1}}+x_{n_1+2}\partial_{y_{n_1+2}})^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+2}^{m_1+m_3}).
\end{aligned}
\tag{3.80}
$$

By Lemma 2.1 with $T_1=\partial_{x_{n_1}}\partial_{x_{n_1+2}}$, $T_1^-=\int_{(x_{n_1})}\int_{(x_{n_1+2})}$ (cf. (2.6)) and $T_2=\partial_{y_{n_1}}\partial_{y_{n_1+2}}$, the polynomial solution space of (3.78) is

$$
\mathrm{Span}\{\phi_{m_1,m_2,m_3}x_{n_1+1}^{m_4}y_{n_1+1}^{m_5},\psi_{m_1+1,m_2,m_3}x_{n_1+1}^{m_4}y_{n_1+1}^{m_5}\mid m_i\in\mathbb{N}\}. \tag{3.81}
$$

Note

$$
\phi_{m_1,m_2,0}x_{n_1+1}^{m_3}y_{n_1+1}^{m_4}=[\prod_{r=1}^{m_2}(m_1+r)]x_{n_1}^{m_1}y_{n_1}^{m_2}x_{n_1+1}^{m_3}y_{n_1+1}^{m_4}\qquad\text{for } m_i\in\mathbb{N} \tag{3.82}
$$

and

$$
x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}\psi_{m_3,0,m_4}=[\prod_{i=1}^{m_3}(m_4+i)]x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}x_{n_1+2}^{m_3}y_{n_1+2}^{m_4}\qquad\text{for } m_i\in\mathbb{N}. \tag{3.83}
$$

In particular, all the polynomials in (3.83) are solutions of the equation (3.75). Now

$$
\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+x_{n_1+1}y_{n_1+1}+\sum_{s=n_1+2}^{n}x_s\partial_{y_s}. \tag{3.84}
$$

Write

$$
\begin{aligned}
h_{m_1,m_2,m_3} &= \frac{(m_1+m_2)!}{m_1!}\sum_{i=0}^{\infty}\frac{x_{n_1}^i x_{n_1+1}^{m_1+i}y_{n_1+1}^{m_3+i}\partial_{y_{n_1}}^i(y_{n_1}^{m_2})}{i!\prod_{r=1}^{i}(m_1+r)}\\
&= \sum_{i=0}^{m_2}\binom{m_2}{i}\frac{(m_1+m_2)!x_{n_1}^i x_{n_1+1}^{m_1+i}y_{n_1+1}^{m_3+i}y_{n_1}^{m_2-i}}{(m_1+i)!}\\
&= \sum_{i=0}^{m_2}\binom{m_2}{m_2-i}\frac{(m_1+m_2)!x_{n_1}^i x_{n_1+1}^{m_1+i}y_{n_1+1}^{m_3+i}y_{n_1}^{m_2-i}}{(m_1+i)!}\\
&= \sum_{i=0}^{m_2}\frac{(m_1+m_2)!(y_{n_1}\partial_{x_{n_1}})^{m_2-i}(x_{n_1}^{m_2})x_{n_1+1}^{m_1+i}y_{n_1+1}^{m_3+i}}{(m_2-i)!(m_1+i)!}\\
&= \eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{m_3-m_1})
\end{aligned}
\tag{3.85}
$$

and calculate

$$\begin{aligned}
&\eta^{m_2}(x_{n_1}^{m_1+m_2})y_{n_1+1}^{m_3}=\eta^{m_2}(x_{n_1}^{m_1+m_2}y_{n_1+1}^{m_3})\\
=&\sum_{i=0}^{m_2}\binom{m_2}{m_2-i}[\prod_{r=1}^{m_2-i}(m_1+i+r)]y_{n_1}^{m_2-i}x_{n_1}^{m_1+i}x_{n_1+1}^{i}y_{n_1+1}^{m_3+i}\\
=&\sum_{i=0}^{m_2}\binom{m_2}{i}\frac{[\prod_{r=1}^{m_2}(m_1+r)]y_{n_1}^{m_2-i}x_{n_1}^{m_1+i}x_{n_1+1}^{i}y_{n_1+1}^{m_3+i}}{\prod_{s=1}^{i}(m_1+s)}\\
=&[\prod_{r=1}^{m_2}(m_1+r)]\sum_{i=0}^{m_2}\frac{x_{n_1}^{m_1+i}x_{n_1+1}^{i}y_{n_1+1}^{m_3+i}\partial_{y_{n_1}}^{i}(y_{n_1}^{m_2})}{i![\prod_{s=1}^{i}(m_1+s)]}. \qquad (3.86)
\end{aligned}$$

Lemma 2.1 with $T_1=\partial_{x_{n_1}}\partial_{x_{n_1+1}}$, $T_1^-=\int_{(x_{n_1})}\int_{(x_{n_1+1})}$ (cf. (2.6)) and $T_2=-y_{n_1+1}\partial_{y_{n_1}}$ tells us that $\mathrm{Span}\{h_{m_1,m_2,m_3},\eta^{m_2}(x_{n_1}^{m_1+m_2})y_{n_1+1}^{m_3}\mid m_i\in\mathbb{N}\}$ is the solution space of (3.75) in $\mathrm{Span}\{x_{n_1}^{m_1}y_{n_1}^{m_2}x_{n_1+1}^{m_3}y_{n_1+1}^{m_4}\mid m_i\in\mathbb{N}\}$. In particular, (3.85) and (3.86) can be viewed as algorithms of solving the equation (3.75).

On the other hand,

$$\partial_{x_{n_1}}(\phi_{0,m_2,m_3})=m_2\psi_{1,m_2,m_3-1}, \qquad (3.87)$$

$$\partial_{x_{n_1}}(\phi_{m_1,m_2,m_3})=(m_1+m_2)\phi_{m_1-1,m_2,m_3}\qquad \text{if } m_1>0, \qquad (3.88)$$

$$\partial_{x_{n_1}}(\psi_{m_1,m_2,m_3})=m_2\psi_{m_1+1,m_2-1,m_3-1}, \qquad (3.89)$$

$$\partial_{y_{n_1}}(\phi_{m_1,m_2,m_3})=m_2(m_1+m_2)\phi_{m_1,m_2-1,m_3}, \qquad (3.90)$$

$$\partial_{y_{n_1}}(\psi_{m_1,m_2,m_3})=m_2(m_1+m_2)\psi_{m_1,m_2-1,m_3}. \qquad (3.91)$$

Applying the algorithm (3.85) to (3.81), we get that $\hat{\phi}_{m_1,m_2,m_3}y_{n+1}^{m_4}$ and $\hat{\psi}_{m_1,m_2,m_3}y_{n+1}^{m_4}$ are the solutions of (3.75) by (3.87)-(3.91), where

$$\hat{\phi}_{m_1,m_2,m_3}=\sum_{i=0}^{\infty}\binom{m_2}{i}\phi_{m_1+i,m_2-i,m_3}(x_{n_1+1}y_{n_1+1})^i=\eta^{m_2}(x_{n_1}^{m_1+m_2}y_{n_1+2}^{m_3}), \qquad (3.92)$$

$$\begin{aligned}
\hat{\psi}_{m_1,m_2,m_3}=&\sum_{i=0}^{m_1}\binom{m_1+m_2}{i}\psi_{m_1-i,m_2,m_3+i}(x_{n_1+1}y_{n_1+1})^i\\
&+\sum_{r=1}^{m_2}\binom{m_1+m_2}{m_1+r}\phi_{m_2-r,r,m_1+m_3}(x_{n_1+1}y_{n_1+1})^{m_1+r}\\
=&\eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+2}^{m_1+m_3}). \qquad (3.93)
\end{aligned}$$

Using the algorithm (3.86), we find that the solution space of (3.75) in (3.81) is

$$\begin{aligned}
&\mathrm{Span}\{x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}x_{n_1+2}^{m_3}y_{n_1+2}^{m_4},\hat{\phi}_{m_1,m_2,m_3}y_{n_1+1}^{m_4},h_{m_1,m_2,m_3},\\
&\hat{\psi}_{m_1+1,m_2+1,m_3}y_{n_1+1}^{m_4}\mid m_i\in\mathbb{N}\}. \qquad (3.94)
\end{aligned}$$

According to (3.84), (3.92) and (3.93),

$$\partial_{x_{n_1+2}}(\hat{\phi}_{m_1,m_2,m_3})=m_2m_3\hat{\phi}_{m_1,m_2-1,m_3-1}, \qquad (3.95)$$

$$\partial_{y_{n_1+1}}(\hat{\phi}_{m_1,m_2,m_3})=m_2x_{n_1+1}\hat{\phi}_{m_1,m_2-1,m_3}, \qquad (3.96)$$

$$\partial_{y_{n_1+2}}(\hat{\phi}_{m_1,m_2,m_3})=m_3\hat{\phi}_{m_1,m_2,m_3-1}, \qquad (3.97)$$

$$\partial_{x_{n_1+2}}(\hat{\psi}_{m_1,m_2,m_3}) = (m_1+m_2)(m_1+m_3)\hat{\psi}_{m_1-1,m_2,m_3}, \tag{3.98}$$

$$\partial_{y_{n_1+1}}(\hat{\psi}_{m_1,m_2,m_3}) = (m_1+m_2)x_{n_1+1}\hat{\psi}_{m_1-1,m_2,m_3+1}, \tag{3.99}$$

$$\partial_{y_{n_1+2}}(\hat{\psi}_{m_1,m_2,m_3}) = (m_1+m_3)\hat{\psi}_{m_1,m_2,m_3-1}. \tag{3.100}$$

Put

$$\begin{aligned} g_{m_1,m_2,m_3} &= \frac{(m_1+m_2)!}{m_1!}\sum_{i=0}^{\infty}\frac{y_{n_1+2}^i y_{n_1+1}^{m_1+i} x_{n_1+1}^{m_3+i}\partial_{x_{n_1+2}}^i(x_{n_1+2}^{m_2})}{i!\prod_{r=1}^i(m_1+r)} \\ &= \eta^{m_1+m_2}(y_{n_1+2}^{m_2}x_{n_1+1}^{m_3-m_1}) \end{aligned} \tag{3.101}$$

and

$$\begin{aligned} g'_{m_1,m_2,m_3} &= \sum_{i=0}^{\infty}\frac{[\prod_{s=1}^{m_2}(m_1+s)]y_{n_1+2}^{m_1+i}y_{n_1+1}^i x_{n_1+1}^{m_3+i}\partial_{x_{n_1+2}}^i(x_{n_1+2}^{m_2})}{i!\prod_{r=1}^i(m_1+r)} \\ &= \eta^{m_2}(y_{n_1+2}^{m_1+m_2}x_{n_1+1}^{m_3}). \end{aligned} \tag{3.102}$$

Symmetrically, $\text{Span}\{g_{m_1,m_2,m_3}, g'_{m_1,m_2,m_3} \mid m_i \in \mathbb{N}\}$ is the solution space of (3.76) in $\text{Span}\{x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}x_{n_1+2}^{m_3}y_{n_1+2}^{m_4} \mid m_i \in \mathbb{N}\}$ by Lemma 2.1 with $T_1 = \partial_{y_{n_1+1}}\partial_{y_{n_1+2}}$, $T_1^- = \int_{(y_{n_1+1})}\int_{(y_{n_1+2})}$ (cf. (2.6)) and $T_2 = -x_{n_1+1}\partial_{x_{n_1+2}}$. Observe that $\{\hat{\phi}_{m_1,m_2,m_3}, \hat{\psi}_{m_1,m_2,m_3}, h_{m_1,m_2,m_3} \mid m_i \in \mathbb{N}\}$ are solutions of (3.76). Thus the solution space of (3.76) in (3.94) is

$$\begin{aligned} &\text{Span}\{g_{m_1,m_2,m_3}, g'_{m_1,m_2,m_3}, h_{m_1,m_2,m_3}, \hat{\phi}_{m_1,m_2,m_3}, \\ &\quad \hat{\phi}_{m_1,m_2,0}y_{n_1+1}^{m_3}, \hat{\psi}_{m_1+1,m_2+1,m_3} \mid m_i \in \mathbb{N}\} \end{aligned} \tag{3.103}$$

by (3.97) and (3.100).

Expressions (3.85), (3.92), (3.93), (3.101) and (3.102) imply that the solution space of the singular vectors in $\mathcal{B}$ is

$$\begin{aligned} &\text{Span}\{\eta^{m_2}(x_i^{m_1}y_j^{m_3}), x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}, \eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{m_3-m_1}), \eta^{m_1+m_2}(y_{n_1+2}^{m_2}x_{n_1+1}^{m_3-m_1}) \\ &\quad \mid m_r \in \mathbb{N}; (i,j) = (n_1, n_1+1), (n_1, n_1+2), (n_1+1, n_1+2)\}. \end{aligned} \tag{3.104}$$

Remind that in this case,

$$\mathcal{D} = -\sum_{i=1}^{n_1} x_i\partial_{y_i} + \partial_{x_{n_1+1}}\partial_{y_{n_1+1}} - \sum_{s=n_1+2}^{n} y_s\partial_{x_s}. \tag{3.105}$$

We have

$$\mathcal{D}[\eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{m_3-m_1})] = (m_1+m_2)m_3\eta^{m_1+m_2-1}(x_{n_1}^{m_2}y_{n_1+1}^{m_3-m_1}) \tag{3.106}$$

by (3.44). Thus we find a singular

$$\eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{-m_1}) \in \mathcal{H}_{\langle m_1,m_2\rangle} \tag{3.107}$$

of new type if $m_1, m_2 \geq 1$. Symmetrically, $\eta^{m_1+m_2}(y_{n_1+2}^{m_2}x_{n_1+1}^{-m_1}) \in \mathcal{H}_{\langle m_2,m_1\rangle}$ is a singular vector.

Recall the singular vectors

$$f_{\langle -m_1,-m_2\rangle} = x_{n_1}^{m_1}y_{n_1+2}^{m_2} \in \mathcal{H}_{\langle -m_1,-m_2\rangle}, \qquad f_{\langle -m_1,m_2\rangle} = x_{n_1}^{m_1}y_{n_1+1}^{m_2} \in \mathcal{H}_{\langle -m_1,m_2\rangle}, \tag{3.108}$$

$$f_{\langle m_1,-m_2\rangle}=x_{n_1+1}^{m_1}y_{n_1+2}^{m_2}\in\mathcal{H}_{\langle m_1,-m_2\rangle}. \tag{3.109}$$

Moreover, we have the singular vectors

$$\eta^{-\ell_1-\ell_2}(f_{\langle \ell_1,\ell_2\rangle})\in\mathcal{H}_{\langle -\ell_2,-\ell_1\rangle}\qquad\text{for }\ \ell_1,\ell_2\in\mathbb{Z}\text{ with }\ell_1+\ell_2\le -1. \tag{3.110}$$

Therefore, any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\begin{aligned}&\text{Span}\{f_{\langle \ell_1,\ell_2\rangle},\eta^{-\ell_1'-\ell_2'}(f_{\langle \ell_1',\ell_2'\rangle}),\eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{-m_1}),\eta^{m_1+m_2}(y_{n_1+2}^{m_2}x_{n_1+1}^{-m_1})\\&\quad\mid \ell_1,\ell_2,\ell_1',\ell_2'\in\mathbb{Z},\ m_1,m_2\in\mathbb{N}+1;\ell_1\le 0\text{ or }\ell_2\le 0;\ell_1'+\ell_2'\le -1\}.\end{aligned} \tag{3.111}$$

Assume $n_2=n$. We similarly find that the solution space of the singular vectors in $\mathcal{B}$ is

$$\text{Span}\{\eta^{m_2}(x_{n-1}^{m_1}y_n^{m_3}),x_n^{m_1}y_n^{m_2},\eta^{m_1+m_2}(x_{n-1}^{m_2}y_n^{m_3-m_1})\mid m_i\in\mathbb{N}\}. \tag{3.112}$$

In particular, any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\begin{aligned}&\text{Span}\{x_{n-1}^{m_1}y_n^{m_2},x_n^{m_1},\eta^{m_1+1}(x_{n-1}^{m_1+m_2+1}y_n^{m_2}),\\&\quad\eta^{m_1+m_2+2}(x_{n-1}^{m_2+1}y_n^{-m_1-1})\mid m_1,m_2\in\mathbb{N}\}.\end{aligned} \tag{3.113}$$

By the arguments of (3.55)-(3.70), we have:

Theorem 3.2. *Suppose $n_1+1=n_2$. For $\ell_1,\ell_2\in\mathbb{Z}$ with $\ell_1+\ell_2\le 0$ or $n_2=n$ and $0\le\ell_2\le\ell_1$, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-module and*

$$\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\bigoplus_{m=0}^{\infty}\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle}) \tag{3.114}$$

is an orthogonal decomposition of irreducible submodules. In particular, $\mathcal{B}_{\langle \ell_1,\ell_2\rangle}=\mathcal{H}_{\langle \ell_1,\ell_2\rangle}\oplus\eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})$. The symmetric bilinear form $(\cdot|\cdot)$ restricted to $\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle})$ is nondegenerate.

Assume $n_2<n$. For $m_1,m_2\in\mathbb{N}+1$, $\mathcal{H}_{\langle m_1,m_2\rangle}$ has exactly three singular vectors. All the submodules $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such $\ell_1+\ell_2>0$ and $\ell_1\ell_2\le 0$ have two singular vectors. Consider $n_2=n$. For $m_1,m_2\in\mathbb{N}$ with $m_1<m_2$, $\mathcal{H}_{\langle m_1,m_2\rangle}$ is also an irreducible highest-weight $sl(n,\mathbb{F})$-module. All submodules $\mathcal{H}_{\langle -m_1,m_1+m_2+1\rangle}$ with $m_1,m_2\in\mathbb{N}$ have have exactly two singular vectors.

Indeed, we have more detailed information. Suppose $n_2<n$. For $m_1,m_2\in\mathbb{N}$, $\mathcal{H}_{\langle -m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}y_{n_1+2}^{m_2}$ of weight $m_1\lambda_{n_1-1}-(m_1+1)\lambda_{n_1}-(m_2+1)\lambda_{n_1+1}+m_2(1-\delta_{n_1,n-2})\lambda_{n_1+2}$. When $m_1,m_2\in\mathbb{N}$ with $m_2-m_1\ge 0$, $\mathcal{H}_{\langle m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1}y_{n_1+2}^{m_2}$ of weight $-(m_1+1)\lambda_{n_1}+(m_1-m_2-1)\lambda_{n_1+1}+m_2(1-\delta_{n_1,n-2})\lambda_{n_1+2}$. If $m_1,m_2\in\mathbb{N}$ with $m_1-m_2\ge 0$, $\mathcal{H}_{\langle -m_1,m_2\rangle}$ is has a highest-weight vector $x_{n_1}^{m_1}y_{n_1+1}^{m_2}$ of weight $m_1\lambda_{n_1-1}+(m_2-m_1-1)\lambda_{n_1}-(m_2+1)\lambda_{n_1+1}$.

