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Irrationality of the Roots of the Yablonskii–Vorob'ev Polynomials and Relations between Them

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Abstract. We study the Yablonskii–Vorob'ev polynomials, which are special polynomials used to represent rational solutions of the second Painlevé equation. Divisibility properties of the coefficients of these polynomials, concerning powers of 4, are obtained and we prove that the nonzero roots of the Yablonskii–Vorob'ev polynomials are irrational. Furthermore, relations between the roots of these polynomials for consecutive degree are found by considering power series expansions of rational solutions of the second Painlevé equation.

Key words: second Painlevé equation; rational solutions; power series expansion; irrational roots; Yablonskii–Vorob'ev polynomials

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1 Introduction

In this paper we study the Yablonskii–Vorob'ev polynomials Q_n, with a special interest in their roots. These polynomials were derived by Yablonskii and Vorob'ev, while examining the hierarchy of rational solutions of the second Painlevé equation. The Yablonskii–Vorob'ev polynomials are defined by the differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4(Q_nQ_n'' - (Q_n')^2), \qquad (1)$$

with $Q_0 = 1$ and $Q_1 = z$. From the recurrence relation, it is clear that the functions Q_n are rational, though it is far from obvious that they are polynomials, since in every iteration one divides by Q_{n-1}. The Yablonskii–Vorob'ev polynomials Q_n are monic polynomials of degree $\frac{1}{2}n(n+1)$, with integer coefficients. The first few are given in Table 1.

Yablonskii [1] and Vorob'ev [2] expressed the rational solutions of the second Painlevé equation,

$$P_{\mathrm{II}}(\alpha): \quad w''(z) = 2w(z)^3 + zw(z) + \alpha,$$

with complex parameter α, in terms of the Yablonskii–Vorob'ev polynomials, as summerized in the following theorem:

Theorem 1. *$P_{\mathrm{II}}(\alpha)$ has a rational solution iff $\alpha = n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ the rational solution is unique and if $n \geq 1$, then it is equal to*

$$w_n = \frac{Q_{n-1}'}{Q_{n-1}} - \frac{Q_n'}{Q_n}.$$

The other rational solutions are given by $w_0 = 0$ and for $n \geq 1$, $w_{-n} = -w_n$.

Table 1.

Yablonskii–Vorob'ev polynomials
$Q_2 = 4 + z^3$
$Q_3 = -80 + 20z^3 + z^6$
$Q_4 = z(11200 + 60z^6 + z^9)$
$Q_5 = -6272000 - 3136000z^3 + 78400z^6 + 2800z^9 + 140z^{12} + z^{15}$
$Q_6 = -38635520000 + 19317760000z^3 + 1448832000z^6 - 17248000z^9 + 627200z^{12}$ $+18480z^{15} + 280z^{18} + z^{21}$
$Q_7 = z(-3093932441600000 - 49723914240000z^6 - 828731904000z^9 + 13039488000z^{12}$ $+62092800z^{15} + 5174400z^{18} + 75600z^{21} + 504z^{24} + z^{27})$
$Q_8 = -991048439693312000000 - 743286329769984000000z^3$ $+371643164884992000000z^6 + 176972935659520000z^9 + 12669653348352000z^{12}$ $+407736096768000z^{15} - 6629855232000z^{18} + 124309785600z^{21} + 2018016000z^{24}$ $+32771200z^{27} + 240240z^{30} + 840z^{33} + z^{36}$

The rational solutions of P_{II} can also be determined, using the Bäcklund transformations, first given by Gambier [3], of the second Painlevé equation, by

$$w_{n+1} = -w_n - \frac{2n+1}{2w_n^2 + 2w_n' + z}, \qquad w_{-n} = -w_n,$$

with "seed solution" $w_0 = 0$; see also Lukashevich [4] and Noumi [5].