Assume $n_2=n$. For $m_1,m_2\in\mathbb{N}$ with $m_2\le m_1$, $\mathcal{H}_{\langle -m_1,m_2\rangle}$ has a highest-weight vector $x_{n-1}^{m_1}y_n^{m_2}$ of weight $m_1\lambda_{n-2}+(m_2-m_1-1)\lambda_{n-1}$. Moreover, $\mathcal{H}_{\langle m,0\rangle}$ has a highest-weight vector x_{n-1}^{m} of weight $m\lambda_{n-2}-(m+1)\lambda_{n-1}$ for $m\in\mathbb{Z}$. For $m_1,m_2\in\mathbb{N}+1$, $\mathcal{H}_{\langle m_1,m_2\rangle}$ has a highest-weight vector $\eta^{m_1+m_2}(x_{n-1}^{m_2}y_n^{-m_1})$ of weight $m_2\lambda_{n-2}+(m_1-m_2-1)\lambda_{n-1}$. Again $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ has a basis of the form (3.72).

4 The $sl(n,\mathbb{F})$-Variation II: $n_1=n_2$

In this section, we continue the discussion from last section. Recall $n\geq 2$.

Case 3. $n_1=n_2$.

In this case, the variated Laplace operator

$$\mathcal{D}=-\sum_{i=1}^{n_1}x_i\partial_{y_i}-\sum_{s=n_1+1}^{n}y_s\partial_{x_s}\tag{4.1}$$

and its dual

$$\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+\sum_{s=n_2+1}^{n}x_s\partial_{y_s}.\tag{4.2}$$

First we consider the subcase $1<n_1<n-1$. Suppose that $f\in\mathcal{Q}$ is a singular vector. According to the arguments in (3.13)-(3.17), f is a rational function in

$$\{x_{n_1},x_{n_1+1},y_{n_1},y_{n_1+1},\zeta_1,\zeta_2\}\tag{4.3}$$

(cf. (3.12)). Note

$$E_{n_1,n_1+1}|_{\mathcal{Q}}=\partial_{x_{n_1}}\partial_{x_{n_1+1}}-\partial_{y_{n_1}}\partial_{y_{n_1+1}}\tag{4.4}$$

by (3.1)-(3.3). Now $E_{n_1,n_1+1}(f)=0$ implies

$$(\partial_{x_{n_1}}\partial_{x_{n_1+1}}-\partial_{y_{n_1}}\partial_{y_{n_1+1}})(f)=0,\tag{4.5}$$

equivalently,

$$(x_{n_1-1}x_{n_1+2}-y_{n_1-1}y_{n_1+2})f_{\zeta_1\zeta_2}-y_{n_1-1}f_{\zeta_1x_{n_1+1}}-x_{n_1-1}f_{\zeta_1y_{n_1+1}}$$
$$+y_{n_1+2}f_{\zeta_2x_{n_1}}+x_{n_1+2}f_{\zeta_2y_{n_1}}+f_{x_{n_1}x_{n_1+1}}-f_{y_{n_1}y_{n_1+1}}=0.\tag{4.6}$$

According to (3.12),

$$y_{n_1-1}=x_{n_1}^{-1}y_{n_1}x_{n_1-1}-x_{n_1}^{-1}\zeta_1,\qquad y_{n_1+2}=x_{n_1+1}^{-1}\zeta_2+x_{n_1+1}^{-1}y_{n_1+1}x_{n_1+2}.\tag{4.7}$$

Substituting (4.7) into (4.6), the coefficient of $x_{n_1-1}x_{n_1+2}$ implies $f_{\zeta_1\zeta_2}=0$. Thus

$$f=g+h\qquad\text{with }\;g_{\zeta_2}=h_{\zeta_1}=0.\tag{4.8}$$

Now (4.6) becomes

$$x_{n_1}^{-1}\zeta_1g_{\zeta_1x_{n_1+1}}-(x_{n_1}^{-1}y_{n_1}g_{\zeta_1x_{n_1+1}}+g_{\zeta_1y_{n_1+1}})x_{n_1-1}+(x_{n_1+1}^{-1}y_{n_1+1}h_{\zeta_2x_{n_1}}+h_{\zeta_2y_{n_1}})x_{n_1+2}$$
$$+x_{n_1+1}^{-1}\zeta_2h_{\zeta_2x_{n_1}}+g_{x_{n_1}x_{n_1+1}}-g_{y_{n_1}y_{n_1+1}}+h_{x_{n_1}x_{n_1+1}}-h_{y_{n_1}y_{n_1+1}}=0,\tag{4.9}$$

which implies

$$x_{n_1}^{-1}y_{n_1}g_{\zeta_1x_{n_1+1}}+g_{\zeta_1y_{n_1+1}}=0,\qquad x_{n_1+1}^{-1}y_{n_1+1}h_{\zeta_2x_{n_1}}+h_{\zeta_2y_{n_1}}=0.\tag{4.10}$$

For the representation purpose, we assume that g is polynomial in ζ_1 with $g|_{\zeta_1=0}=0$ and h is polynomial in ζ_2. Set

$$\zeta_3=x_{n_1}y_{n_1+1}-x_{n_1+1}y_{n_1}.\tag{4.11}$$

By (4.10),

$$g \text{ is a function in } x_{n_1}, y_{n_1}, \zeta_1, \zeta_3. \tag{4.12}$$

Moreover, (4.9) says

$$-x_{n_1}^{-1} y_{n_1} \zeta_1 g_{\zeta_1\zeta_3} - y_{n_1} g_{x_{n_1}\zeta_3} - x_{n_1} g_{y_{n_1}\zeta_3} = 0. \tag{4.13}$$

Again we can assume that $g = \hat{g} + \tilde{g}$ is polynomial in $x_{n_1}, y_{n_1}, \zeta_3$ with $\hat{g}|_{\zeta_3=0} = 0$ and $\tilde{g}_{\zeta_3} = 0$. Then (4.13) is equivalent to

$$y_{n_1} \zeta_1 \hat{g}_{\zeta_1} + x_{n_1} y_{n_1} \hat{g}_{x_{n_1}} + x_{n_1}^2 \hat{g}_{y_{n_1}} = 0. \tag{4.14}$$

This shows that $\hat{g}$ is a function in $\zeta_1/x_{n_1}, x_{n_1}^2 - y_{n_1}^2, \zeta_3$. If $\hat{g}$ is a polynomial, then $\hat{g} = 0$. So the polynomial solution of g must be a polynomial in $x_{n_1}, y_{n_2}, \zeta_1$ with $g_{\zeta_1} \neq 0$. Similarly, if $h_{\zeta_2} \neq 0$ and $h|_{\zeta_2=0} = 0$, the polynomial solution of h must be a polynomial in $x_{n+1}, y_{n+1}, \zeta_2$. Assume $h_{\zeta_2} = 0$. Then

$$h_{x_{n_1} x_{n_1+1}} - h_{y_{n_1} y_{n_1+1}} = 0 \tag{4.15}$$

by (4.9).

By Lemma 2.1, (3.78)-(3.80) and (4.2), the polynomial solution of h must be in

$$\text{Span}\{\eta^{m_3}(x_{n_1}^{m_1} y_{n_1+1}^{m_2}) \mid m_1, m_2, m_3 \in \mathbb{N}\}. \tag{4.16}$$

Therefore, a singular vector in $\mathcal{B}$ must be a nonzero weight vector in

$$\text{Span}\{x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1}, x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1}, \eta^{m_3}(x_{n_1}^{m_1} y_{n_1+1}^{m_2}) \mid m_i \in \mathbb{N}\}. \tag{4.17}$$

Note

$$x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1} \in \mathcal{B}_{\langle -m_1-m_3-1, m_2+m_3+1\rangle}, \tag{4.18}$$

$$x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1} \in \mathcal{B}_{\langle m_1+m_3+1, -m_2-m_3-1\rangle}. \tag{4.19}$$

Moreover,

$$\mathcal{D}(x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1}) = -m_2 x_{n_1}^{m_1+1} y_{n_1}^{m_2-1} \zeta_1^{m_3+1} = 0 \Longleftrightarrow m_2 = 0 \tag{4.20}$$

and

$$\mathcal{D}(x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1}) = -m_1 x_{n_1+1}^{m_1-1} y_{n_1+1}^{m_2+1} \zeta_2^{m_3+1} = 0 \Longleftrightarrow m_1 = 0 \tag{4.21}$$

by (3.12) and (4.1). Furthermore,

$$x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1} = \frac{\eta^{m_2}(x_{n_1}^{m_1+m_2} \zeta_1^{m_3+1})}{\prod_{r=1}^{m_2}(m_1+r)}, \quad x_{n_1+}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1} = \frac{\eta^{m_1}(y_{n_1+1}^{m_1+m_2} \zeta_2^{m_3+1})}{\prod_{r=1}^{m_1}(m_2+r)} \tag{4.21}$$

by (4.2). Indeed,

$$\eta^{m_1+1}(x_{n_1}^{m_1} \zeta_1^{m_2}) = \eta^{m_1+1}(y_{n_1+1}^{m_1} \zeta_2^{m_2}) = 0 \qquad \text{for } m_1, m_2 \in \mathbb{N}. \tag{4.22}$$

Since $x_{n_1}^{m_1} y_{n_1+1}^{m_2} \in \mathcal{H}_{\langle -m_1, -m_2\rangle}$, (3.45) says that $\eta^m(x_{n_1}^{m_1} y_{n_1+1}^{m_2})$ with $m > 0$ is a singular vector only if $m = m_1 + m_2 + 1$. But $\eta^{m_1+m_2+1}(x_{n_1}^{m_1} y_{n_1+1}^{m_2}) = 0$ by (4.2). Thus any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\text{Span}\{x_{n_1}^{m_1} \zeta_1^{m_2+1}, y_{n_1+1}^{m_1} \zeta_2^{m_2+1}, x_{n_1}^{m_1} y_{n_1+1}^{m_2} \mid m_1, m_2 \in \mathbb{N}\}. \tag{4.23}$$

Since $\mathcal{B}$ is nilpotent with respect to $sl(n,\mathbb{F})_+$ (cf. (2.30)), any nonzero submodule of $\mathcal{B}$ has a singular vector. The above fact implies $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}=\{0\}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such that $\ell_1+\ell_2>0$. Observe that

$$\begin{aligned}
&(x_{n_1}^{m_1}\zeta_1^{m_2}|x_{n_1}^{m_1}\zeta_1^{m_2})\\
=\;&(\sum_{i=0}^{m_2}\binom{m_2}{i}(-1)^i x_{n_1-1}^{m_2-i}x_{n_1}^{m_1+i}y_{n_1-1}^{i}y_{n_1}^{m_2-i}|\sum_{i=0}^{m_2}\binom{m_2}{i}(-1)^i x_{n_1-1}^{m_2-i}x_{n_1}^{m_1+i}y_{n_1-1}^{i}y_{n_1}^{m_2-i})\\
=\;&(-1)^{m_1+m_2}m_2!\sum_{i=0}^{m_2}\binom{m_2}{i}(m_1+i)!(m_2-i))!\neq 0 \qquad (4.24)
\end{aligned}$$

by (3.55). Similarly, $(y_{n_1+1}^{m_1}\zeta_2^{m_2}|y_{n_1}^{m_1}\zeta_2^{m_2})\neq 0$.

Next we assume $n_1=n_2=1$ and $n\geq 3$. By the arguments in the above, a singular vector in $\mathcal{B}$ must be a nonzero weight vector in

$$\mathrm{Span}\{x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}\zeta_2^{m_3+1},\eta^{m_3}(x_{n_1}^{m_1}y_{n_1+1}^{m_2})\mid m_i\in\mathbb{N}\}. \qquad (4.25)$$

Thus any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\mathrm{Span}\{y_{n_1+1}^{m_1}\zeta_2^{m_2+1},x_{n_1}^{m_1}y_{n_1+1}^{m_2}\mid m_1,m_2\in\mathbb{N}\}. \qquad (4.26)$$

The above fact implies $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}=\{0\}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such that $\ell_1+\ell_2>0$ or $\ell_2>0$.

Consider the subcase $n_1=n_2=n-1$ and $n\geq 3$. A singular vector in $\mathcal{B}$ must be a nonzero weight vector in

$$\mathrm{Span}\{x_{n_1}^{m_1}y_{n_1}^{m_2}\zeta_1^{m_3+1},\eta^{m_3}(x_{n_1}^{m_1}y_{n_1+1}^{m_2})\mid m_i\in\mathbb{N}\}. \qquad (4.27)$$

Thus any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\mathrm{Span}\{x_{n_1}^{m_1}\zeta_1^{m_2+1},x_{n_1}^{m_1}y_{n_1+1}^{m_2}\mid m_1,m_2\in\mathbb{N}\}. \qquad (4.28)$$

The above fact implies $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}=\{0\}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such that $\ell_1+\ell_2>0$ or $\ell_1>0$.

Suppose $n_1=n_2=1$ and $n=2$. A singular vector in $\mathcal{B}$ must be a nonzero weight vector in

$$\mathrm{Span}\{\eta^{m_3}(x_1^{m_1}y_2^{m_2})\mid m_i\in\mathbb{N}\}. \qquad (4.29)$$

Thus any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\mathrm{Span}\{x_1^{m_1}y_2^{m_2}\mid m_1,m_2\in\mathbb{N}\}. \qquad (4.30)$$

The above fact implies $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}=\{0\}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such that $\ell_1>0$ or $\ell_2>0$.

Finally, we assume $n_1=n_2=n$. A singular vector in $\mathcal{B}$ must be a nonzero weight vector in

$$\mathrm{Span}\{x_{n_1}^{m_1}y_{n_1}^{m_2}\zeta_1^{m_3}\mid m_i\in\mathbb{N}\}. \qquad (4.31)$$

Thus any singular vector in $\mathcal{H}$ (cf. (3.38)) is a nonzero weight vector in

$$\mathrm{Span}\{x_{n_1}^{m_1}\zeta_1^{m_2}\mid m_1,m_2\in\mathbb{N}\}. \qquad (4.32)$$

The above fact implies $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}=\{0\}$ for $\ell_1,\ell_2\in\mathbb{Z}$ such that $\ell_1+\ell_2>0$. Indeed, all $\mathcal{B}_{\langle -m_1,m_2\rangle}$ with $m_1,m_2\in\mathbb{N}$ are finite-dimensional and completely reducible by Weyl's

Theorem of complete reducibility. Moreover, their irreducible summands are completely determined by (4.31).

By the arguments of (3.55)-(3.70), we obtain:

Theorem 4.1. *Suppose $n_1 = n_2$. Let $\ell_1, \ell_2 \in \mathbb{Z}$ such that $\ell_2 \geq 0$ when $n_1 = n$. Assume $\ell_1 + \ell_2 \leq 0$ and: (a) $\ell_2 \leq 0$ if $n_1 = 1$ and $n \geq 3$; (b) $\ell_1 \leq 0$ if $n_1 = n-1$ and $n \geq 3$; (c) $\ell_1, \ell_2 \leq 0$ when $n_1 = 1$ and $n = 2$. Then $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ is an irreducible highest-weight $sl(n,\mathbb{F})$-module and*

$$\mathcal{B}_{\langle \ell_1,\ell_2\rangle} = \bigoplus_{m=0}^{\infty} \eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle}) \tag{4.33}$$

is an orthogonal decomposition of irreducible submodules. The symmetric bilinear form restricted to $\eta^m(\mathcal{H}_{\langle \ell_1-m,\ell_2-m\rangle})$. In particular, $\mathcal{B}_{\langle \ell_1,\ell_2\rangle} = \mathcal{H}_{\langle \ell_1,\ell_2\rangle} \oplus \eta(\mathcal{B}_{\langle \ell_1-1,\ell_2-1\rangle})$. If the conditions fails, $\mathcal{H}_{\langle \ell_1,\ell_2\rangle} = \{0\}$. When $n_1 = n_2 = n$, all the above irreducible modules are of finite-dimensional.

Suppose $n_1 < n-1$. Let $m_1, m_2 \in \mathbb{N}$. The subspace $\mathcal{H}_{\langle -m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+1}^{m_2}$ of weight $m_1(1-\delta_{1,n_1})\lambda_{n_1-1} - (m_1+m_2+2)\lambda_{n_1} + m_2\lambda_{n_1+1}$. If $n_1 \geq 2$, the subspace $\mathcal{H}_{\langle -m_1-m_2-1,m_2+1\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$ of weight $(m_2+1)\lambda_{n_1-2} - m_1\lambda_{n_1-1} - (m_1+m_2+3)\lambda_{n_1}$. The subspace $\mathcal{H}_{\langle m_1+1,-m_2-m_1-1\rangle}$ has a highest-weight vector $y_{n_1+1}^{m_2}\zeta_2^{m_1+1}$ of weight $-(m_1+m_2+3)\lambda_{n_1} + m_2\lambda_{n_1+1} - (m_1+1)(1-\delta_{n_1,n-2})\lambda_{n_1+2}$.

Consider $n_1 = n-1$. The subspace $\mathcal{H}_{\langle -m_1,-m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+1}^{m_2}$ of weight $m_1(1-\delta_{n,2})\lambda_{n-2} - (m_1+m_2+2)\lambda_{n-1}$. If $n \geq 3$, the subspace $\mathcal{H}_{\langle -m_1-m_2-1,m_2+1\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$ of weight $(m_2+1)(1-\delta_{n,3})\lambda_{n-3} - m_1\lambda_{n-2} - (m_1+m_2+3)\lambda_{n-1}$.

Assume $n_1 = n$. The subspace $\mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$ has a highest-weight vector $x_n^{m_1}\zeta_1^{m_2}$ of weight $m_2(1-\delta_{n,2})\lambda_{n-2} + m_1\lambda_{n-1}$.

Now we want to find an explicit expression for $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}$ when it is irreducible. Set

$$\mathcal{G}' = \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} \mathbb{F}E_{j,i}, \tag{4.34}$$

$$\hat{\mathcal{G}} = H + \sum_{r,s\in\overline{1,n_1} \text{ or } r,s\in\overline{n_1+1,n}; r\neq s} \mathbb{F}E_{r,s} + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} \mathbb{F}E_{i,j}. \tag{4.35}$$

Then $\mathcal{G}'$ and $\hat{\mathcal{G}}$ are Lie subalgebras of $sl(n,\mathbb{F})$ and $sl(n,\mathbb{F}) = \mathcal{G}' \oplus \hat{\mathcal{G}}$. By PBW Theorem, $U(sl(n,\mathbb{F})) = U(\mathcal{G}')U(\hat{\mathcal{G}})$. According to (1.6)-(1.8),

$$E_{r,s}|_{\mathcal{B}} = -x_s\partial_{x_r} - y_s\partial_{y_r}, \qquad E_{p,q}|_{\mathcal{B}} = x_p\partial_{x_q} + y_p\partial_{y_q}, \tag{4.36}$$

$$E_{r,p}|_{\mathcal{B}} = \partial_{x_r}\partial_{x_p} - \partial_{y_r}\partial_{y_p}, \qquad E_{p,r}|_{\mathcal{B}} = -x_rx_p + y_ry_p \tag{4.37}$$

for $r,s \in \overline{1,n_1}$ and $p,q \in \overline{n_1+1,n}$.

First we assume $n_1 < n$. For $m_1, m_2 \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{H}_{\langle -m_1,-m_2\rangle} &= U(sl(n,\mathbb{F}))(x_{n_1}^{m_1}y_{n_1+1}^{m_2}) = U(\mathcal{G}')U(\hat{\mathcal{G}})(x_{n_1}^{m_1}y_{n_1+1}^{m_2}) \\ &= \text{Span}\{[\prod_{r=1}^{n_1} x_r^{l_r}][\prod_{s=1}^{n-n_1} y_{n_1+s}^{k_s}][\prod_{r=1}^{n_1}\prod_{s=1}^{n-n_1}(x_r x_{n_1+s} - y_r y_{n_1+s})^{l_{r,s}}] \\ &\qquad \mid l_r, k_s, l_{r,s} \in \mathbb{N}; \sum_{r=1}^{n_1} l_r = m_1; \sum_{s=1}^{n-n_1} k_s = m_2\} \end{aligned} \tag{4.38}$$

by (4.36) and (4.37). Furthermore, we assume $n_1 > 1$. We let

$$\begin{aligned}&\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} \\ &= \text{Span}\{[\prod_{r=1}^{n_1} x_r^{l_r}][\prod_{1\le p<q\le n_1}(x_p y_q - x_q y_p)^{k_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=1}^{n-n_1}(x_r x_{n_1+s} - y_r y_{n_1+s})^{l_{r,s}}] \\ &\qquad \mid l_r, k_{p,q}, l_{r,s} \in \mathbb{N}; \sum_{r=1}^{n_1} l_r = m_1; \sum_{1\le p<q\le n_1} k_{p,q} = m_2\}. \end{aligned} \tag{4.39}$$

By (3.38), (3.40) and (4.1), we have $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} \subset \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$. Moreover, (4.37) and (4.38) yield

$$\mathcal{H}_{\langle -m_1-m_2,m_2\rangle} = U(sl(n,\mathbb{F}))(x_{n_1}^{m_1}\zeta_1^{m_2}) = U(\mathcal{G}')U(\hat{\mathcal{G}})(x_{n_1}^{m_1}\zeta_1^{m_2}) \subset \mathcal{H}'_{\langle -m_1-m_2,m_2\rangle}. \tag{4.40}$$

Thus $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$. Symmetrically, if $n_1 = n_2 < n-1$,

$$\begin{aligned}\mathcal{H}_{\langle m_2,-m_1-m_2\rangle} &= \text{Span}\{[\prod_{n_1+1\le p<q\le n}(x_p y_q - x_q y_p)^{k_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=n_1+1}^{n}(x_r x_s - y_r y_s)^{l_{r,s}}] \\ &\times[\prod_{r=1}^{n-n_1} y_{n_1+r}^{l_r}] \mid l_r, k_{p,q}, l_{r,s} \in \mathbb{N}; \sum_{r=1}^{n_1} l_r = m_1; \sum_{n_1+1\le p<q\le n} k_{p,q} = m_2\}. \end{aligned} \tag{4.41}$$

When $n_1 = n_2 = n$, by the arguments between (4.39) and(4.40),

$$\begin{aligned}\mathcal{H}_{\langle -m_1-m_2,m_2\rangle} &= \text{Span}\{[\prod_{r=1}^{n} x_r^{l_r}][\prod_{1\le p<q\le n}(x_p y_q - x_q y_p)^{k_{p,q}}] \\ &\qquad \mid l_r, k_{p,q} \in \mathbb{N}; \sum_{r=1}^{n} l_r = m_1; \sum_{1\le p<q\le n} k_{p,q} = m_2\}, \end{aligned} \tag{4.42}$$

which is of finite-dimensional.

5 The $o(2n,\mathbb{F})$-Variation

Recall that $\mathcal{B} = \mathbb{F}[x_1,...,x_n,y_1,...,y_n]$ and the representation of $o(2n,\mathbb{F})$ on $\mathcal{B}$ defined by (1.14)-(1.16). It is easy to verify

$$T\xi = \xi T \ \text{ on } \ \mathcal{B} \qquad \text{for } \ \xi \in o(2n,\mathbb{F}); T = \flat, \flat', \mathcal{D}, \eta \tag{5.1}$$

by (1.9), (1.10), (2.13) and (3.4). Recall the notions $\mathcal{B}_{\langle k\rangle}$ and $\mathcal{H}_{\langle k\rangle}$ defined in (1.17). The $\mathcal{B} = \bigoplus_{k\in\mathbb{Z}}\mathcal{B}_{\langle k\rangle}$ forms a $\mathbb{Z}$-graded algebra and

$$\mathcal{H}_{\langle k\rangle} = \bigoplus_{\ell_1,\ell_2\in\mathbb{Z};\ell_1+\ell_2=k}\mathcal{H}_{\langle \ell_1,\ell_2\rangle}. \tag{5.2}$$

Moreover, $\mathcal{B}_{\langle k\rangle}$ and $\mathcal{H}_{\langle k\rangle}$ are $o(2n,\mathbb{F})$-submodules. Recall $\mathcal{K}=\sum_{i,j=1}^n\mathbb{F}(E_{i,j}-E_{n+j,n+i})$.

Theorem 5.1. *For any $n_1-n_2+1-\delta_{n_1,n_2}\geq k\in\mathbb{Z}$, $\mathcal{H}_{\langle k\rangle}$ is an irreducible $o(2n,\mathbb{F})$-submodule and*

$$\mathcal{B}_{\langle k\rangle}=\bigoplus_{i=0}^{\infty}\eta^i(\mathcal{H}_{\langle k-2i\rangle}) \tag{5.3}$$

is an orthogonal decomposition of irreducible submodules. In particular, $\mathcal{B}_{\langle k\rangle}=\mathcal{H}_{\langle k\rangle}\oplus\eta(\mathcal{B}_{\langle k-2\rangle})$. Moreover, the bilinear form $(\cdot|\cdot)$ restricted to $\eta^i(\mathcal{H}_{\langle k-2i\rangle})$ is nondegenerate. Furthermore, $\mathcal{H}_{\langle k\rangle}$ has a basis

$$\begin{aligned}&\{\sum_{i=0}^{\infty}\frac{(x_{n_1+1}y_{n_1+1})^i(\mathcal{D}-\partial_{x_{n_1+1}}\partial_{y_{n_1+1}})^i(x^\alpha y^\beta)}{\prod_{r=1}^i(\alpha_{n_1+1}+r)(\beta_{n_1+1}+r)}\mid\alpha,\beta\in\mathbb{N}^{\,n};\\&\alpha_{n_1+1}\beta_{n_1+1}=0;-\sum_{i=1}^{n_1}\alpha_i+\sum_{r=n_1+1}^{n}\alpha_r+\sum_{i=1}^{n_2}\beta_i-\sum_{r=n_2+1}^{n}\beta_r=k\}\end{aligned} \tag{5.4}$$

when $n_1<n_2$. The module $\mathcal{H}_{\langle k\rangle}$ under the assumption is of highest-weight type only if $n_2=n$, in which case $x_{n_1}^{-k}$ is a highest-weight vector with weight $-k\lambda_{n_1-1}+(k-1)\lambda_{n_1}+[(k-1)\delta_{n_1,n-1}-2k\delta_{n_1,n}]\lambda_n$. When $n_1=n_2=n$, all the irreducible modules $\mathcal{H}_{\langle k\rangle}$ with $0\geq k\in\mathbb{Z}$ are of $(\mathcal{G},\mathcal{K})$-type.

Proof. Let $n_1-n_2+1\geq k\in\mathbb{Z}$. Note $sl(n,\mathbb{F})|_{\mathcal{B}}$ is a subalgebra of $o(2n,\mathbb{F})|_{\mathcal{B}}$. Suppose $n_1+1<n_2<n$. By (5.2), Theorem 3.1 and the paragraph below, the $sl(n,\mathbb{F})$-singular vectors in $\mathcal{H}_{\langle k\rangle}$ are: for $m_1,m_2\in\mathbb{N}$,

$$x_{n_1}^{m_1}y_{n_2+1}^{m_2}\qquad\text{with }-(m_1+m_2)=k, \tag{5.5}$$

$$x_{n_1+1}^{m_1}y_{n_2+1}^{m_2}\qquad\text{with }m_1-m_2=k, \tag{5.6}$$

$$x_{n_1}^{m_1}y_{n_2}^{m_2}\qquad\text{with }-m_1+m_2=k. \tag{5.7}$$

Note

$$(E_{n+n_2+1,n_1}-E_{n+n_1,n_2+1})|_{\mathcal{B}}=-x_{n_1}\partial_{y_{n_2+1}}-y_{n_1}\partial_{x_{n_2+1}} \tag{5.8}$$

by (1.16). So

$$(E_{n+n_2+1,n_1}-E_{n+n_1,n_2+1})^{m_2}(x_{n_1}^{m_1}y_{n_2+1}^{m_2})=(-1)^{m_2}m_2!x_{n_1}^{-k} \tag{5.9}$$

for the vectors in (5.5). Moreover,

$$(E_{n+n_2+1,n_1+1}-E_{n+n_1+1,n_2+1})|_{\mathcal{B}}=\partial_{x_{n_1+1}}\partial_{y_{n_2+1}}-y_{n_1+1}\partial_{x_{n_2+1}} \tag{5.10}$$

again by (1.16), which implies

$$(E_{n+n_2+1,n_1+1}-E_{n+n_1+1,n_2+1})^{m_2}(x_{n_1+1}^{m_1}y_{n_2+1}^{m_2})=m_1![\prod_{r=0}^{m_1-1}(m_2-r)]y_{n_2+1}^{-k} \tag{5.11}$$

for the vectors in (5.6). Furthermore,

$$(E_{n_1,n+n_2}-E_{n_2,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_2}}-x_{n_2}\partial_{y_{n_1}} \tag{5.12}$$

by (1.15), which implies

$$(E_{n_1,n+n_2}-E_{n_2,n+n_1})^{m_2}(x_{n_1}^{m_1}y_{n_2}^{m_2})=m_2![\prod_{r=0}^{m_2-1}(m_1-r)]x_{n_1}^{-k} \tag{5.13}$$

for the vectors in (5.7).

On the other hand,

$$(E_{n_1,n+n_2+1}-E_{n_2+1,n+n_1})|_{\mathcal{B}}=-y_{n_2+1}\partial_{x_{n_1}}-x_{n_2+1}\partial_{y_{n_1}} \tag{5.14}$$

by (1.15), which implies

$$(E_{n_1,n+n_2+1}-E_{n_2+1,n+n_1})^{m_2}(x_{n_1}^{-k})=(-1)^{m_2}[\prod_{r=0}^{m_2-1}(-k-r)]x_{n_1}^{m_1}y_{n_2+1}^{m_2} \tag{5.15}$$

for the vectors in (5.5). Moreover,

$$(E_{n_1+1,n+n_2+1}-E_{n_2+1,n+n_1+1})|_{\mathcal{B}}=-x_{n_1+1}y_{n_2+1}-x_{n_2+1}\partial_{y_{n_1+1}} \tag{5.16}$$

by (1.15), which implies

$$(E_{n_1+1,n+n_2+1}-E_{n_2+1,n+n_1+1})^{m_2}(y_{n_2+1}^{-k})=(-1)^{m_2}x_{n_1}^{m_1}y_{n_2+1}^{m_2} \tag{5.17}$$

for the vectors in (5.6). Furthermore,

$$(E_{n+n_2,n_1}-E_{n+n_1,n_2})|_{\mathcal{B}}=-x_{n_1}y_{n_2}-y_{n_1}\partial_{x_{n_2}} \tag{5.18}$$

by (1.16), which implies

$$(E_{n+n_2,n_1}-E_{n+n_1,n_2})^{m_2}(x_{n_1}^{-k})=(-1)^{m_2}x_{n_1}^{m_1}y_{n_2}^{m_2} \tag{5.19}$$

for the vectors in (5.7). Thus for any two vectors in (5.5)-(5.7), there exists an element in the universal enveloping algebra $U(o(2n,\mathbb{F}))$ which carries one to another. On the other hand, the vectors in (5.5)-(5.7) have distinct weights (see the paragraph below Theorem 3.1). Thus any nonzero submodule of $\mathcal{H}_{\langle k\rangle}$ must contain one of the vectors in (5.5)-(5.7). Hence all the vectors in (5.5)-(5.7) are in the submodule by (5.8)-(5.19). Therefore, the submodule must be equal to $\mathcal{H}_{\langle k\rangle}$, that is, $\mathcal{H}_{\langle k\rangle}$ is irreducible. By (5.16) and (5.18), $\mathcal{H}_{\langle k\rangle}$ is not of highest-weight type. The equation (5.3) follows from Theorem 3.1 and (5.2).

Assume $n_1+1=n_2<n$. By Theorem 3.2 and the paragraph below, the $sl(n,\mathbb{F})$-singular vectors in $\mathcal{H}_{\langle k\rangle}$ are those in (5.5)-(5.7). So the theorem holds. Suppose $n_1<n_2=n$. By Theorems 3.1, 3.2 and the paragraph below them, the $sl(n,\mathbb{F})$-singular vectors in $\mathcal{H}_{\langle k\rangle}$ are those in (5.7). Expressions (5.13) and (5.19) imply the conclusions in the theorem.

Recall

$$\zeta_1=x_{n_1-1}y_{n_1}-x_{n_1}y_{n_1-1},\quad \zeta_2=x_{n_2+1}y_{n_2+2}-x_{n_2+2}y_{n_2}. \tag{5.20}$$

In the case $n_1=n_2<n-1$, Theorem 4.1 tell us that the $sl(n,\mathbb{F})$-singular vectors in $\mathcal{H}_{\langle k\rangle}$ are those in (5.5) and

$$x_{n_1}^{-k}\zeta_1^{m+1}\qquad \text{for }\ m\in\mathbb{N}, \tag{5.21}$$

$$y_{n_1+1}^{-k}\zeta_2^{m+1}\qquad \text{for}\;\; m\in\mathbb{N}. \tag{5.22}$$

Again all the singular vectors have distinct weights. If N is a nonzero submodule of $\mathcal{H}_{\langle k\rangle}$, then N must contain one of the above $sl(n,\mathbb{F})$-singular vectors. If N contains a singular vector in (5.5), then $x_{n_1}^{-k}\in N$ by (5.9). Suppose $x_{n_1}^{-k}\zeta_1^{m+1}\in N$ for some $m\in\mathbb{N}$. Note

$$(E_{n_1-1,n+n_1}-E_{n_1,n+n_1-1})|_{\mathcal{B}}=\partial_{x_{n_1-1}}\partial_{y_{n_1}}-\partial_{x_{n_1}}\partial_{y_{n_1-1}} \tag{5.22}$$

by (1.15). Thus

$$\begin{aligned}
& (E_{n_1-1,n+n_1}-E_{n_1,n+n_1-1})^{m+1}(x_{n_1}^{-k}\zeta_1^{m+1})\\
= & \left[\sum_{r=0}^{m+1}(-1)^r\binom{m+1}{r}(\partial_{x_{n_1-1}}\partial_{y_{n_1}})^{m+1-r}(\partial_{x_{n_1}}\partial_{y_{n_1-1}})^r\right]\\
& \left[\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s}(x_{n_1-1}y_{n_1})^{m+1-s}x_{n_1}^{-k+s}y_{n_1-1}^{s}\right]\\
= & \left(\sum_{r=0}^{m+1}\binom{m+1}{r}^2[(m+1-r!)]^2r![\prod_{i=1}^{r}(-k+i)]\right)x_{n_1}^{-k}\\
= & [(m+1)!]^2\left(\sum_{r=0}^{m+1}\binom{-k+r}{r}\right)x_{n_1}^{-k}\in N.
\end{aligned} \tag{5.23}$$

So we have $x_{n_1}^{-k}\in N$ again. Symmetrically, it holds if $y_{n_1+1}^{-k}\zeta_2^{m+1}\in N$ for some $m\in\mathbb{N}$. Therefore, we always have $x_{n_1}^{-k}\in N$.

According to (5.15), N contains all the singular vectors in (5.5). Observe

$$(E_{n+n_1-1,n_1}-E_{n+n_1,n_1-1})|_{\mathcal{B}}=\zeta_1,\;\;(E_{n_1+2,n+n_1+1}-E_{n_1+1,n+n_1+2})|_{\mathcal{B}}=\zeta_2 \tag{5.24}$$

as multiplication operators on $\mathcal{B}$ by (1.15) and (1.16). Thus

$$(E_{n+n_1-1,n_1}-E_{n+n_1,n_1-1})^{m+1}(x_{n_1}^{-k})=x_{n_1}^{-k}\zeta_1^{m+1}, \tag{5.25}$$

$$(E_{n_1+2,n+n_1+1}-E_{n_1+1,n+n_1+2})^{m+1}(x_{n_1}^{-k})=x_{n_1}^{-k}\zeta_2^{m+1}\in N. \tag{5.26}$$

Thus N contains all the $sl(n,\mathbb{F})$-singular vectors in $\mathcal{H}_{\langle k\rangle}$, which implies that it contains all $\mathcal{H}_{\langle \ell_1,\ell_2\rangle}\subset\mathcal{H}_{\langle k\rangle}$. So $N=\mathcal{H}_{\langle k\rangle}$, that is, $\mathcal{H}_{\langle k\rangle}$ is an irreducible $o(2n,\mathbb{F})$-module, which is of $(\mathcal{G},\mathcal{K})$-type if $n_1=n_2=n$ by (5.2). The basis (5.4) is obtained by (3.72) and (5.2). $\square$

Finally, we want to find an expression for $\mathcal{H}_{\langle k\rangle}$ for $0\geq k\in\mathbb{Z}$ when $n_1=n_2$.