We note that the Yablonskii–Vorob'ev polynomials find many applications in physics. For instance, solutions of the Korteweg–de Vries equation (Airault, McKean and Moser [6]) and the Boussinesq equation (Clarkson [7]) can be expressed in terms of these polynomials. Clarkson and Mansfield [8] studied the structure of the roots of the Yablonskii–Vorob'ev polynomials Q_n and observed that the roots, of each of these polynomials, form a highly regular triangular-like pattern, for $n \leq 7$, suggesting that they have interesting properties. This further motivates studying the zeros of the Yablonskii–Vorob'ev polynomials.

In Section 2 the divisibility of the coefficients of the Yablonskii–Vorob'ev polynomials by powers of 4 is examined. From the divisibility properties found, we conclude that nonzero roots of the Yablonskii–Vorob'ev polynomials are irrational. In Section 3 we study power series expansions of (functions related to) the rational solution w_n of $P_{\mathrm{II}}(n)$, around poles of w_n. This leads to relations between the roots of Q_{n-1} and Q_n. These relations suggest deeper connections between the zeros of Q_{n-1} and Q_n. Similarly, we look at power series expansions of (functions related to) the rational solution w_n of $P_{\mathrm{II}}(n)$ around 0, in Section 4. We obtain polynomial expressions in n, with rational coefficients, for sums of fixed negative powers of the nonzero roots of Q_n.

2 Nonzero roots are irrational

The Yablonskii–Vorob'ev polynomials Q_n are monic polynomials of degree $\frac{1}{2}n(n+1)$, and Taneda [9] proved:

- if $n \equiv 1 \pmod 3$, then $\frac{Q_n}{z} \in \mathbb{Z}[z^3]$;
- if $n \not\equiv 1 \pmod 3$, then $Q_n \in \mathbb{Z}[z^3]$.

Therefore, we have

$$Q_n = z^{\frac{1}{2}n(n+1)} + a_1^n z^{\frac{1}{2}n(n+1)-3} + a_2^n z^{\frac{1}{2}n(n+1)-6} + \cdots + a^n_{\left[\frac{1}{6}n(n+1)\right]} z^{\frac{1}{2}n(n+1)-3\left[\frac{1}{6}n(n+1)\right]}, \qquad (2)$$

for certain $a_s^n \in \mathbb{Z}$, with convention $a_0^n = 1$, where $[\cdot]$ denotes the floor function.

Lemma 1. *For every* $0 \le m \le \left[\frac{1}{6}n(n+1)\right]$, *we have* $4^m \mid a_m^n$.

Proof. We proceed by proving the following statement, by induction, for all $M \in \mathbb{N}$:

For every $1 \le m \le M$, for all $n \in \mathbb{N}$, whenever $m \le \left[\frac{1}{6}n(n+1)\right]$, we have $4^m \mid a_m^n$, and

$$4^M \mid a_{M+1}^n, \qquad 4^M \mid a_{M+2}^n, \qquad \ldots, \qquad 4^M \mid a_{\left[\frac{1}{6}n(n+1)\right]}^n.$$

Observe that the case $M = 0$ is trivial. Now suppose the statement is true for $M \in \mathbb{N}$. Then there are $b_s^n \in \mathbb{Z}$, such that for every $n \in \mathbb{N}$,

$$Q_n = z^{\frac{1}{2}n(n+1)} + 4b_1^n z^{\frac{1}{2}n(n+1)-3} + 4^2 b_2^n z^{\frac{1}{2}n(n+1)-6} + \cdots + 4^M b_M^n z^{\frac{1}{2}n(n+1)-3M} + 4^M P_n,$$

where $P_n \in \mathbb{Z}[z]$ is zero or has degree less or equal to $\frac{1}{2}n(n+1)-3(M+1)$, and if $m > \left[\frac{1}{6}n(n+1)\right]$, then $b_m^n = 0$.