First we assume $n_1=n_2=1$ and $n\geq 3$. According to (4.26), (4.38) and (4.41)

$$\begin{aligned}
& \mathcal{H}_{\langle -k\rangle}\\
= & \;\text{Span}\{[\prod_{r=2}^{n}y_r^{\hat{l}_r}][\prod_{2\leq p<q\leq n}(x_py_q-x_qy_p)^{\hat{k}_{p,q}}][\prod_{s=2}^{n}(x_1x_s-y_1y_s)^{\hat{l}_s}],x_1^l[\prod_{s=2}^{n}y_s^{k_s}]\\
& \times[\prod_{s=2}^{n}(x_1x_s-y_1y_s)^{l_s}]\mid l,k_s,l_s,\hat{l},\hat{k}_{p,q},\hat{l}_s\in\mathbb{N};l+\sum_{s=2}^{n}k_s=\sum_{r=2}^{n}\hat{l}_r=k\}.
\end{aligned} \tag{5.27}$$

Next we consider the subcase $1 < n_1 = n_2 < n-1$. By (4.23), (4.38), (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$) and (4.41), we have

$$\begin{aligned} & \mathcal{H}_{\langle -k\rangle} \\ = \ & \mathrm{Span}\{[\prod_{r=1}^{n_1} x_r^{l'_r}][\prod_{1\leq p<q\leq n_1}(x_py_q-x_qy_p)^{k'_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=n_1+1}^{n}(x_rx_s-y_ry_s)^{l'_{r,s}}], \\ & [\prod_{r=1}^{n-n_1} y_{n_1+r}^{\hat{l}_r}][\prod_{n_1+1\leq p<q\leq n}(x_py_q-x_qy_p)^{\hat{k}_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=n_1+1}^{n}(x_rx_s-y_ry_s)^{\hat{l}_{r,s}}], \\ & [\prod_{r=1}^{n_1} x_r^{l_r}][\prod_{s=1}^{n-n_1} y_{n_1+s}^{k_s}][\prod_{r=1}^{n_1}\prod_{s=1}^{n-n_1}(x_rx_{n_1+s}-y_ry_{n_1+s})^{l_{r,s}}] \mid l_r,k_s,l_{r,s},l'_r,k'_{p,q}, \\ & l'_{r,s},\hat{l}_r,\hat{k}_{p,q},\hat{l}_{r,s}\in\mathbb{N}; \sum_{r=1}^{n_1} l_r+\sum_{s=1}^{n-n_1}k_s=\sum_{r=1}^{n_1}l'_r=\sum_{r=1}^{n-n_1}\hat{l}_r=k\}. \end{aligned} \tag{5.28}$$

Consider the subcase $n_1 = n_2 = n-1$ and $n \geq 3$. By (4.28), (4.38) and (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$), we obtain

$$\begin{aligned} & \mathcal{H}_{\langle -k\rangle} \\ = \ & \mathrm{Span}\{[\prod_{r=1}^{n-1} x_r^{l'_r}][\prod_{1\leq p<q\leq n-1}(x_py_q-x_qy_p)^{k'_{p,q}}][\prod_{r=1}^{n-1}(x_rx_n-y_ry_n)^{\bar{l}'_r}], [\prod_{r=1}^{n-1} x_r^{l_r}]y_n^{\hat{k}} \\ & \times[\prod_{r=1}^{n-1}(x_rx_n-y_ry_n)^{\bar{l}_r}] \mid l_r,\hat{k},\bar{l}_r,l'_r,k'_{p,q},\bar{l}'_r\in\mathbb{N}; \sum_{r=1}^{n-1}l_r+\hat{k}=\sum_{r=1}^{n-1}l'_r=k\}. \end{aligned} \tag{5.29}$$

Suppose $n_1 = n_2 = 1$ and $n = 2$. According to (4.30) and (4.38),

$$\mathcal{H}_{\langle -k\rangle} = \mathrm{Span}\{[x_1^r y_2^s (x_1x_2-y_1y_2)^l \mid r,s,l\in\mathbb{N}; r+s=k\}. \tag{5.30}$$

Finally we assume $n_1 = n_2 = n$. By (4.32) and (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$),

$$\mathcal{H}_{\langle -k\rangle} = \mathrm{Span}\{\prod_{r=1}^{n} x_r^{l_r}][\prod_{1\leq p<q\leq n}(x_py_q-x_qy_p)^{k_{p,q}}] \mid l_r,k_{p,q}\in\mathbb{N}; \sum_{r=1}^{n} l_r = k\}, \tag{5.31}$$

whose $(\mathcal{G},\mathcal{K})$-module structure is given by $\mathcal{H}_{\langle -k\rangle} = \bigoplus_{m=0}^{\infty}\mathcal{H}_{\langle -k-m,m\rangle}$ with $\mathcal{H}_{\langle -k-m,m\rangle}$ given in (4.42).

6 The $o(2n+1,\mathbb{F})$-Variation

Recall

$$o(2n+1,\mathbb{F}) = o(2n,\mathbb{F}) \oplus \bigoplus_{i=1}^{n}[\mathbb{F}(E_{0,i}-E_{n+i,0})+\mathbb{F}(E_{0,n+i}-E_{i,0})] \tag{6.1}$$

and $\mathcal{B}' = \mathbb{F}[x_0,x_1,...,x_n,y_1,...,y_n]$.

Fix $n_1,n_2 \in \overline{1,n}$ such that $n_1 \leq n_2$. The representation of $o(2n+1,\mathbb{F})$ on $\mathcal{B}'$ by the differential operators in (1.14)-(1.16), (1.19) and (1.20). Recall $\mathcal{B}'_{\langle k\rangle} = \sum_{i=0}^{\infty}\mathcal{B}_{\langle k-i\rangle}x_0^i$. Then all $\mathcal{B}'_{\langle k\rangle}$ with $k\in\mathbb{Z}$ are $o(2n+1,\mathbb{F})$-submodules and $\mathcal{B}' = \bigoplus_{k\in\mathbb{Z}}\mathcal{B}'_{\langle k\rangle}$ forms a $\mathbb{Z}$-graded algebra. Moreover, the variated Laplace operator $\mathcal{D}' = \partial_{x_0}^2 + 2\mathcal{D}$ by (1.21) and its dual $\eta' = x_0^2 + 2\eta$ by (1.22).

A straightforward verification shows

$$\mathcal{D}'\xi=\xi\mathcal{D}',\ \xi\eta'=\eta'\xi \text{ on } \mathcal{B}' \qquad \text{for } \xi\in o(2n+1,\mathbb{F}). \tag{6.2}$$

As in the introduction, $\mathcal{H}'_{\langle k\rangle}=\{f\in\mathcal{B}'\langle k\rangle\mid \mathcal{D}'(f)=0\}$. According to (6.2), $\mathcal{H}'_{\langle k\rangle}$ is an $o(2n+1,\mathbb{F})$-submodule. By Lemma 2.1 with $T_1=\partial_{x_0}^2$, $T_1^-=\int_{(x_0)}^{(2)}$ (cf. (2.6) and (2.7)) and $T_2=2\mathcal{D}$, we obtain

$$\mathcal{H}'_{\langle k\rangle}=\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\mathcal{B}_{\langle k\rangle})\oplus\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\mathcal{B}_{\langle k-1\rangle}). \tag{6.3}$$

Recall $\mathcal{K}=\sum_{i,j=1}^{n}\mathbb{F}(E_{i,j}-E_{n+j,n+i})$.

Theorem 6.1. *For any $n_1-n_2+1-\delta_{n_1,n_2}\geq k\in\mathbb{Z}$, $\mathcal{H}'_{\langle k\rangle}$ is an irreducible $o(2n+1,\mathbb{F})$-submodule and*

$$\mathcal{B}'_{\langle k\rangle}=\bigoplus_{i=0}^{\infty}(\eta')^i(\mathcal{H}'_{\langle k-2i\rangle}) \tag{6.4}$$

is an orthogonal decomposition of irreducible submodules. In particular, $\mathcal{B}'_{\langle k\rangle}=\mathcal{H}'_{\langle k\rangle}\oplus\eta'(\mathcal{B}'_{\langle k-2\rangle})$. Moreover, the bilinear form $(\cdot|\cdot)$ restricted to $(\eta')^i(\mathcal{H}'_{\langle k-2i\rangle})$ is nondegenerate. Furthermore, $\mathcal{H}_{\langle k\rangle}$ has a basis

$$\{\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+\iota}\mathcal{D}^i(x^\alpha y^\beta)}{(2i+\iota)!}\mid \alpha,\beta\in\mathbb{N}^{\,n};\iota=0,1;$$
$$-\sum_{i=1}^{n_1}\alpha_i+\sum_{r=n_1+1}^{n}\alpha_r+\sum_{i=1}^{n_2}\beta_i-\sum_{r=n_2+1}^{n}\beta_r=k-\iota\}. \tag{6.5}$$

The module $\mathcal{H}'_{\langle k\rangle}$ under the assumption is of highest-weight type only if $n_2=n$, in which case $x_{n_1}^{-k}$ is a highest-weight vector with weight $-k\lambda_{n_1-1}+(k-1)\lambda_{n_1}+[(k-1)\delta_{n_1,n-1}-2k\delta_{n_1,n}]\lambda_n$. When $n_1=n_2=n$, all the irreducible modules $\mathcal{H}_{\langle k\rangle}$ with $0\geq k\in\mathbb{Z}$ are of $(\mathcal{G},\mathcal{K})$-type.

Proof. Observe that

$$(x_0^r x^\alpha y^\beta|x_0^s x^{\alpha_1}y^{\beta_1})=\delta_{r,s}\delta_{\alpha,\alpha_1}\delta_{\beta,\beta_1}(-1)^{\sum_{i=1}^{n_1}\alpha_i+\sum_{r=n_2+1}^{n}\beta_r}r!\alpha!\beta! \tag{6.6}$$

for $r,s\in\mathbb{N}$ and $\alpha,\beta,\alpha_1,\beta_1\in\mathbb{N}^{\,n}$. By (1.21) and (1.22),

$$(\mathcal{D}'(f)|g)=(f|\eta'(g)) \qquad \text{for } f,g\in\mathcal{B}'. \tag{6.7}$$

Let $n_1-n_2+1\geq k\in\mathbb{Z}$. First by (5.3) and (6.3),

$$\mathcal{H}'_{\langle k\rangle}=\bigoplus_{r=0}^{\infty}\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^r(\mathcal{H}_{\langle k-2r\rangle}))\oplus\bigoplus_{s=0}^{\infty}\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^s(\mathcal{H}_{\langle k-2s-1\rangle})). \tag{6.8}$$

Let N be a nonzero submodule of $\mathcal{H}'_{\langle k\rangle}$. By comparing weights and the arguments in (5.5)-(5.13) and (5.21)-(5.23), we have

$$\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^{m_1}(x_{n_1}^{-k+2m_1}))\in N \tag{6.9}$$

for some $m_1\in\mathbb{N}$ or

$$\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_2}(x_{n_1}^{-k+2m_2+1}))\in N \tag{6.10}$$

for some $m_2\in\mathbb{N}$.

Note

$$(E_{n_1,0}-E_{0,n+n_1})=\partial_{x_0}\partial_{x_{n_1}}-x_0\partial_{y_{n_1}} \tag{6.11}$$

by (1.19) and (1.20). Recall

$$\mathcal{D}=-\sum_{i=1}^{n_1}x_i\partial_{y_i}+\sum_{r=n_1+1}^{n_2}\partial_{x_r}\partial_{y_r}-\sum_{s=n_2+1}^{n}y_s\partial_{x_s} \tag{6.12}$$

and

$$\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+\sum_{r=n_1+1}^{n_2}x_ry_r+\sum_{s=n_2+1}^{n}x_s\partial_{y_s}. \tag{6.13}$$

Then (3.44) gives

$$\begin{aligned}
&(E_{n_1,0}-E_{0,n+n_1})\left[\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^{m_1}(x_{n_1}^{-k+2m_1}))\right]\\
=&\left(\sum_{i=0}^{\infty}\frac{(i+1)m_1(-k+2m_1)(-2)^{i+1}x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_1-1}(x_{n_1}^{-k+2m_1-1}))\\
&+\left(\sum_{i=0}^{\infty}\frac{(-k+2m_1)(-2)^{i+1}x_0^{2i+1}\mathcal{D}^{i+1}}{(2i+1)!}\right)(\eta^{m_1}(x_{n_1}^{-k+2m_1-1}))\\
&-m_1(-k+2m_1)\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i+1}\mathcal{D}^i}{(2i)!}\right)(\eta^{m_1-1}(x_{n_1}^{-k+2m_1-1}))\\
=&\;m_1(-k+2m_1)\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_1-1}(x_{n_1}^{-k+2m_1-1}))\\
&+m_1(-k+2m_1)(m_1-k+n_1-n_2)\left(\sum_{i=0}^{\infty}\frac{(-2)^{i+1}x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_1-1}(x_{n_1}^{-k+2m_1-1}))\\
=&\;m_1(-k+2m_1)(2m_1-2k+2n_1-2n_2+1)\\
&\times\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_1-1}(x_{n_1}^{-k+2m_1-1})).
\end{aligned} \tag{6.14}$$

Moreover,

$$\begin{aligned}
&(E_{n_1,0}-E_{0,n+n_1})\left[\left(\sum_{i=0}^{\infty}\frac{(-2)^ix_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_2}(x_{n_1}^{-k+2m_2+1}))\right]\\
=&\left(\sum_{i=0}^{\infty}\frac{im_2(-k+2m_2+1)(-2)^ix_0^{2i}\mathcal{D}^{i-1}}{(2i)!}\right)(\eta^{m_2-1}(x_{n_1}^{-k+2m_2}))\\
&+\left(\sum_{i=0}^{\infty}\frac{(-k+2m_2+1)(-2)^ix_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^{m_2}(x_{n_1}^{-k+2m_2}))
\end{aligned}$$

$$
\begin{aligned}
&-m_2(-k+2m_2+1)\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+2}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{m_2-1}(x_{n_1}^{-k+2m_2}))\\
= &\;(-k+2m_2+1)\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^{m_2}(x_{n_1}^{-k+2m_2})). \qquad (6.15)
\end{aligned}
$$

Note $k\leq 0$ by our assumption. Using (6.9), (6.10), (6.14), (6.15) and induction, we obtain $x_{n_1}^{-k}\in N$.

Observe

$$
(E_{n+n_1,0}-E_{0,n_1})|_{\mathcal{B}'}=x_0x_{n_1}+y_{n_1}\partial_{x_0} \qquad (6.16)
$$

by (1.19) and (1.20). Then

$$
(E_{n+n_1,0}-E_{0,n_1})^m(x_{n_1}^{-k})=x_0^m x_{n_1}^{-k+m}+P_m\in N, \qquad (6.17)
$$

where the degree of P_m with respect to x_0 is less than m. For any $f\in\mathcal{H}_{\langle k-2m\rangle}$ and $g\in\mathcal{H}_{\langle k-2m-1\rangle}$, (3.44) and (5.2) says that

$$
\begin{aligned}
&\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^m(f))\\
= &\sum_{i=0}^{m}\frac{2^i x_0^{2i}\prod_{r=1}^{i}(m-r)(m-k+n_1-n_2+1+r)}{(2i)!}\eta^{m-i}(f) \qquad (6.18)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+1}\mathcal{D}^i}{(2i)!}\right)(\eta^m(g))\\
= &\sum_{i=0}^{m}\frac{2^i x_0^{2i+1}\prod_{r=1}^{i}(m-r)(m-k+n_1-n_2+2+r)}{(2i+1)!}\eta^{m-i}(g). \qquad (6.19)
\end{aligned}
$$

This shows that if x_0^m is the highest x_0-power of a nonzero element in $\mathcal{H}'_{\langle k\rangle}$, then its coefficient must be in $\mathcal{H}_{\langle k-m\rangle}$ by (6.8).

On the other hand, (6.17) implies that

$$
\text{the coefficients of } x_0^m \text{ in } U(o(2n,\mathbb{F}))[(E_{n+n_1,0}-E_{0,n_1})^m(x_{n_1}^{-k})]=\mathcal{H}_{\langle k-m\rangle}, \qquad (6.20)
$$

because it is an irreducible $o(2n,\mathbb{F})$-module by Theorem 5.1. By induction on m, we can prove

$$
\mathcal{H}'_{\langle k\rangle}\subset\sum_{m=0}^{\infty}U(o(2n,\mathbb{F}))[(E_{n+n_1,0}-E_{0,n_1})^m(x_{n_1}^{-k})]\subset N. \qquad (6.21)
$$

Thus $N=\mathcal{H}'_{\langle k\rangle}$. This shows that $\mathcal{H}'_{\langle k\rangle}$ is irreducible. Since the bilinear form $(\cdot|\cdot)$ restricted to $\mathcal{H}_{\langle k\rangle}\subset\mathcal{H}'_{\langle k\rangle}$ is nondegenerate, the irreducibility of $\mathcal{H}'_{\langle k\rangle}$ implies that the symmetric bilinear form $(\cdot|\cdot)$ restricted to $\mathcal{H}'_{\langle k\rangle}$ is nondegenerate.

Next want to prove

$$
(\mathcal{H}'_{\langle k\rangle}|\mathcal{H}'_{\langle k'\rangle})=\{0\}\qquad\text{for } n_1-n_2+1-\delta_{n_1,n_2}\geq k,k'\in\mathbb{Z}\text{ such that } k\neq k'. \qquad (6.22)
$$

For any $f \in \mathcal{H}_{\langle k-2m\rangle}$ and $f' \in \mathcal{H}_{\langle k'-2m'\rangle}$, (3.64). (5.2), (6.6) and (6.18) yield

$$\begin{aligned}
&\left(\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^{2m}(f))|\left(\sum_{r=0}^{\infty}\frac{(-2)^r x_0^{2r}\mathcal{D}^r}{(2r)!}\right)(\eta^{2m'}(f'))\right)\\
= &\sum_{i=0}^{m}\frac{2^{2i}}{(2i)!}[\prod_{s=1}^{i}(m-s)(m-k+n_1-n_2+1+s)]\\
&\times[\prod_{s'=1}^{i}(m'-s')(m'-k'+n_1-n_2+1+s')](\eta^{m-i}(f)|\eta^{m'-i}(f))\\
= &\ 0 \qquad \text{if } (m,k-2m)\neq(m',k'-2m'). \qquad (6.23)
\end{aligned}$$

Let $g \in \mathcal{H}_{\langle k-2m-1\rangle}$ and $g' \in \mathcal{H}_{\langle k'-2m'-1\rangle}$. By (3.60). (5.2), (6.6) and (6.19), we have

$$\begin{aligned}
&\left(\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^{2m+1}(g))|\left(\sum_{r=0}^{\infty}\frac{(-2)^r x_0^{2r+1}\mathcal{D}^r}{(2r+1)!}\right)(\eta^{2m'+1}(g'))\right)\\
= &\sum_{i=0}^{m}\frac{2^{2i}}{(2i+1)!}[\prod_{s=1}^{i}(m-s)(m-k+n_1-n_2+2+s)]\\
&\times[\prod_{s'=1}^{i}(m'-s')(m'-k'+n_1-n_2+2+s')](\eta^{m-i}(g)|\eta^{m'-i}(g'))\\
= &\ 0 \qquad \text{if } (m,k-2m-1)\neq(m',k'-2m'-1). \qquad (6.24)
\end{aligned}$$

Since $(x_0^{2i}|x_0^{2i'+1})=0$ for $i,i'\in\mathbb{N}$, the elements of the form (6.18) are orthogonal to those of the form (6.19). Hence (6.22) holds by (6.8).

For $g \in \mathcal{H}'_{\langle k\rangle}$ and $m \in \mathbb{N}+1$,

$$\mathcal{D}'[(\eta')^m(g)] = 2m[2(k+n_2-n_1+m-1)+1](\eta')^{m-1}(g) \qquad (6.25)$$

by (3.44) and the facts $\mathcal{D}'=\partial_{x_0}^2-2\mathcal{D}$ and its dual $\eta'=x_0^2+2\eta$. This shows that

$$((\eta')^m(\mathcal{H}'_{\langle k\rangle})|(\eta')^{m'}(\mathcal{H}'_{\langle k'\rangle}))=\{0\} \qquad \text{if } (m,k)\neq(m',k') \qquad (6.26)$$

for $n_1-n_2+1-\delta_{n_1,n_2}\geq k,k'\in\mathbb{Z}$ and $m,m'\in\mathbb{N}$ by (6.7). Moreover, the symmetric bilinear form $(\cdot|\cdot)$ restricted to $(\eta')^m(\mathcal{H}'_{\langle k\rangle})$ is nondegenerate.

Fix $n_1-n_2+1-\delta_{n_1,n_2}\geq k\in\mathbb{Z}$. Denote

$$\hat{\mathcal{B}}'_{\langle k\rangle}=\bigoplus_{i=0}^{\infty}(\eta')^i(\mathcal{H}'_{\langle k-2i\rangle}) \qquad (6.27)$$

Then the symmetric bilinear form $(\cdot|\cdot)$ restricted to $\hat{\mathcal{B}}'_{\langle k\rangle}$ is nondegenerate. Thus

$$\mathcal{B}'_{\langle k\rangle}=\hat{\mathcal{B}}'_{\langle k\rangle}\oplus(\hat{\mathcal{B}}'_{\langle k\rangle})^{\perp}\bigcap\mathcal{B}'_{\langle k\rangle}. \qquad (6.28)$$

According to Lemma 3.2, $(\hat{\mathcal{B}}'_{\langle k\rangle})^{\perp}\bigcap\mathcal{B}'_{\langle k\rangle}$ is an $o(2n+1,\mathbb{F})$-module. Assume $(\hat{\mathcal{B}}'_{\langle k\rangle})^{\perp}\bigcap\mathcal{B}'_{\langle k\rangle}\neq\{0\}$. By (5.2), (5.3), (5.8)-(5.13), (5.23) and (1.23), there exists a nonzero element in $(\hat{\mathcal{B}}'_{\langle k\rangle})^{\perp}\bigcap\mathcal{B}'_{\langle k\rangle}$ of the form:

$$f=\sum_{i=0}^{m}a_i x_0^{2i}(2\eta)^{m-i}(x_{n_1}^{-k+2m}) \qquad (6.29)$$

or

$$g=\sum_{i=0}^{m}b_ix_0^{2i+1}(2\eta)^{m-i}(x_{n_1}^{-k+2m+1}) \tag{6.30}$$

for some $m\in\mathbb{N}+1$. Moreover, we assume that the exponent of x_{n_1} is minimal.

If (6.29) holds, then (6.11)) and (6.13) give

$$\begin{aligned}
&(E_{n_1,0}-E_{0,n+n_1})(f)=(\partial_{x_0}\partial_{x_{n_1}}-x_0\partial_{y_{n_1}})(f)\\
=\ &\sum_{i=1}^{m}2i(-k+2m)a_ix_0^{2i-1}(2\eta)^{m-i}(x_{n_1}^{-k+2m-1})\\
&-\sum_{i=0}^{m-1}2(m-i)(-k+2m)a_ix_0^{2i+1}(2\eta)^{m-i-1}(x_{n_1}^{-k+2m-1})\\
=\ &2(-k+2m)\sum_{i=0}^{m-1}[(i+1)a_{i+1}-(m-i)a_i]x_0^{2i+1}(2\eta)^{m-i-1}(x_{n_1}^{-k+2m-1})\\
=\ &0
\end{aligned} \tag{6.31}$$

by the minimality of the exponent of x_{n_1}, equivalently

$$(i+1)a_{i+1}=(m-i)a_i\qquad\text{for }\ i\in\overline{0,m-1}. \tag{6.32}$$

Thus

$$a_i=a_0\binom{m}{i}\qquad\text{for }\ i\in\overline{0,m}. \tag{6.33}$$

So

$$f=\sum_{i=0}^{m}a_0\binom{m}{i}x_0^{2i}(2\eta)^{m-i}(x_{n_1}^{-k+2m})=a_0(\eta')^m(x_{n_1}^{-k+2m})\in\hat{\mathcal{B}}'_{\langle k\rangle}, \tag{6.34}$$

which contradicts (6.28).

Suppose that (6.30) holds. Note $x_0x_{n_1}^{-k+2m+1}\in\mathcal{H}'_{\langle k-2m\rangle}$ by (1.23). Expressions (6.11) and (6.13) deduce

$$\begin{aligned}
&(E_{n_1,0}-E_{0,n+n_1})(g)=(\partial_{x_0}\partial_{x_{n_1}}-x_0\partial_{y_{n_1}})(g)\\
=\ &\sum_{i=0}^{m}(2i+1)(-k+2m+1)b_ix_0^{2i}(2\eta)^{m-i}(x_{n_1}^{-k+2m})\\
&-\sum_{i=0}^{m-1}2(m-i)(-k+2m+1)b_ix_0^{2i+2}(2\eta)^{m-i-1}(x_{n_1}^{-k+2m})\\
=\ &(-k+2m+1)\{\sum_{i=0}^{m-1}[(2i+3)b_{i+1}-2(m-i)b_i]x_0^{2i+2}(2\eta)^{m-i-1}(x_{n_1}^{-k+2m})\\
&+b_0(2\eta)^m(x_{n_1}^{-k+2m})\}=0
\end{aligned} \tag{6.35}$$

by the minimality of the exponent of x_{n_1}, equivalently

$$b_0=0,\ \ (2i+3)b_{i+1}=2(m-i)b_i\qquad\text{for }\ i\in\overline{0,m-1}. \tag{6.36}$$

Thus $b_i=0$ for $i\in\overline{0,m}$, that is, $g=0$. This contradicts our choice of nonzero element. Hence $(\hat{\mathcal{B}}'_{\langle k\rangle})^{\perp}\bigcap\mathcal{B}'_{\langle k\rangle}=\{0\}$. Then (6.28) gives (6.4). Furthermore, (6.5) is obtained by Lemma 3.1 with $T_1=\partial_{x_0}^2$, $T_1^-=\int_{(x_0)}^{(2)}$ (cf. (2.6) and (2.7)) and $T_2=2\mathcal{D}$.

When $n_1=n_2$, an expression of $\mathcal{H}'_{\langle k\rangle}$ can be obtained via (5.3), (5.27)-(5.31), (6.8), (6.18) and (6.19). In particular, when $n_1=n_2=n$, the $(\mathcal{G},\mathcal{K})$-module structure is given by

$$\mathcal{H}'_{\langle -k\rangle} = \bigoplus_{m,r=0}^{\infty}\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i}\mathcal{D}^i}{(2i)!}\right)(\eta^r(\mathcal{H}_{\langle -k-2r-m,m\rangle})) \oplus \bigoplus_{l,s=0}^{\infty}\left(\sum_{i=0}^{\infty}\frac{(-2)^i x_0^{2i+1}\mathcal{D}^i}{(2i+1)!}\right)(\eta^s(\mathcal{H}_{\langle -k-2s-1-l,l\rangle})), \tag{6.37}$$

where $\mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$ given in (4.42). $\square$

7 Noncanonical Representations of $sp(2n,\mathbb{F})$

In this section, we use the results in Sections 3 and 4 to study noncanonical polynomial representation of $sp(2n,\mathbb{F})$.

Recall the symplectic Lie algebra

$$sp(2n,\mathbb{F}) = \sum_{i,j=1}^{n}\mathbb{F}(E_{i,j}-E_{n+j,n+i})+\sum_{i=1}^{n}(\mathbb{F}E_{i,n+i}+\mathbb{F}E_{n+i,i}) + \sum_{1\le i<j\le n}[\mathbb{F}(E_{i,n+j}+E_{n+j,i})+\mathbb{F}(E_{n+i,j}+E_{n+j,i})]. \tag{7.1}$$

Again we take the Cartan subalgebra $H=\sum_{i=1}^{n}\mathbb{F}(E_{i,i}-E_{n+i,n+i})$ and the subspace spanned by positive root vectors

$$sp(2n,\mathbb{F})_+ = \sum_{1\le i<j\le n}[\mathbb{F}(E_{i,j}-E_{n+j,n+i})+\mathbb{F}(E_{i,n+j}+E_{n+j,i})]+\sum_{i=1}^{n}\mathbb{F}E_{i,n+i}. \tag{7.2}$$

Fix $1\le n_1\le n_2\le n$. The noncanonical oscillator representation of $sp(2n,\mathbb{F})$ on $\mathcal{B}=\mathbb{F}[x_1,...,x_n,y_1,...,y_n]$ is defined via (1.14)-(1.16). Recall $\mathcal{K}=\sum_{i,j=1}^{n}\mathbb{F}(E_{i,j}-E_{n+j,n+i})$.

Theorem 7.1. *Let $k\in\mathbb{Z}$. If $n_1<n_2$ or $k\neq 0$, the subspace $\mathcal{B}_{\langle k\rangle}$ (cf. (1.17)) is an irreducible $sp(2n,\mathbb{F})$-module. Moreover, it is a highest-weight module only if $n_2=n$, in which case for $m\in\mathbb{N}$, $x_{n_1}^{-m}$ is a highest-weight vector of $\mathcal{B}_{\langle -m\rangle}$ with weight $-m\lambda_{n_1-1}+(m-1)\lambda_{n_1}$, $x_{n_1+1}^{m+1}$ is a highest-weight vector of $\mathcal{B}_{\langle m+1\rangle}$ with weight $-(m+2)\lambda_{n_1}+(m+1)\lambda_{n_1+1}+(m+1)\delta_{n_1,n-1}\lambda_n$ if $n_1<n$ and y_n^{m+1} is a highest-weight vector of $\mathcal{B}_{\langle m+1\rangle}$ with weight $(m+1)\lambda_{n-1}-2(m+1)\lambda_n$ when $n_1=n$.*

When $n_1=n_2$, the subspace $\mathcal{B}_{\langle 0\rangle}$ is a direct sum of two irreducible $sp(2n,\mathbb{F})$-submodules. If $n_1=n_2=n$, they are highest-weight modules with a highest-weight vector 1 of weight $-2\lambda_n$ and with a highest-weight vector $x_{n-1}y_n-x_ny_{n-1}$ of weight $\delta_{n,2}\lambda_{n-2}-4\lambda_n$, respectively. If $n_1=n_2=n$, all the irreducible modules are of $(\mathcal{G},\mathcal{K})$-type.

Proof. Recall that we embed $sl(n,\mathbb{F})$ into $sp(2n,\mathbb{F})$ via $E_{i,j}\mapsto E_{i,j}-E_{n+j,n_i}$. Moreover, $\mathcal{B}$ is nilpotent with respect to $sl(n,\mathbb{F})_+$ (cf. (2.30)) and

$$\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+\sum_{r=n_1+1}^{n_2}x_ry_r+\sum_{s=n_2+1}^{n}x_s\partial_{y_s}. \tag{7.3}$$

Note

$$(E_{i,n+j}+E_{j,n+i})|_{\mathcal B}=\partial_{x_i}\partial_{y_j}+\partial_{x_j}\partial_{y_i},\;\;(E_{i,n+r}+E_{r,n+i})|_{\mathcal B}=\partial_{x_i}\partial_{y_r}+x_r\partial_{y_i},\tag{7.4}$$

$$(E_{r,n+s}+E_{s,n+r})|_{\mathcal B}=x_r\partial_{y_s}+x_s\partial_{y_r}\tag{7.5}$$

for $i,j\in\overline{1,n_1}$ and $r,s\in\overline{n_1+1,n_2}$ by (1.15). Moreover,

$$(E_{i,j}-E_{n+j,n+i})|_{\mathcal B}=-x_j\partial_{x_i}-y_j\partial_{y_i}-\delta_{i,j}\tag{7.6}$$

and

$$(E_{i,r}-E_{n+r,n+i})|_{\mathcal B}=\partial_{x_i}\partial_{x_r}-y_r\partial_{y_i}\tag{7.7}$$

for $i,j\in\overline{1,n_1}$ and $r\in\overline{n_1+1,n_2}$ by (1.7), (1.8) and (1.14). We will process our arguments in two steps.

Step 1. $n_2=n$.

Under the assumption, $\mathcal B$ is nilpotent with respect to $sp(2n,\mathbb F)_+$ by (7.4)-(7.7).

First we assume $n_1+1<n$. According to (3.37), the nonzero weight vectors in

$$\text{Span}\{\eta^{m_3}(x_i^{m_1}y_n^{m_2})\mid m_r\in\mathbb N;i=n_1,n_1+1\}\tag{7.8}$$

are all the singular vectors of $sl(n,\mathbb F)$ in $\mathcal B$. The singular vectors of $sp(2n,\mathbb F)$ in $\mathcal B$ must be among them. Moreover, the subalgebra $sp(2n,\mathbb F)_+$ is generated by $sl(n,\mathbb F)_+$ and $E_{n,2n}$. According to (7.5), $E_{n,2n}|_{\mathcal B}=x_n\partial_{y_n}$. Hence

$$E_{n,2n}(\eta^{m_3}(x_i^{m_1}y_n^{m_2}))=x_n[m_3x_n\eta^{m_3-1}(x_i^{m_1}y_n^{m_2})+m_2\eta^{m_3}(x_i^{m_1}y_n^{m_2-1})]\tag{7.9}$$

for $i=n_1,n_1+1$ by (7.3). Considering weights, we conclude that the vectors $\{x_{n_1}^m,x_{n_1+1}^{m+1}\mid m\in\mathbb N$ are all the singular vectors of $sp(2n,\mathbb F)$ in $\mathcal B$. Furthermore,

$$x_{n_1}^m\in\mathcal B_{\langle -m\rangle}\;\;\text{and}\;\;x_{n_1+1}^{m+1}\in\mathcal B_{\langle m+1\rangle}\qquad\text{for}\;\;m\in\mathbb N.\tag{7.10}$$

Thus each $\mathcal B_{\langle k\rangle}$ has a unique non-isotropic singular vector for $k\in\mathbb Z$. By Lemma 3.3, all $\mathcal B_{\langle k\rangle}$ with $k\in\mathbb Z$ are irreducible highest-weight $sp(2n,\mathbb F)$-submodules.

Consider the case $n_1+1=n$. According to (3.112), the nonzero weight vectors in

$$\text{Span}\{\eta^{m_2}(x_{n-1}^{m_1}y_n^{m_3}),x_n^{m_1}y_n^{m_2},\eta^{m_1+m_2}(x_{n-1}^{m_2}y_n^{m_3-m_1})\mid m_i\in\mathbb N\}\tag{7.11}$$

are all the singular vectors of $sl(n,\mathbb F)$ in $\mathcal B$. Recall $E_{n,2n}|_{\mathcal B}=x_n\partial_{y_n}$. We have

$$E_{n,2n}(x_n^{m_1}y_n^{m_2})=m_2x_n^{m_1+1}y_n^{m_2-1}.\tag{7.12}$$

By (7.11) and considering weights, we again conclude that the vectors $\{x_{n-1}^m,x_n^{m+1}\mid m\in\mathbb N$ are all the singular vectors of $sp(2n,\mathbb F)$ in $\mathcal B$. Again all $\mathcal B_{\langle k\rangle}$ with $k\in\mathbb Z$ are irreducible highest-weight $sp(2n,\mathbb F)$-submodules.

Suppose $n_1=n$. By (7.4), we have $E_{n,2n}=\partial_{x_n}\partial_{y_n}$ in this case. According to (4.31), the nonzero weight vectors in

$$\text{Span}\{x_n^{m_1}y_n^{m_2}\zeta_1^{m_3}\mid m_i\in\mathbb N\}\tag{7.13}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}$, where $\zeta_1 = x_{n-1}y_n - x_n y_{n-1}$ in this case.

$$\begin{aligned} & E_{n,2n}(x_n^{m_1} y_n^{m_2} \zeta_1^{m_3}) \\ = \; & m_1 m_2 x_n^{m_1-1} y_n^{m_2-1} \zeta_1^{m_3} + m_1 m_3 x_{n-1} x_n^{m_1-1} y_n^{m_2} \zeta_1^{m_3-1} \\ & - m_2 m_3 y_{n-1} x_n^{m_1} y_n^{m_2-1} \zeta_1^{m_3-1} - m_3(m_3-1) x_{n-1} y_{n-1} x_n^{m_1} y_n^{m_2} \zeta_1^{m_3-2} \\ = \; & m_1(m_2+m_3) x_n^{m_1-1} y_n^{m_2-1} \zeta_1^{m_3} + m_3(m_1-m_2-m_3+1) y_{n-1} x_n^{m_1} y_n^{m_2-1} \zeta_1^{m_3-1} \\ & - m_3(m_3-1) y_{n-1}^2 x_n^{m_1+1} y_n^{m_2-1} \zeta_1^{m_3-2}. \end{aligned} \tag{7.14}$$

Considering weights, we again conclude that the vectors $\{x_n^m, y_n^{m+1}, \zeta_1 \mid m \in \mathbb{N}\}$ are all the singular vectors of $sp(2n,\mathbb{F})$ in $\mathcal{B}$. Moreover,

$$x_n^m \in \mathcal{B}_{\langle -m \rangle}, \;\; \zeta_1 \in \mathcal{B}_{\langle 0 \rangle} \;\text{ and }\; y_n^{m+1} \in \mathcal{B}_{\langle m+1 \rangle} \qquad \text{for } m \in \mathbb{N}. \tag{7.15}$$

Thus each $\mathcal{B}_{\langle k \rangle}$ with $k \neq 0$ has a unique non-isotropic singular vector for $k \in \mathbb{Z}$. By Lemma 3.3, all $\mathcal{B}_{\langle k \rangle}$ with $0 \neq k \in \mathbb{Z}$ are irreducible highest-weight $sp(2n,\mathbb{F})$-submodules.