To complete the induction, we need to show that for every $n \in \mathbb{N}$, $4 \mid P_n$. We prove this by induction with respect to n. Observe that $P_0 = 0$ and $P_1 = 0$, therefore, indeed $4 \mid P_0$ and $4 \mid P_1$. Assume $4 \mid P_{n-1}$ and $4 \mid P_n$. Then $4^M P_n \equiv 0 \pmod{4^{M+1}}$, therefore, modulo 4^{M+1}, we have:

$$z^{\max(0,n(n+1)-3M+1)} \mid zQ_n^2, \qquad z^{\max(0,n(n+1)-3M+1)} \mid 4Q_nQ_n'',$$
$$z^{\max(0,n(n+1)-3M+1)} \mid 4(Q_n')^2.$$

By the definition of Q_{n+1} (1),

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\big(Q_nQ_n'' - (Q_n')^2\big),$$

so

$$z^{\max(0,n(n+1)-3M+1)} \mid Q_{n+1}Q_{n-1} \pmod{4^{M+1}}. \tag{3}$$

Let us consider $Q_{n+1}Q_{n-1}$. Since $4 \mid P_{n-1}$, we have

$$4^M P_{n-1} \equiv 0 \pmod{4^{M+1}},$$

therefore, modulo 4^{M+1},

$$Q_{n+1}Q_{n-1} \equiv Q_{n+1}z^{\frac{1}{2}n(n-1)} + Q_{n+1}\big(4b_1^{n-1}z^{\frac{1}{2}n(n-1)-3} + 4^2 b_2^{n-1} z^{\frac{1}{2}n(n-1)-6} + \cdots + 4^M b_M^{n-1} z^{\frac{1}{2}n(n-1)-3M}\big). \tag{4}$$

Since

$$Q_{n+1} = z^{\frac{1}{2}(n+1)(n+2)} + 4b_1^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3} + 4^2 b_2^{n+1} z^{\frac{1}{2}(n+1)(n+2)-6} + \cdots + 4^M b_M^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3M} + 4^M P_{n+1},$$

we have, modulo 4^{M+1},

$$z^{\max(0,n(n+1)-3M+1)} \mid Q_{n+1}\big(4b_1^{n-1}z^{\frac{1}{2}n(n-1)-3} + 4^2 b_2^{n-1} z^{\frac{1}{2}n(n-1)-6} + \cdots + 4^M b_M^{n-1} z^{\frac{1}{2}n(n-1)-3M}\big).$$

Hence, by (3) and (4),

$$z^{\max(0,n(n+1)-3M+1)} \mid Q_{n+1}z^{\frac{1}{2}n(n-1)} \pmod{4^{M+1}},$$

which implies

$$z^{\max(0,\frac{1}{2}(n+1)(n+2)-3M)} \mid Q_{n+1} \pmod{4^{M+1}}.$$

Since

$$\begin{aligned}Q_{n+1} &= z^{\frac{1}{2}(n+1)(n+2)} + 4b_1^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3} \\ &\quad + 4^2 b_2^{n+1} z^{\frac{1}{2}(n+1)(n+2)-6} + \cdots + 4^M b_M^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3M} + 4^M P_{n+1},\end{aligned}$$

we have, therefore, $4 \mid P_{n+1}$. Hence, by induction, for all $n \in \mathbb{N}$, $4 \mid P_n$.

The lemma follows by induction on M. ■

Let us denote the coefficient of the lowest degree term in Q_n by

$$x_n := a^n_{\left[\frac{1}{6}n(n+1)\right]},$$

i.e. x_n is the constant coefficient in Q_n if $n \not\equiv 1 \pmod 3$, and x_n is the coefficient of z in Q_n if $n \equiv 1 \pmod 3$. Fukutani, Okamoto, and Umemura [10] proved that the roots of the Yablonskii–Vorob'ev polynomials are simple, hence x_n is nonzero. Let p_n be the multiplicity of 2 in the prime factorization of x_n. As a consequence of Lemma 1, we obtain that $p_n \geq 2\left[\frac{1}{6}n(n+1)\right]$. We prove

$$p_n = \left[\frac{1}{3}n(n+1)\right].$$

Observe that $x_n = Q_n(0)$ if $n \not\equiv 1 \pmod 3$, and $x_n = Q_n'(0)$ if $n \equiv 1 \pmod 3$. Fukutani, Okamoto, and Umemura [10] derived the following identity for the Yablonskii–Vorob'ev polynomials:

$$Q_{n+1}'Q_{n-1} - Q_{n+1}Q_{n-1}' = (2n+1)Q_n^2.$$

Using this identity at 0, we obtain

$$x_{n+1}x_{n-1} = \begin{cases} (2n+1)x_n^2 & \text{if } n \equiv 0 \pmod 3, \\ -(2n+1)x_n^2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