Set

$$\mathcal{B}_{\langle 0,1 \rangle} = \text{Span}\{[\prod_{1 \leq r \leq s \leq n} (x_r y_s + x_s y_r)^{l_{r,s}}] \mid l_{r,s} \in \mathbb{N}\} \tag{7.18}$$

and

$$\mathcal{B}_{\langle 0,2 \rangle} = \text{Span}\{[\prod_{1 \leq r \leq s \leq n} (x_r y_s + x_s y_r)^{l_{r,s}}](x_p y_q - x_q y_p) \mid l_{r,s} \in \mathbb{N}; 1 \leq p < q \leq n\}. \tag{7.19}$$

Let

$$\mathcal{G}' = \sum_{1 \leq r \leq s \leq n} \mathbb{F}(E_{n+s,r} + E_{n+r,s}) \tag{7.20}$$

and

$$\hat{\mathcal{G}} = \sum_{i,j=1}^{n} \mathbb{F}(E_{i,j} - E_{n+j,n+i}) + \sum_{1 \leq r \leq s \leq n} \mathbb{F}(E_{r,n+s} + E_{s,n+r}). \tag{7.21}$$

Then $\mathcal{G}'$ and $\hat{\mathcal{G}}$ are Lie subalgebras of $sp(2n,\mathbb{F})$ and $sp(2n,\mathbb{F}) = \mathcal{G}' \oplus \hat{\mathcal{G}}$. By PBW Theorem

$$U(sp(2n,\mathbb{F})) = U(\mathcal{G}')U(\hat{\mathcal{G}}). \tag{7.22}$$

Note

$$(E_{n+s,r} + E_{n+r,s})_{\mathcal{B}} = -(x_r y_s + x_s y_r) \qquad \text{for } r, s \in \overline{1,n} \tag{7.23}$$

by (1.16). According to (7.4), (7.6) and (7.23),

$$\mathcal{B}_{\langle 0,1 \rangle} = U(\mathcal{G}')(1) = U(sp(2n,\mathbb{F}))(1) \tag{7.24}$$

and

$$\mathcal{B}_{\langle 0,2 \rangle} = \sum_{1 \leq p < q \leq n} U(\mathcal{G}')(x_p y_q - x_q y_p) = U(sp(2n,\mathbb{F}))(\zeta_1) \tag{7.25}$$

are $sp(2n,\mathbb{F})$-submodules.

It is obvious, $1 \notin \mathcal{B}_{\langle 0,2 \rangle}$. On the other hand, $(\mathcal{B}_{\langle 0,1 \rangle} | x_{n-1}y_n - x_n y_{n-1}) = \{0\}$. Hence $x_{n-1}y_n - x_n y_{n-1} \notin \mathcal{B}_{\langle 0,1 \rangle}$. Thus $\mathcal{B}_{\langle 0,1 \rangle}$ and $\mathcal{B}_{\langle 0,0 \rangle}$ have a unique non-isotropic singular vector. By Lemma 3.3, they are irreducible. Since 1 and $x_{n-1}y_n - x_ny_{n-1}$ are the only singular vectors in $\mathcal{B}_{\langle 0\rangle}$ which is nilpotent with respect to $sp(2n,\mathbb{F})_+$, Lemma 2.3 yields

$$\mathcal{B}_{\langle 0\rangle} = \mathcal{B}_{\langle 0,1\rangle} \oplus \mathcal{B}_{\langle 0,2\rangle} \tag{7.26}$$

by the similar arguments as those from (3.67) to (3.69).

Step 2. $n_2 < n$.

We set

$$\begin{aligned}\mathcal{G}_1 &= \sum_{i,j=1}^{n_2} \mathbb{F}(E_{i,j} - E_{n+j,n+i}) + \sum_{i=1}^{n_2} (\mathbb{F}E_{i,n+i} + \mathbb{F}E_{n+i,i}) \\ &+ \sum_{1\leq i<j\leq n_2} [\mathbb{F}(E_{i,n+j} + E_{n+j,i}) + \mathbb{F}(E_{n+i,j} + E_{n+j,i})]\end{aligned} \tag{7.27}$$

and

$$\begin{aligned}\mathcal{G}_2 &= \sum_{i,j=n_1+1}^{n} \mathbb{F}(E_{i,j} - E_{n+j,n+i}) + \sum_{i=n_1+1}^{n} (\mathbb{F}E_{i,n+i} + \mathbb{F}E_{n+i,i}) \\ &+ \sum_{n_1+1\leq i<j\leq n} [\mathbb{F}(E_{i,n+j} + E_{n+j,i}) + \mathbb{F}(E_{n+i,j} + E_{n+j,i})].\end{aligned} \tag{7.28}$$

Then $\mathcal{G}_1 = sp(2n_2,\mathbb{F})$ and $\mathcal{G}_2 \cong sp(2(n-n_1),\mathbb{F})$ are Lie subalgebras of $sp(2n,\mathbb{F})$. Denote

$$\mathcal{M}^1 = \mathbb{F}[x_1,...,x_{n_2},y_1,...,y_{n_2}], \qquad \mathcal{M}^2 = \mathbb{F}[x_{n_1+1},...,x_n,y_{n_1+1},...,y_n]. \tag{7.29}$$

Observe that $\mathcal{M}^1$ is exactly the $\mathcal{G}_1$-module as $\mathcal{B}$ in Step 1 with $n \to n_2$ and $\mathcal{M}^2$ is exactly the $\mathcal{G}_1$-module as $\mathcal{B}$ in Step 1 with $n_1 = n_2$ and $n \to n - n_1$. Moreover, we set

$$\mathcal{M}^3 = \mathbb{F}[x_1,...,x_{n_1},y_1,...,y_{n_1}], \qquad \mathcal{M}^4 = \mathbb{F}[x_{n_1+1},...,x_{n_2},y_{n_1+1},...,y_{n_2}]. \tag{7.30}$$

Let

$$\mathcal{M}^i_{\langle k\rangle} = \mathcal{M}^i \bigcap \mathcal{B}_{\langle k\rangle} \qquad \text{for } i \in \overline{1,4},\ k \in \mathbb{Z}. \tag{7.31}$$

Then

$$\mathcal{M}^1_{\langle k\rangle} = \bigoplus_{r\in\mathbb{Z}} \mathcal{M}^3_{\langle r\rangle}\mathcal{M}^4_{\langle k-r\rangle} \qquad \text{for } k \in \mathbb{Z}. \tag{7.32}$$

Next we prove the theorem case by case.

Case 1. $n_1 + 1 < n_2$

According to (3.36), the nonzero weight vectors in

$$\text{Span}\{\eta^{m_3}(x_i^{m_1}y_j^{m_2}) \mid m_r \in \mathbb{N}; i = n_1, n_1+1; j = n_2, n_2+1\} \tag{7.33}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}$. Fix $k \in \mathbb{N}$. Then the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle -k\rangle}$ are

$$\begin{aligned}&\{\eta^{m_3}(x_{n_1}^{k+m_2+2m_3}y_{n_2}^{m_2}), \eta^{m_3}(x_{n_1+1}^{m_1}y_{n_2+1}^{k+m_1+2m_3}),\\ &\eta^{m_3}(x_{n_1}^{m_4}y_{n_2+1}^{m_5}) \mid m_i \in \mathbb{N}; m_4 + m_5 - 2m_3 = k\}.\end{aligned} \tag{7.34}$$

Let M be a nonzero $sp(2n,\mathbb{F})$-submodule of $\mathcal{B}_{\langle -k\rangle}$. Then M contains a singular of $sl(n,\mathbb{F})$. Suppose some $\eta^{m_3}(x_{n_1}^{k+m_2+2m_3}y_{n_2}^{m_2})\in M$. We have $E_{n_1,n+n_1}|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_1}}$ and

$$E_{n_1,n+n_1}^{m_3}[\eta^{m_3}(x_{n_1}^{k+m_2+2m_3}y_{n_2}^{m_2})]=m_3![\prod_{r=1}^{2m_3}(k+m_2+r)]x_{n_1}^{k+m_2}y_{n_2}^{m_2}\in M \tag{7.35}$$

by (7.3) and (7.4). Moreover, $(E_{n_1,n+n_2}+E_{n_2,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_2}}+x_{n_2}\partial_{y_{n_1}}$ and

$$(E_{n_1,n+n_2}+E_{n_2,n+n_1})^{m_2}(x_{n_1}^{k+m_2}y_{n_2}^{m_2})=m_2![\prod_{r=1}^{m_2}(k+r)]x_{n_1}^{k}\in M \tag{7.36}$$

by (7.4). Thus

$$x_{n_1}^k\in M. \tag{7.37}$$

Assume some $\eta^{m_3}(x_{n_1+1}^{m_1}y_{n_2+1}^{k+m_1+2m_3})\in M$. According to (1.16),

$$(E_{n+i,j}+E_{n+j,i})|_{\mathcal{B}}=\partial_{x_i}\partial_{y_j}+\partial_{x_j}\partial_{y_i}\qquad\text{for }\ i\in\overline{n_2+1,n}. \tag{7.38}$$

So

$$E_{n+n_2+1,n_2+1}^{m_3}[\eta^{m_3}(x_{n_1+1}^{m_1}y_{n_2+1}^{k+m_1+2m_3})]=m_3![\prod_{r=1}^{2m_3}(k+m_1+r)]x_{n_1+1}^{m_1}y_{n_2+1}^{k+m_1}\in M. \tag{7.39}$$

Moreover,

$$(E_{n+n_2+1,n_1+1}+E_{n+n_1+1,n_2+1})|_{\mathcal{B}}=\partial_{x_{n_1+1}}\partial_{y_{n_2+1}}+y_{n_1+1}\partial_{x_{n_2+1}} \tag{7.40}$$

by (1.16). Hence

$$(E_{n+n_2+1,n_1+1}+E_{n+n_1+1,n_2+1})^{m_1}(x_{n_1+1}^{m_1}y_{n_2+1}^{k+m_1})=m_1![\prod_{r=1}^{m_1}(k+r)]y_{n_2+1}^{k}\in M. \tag{7.41}$$

Furthermore,

$$(E_{n+n_2+1,n_1}+E_{n+n_1,n_2+1})|_{\mathcal{B}}=-x_{n_1}\partial_{y_{n_2+1}}+y_{n_1}\partial_{x_{n_2+1}} \tag{7.42}$$

by (1.16). Thus

$$(E_{n+n_2+1,n_1}+E_{n+n_1,n_2+1})^{k}(y_{n_2+1}^{k})=(-1)^k k!x_{n_1}^{k}\in M. \tag{7.43}$$

Thus (7.37) holds again.

Consider $\eta^{m_3}(x_{n_1}^{m_4}y_{n_2+1}^{m_5})$ for some $m_3,m_3,m_4\in\mathbb{N}$ such that $m_4+m_5-2m_3=k$. Note that $E_{n_1+1,n+n_1+1}|_{\mathcal{B}}=x_{n_1+1}\partial_{y_{n_1+1}}$ by (7.5) and

$$E_{n_1+1,n+n_1+1}^{m_3}[\eta^{m_3}(x_{n_1}^{m_4}y_{n_2+1}^{m_5})]=m_3!x_{n_1+1}^{2m_3}x_{n_1}^{m_4}y_{n_2+1}^{m_5}\in M. \tag{7.44}$$

There exists $r_1,r_2\in\mathbb{N}$ such that $r_1+r_2=2m_3$ and $r_1\le m_4,\ r_2\le m_5$. Moreover,

$$(E_{n_1,n_1+1}-E_{n+n_1+1,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{x_{n_1+1}}-y_{n_1+1}\partial_{y_{n_1}} \tag{7.45}$$

by (1.7), (1.8) and (1.14). Moreover, (7.40) and (7.45) yield

$$\begin{aligned}&(E_{n_1,n_1+1}-E_{n+n_1+1,n+n_1})^{r_1}(E_{n+n_2+1,n_1+1}+E_{n+n_1+1,n_2+1})^{r_2}(x_{n_1+1}^{2m_3}x_{n_1}^{m_4}y_{n_2+1}^{m_5})\\ =\ &(2m_3)![\prod_{s_1=0}^{r_1-1}(m_4-s_1)][\prod_{s_2=0}^{r_2-1}(m_5-s_2)]x_{n_1}^{m_4-r_1}y_{n_2+1}^{m_5-r_2}\in M.\end{aligned} \tag{7.46}$$

Furthermore, (7.42) yields

$$\begin{aligned}&(E_{n+n_2+1,n_1}+E_{n+n_1,n_2+1})^{m_5-r_2}(x_{n_1}^{m_4-r_1}y_{n_2+1}^{m_5-r_2})\\ =\ &(-1)^{m_5-r_2}(m_5-r_2)!x_{n_1}^k\in M.\end{aligned}\tag{7.47}$$

Thus we always have $x_{n_1}^k\in M$.

Note that $\mathcal{M}^1_{\langle -k\rangle}\ni x_{n_1}^k$ is an irreducible $\mathcal{G}_1$-module (cf. (7.27) and (7.29)) by Step 1. So

$$\mathcal{M}^1_{\langle -k\rangle}\subset M.\tag{7.48}$$

Let $r\in\mathbb{Z}$. According to (7.32),

$$\mathcal{M}^3_{\langle r\rangle}\mathcal{M}^4_{\langle -k-r\rangle}\subset\mathcal{M}^1_{\langle -k\rangle}\subset M.\tag{7.49}$$

Moreover, $\mathcal{M}^2_{\langle -k-r\rangle}\supset\mathcal{M}^4_{\langle -k-r\rangle}$ is an irreducible $\mathcal{G}_2$-module (cf. (7.28) and (7.29)) by Step 1. Thus

$$\mathcal{M}^3_{\langle r\rangle}\mathcal{M}^2_{\langle -k-r\rangle}=U(\mathcal{G}_2)(\mathcal{M}^3_{\langle r\rangle}\mathcal{M}^4_{\langle -k-r\rangle})\subset M.\tag{7.50}$$

Then

$$\mathcal{B}_{\langle -k\rangle}=\bigoplus_{r\in\mathbb{Z}}\mathcal{M}^3_{\langle r\rangle}\mathcal{M}^2_{\langle -k-r\rangle}\subset M\tag{7.51}$$

by (7.29) and (7.30). Therefore, $M=\mathcal{B}_{\langle -k\rangle}$, that is, $\mathcal{B}_{\langle -k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-submodule.

Fix $0<k\in\mathbb{N}$. Then the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle k\rangle}$ are

$$\begin{aligned}&\{\eta^{m_2}(x_{n_1+1}^{k+m_1-2m_2}y_{n_2+1}^{m_1}),\eta^{m_2}(x_{n_1}^{m_1}y_{n_2}^{k+m_1-2m_2}),\eta^{m_3}(x_{n_1+1}^{m_4}y_{n_2}^{m_5})\\ &\mid m_i\in\mathbb{N};2m_2\leq k+m_1;m_4+m_5+2m_3=k\}\end{aligned}\tag{7.52}$$

by (7.33). Let M be a nonzero $sp(2n,\mathbb{F})$-submodule of $\mathcal{B}_{\langle k\rangle}$. Then M contains a singular of $sl(n,\mathbb{F})$. Suppose some $\eta^{m_2}(x_{n_1+1}^{k+m_1-2m_2}y_{n_2+1}^{m_1})\in M$ with $2m_2\leq k+m_1$. We have $E_{n_1+1,n+n_1+1}|_{\mathcal{B}}=x_{n_1+1}\partial_{y_{n_1+1}}$ and

$$E_{n_1+1,n+n_1+1}^{m_2}[\eta^{m_2}(x_{n_1+1}^{k+m_1-2m_2}y_{n_2}^{m_1})]=m_2!x_{n_1+1}^{k+m_1}y_{n_2+1}^{m_1}\in M\tag{7.53}$$

by (7.3) and (7.5). Moreover, (7.40) gives

$$(E_{n+n_2+1,n_1+1}+E_{n+n_1+1,n_2+1})^{m_1}(x_{n_1+1}^{k+m_1}y_{n_2+1}^{m_1})=m_1![\prod_{r=1}^{m_1}(k+r)]x_{n_1+1}^k\in M.\tag{7.54}$$

Thus

$$x_{n_1+1}^k\in M.\tag{7.55}$$

Assume some $\eta^{m_2}(x_{n_1}^{m_1}y_{n_2}^{k+m_1-2m_2})\in M$ with $2m_2\leq k+m_1$. Observe $E_{n+n_2,n_2}=y_{n_2}\partial_{x_{n_2}}$ by (1.16). So

$$E_{n+n_2,n_2}^{m_2}[\eta^{m_2}(x_{n_1}^{m_1}y_{n_2}^{k+m_1-2m_2})]=m_2!x_{n_1}^{m_1}y_{n_2}^{k+m_1}\in M.\tag{7.56}$$

Moreover, (7.4) gives that $(E_{n_1,n+n_2}+E_{n_2,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_2}}+x_{n_2}\partial_{y_{n_1}}$ and

$$(E_{n_1,n+n_2}+E_{n_2,n+n_1})^{m_1}(x_{n_1}^{m_1}y_{n_2}^{k+m_1})=m_1![\prod_{r=1}^{m_1}(k+r)]y_{n_2}^k\in M.\tag{7.57}$$

Furthermore, (7.5) yields that $(E_{n_1+1,n+n_2}+E_{n_2,n+n_1+1})|_{\mathcal{B}}=x_{n_1+1}\partial_{y_{n_2}}+x_{n_2}\partial_{y_{n_1+1}}$ and

$$(E_{n_1+1,n+n_2}+E_{n_2,n+n_1+1})^k(y_{n_2}^k)=k!x_{n_1+1}^k\in M. \tag{7.58}$$

Thus (7.55) holds again.

Consider $\eta^{m_3}(x_{n_1+1}^{m_4}y_{n_2}^{m_5})$ for some $m_3,m_3,m_4\in\mathbb{N}$ such that $m_4+m_5+2m_3=k$. Note $E_{n_1+1,n+n_1+1}=x_{n_1+1}\partial_{y_{n_1+1}}$ by (7.5). So

$$E_{n_1+1,n+n_1+1}^{m_3}[\eta^{m_3}(x_{n_1+1}^{m_4}y_{n_2}^{m_5})]=m_3!x_{n_1+1}^{m_4+2m_3}y_{n_2}^{m_5}\in M. \tag{7.59}$$

According to (7.5),

$$(E_{n_1+1,n+n_2}+E_{n_2,n+n_1+1})^{m_5}(x_{n_1+1}^{m_4+2m_3}y_{n_2}^{m_5})=m_5!x_{n_1+1}^k\in M. \tag{7.60}$$

Therefore, we always have $x_{n_1+1}^k\in M$.