By evaluating equation (1) at 0,

$$x_{n+1}x_{n-1} = 4x_n^2, \qquad \text{if } n \equiv 1 \pmod 3.$$

Therefore, we have the following recursion for $(x_n)_n$:

$$x_0 = 1, \qquad x_1 = 1 \qquad \text{and}$$

$$x_{n+1}x_{n-1} = \begin{cases} (2n+1)x_n^2 & \text{if } n \equiv 0 \pmod 3, \\ 4x_n^2 & \text{if } n \equiv 1 \pmod 3, \\ -(2n+1)x_n^2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

So, we obtain the following recursion for $(p_n)_n$:

$$p_0 = 0, \qquad p_1 = 0 \qquad \text{and}$$

$$p_{n+1} = \begin{cases} 2p_n - p_{n-1} & \text{if } n \not\equiv 1 \pmod 3, \\ 2 + 2p_n - p_{n-1} & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Using this recursion, the formula $p_n = \left[\frac{1}{3}n(n+1)\right]$, can be proven directly, by induction.

Remark 1. Kaneko and Ochiai [11] found an explicit expression for the coefficients x_n. But deriving the formula $p_n = \left[\frac{1}{3}n(n+1)\right]$ directly from this expression seems to be a difficult task.

Theorem 2. *The nonzero roots of the Yablonskii–Vorob'ev polynomials are irrational.*

Proof. Let $n \not\equiv 1 \pmod 3$. Suppose x is a rational root of Q_n. Since $Q_n \in \mathbb{Z}[z]$ is monic, by Gauss's lemma, $x \in \mathbb{Z}$. By Lemma 1,

$$Q_n \equiv z^{\frac{1}{2}n(n+1)} \pmod 4,$$

so x is even. Let $y := \frac{x}{2}$, then, by equation (2),

$$0 = (2y)^{\frac{1}{2}n(n+1)} + a_1^n (2y)^{\frac{1}{2}n(n+1)-3} + a_2^n (2y)^{\frac{1}{2}n(n+1)-6} + \cdots + a^n_{\frac{1}{6}n(n+1)-1}(2y)^3 + a^n_{\frac{1}{6}n(n+1)}.$$

By Lemma 1, for every $m \le \frac{1}{6}n(n+1)$, we have $4^m \mid a_m^n$. Hence

$$\begin{aligned} &2^{\frac{1}{2}n(n+1)} \mid (2y)^{\frac{1}{2}n(n+1)}, \qquad 2^{\frac{1}{2}n(n+1)-1} \mid a_1^n (2y)^{\frac{1}{2}n(n+1)-3},\\ &2^{\frac{1}{2}n(n+1)-2} \mid a_2^n (2y)^{\frac{1}{2}n(n+1)-6}, \qquad \ldots, \qquad 2^{\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)+1} \mid a_{\frac{1}{6}n(n+1)-1}(2y)^3. \end{aligned}$$

So

$$2^{\frac{1}{3}n(n+1)+1} \mid a^n_{\frac{1}{6}n(n+1)} = x_n,$$

which implies

$$p_n \ge \frac{1}{3}n(n+1) + 1.$$

But $p_n = \frac{1}{3}n(n+1)$, a contradiction, hence roots of Q_n are irrational.