Observe that $\mathcal{M}_{\langle k\rangle}^2\ni x_{n_1+1}^k$ is an irreducible $\mathcal{G}_2$-module (cf. (7.28) and (7.29)) by Step 1. So

$$\mathcal{M}_{\langle k\rangle}^2\subset M. \tag{7.61}$$

Let $r\in\mathbb{Z}$. Denote

$$\mathcal{M}^5=\mathbb{F}[x_{n_2+1},...,x_n,y_{n_2+1},...,y_n],\qquad \mathcal{M}_{\langle k\rangle}^5=\mathcal{M}^5\bigcap\mathcal{B}_{\langle k\rangle},\quad k\in\mathbb{Z}. \tag{7.62}$$

Then

$$\mathcal{M}_{\langle k\rangle}^2=\bigoplus_{r\in\mathbb{Z}}\mathcal{M}_{\langle r\rangle}^4\mathcal{M}_{\langle k-r\rangle}^5 \tag{7.63}$$

(cf. (7.30)). Fix $r\in\mathbb{Z}$.

$$\mathcal{M}_{\langle r\rangle}^4\mathcal{M}_{\langle k-r\rangle}^5\subset\mathcal{M}_{\langle k\rangle}^2\subset M. \tag{7.64}$$

Moreover, $\mathcal{M}_{\langle r\rangle}^1\supset\mathcal{M}_{\langle r\rangle}^4$ is an irreducible $\mathcal{G}_1$-module (cf. (7.27) and (7.29)) by Step 1. Thus

$$\mathcal{M}_{\langle r\rangle}^1\mathcal{M}_{\langle k-r\rangle}^5=U(\mathcal{G}_1)(\mathcal{M}_{\langle r\rangle}^4\mathcal{M}_{\langle k-r\rangle}^5)\subset M. \tag{7.65}$$

Furthermore,

$$\mathcal{B}_{\langle k\rangle}=\bigoplus_{r\in\mathbb{Z}}\mathcal{M}_{\langle r\rangle}^1\mathcal{M}_{\langle k-r\rangle}^5\subset M \tag{7.66}$$

by (7.27) and (7.65). Therefore, $M=\mathcal{B}_{\langle k\rangle}$, that is, $\mathcal{B}_{\langle k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-submodule.

Case 2. $n_2=n_1+1$.

According to (3.104), the nonzero weight vectors in

$$\begin{aligned}&\mathrm{Span}\{\eta^{m_2}(x_i^{m_1}y_j^{m_3}),x_{n_1+1}^{m_1}y_{n_1+1}^{m_2},\eta^{m_1+m_2}(x_{n_1}^{m_2}y_{n_1+1}^{m_3-m_1}),\eta^{m_1+m_2}(y_{n_1+2}^{m_2}x_{n_1+1}^{m_3-m_1})\\&\qquad\mid m_r\in\mathbb{N};(i,j)=(n_1,n_1+1),(n_1,n_1+2),(n_1+1,n_1+2)\}.\end{aligned} \tag{7.67}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}$. Fix $k\in\mathbb{N}$. Then the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle -k\rangle}$ are those in (7.34). According to the arguments in Case 1, $\mathcal{B}_{\langle -k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-submodule. Let $0<k\in\mathbb{N}$. Then the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle k\rangle}$ are

$$\{\eta^{m_2}(x_{n_1+1}^{k+m_1-2m_2}y_{n_1+2}^{m_1}),\eta^{m_2}(x_{n_1}^{m_1}y_{n_1+1}^{k+m_1-2m_2}),\eta^{m_5+m_6}(x_{n_1}^{m_6}y_{n_1+1}^{m_7-m_5}),\eta^{m_5+m_6}(y_{n_1+2}^{m_6}x_{n_1+1}^{m_7-m_5}),$$
$$x_{n_1+1}^{m_3}y_{n_1+1}^{m_4}\mid m_i\in\mathbb{N};2m_2\leq k+m_1;m_3+m_4=k=m_5+m_5+m_7\} \tag{7.68}$$

by (7.67). Let M be a nonzero $sp(2n,\mathbb{F})$-submodule of $\mathcal{B}_{\langle k\rangle}$. As an $sl(n,\mathbb{F})$-module, M contains a singular of $sl(n,\mathbb{F})$. If $x_{n_1+1}^{m_3}y_{n_1+1}^{m_4}\in M$ with $m_3+m_4=k$, then $E_{n_1+1,n+n_1+1}|_{\mathcal{B}}=x_{n_1+1}\partial_{y_{n_1+1}}$ and

$$E_{n_1+1,n+n_1+1}^{m_4}(x_{n_1+1}^{m_3}y_{n_1+1}^{m_4})=m_4!x_{n_1+1}^k\in M\Longrightarrow x_{n_1+1}^k\in M \tag{7.69}$$

by (7.5). Suppose some $\eta^{m_5+m_6}(x_{n_1}^{m_6}y_{n_1+1}^{m_7-m_5})\in M$ with $m_5+m_5+m_7=k$. According to (1.16), $E_{n+n_1+1,n_1+1}=y_{n_1+1}\partial_{x_{n_1+1}}$. So

$$E_{n+n_1+1,n_1+1}^{m_5+m_6}[\eta^{m_5+m_6}(x_{n_1}^{m_6}y_{n_1+1}^{m_7-m_5})]=(m_5+m_6)!x_{n_1}^{m_6}y_{n_1+1}^{k+m_6}\in M. \tag{7.70}$$

Moreover, (7.4) yields that $(E_{n_1,n+n_1+1}+E_{n_1+1,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_1+1}}+x_{n_1+1}\partial_{y_{n_1}}$ and

$$(E_{n_1,n+n_1+1}+E_{n_1+1,n+n_1})^{m_6}(x_{n_1}^{m_6}y_{n_1+1}^{k+m_6})=m_6![\prod_{r=1}^{m_6}(k+r)]y_{n_1+1}^k\in M. \tag{7.71}$$

Assume some $\eta^{m_5+m_6}(y_{n_1+2}^{m_6}x_{n_1+1}^{m_7-m_5})\in M$ with $m_5+m_5+m_7=k$. By (7.3) and (7.5),

$$E_{n_1+1,n+n_1+1}^{m_5+m_6}[\eta^{m_5+m_6}(y_{n_1+2}^{m_6}x_{n_1+1}^{m_7-m_5})]=(m_5+m_6)!y_{n_1+2}^{m_6}x_{n_1+1}^{k+m_6}\in M. \tag{7.72}$$

Observe

$$(E_{n+n_1+2,n_1+1}+E_{n+n_1+1,n_1+2})|_{\mathcal{B}}=\partial_{x_{n_1+1}}\partial_{y_{n_1+2}}+y_{n_1+1}\partial_{x_{n_1+2}} \tag{7.73}$$

by (1.16). Hence

$$(E_{n+n_1+2,n_1+1}+E_{n+n_1+1,n_1+2})^{m_6}(y_{n_1+2}^{m_6}x_{n_1+1}^{k+m_6})=m_6![\prod_{r=1}^{m_6}(k+r)]x_{n_1+1}^k\in M. \tag{7.74}$$

Expressions (7.53)-(7.60), (7.69), (7.71) and (7.74) show that we always have $x_{n_1+1}^k\in M$. Furthermore, (7.61)-(7.66) imply that $\mathcal{B}_{\langle k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-module.

Case 3. $n_1=n_2$.

In this case,

$$\eta=\sum_{i=1}^{n_1}y_i\partial_{x_i}+\sum_{s=n_2+1}^{n}x_s\partial_{y_s}. \tag{7.75}$$

First we consider the subcase $1<n_1<n-1$. Expression (4.17) says that the nonzero weight vectors in

$$\text{Span}\{x_{n_1}^{m_1}y_{n_1}^{m_2}\zeta_1^{m_3+1},x_{n_1+1}^{m_1}y_{n_1+1}^{m_2}\zeta_2^{m_3+1},\eta^{m_3}(x_{n_1}^{m_1}y_{n_1+1}^{m_2})\mid m_i\in\mathbb{N}\} \tag{7.76}$$

are all the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}$, where

$$\zeta_1=x_{n_1-1}y_{n_1}-x_{n_1}y_{n_1-1},\;\;\zeta_2=x_{n_1+1}y_{n_1+2}-x_{n_1+2}y_{n_1+1}. \tag{7.77}$$

Fix $k\in\mathbb{N}+1$. Then the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle -k\rangle}$ are

$$\{x_{n_1}^{k+m_1}y_{n_1}^{m_1}\zeta_1^{m_2+1},x_{n_1+1}^{m_1}y_{n_1+1}^{k+m_1}\zeta_2^{m_2+1},\eta^{m_3}(x_{n_1}^{m_4}y_{n_1+1}^{m_5}) \mid m_i\in\mathbb{N};m_4+m_5-2m_3=k\}. \tag{7.78}$$

Let M be a nonzero $sp(2n,\mathbb{F})$-submodule of $\mathcal{B}_{\langle -k\rangle}$. As an $sl(n,\mathbb{F})$-module, M contains a singular vector of $sl(n,\mathbb{F})$. Suppose some $x_{n_1}^{k+m_1}y_{n_1}^{m_1}\zeta_1^{m_2+1}\in M$. Note $E_{n_1,n+n_1}|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_1}}$ by (7.4), and so

$$\begin{aligned}&E_{n_1,n+n_1}(x_{n_1}^{k+m_1}y_{n_1}^{m_1}\zeta_1^{m_2})\\ =\;&(k+m_1)m_1x_{n_1}^{k+m_1-1}y_{n_1}^{m_1-1}\zeta_1^{m_2}-m_2(m_2-1)x_{n_1}^{k+m_1}y_{n_1}^{m_1}x_{n_1-1}y_{n_1-1}\zeta_1^{m_2-2}\\ &+(k+m_1)m_2x_{n_1}^{k+m_1-1}y_{n_1}^{m_1}x_{n_1-1}\zeta_1^{m_2-1}-m_1m_2x_{n_1}^{k+m_1}y_{n_1}^{m_1-1}y_{n_1-1}\zeta_1^{m_2-1}.\end{aligned} \tag{7.79}$$

Moreover,

$$(E_{n_1-1,n_1}-E_{n+n_1,n+n_1-1})|_{\mathcal{B}}=-(x_{n_1}\partial_{x_{n_1-1}}+y_{n_1}\partial_{y_{n_1-1}}) \tag{7.80}$$

by (1.7), (1.8) and (1.14). Thus

$$\begin{aligned}&(E_{n_1-1,n_1}-E_{n+n_1,n+n_1-1})^2E_{n_1,n+n_1}(x_{n_1}^{k+m_1}y_{n_1}^{m_1}\zeta_1^{m_2})\\ =\;&-2m_2(m_2-1)x_{n_1}^{k+m_1+1}y_{n_1}^{m_1+1}\zeta_1^{m_2-2}\in M.\end{aligned} \tag{7.81}$$

Hence

$$x_{n_1}^{k+m_1+1}y_{n_1}^{m_1+1}\zeta_1^{m_2-2}\in M\qquad\text{if }\; m_2>1. \tag{7.82}$$

Furthermore,

$$(E_{n_1-1,n_1}-E_{n+n_1,n+n_1-1})E_{n_1,n+n_1}(x_{n_1}^{k+m_1}y_{n_1}^{m_1}\zeta_1)=-kx_{n_1}^{k+m_1}y_{n_1}^{m_1}\in M. \tag{7.83}$$

So we always have $x_{n_1}^{k+m}y_{n_1}^{m}\in M$ for some $m\in\mathbb{N}$ by induction on m_2.

Observe

$$E_{n_1,n+n_1}(x_{n_1}^{k+m}y_{n_1}^{m})=\partial_{x_{n_1}}\partial_{y_{n_1}}(x_{n_1}^{k+m}y_{n_1}^{m})=m![\prod_{r=1}^{m}(k+r)]x_{n_1}^{k} \tag{7.84}$$

by (7.4). Thus

$$x_{n_1}^{k}\in M. \tag{7.85}$$

Symmetrically, if some $x_{n_1+1}^{m_1}y_{n_1+1}^{k+m_1}\zeta_2^{m_2+1}\in M$, we have $y_{n_1+1}^{k}\in M$. But

$$(E_{n+n_1+1,n_1}+E_{n+n_1,n_1+1})|_{\mathcal{B}}=-x_{n_1}\partial_{y_{n_1+1}}+y_{n_1}\partial_{x_{n_1+1}} \tag{7.86}$$

by (1.16), which gives

$$(E_{n+n_1+1,n_1}+E_{n+n_1,n_1+1})^k(y_{n_1+1}^{k})=(-1)^kk!x_{n_1}^{k}\in M. \tag{7.87}$$

Thus (7.85) holds again.

Assume that some $\eta^{m_3}(x_{n_1}^{m_4}y_{n_1+1}^{m_5})\in M$ with $m_4+m_5-2m_3=k$. Note there exists $r_1,r_2\in\mathbb{N}$ such that $r_1+r_2=m_3$ and $2r_1\leq m_4$, $2r_2\leq m_5$. Moreover, $E_{n_1,n+n_1}|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_1}}$ by (7.4) and $E_{n+n_1+1,n_1+1}|_{\mathcal{B}}=\partial_{x_{n_1+1}}\partial_{y_{n_1+1}}$ by (1.16). Thus

$$\begin{aligned}&E_{n_1,n+n_1}^{r_1}E_{n+n_1+1,n_1+1}^{r_2}[\eta^{m_3}(x_{n_1}^{m_4}y_{n_1+1}^{m_5})]\\ =\;&m_3![\prod_{s_1=0}^{2r_1-1}(m_4-s_1)][\prod_{s_2=0}^{2r_2-1}(m_5-s_2)]x_{n_1}^{m_4-2r_1}y_{n_1+1}^{m_5-2r_2}\in M.\end{aligned} \tag{7.88}$$

Furthermore, (1.16) gives $(E_{n+n_1+1,n_1}+E_{n+n_1,n_1+1})|_{\mathcal{B}}=-x_{n_1}\partial_{y_{n_1+1}}+y_{n_1}\partial_{x_{n_1+1}}$, and so

$$\begin{aligned}&(E_{n+n_1+1,n_1}+E_{n+n_1,n_1+1})^{m_5-2r_2}(x_{n_1}^{m_4-2r_1}y_{n_1+1}^{m_5-2r_2})\\ =\;&(-1)^{m_5-2r_2}(m_5-2r_2)!x_{n_1}^k\in M.\end{aligned}\tag{7.89}$$

Thus we always have $x_{n_1}^k\in M$.

Now

$$(E_{n_1,n+n_1+1}+E_{n_1+1,n+n_1})|_{\mathcal{B}}=-y_{n_1+1}\partial_{x_{n_1}}+x_{n_1+1}\partial_{y_{n_1}}\tag{7.90}$$

by (7.4). For any $r\in\mathbb{N}+1$,

$$\frac{(-1)^r}{\prod_{s=0}^{r-1}(k-r)}(E_{n_1,n+n_1+1}+E_{n_1+1,n+n_1})^r(x_{n_1}^k)=x_{n_1}^{k-r}y_{n_1+1}^r\in M.\tag{7.91}$$

If $k\geq 2$ and $r\in\overline{1,k-1}$, then

$$\mathcal{M}^1_{\langle -k+r\rangle}\mathcal{M}^2_{\langle -r\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^{k-r}y_{n_1+1}^r)\subset M\tag{7.92}$$

because $\mathcal{M}^1_{\langle -k+r\rangle}$ is an irreducible $\mathcal{G}_1$-module and $\mathcal{M}^2_{\langle -r\rangle}$ is an irreducible $\mathcal{G}_2$-module by Step 1. Moreover,

$$\mathcal{M}^1_{\langle -k\rangle}=U(\mathcal{G}_1)(x_{n_1}^k),\;\mathcal{M}^2_{\langle -k\rangle}=U(\mathcal{G}_2)(y_{n_1+1}^k)\subset M.\tag{7.93}$$

Furthermore,

$$\mathcal{M}^1_{\langle -k\rangle}\mathcal{M}^2_{\langle 0\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^k)\subset M\;\;\text{if}\;\;n_1=n-1\tag{7.94}$$

and

$$\mathcal{M}^1_{\langle 0\rangle}\mathcal{M}^2_{\langle -k\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(y_{n_1+1}^k)\subset M\;\;\text{if}\;\;n_1=1.\tag{7.95}$$

Note

$$(E_{r,i}-E_{n+i,n+r})|_{\mathcal{B}}=y_iy_r-x_ix_r\qquad\text{for}\;\;i\in\overline{1,n_1},\;r\in\overline{n_1+1,n}\tag{7.96}$$

by (1.7), (1.8) and (1.14). In particular, if $k>1$ or $n_1=1$, we have

$$(E_{n_1+1,n_1}-E_{n+n_1,n+n_1+1})(x_{n_1}^k)=y_{n_1}x_{n_1}^ky_{n_1+1}-x_{n_1}^{k+1}x_{n_1+1}\in M.\tag{7.97}$$

Since

$$y_{n_1}x_{n_1}^ky_{n_1+1}\in\mathcal{M}^1_{\langle -k+1\rangle}\mathcal{M}^2_{\langle -1\rangle}\subset M,\tag{7.98}$$

we get

$$x_{n_1}^{k+1}x_{n_1+1}\in M.\tag{7.99}$$

Suppose $k=1$ and $n_1>1$. By (7.93),

$$\zeta_1x_{n_1}=(x_{n_1-1}y_{n_1}-x_{n_1}y_{n_1-1})x_{n_1}\in M.\tag{7.100}$$

Observe

$$(E_{n_1+1,n+n_1-1}+E_{n_1-1,n+n_1+1})|_{\mathcal{B}}=x_{n_1+1}\partial_{y_{n_1-1}}-y_{n_1+1}\partial_{x_{n_1-1}}\tag{7.101}$$

by (1.15). So

$$-(E_{n_1+1,n+n_1-1}+E_{n_1-1,n+n_1+1})(\zeta_1x_{n_1})=x_{n_1}^2x_{n_1+1}-x_{n_1}y_{n_1}y_{n_1+1}\in M.\tag{7.102}$$

On the other hand, (1.16) gives

$$(E_{n+i,j}+E_{n+j,i})|_{\mathcal{B}}=-(x_iy_j+x_jy_i)\qquad\text{for }\; i,j\in\overline{1,n_1}, \tag{7.103}$$

which implies

$$-E_{n+n_1,n_1}(y_{n_1+1})=x_{n_1}y_{n_1}y_{n_1+1}\in M. \tag{7.104}$$

By (7.102), we have $x_{n_1}^2x_{n_1+1}\in M$. So (7.99) always holds.