If $n \equiv 1 \pmod 3$, we can apply the same reasoning to $\frac{Q_n}{z}$, and show that roots of $\frac{Q_n}{z}$ are irrational. Therefore, nonzero roots of Q_n are irrational. ∎

This result raises the question whether the Yablonskii–Vorob'ev polynomials, excluding the trivial factor z in case $n \equiv 1 \pmod 3$, are irreducible in $\mathbb{Q}[z]$. Kametaka [12] showed that for $n \le 23$, the Yablonskii–Vorob'ev polynomials Q_n are indeed irreducible.

3 Relations between roots of the Yablonskii–Vorob'ev polynomials

By Theorem 1, for $n \ge 1$, the unique rational solution of $P_{\rm II}(n)$ is given by

$$w_n = \frac{Q'_{n-1}}{Q_{n-1}} - \frac{Q'_n}{Q_n}.$$

Fukutani, Okamoto, and Umemura [10] proved that the roots of the Yablonskii–Vorob'ev polynomials are simple, hence

$$w_n = \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z - z_{n-1,k}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z - z_{n,k}}, \tag{5}$$

where the $z_{m,k}$ are the roots of Q_m. From equation (5) and the fact that w_n is the rational solution of $P_{\rm II}(n)$, we obtain relations between the zeros of Q_{n-1} and Q_n.

Theorem 3. *For* $1 \le j \le \frac{1}{2}n(n-1)$*:*

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,j}-z_{n-1,k}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n-1,j}-z_{n,k}} = 0,$$

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j}-z_{n-1,k})^2} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j}-z_{n,k})^2} = \frac{z_{n-1,j}}{6},$$

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j}-z_{n-1,k})^3} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j}-z_{n,k})^3} = -\frac{n+1}{4},$$

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j}-z_{n-1,k})^5} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j}-z_{n,k})^5} = z_{n-1,j}\left(\frac{n+1}{24} - \frac{1}{36}\right).$$

For $1 \le j \le \frac{1}{2}n(n+1)$*:*

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n,j}-z_{n-1,k}} - \sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,j}-z_{n,k}} = 0,$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j}-z_{n-1,k})^2} - \sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j}-z_{n,k})^2} = -\frac{z_{n,j}}{6},$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j}-z_{n-1,k})^3} - \sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j}-z_{n,k})^3} = -\frac{n-1}{4},$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j}-z_{n-1,k})^5} - \sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j}-z_{n,k})^5} = z_{n,j}\left(\frac{n-1}{24} + \frac{1}{36}\right).$$

Proof. Let $1 \le j \le \frac{1}{2}n(n-1)$ and define $\omega := z_{n-1,j}$ and $u := w_n - \frac{1}{z-\omega}$. Since $\gcd(Q_{n-1}, Q_n) = 1$, see Fukutani, Okamoto, and Umemura [10], equation (5) shows that u is holomorphic in a neighbourhood of ω. Hence u has a power series expansion, say

$$\sum_{m=0}^{\infty} a_m (z-\omega)^m,$$

which converges in an open disc centered at ω.

Since w_n is a solution of $P_{\mathrm{II}}(n)$, u satisfies

$$(z-\omega)^2 u'' = 6u + 6(z-\omega)u^2 + 2(z-\omega)^2 u^3 + (n+1)(z-\omega)^2 + \omega(z-\omega) + (z-\omega)^3 u + \omega(z-\omega)^2 u.$$

Hence we have the following identity in an open disc centered at ω:

$$\sum_{m=2}^{\infty} (m-1)m a_m (z-\omega)^m = 6\sum_{m=0}^{\infty} a_m (z-\omega)^m + 6(z-\omega)\left(\sum_{m=0}^{\infty} a_m (z-\omega)^m\right)^2 + 2(z-\omega)^2\left(\sum_{m=0}^{\infty} a_m (z-\omega)^m\right)^3 + (n+1)(z-\omega)^2 + \omega(z-\omega)$$

$$+ (z-\omega)^3 \sum_{m=0}^{\infty} a_m (z-\omega)^m + \omega (z-\omega)^2 \sum_{m=0}^{\infty} a_m (z-\omega)^m.$$

By considering coefficients of $(z-\omega)^n$, $n = 0, 1, 2, 4$, it is easy to deduce that $a_0 = 0$, $a_1 = -\frac{\omega}{6}$, $a_2 = -\frac{n+1}{4}$ and $a_4 = \omega\left(\frac{n+1}{24} - \frac{1}{36}\right)$. Note that a_3 does not follow from considering coefficients of $(z-\omega)^3$.

By Taylor's theorem and equation (5),

$$a_m = \frac{u^{(m)}(z_{n-1,j})}{m!} = (-1)^m \left(\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j} - z_{n-1,k})^{m+1}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^{m+1}} \right).$$

The first half of the theorem follows, the second half is proved analogously. ■

Note that countably many nontrivial relations can be found between the a_m in the above proof, by considering the coefficient of $(z-\omega)^n$, for $n \in \mathbb{N}$.

In Kudryashov and Demina [13] similar relations for the roots of Q_n are obtained using the Korteweg–de Vries equation. In particular, the following results are presented in [13] for $1 \leq j \leq \frac{1}{2}n(n+1)$:

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^2} = -\frac{z_{n,j}}{12}, \qquad \sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^3} = 0,$$

$$\sum_{k=1,\, k\neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^5} = -\frac{z_{n,j}}{144}.$$

From these relations and Theorem 3, we obtain the following corollary:

Corollary 1. *For $1 \leq j \leq \frac{1}{2}n(n-1)$:*

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^2} = -\frac{z_{n-1,j}}{4}, \qquad \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^3} = \frac{n+1}{4},$$

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^5} = -z_{n-1,j}\left(\frac{n+1}{24} - \frac{1}{48}\right).$$

For $1 \leq j \leq \frac{1}{2}n(n+1)$:

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^2} = -\frac{z_{n,j}}{4}, \qquad \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^3} = -\frac{n-1}{4},$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^5} = z_{n,j}\left(\frac{n-1}{24} + \frac{1}{48}\right).$$

In Theorem 3, we have obtained 4 times $\frac{1}{2}n(n-1)$ plus 4 times $\frac{1}{2}n(n+1)$ equations satisfied by the $\frac{1}{2}n(n+1)$ roots of Q_n, suggesting that these equations can be used to determine the roots of the polynomials Q_n recursively. If so, then these equations may be of use to derive properties of the roots of the Yablonskii–Vorob'ev polynomials. We shall not pursue this issue further here.

4 Sums of negative powers of roots

In Section 2, the rational solutions w_n of $P_{\mathrm{II}}(n)$ were studied around roots of the Yablonskii–Vorob'ev polynomials. In this section, we consider w_n at 0.

Let $n \equiv 0 \pmod 3$, then 0 is not a root of Q_{n-1} or Q_n. Therefore, by equation (5), w_n is holomorphic in a neighbourhood of 0. So w_n has a power series expansion, say

$$\sum_{m=0}^{\infty} a_m z^m,$$

which converges on an open disc centered at 0.

By Taylor's theorem and equation (5), we have

$$a_m = -\left(\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{m+1}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} \right).$$

Let $\omega := e^{\frac{2\pi i}{3}}$. Since $n \equiv 0 \pmod 3$, $Q_n \in \mathbb{Z}[z^3]$. Therefore, the roots of Q_n are invariant under multiplication by ω. Hence

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} = \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(\omega z_{n,k})^{m+1}} = \frac{1}{\omega^{m+1}} \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}},$$

therefore, if $m \not\equiv 2 \pmod 3$,

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} = 0. \tag{6}$$

By the same reason, if $m \not\equiv 2 \pmod 3$,

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{m+1}} = 0.$$

So $a_m = 0$, if $m \not\equiv 2 \pmod 3$, and in an open disc centered at 0,

$$w_n(z) = \sum_{m=0}^{\infty} a_{3m+2} z^{3m+2}.$$

Since w_n is a solution of $P_{\mathrm{II}}(n)$, we have the following identity in an open disc centered at 0:

$$\sum_{m=0}^{\infty} (3m+1)(3m+2) a_{3m+2} z^{3m} = 2 \left(\sum_{m=0}^{\infty} a_{3m+2} z^{3m+2} \right)^3 + \sum_{m=0}^{\infty} a_{3m+2} z^{3m+3} + n.$$