By Step 1,

$$\mathcal{M}^1_{\langle -k-1\rangle}\mathcal{M}^2_{\langle 1\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^{k+1}x_{n_1+1})\subset M. \tag{7.105}$$

Suppose

$$\mathcal{M}^1_{\langle -k-i\rangle}\mathcal{M}^2_{\langle i\rangle}\subset M \tag{7.106}$$

for $1\leq i\leq m$. Then

$$\begin{aligned}&(E_{n_1+1,n_1}-E_{n+n_1,n+n_1+1})(x_{n_1}^{k+m}x_{n_1+1}^m)\\ =\;& y_{n_1}x_{n_1}^{k+m}x_{n_1+1}^my_{n_1+1}-x_{n_1}^{k+m+1}x_{n_1+1}^{m+1}\in M\end{aligned} \tag{7.107}$$

by (7.96). If $m>1$, we have

$$y_{n_1}x_{n_1}^{k+m}x_{n_1+1}^my_{n_1+1}\in\mathcal{M}^1_{\langle -k-(m-1)\rangle}\mathcal{M}^2_{\langle m-1\rangle}\subset M. \tag{7.108}$$

Note

$$(E_{r,n+s}+E_{s,n+r})|_{\mathcal{B}}=-(x_ry_s+x_sy_r)\qquad\text{for }\; r,s\in\overline{n_1+1,n} \tag{7.109}$$

by (1.15). If $m=1$, we have

$$y_{n_1}x_{n_1}^{k+1}x_{n_1+1}y_{n_1+1}=-E_{n_1+1,n+n_1+1}(y_{n_1}x_{n_1}^{k+1})\subset E_{n_1+1,n+n_1+1}(\mathcal{M}^1_{\langle -k\rangle})\subset M. \tag{7.110}$$

Then (7.107), (7.108) and (7.110) give

$$x_{n_1}^{k+m+1}x_{n_1+1}^{m+1}\in M. \tag{7.111}$$

Furthermore,

$$\mathcal{M}^1_{\langle -k-m-1\rangle}\mathcal{M}^2_{\langle m+1\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^{k+m+1}x_{n_1+m+1})\subset M. \tag{7.112}$$

Thus (7.106) holds for any $i\in\mathbb{N}+1$. Symmetrically, we have

$$\mathcal{M}^1_{\langle i\rangle}\mathcal{M}^2_{\langle -k-i\rangle}\subset M\qquad\text{for }\; i\in\mathbb{N}+1. \tag{7.113}$$

Suppose $n_1<n-1$. Then $x_{n_1}^{k+1}x_{n_1+1}\zeta_2\in M$ by (7.105). Moreover,

$$(k+1)y_{n_1}x_{n_1}^ky_{n_1+1}\zeta_2=-(k+1)E_{n+n_1,n_1}(x_{n_1}^{k-1}y_{n_1+1}\zeta_2)\in M \tag{7.114}$$

by (7.92) and (7.103). According (1.7), (1.8) and (1.14),

$$(E_{n_1,n_1+1}-E_{n+n_1+1,n+n_1})|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{x_{n_1+1}}-\partial_{y_{n_1}}\partial_{y_{n_1+1}}. \tag{7.115}$$

Thus

$$\begin{aligned}&(E_{n_1,n_1+1}-E_{n+n_1+1,n+n_1})[(x_{n_1}^{k+1}x_{n_1+1}-(k+1)y_{n_1}x_{n_1}^ky_{n_1+1})\zeta_2]\\ =\;& 3(k+1)x_{n_1}^k\zeta_2\in M\end{aligned} \tag{7.116}$$

by (7.77). Hence

$$\mathcal{M}^1_{\langle -k\rangle}\mathcal{M}^2_{\langle 0\rangle}=U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^k)+U(\mathcal{G}_1)U(\mathcal{G}_2)(x_{n_1}^k\zeta_2)\subset M \tag{7.117}$$

by (7.26) and(7.85). Symmetrically,

$$\mathcal{M}^1_{\langle 0\rangle}\mathcal{M}^2_{\langle -k\rangle}\subset M. \tag{7.118}$$

By (7.92)-(7.95), (7.106), (7.112), (7.113), (7.117) and (7.118),

$$\mathcal{M}^1_{\langle -k-r\rangle}\mathcal{M}^2_{\langle r\rangle}\subset M\qquad \text{for}\;\; r\in\mathbb{Z}. \tag{7.119}$$

Therefore,

$$\mathcal{B}_{\langle -k\rangle}=\bigoplus_{r\in\mathbb{Z}}\mathcal{M}^1_{\langle -k-r\rangle}\mathcal{M}^2_{\langle r\rangle}\subset M. \tag{7.120}$$

We get $M=\mathcal{B}_{\langle -k\rangle}$, that is, $\mathcal{B}_{\langle -k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-module. We can similarly prove that $\mathcal{B}_{\langle k\rangle}$ is an irreducible $sp(2n,\mathbb{F})$-module.

Finally, we study $\mathcal{B}_{\langle 0\rangle}$. We first consider the generic case $1<n_1<n-1$. Set

$$\begin{aligned}\mathcal{B}_{\langle 0,1\rangle} &= \text{Span}\{[\prod_{1\leq r\leq s\leq n_1\text{ or } n_1+1\leq r\leq s\leq n}(x_ry_s+x_sy_r)^{l_{r,s}}]\\ &\qquad\times[\prod_{p=1}^{n_1}\prod_{q=n_1+1}^{n}(x_px_q-y_py_q)^{k_{p,q}}]\mid l_{r,s},k_{p,q}\in\mathbb{N}\}\end{aligned} \tag{7.121}$$

and

$$\mathcal{B}_{\langle 0,2\rangle}=\sum_{1\leq r<s\leq n_1\text{ or } n_1+1\leq r<s\leq n}\mathcal{B}_{\langle 0,1\rangle}(x_ry_s-x_sy_r)+\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}\mathcal{B}_{\langle 0,1\rangle}(x_px_q+y_py_q). \tag{7.122}$$

We want to prove that $\mathcal{B}_{\langle 0,1\rangle}$ and $\mathcal{B}_{\langle 0,2\rangle}$ forms $sp(2n,\mathbb{F})$-submodules.

Let

$$\begin{aligned}\mathcal{G}' &= \sum_{1\leq r\leq s\leq n_1}\mathbb{F}(E_{n+s,r}+E_{n+r,s})+\sum_{n_1+1\leq p\leq q\leq n}\mathbb{F}(E_{p,n+q}+E_{q,n+p})\\ &\quad+\sum_{r=1}^{n_1}\sum_{p=n_1+1}^{n}\mathbb{F}(E_{p,r}-E_{n+r,n+p})\end{aligned} \tag{7.123}$$

and

$$\begin{aligned}\hat{\mathcal{G}} &= \sum_{i,j=1}^{n_1}\mathbb{F}(E_{i,j}-E_{n+j,n+i})+\sum_{r,s=n_1+1}^{n}\mathbb{F}(E_{r,s}-E_{n+s,n+r})+\sum_{1\leq r\leq s\leq n_1}\mathbb{F}(E_{r,n+s}+E_{s,n+r})\\ &\quad+\sum_{n_1+1\leq p\leq q\leq n}\mathbb{F}(E_{n+q,p}+E_{n+p,q})+\sum_{r=1}^{n_1}\sum_{p=n_1+1}^{n}[\mathbb{F}(E_{r,p}-E_{n+p,n+r})\\ &\quad+\mathbb{F}(E_{r,n+p}+E_{p,n+r})+\mathbb{F}(E_{n+r,p}-E_{n+p,r})].\end{aligned} \tag{7.124}$$

Then $\mathcal{G}'$ and $\hat{\mathcal{G}}$ are Lie subalgebras of $sp(2n,\mathbb{F})$ and $sp(2n,\mathbb{F})=\mathcal{G}'\oplus\hat{\mathcal{G}}$. By PBW Theorem $U(sp(2n,\mathbb{F}))=U(\mathcal{G}')U(\hat{\mathcal{G}})$.

By (7.96), (7.103) and (7.109),

$$U(\mathcal{G}')|_{\mathcal{B}}=\mathcal{B}_{\langle 0,1\rangle}\;\text{ as multiplication operators on }\;\mathcal{B}. \tag{7.125}$$

Moreover,

$$(E_{r,s}-E_{n+s,n+r})|_{\mathcal{B}}=x_r\partial_{x_s}+y_r\partial_{y_s}+\delta_{r,s}, \tag{7.126}$$

$$(E_{n+r,s}+E_{n+s,r})|_{\mathcal{B}}=\partial_{x_r}\partial y_s+\partial_{x_s}\partial y_r, \tag{7.127}$$

$$(E_{n+r,i}+E_{n+i,r})|_{\mathcal{B}}=-x_i\partial y_r+y_i\partial x_r, \tag{7.128}$$

$$(E_{i,n+r}+E_{r,n+i})|_{\mathcal{B}}=-y_r\partial x_i+x_r\partial y_i, \tag{7.129}$$

$$(E_{i,r}-E_{n+r,n+i})|_{\mathcal{B}}=\partial_{x_i}\partial x_r-\partial_{y_i}\partial y_r \tag{7.130}$$

for $i\in\overline{1,n_1}$ and $r,s\in\overline{n_1+1,n}$. According to (7.4), (7.6), (7.124) and (7.126)-(7.130), $U(\hat{\mathcal{G}})(1)=\mathbb{F}$. Thus

$$\mathcal{B}_{\langle 0,1\rangle}=U(\mathcal{G}')(1)=U(sp(2n,\mathbb{F}))(1) \tag{7.131}$$

forms an $sp(2n,\mathbb{F})$-submodule.

Let

$$W=\sum_{1\le r<s\le n_1\text{ or }n_1+1\le r<s\le n}\mathbb{F}(x_ry_s-x_sy_r)+\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}\mathbb{F}(x_px_q+y_py_q). \tag{7.132}$$

By (7.4), (7.6) and (7.126)-(7.130), we can verify that W forms an irreducible $\hat{\mathcal{G}}$-submodule. Hence

$$\mathcal{B}_{\langle 0,2\rangle}=U(\mathcal{G}')(W)=U(sp(2n,\mathbb{F}))(W) \tag{7.133}$$

forms an $sp(2n,\mathbb{F})$-submodule. Moreover,

$$\mathcal{B}_{\langle 0,1\rangle}\bigcap W=\{0\}. \tag{7.134}$$

Next we want to prove that $\mathcal{B}_{\langle 0,1\rangle}$ and $\mathcal{B}_{\langle 0,2\rangle}$ are irreducible $sp(2n,\mathbb{F})$-submodules. According to (7.78), the singular vectors of $sl(n,\mathbb{F})$ in $\mathcal{B}_{\langle 0\rangle}$ are

$$\{x_{n_1}^{m_1}y_{n_1}^{m_1}\zeta_1^{m_2+1},x_{n_1+1}^{m_1}y_{n_1+1}^{m_1}\zeta_2^{m_2+1},\eta^{m_3}(x_{n_1}^{m_4}y_{n_1+1}^{m_5})\mid m_i\in\mathbb{N};m_4+m_5=2m_3\}. \tag{7.135}$$

Let M be a nonzero submodule of $\mathcal{B}_{\langle 0,1\rangle}$. Then M contains a singular vector of $sl(n,\mathbb{F})$. Suppose some $x_{n_1}^{m_1}y_{n_1}^{m_1}\zeta_1^{m_2}\in M$. By (7.79)-(7.82), we can assume $m_2=0,1$. If $m_2=0$, (7.84) yields $1\in M$. Then $M=\mathcal{B}_{\langle 0,1\rangle}$ by (7.131). Suppose $m_2=1$. We have $E_{n_1,n+n_1}|_{\mathcal{B}}=\partial_{x_{n_1}}\partial_{y_{n_1}}$ by (7.4), and

$$E_{n_1,n+n_1}[x_{n_1}^{m_1}y_{n_1}^{m_1}\zeta_1]=m_1(m_1+1)x_{n_1}^{m_1-1}y_{n_1}^{m_1-1}\zeta_1 \tag{7.136}$$

by (7.79). By induction on m_1, we have $\zeta_1\in M\subset\mathcal{B}_{\langle 0,1\rangle}$, which contradicts (7.134). Similarly, if some $x_{n_1+1}^{m_1}y_{n_1+1}^{m_1}\zeta_2^{m_2+1}\in M$, we have $M=\mathcal{B}_{\langle 0,1\rangle}$. Assume some $\eta^{m_3}(x_{n_1}^{m_4}y_{n_1+1}^{m_5})\in M$ with $m_4+m_5=2m_3$. Note m_4 and m_5 are both even or odd. If $m_4=2r_1$ and $m_5=2r_2$ are even, then (7.88) gives $1\in M$, equivalently $M=\mathcal{B}_{\langle 0,1\rangle}$. Suppose that $m_4=2r_1+1$ and $m_5=2r_2+1$ are odd. Expression (7.75) yields

$$\eta(x_{n_1}y_{n_1+1})=x_{n_1}x_{n_1+1}+y_{n_1}y_{n_1+1}\in M\subset\mathcal{B}_{\langle 0,1\rangle}, \tag{7.137}$$

which contradicts (7.134) again. Thus we always have $M=\mathcal{B}_{\langle 0,1\rangle}$, that is, $\mathcal{B}_{\langle 0,1\rangle}$ is irreducible. Similarly, we can prove that $\mathcal{B}_{\langle 0,2\rangle}$ is irreducible.

If $n_1=1$ and $n=2$, we let

$$\mathcal{B}_{\langle 0,1\rangle}=\text{Span}\{[\prod_{i=1}^{n}(x_iy_i)^{m_i}](x_1x_2-y_1y_2)^{m_3}]\mid m_i\in\mathbb{N}\} \tag{7.138}$$

and $\mathcal{B}_{\langle 0,2\rangle}=\mathcal{B}_{\langle 0,1\rangle}(x_1x_2+y_1y_2)$. When $n_1=1$ and $n>2$, we set

$$\mathcal{B}_{\langle 0,1\rangle}=\text{Span}\{[(x_1y_1)^l\prod_{2\leq r\leq s\leq n}(x_ry_s+x_sy_r)^{l_{r,s}}][\prod_{q=2}^{n}(x_1x_q-y_1y_q)^{k_q}]\mid l,l_{r,s},k_q\in\mathbb{N}\} \tag{7.139}$$

and

$$\mathcal{B}_{\langle 0,2\rangle}=\sum_{2\leq r<s\leq n}\mathcal{B}_{\langle 0,1\rangle}(x_ry_s-x_sy_r)+\sum_{q=2}^{n}\mathcal{B}_{\langle 0,1\rangle}(x_1x_q+y_1y_q). \tag{7.140}$$

In the case $1<n_1=n-1$, we put

$$\begin{aligned}\mathcal{B}_{\langle 0,1\rangle} &= \text{Span}\{(x_ny_n)^l[\prod_{1\leq r\leq s\leq n-1}(x_ry_s+x_sy_r)^{l_{r,s}}]\\ &\quad\times[\prod_{p=1}^{n-1}(x_px_n-y_py_n)^{k_p}]\mid l,l_{r,s},k_p\in\mathbb{N}\}\end{aligned} \tag{7.141}$$

and

$$\mathcal{B}_{\langle 0,2\rangle}=\sum_{1\leq r<s\leq n_1}\mathcal{B}_{\langle 0,1\rangle}(x_ry_s-x_sy_r)+\sum_{p=1}^{n_1}\mathcal{B}_{\langle 0,1\rangle}(x_px_n+y_py_n). \tag{7.142}$$

The above corresponding partial arguments show that $\mathcal{B}_{\langle 0,1\rangle}$ and $\mathcal{B}_{\langle 0,2\rangle}$ are irreducible in the corresponding case.

Now 1 is a non-isotropic element in $\mathcal{B}_{\langle 0,1\rangle}$ and $x_{n_1}x_{n_1+1}+y_{n_1}y_{n_1+1}$ a non-isotropic element in $\mathcal{B}_{\langle 0,2\rangle}$ by (3.54). By Lemma 2.3, the symmetric bilinear form $(\cdot|\cdot)$ restricted to them are nondegenrate. Since $(1|\mathcal{B}_{\langle 0,2\rangle})=\{0\}$, $\mathcal{B}_{\langle 0,1\rangle}$ is orthogonal to $\mathcal{B}_{\langle 0,2\rangle}$. Thus the symmetric bilinear form $(\cdot|\cdot)$ restricted $\mathcal{B}_{\langle 0,1\rangle}+\mathcal{B}_{\langle 0,2\rangle}$ is nondegenerate. Then

$$\mathcal{B}_{\langle 0\rangle}=(\mathcal{B}_{\langle 0,1\rangle}+\mathcal{B}_{\langle 0,2\rangle})\oplus(\mathcal{B}_{\langle 0,1\rangle}+\mathcal{B}_{\langle 0,2\rangle})^{\perp}\bigcap\mathcal{B}_{\langle 0\rangle}. \tag{7.143}$$

If $(\mathcal{B}_{\langle 0,1\rangle}+\mathcal{B}_{\langle 0,2\rangle})^{\perp}\bigcap\mathcal{B}_{\langle 0\rangle}\neq\{0\}$, then it contains a singular vector of $sl(n,\mathbb{F})$. Our above arguments in proving the irreducibility of $\mathcal{B}_{\langle 0,1\rangle}$ show that it contains either $\mathcal{B}_{\langle 0,1\rangle}$ or $\mathcal{B}_{\langle 0,2\rangle}$, which is absurd. Therefore, $\mathcal{B}_{\langle 0\rangle}=\mathcal{B}_{\langle 0,1\rangle}\oplus\mathcal{B}_{\langle 0,2\rangle}$ is an orthogonal decomposition of irreducible $sp(2n,\mathbb{F})$-submodules.

Suppose $n_1=n_2=n$. For $k\in\mathbb{N}+1$, (4.21) and (4.31) imply that

$$\mathcal{B}_{\langle k\rangle}=\bigoplus_{m=0}^{\infty}\bigoplus_{r=[\![(k+1)/2]\!]}^{\infty}\eta^r(\mathcal{H}_{\langle k-2r-m,m\rangle}) \tag{7.144}$$

and

$$\mathcal{B}_{\langle -k\rangle}=\bigoplus_{m,r=0}^{\infty}\eta^r(\mathcal{H}_{\langle -k-2r-m,m\rangle}) \tag{7.145}$$

are $(\mathcal{G},\mathcal{K})$-structures, where $\mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$ is given in (4.42). Moreover,

$$\mathcal{B}_{\langle 0,1\rangle}=\bigoplus_{m,r=0}^{\infty}\eta^r(\mathcal{H}_{\langle -2r-2m,2m\rangle}) \tag{7.146}$$

and

$$\mathcal{B}_{\langle 0,2\rangle}=\bigoplus_{m,r=0}^{\infty}\eta^r(\mathcal{H}_{\langle -2r-2m-1,2m+1\rangle}) \tag{7.147}$$

are $(\mathcal{G},\mathcal{K})$-structures by the arguments in (7.79)-(7.82), (7.84) and (7.136) (cf. (7.24), (7.25)). $\Box$

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