Comparing coefficients gives $a_2 = \frac{1}{2}n$, $a_5 = \frac{1}{40}n$ and $a_8 = \frac{1}{2240}n + \frac{1}{224}n^3$. We have obtained the following relations for $n \equiv 0 \pmod 3$:

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{3}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{3}} = -\frac{n}{2}, \qquad \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{6}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{6}} = -\frac{n}{40},$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = -\frac{1}{2240}n - \frac{1}{224}n^3.$$

If $n \equiv 1 \pmod 3$, then $u := w_n + \frac{1}{z}$ is holomorphic at 0 and satisfies

$$z^2u'' = 6u - 6zu^2 + 2z^2u^3 + z^3u + (n-1)z^2.$$

By considering the power series expansion of $u = w_n + \frac{1}{z}$ around 0, the following relations are found:

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^3} - \sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} = \frac{1}{4}(n-1),$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^6} - \sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} = \frac{1}{56}(n-1) + \frac{3}{112}(n-1)^2,$$

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = \frac{1}{2800}(n-1) + \frac{9}{5600}(n-1)^2 + \frac{1}{448}(n-1)^3.$$

If $n \equiv 2 \pmod 3$, then $u := w_n - \frac{1}{z}$ is holomorphic at 0 and satisfies

$$z^2u'' = 6u - 6zu^2 + 2z^2u^3 + z^3u + (n+1)z^2.$$

By considering the power series expansion of $u = w_n - \frac{1}{z}$ around 0, the following relations are found:

$$\sum_{k=1,\, z_{n-1,k}\neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^3} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} = \frac{1}{4}(n+1),$$

$$\sum_{k=1,\, z_{n-1,k}\neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^6} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} = \frac{1}{56}(n+1) - \frac{3}{112}(n+1)^2,$$

$$\sum_{k=1,\, z_{n-1,k}\neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = \frac{1}{2800}(n+1) - \frac{9}{5600}(n+1)^2 + \frac{1}{448}(n+1)^3.$$

Remark 2. Considering higher order coefficients, we see that for every threefold $m \geq 3$, polynomial expressions in n, with rational coefficients, depending on $n \pmod 3$, exist for

$$\sum_{k=1,\, z_{n-1,k}\neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^m} - \sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m}.$$

As a corollary of these relations, by induction, we obtain:

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} = \begin{cases} \dfrac{n}{4} & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \equiv 1 \pmod 3, \\ -\dfrac{n+1}{4} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} = \begin{cases} \frac{1}{40}n^2 + \frac{1}{80}n & \text{if } n \equiv 0 \pmod 3, \\ -\frac{1}{560}n^2 - \frac{1}{560}n + \frac{1}{280} & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{40}n^2 + \frac{3}{80}n + \frac{1}{80} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = \begin{cases} \frac{n + 7n^2 + 10n^3}{4480} & \text{if } n \equiv 0 \pmod 3, \\ \frac{2 - n - n^2}{22400} & \text{if } n \equiv 1 \pmod 3, \\ \frac{-20 - 85n - 115n^2 - 50n^3}{22400} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

By Remark 2, for every threefold $m \geq 3$, polynomial expressions in n, with rational coefficients, depending on $n \pmod 3$, exist for

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m}.$$

If $m \not\equiv 0 \pmod 3$, see equation (6), then

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m} = 0.$$

So, for all $n, m \in \mathbb{N}$,

$$\sum_{k=1,\, z_{n,k}\neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m} \in \mathbb{Q},$$

even though the nonzero roots of the Yablonskii–Vorob'ev polynomials are irrational.

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