

**FONTAINE'S PROPERTY (P_m)
AT THE MAXIMAL RAMIFICATION BREAK**

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ABSTRACT. We completely determine which extension of local fields satisfies Fontaine's property (P_m) for a given real number m .

1. INTRODUCTION

Let K be a complete discrete valuation field with perfect residue field k . All algebraic extensions of K are taken inside a fixed algebraic closure of K . We denote by v_K the normalized valuation of K and its extension to any algebraic extension of K . For an algebraic extension E/K and a non-negative real number m , we denote by \mathcal{O}_E the ring of integers of E and by $\mathfrak{a}_{E/K}^m$ the set of elements $x \in \mathcal{O}_E$ with $v_K(x) \geq m$. In [2], Fontaine gave upper bounds for ramification of Galois representations of K appearing in the generic fibers of finite flat group schemes over \mathcal{O}_K . For this, he considered the following property (P_m) for a finite Galois extension L/K for each non-negative real number m :

(P_m) If E is an algebraic extension of K and if there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then $L \subset E$.

By results of Fontaine ([2, Proposition 1.5]) and the second author ([9, Theorem 5.2]), every finite Galois extension L/K satisfies (P_m) for $m > u_{L/K}$ and does not satisfy (P_m) for $m < u_{L/K}$, where $u_{L/K}$ is the maximal upper ramification break for L/K (see Notation below for the definition). For $m = u_{L/K}$, it is easy to see that trivial L/K satisfies (P_m) and non-trivial tame L/K does not satisfy (P_m) . Here we call L/K tame if $u_{L/K} \leq 1$ and wild if $u_{L/K} > 1$.

In this paper, we determine which wild L/K satisfies (P_m) for $m = u_{L/K}$. The result is the following, which shows that the answer to the question depends (only) on the residue field of K .

Theorem 1.1. *Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$ and let L be a finite wild Galois extension of K . Then L/K satisfies (P_m) for $m = u_{L/K}$ if and only if any finite extension of k has no Galois extension of degree p .*

Even if there exists a finite extension of k having a Galois extension of degree p (for example, if k is finite), the extension L/K still satisfies a property that is weaker than (P_m) (see Remark 5.3).

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The proof consists of four parts. In section 2, we reduce the theorem to the case where L/K has only one jump in the ramification filtration of its Galois group. In this case, L/K is an abelian extension. In section 3, we reduce the theorem so that we need only totally ramified E/K for the extensions appearing in the definition of (P_m) . In section 4, we prove some facts on the local class field theory of Serre and Hazewinkel ([6], [1]). In section 5, we prove the theorem for the reduced case by using the local class field theory of Serre and Hazewinkel. Note that this technique works for more general abelian cases; see Remark 5.4.

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Notation. For a complete discrete valuation field K with perfect residue field k , we denote by v_K the discrete valuation with $v_K(K^\times) = \mathbb{Z}$, by \mathcal{O}_K the ring of integers, by \mathfrak{p}_K the maximal ideal, by U_K the group of units, and by U_K^n the group of n -th principal units ($n \geq 0$). All algebraic extensions of K are taken inside a fixed algebraic closure \overline{K} of K . We denote by $K^{\text{sep}} (\subset \overline{K})$ the separable closure of K , by K^{ur} the maximal unramified extension of K and by K^{ab} the maximal abelian extension of K . The same notation and convention are applied to k . The absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ (resp. $\text{Gal}(\overline{k}/k)$) is denoted by G_K (resp. G_k). We extend v_K to \overline{K} . Note that $v_K(\overline{K}^\times) = \mathbb{Q} \supset \mathbb{Z}$. For an algebraic extension E/K , we denote by \mathcal{O}_E the ring of integers, by \hat{E} the completion of E , and by $e_{E/K}$ the ramification index of E/K . We define $\mathfrak{a}_{E/K}^m = \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}$ for a non-negative real number m . If E/K is totally ramified, an extension of the form EK'/K' for an unramified extension K'/K will be called an unramified base change of E/K . For a finite Galois extension L/K , let $\text{Gal}(L/K)_i$, $\text{Gal}(L/K)^u$ (resp. $\varphi_{L/K}$, $\psi_{L/K}$) be the ramification groups (resp. the Herbrand functions) that are studied in [7]. Also, let $\text{Gal}(L/K)_{(i)}$, $\text{Gal}(L/K)^{(u)}$ (resp. $\tilde{\varphi}_{L/K}$, $\tilde{\psi}_{L/K}$) be the ramification groups (resp. the Herbrand functions) that are defined in [2]. Their relations are as follows: for real numbers $i, u \geq -1$, we have

$$\begin{aligned} \text{Gal}(L/K)_i &= \text{Gal}(L/K)_{((i+1)/e_{L/K})}, & \text{Gal}(L/K)^u &= \text{Gal}(L/K)^{(u+1)}, \\ \varphi_{L/K}(i) &= \tilde{\varphi}_{L/K}((i+1)/e_{L/K}) - 1, & \psi_{L/K}(u) &= e_{L/K}\tilde{\psi}_{L/K}(u+1) - 1. \end{aligned}$$

We define the maximal upper ramification break for L/K to be the maximal real number $u_{L/K}$ with non-trivial $\text{Gal}(L/K)^{(u_{L/K})}$. Set $i_{L/K} = \tilde{\psi}_{L/K}(u_{L/K})$; it is the maximal real number with non-trivial $\text{Gal}(L/K)_{(i_{L/K})}$.

2. REDUCTION TO ABELIAN L/K

The following proposition allows us to reduce the proof of Theorem 1.1 to the case where L/K has only one jump in the ramification filtration of its Galois group.

Proposition 2.1. *Let L/K be a finite Galois extension of complete discrete valuation fields with perfect residue fields. Let M be the $\text{Gal}(L/K)^{(u_{L/K})}$ -fixed subfield of L . Then L/K satisfies (P_m) for $m = u_{L/K}$ if and only if L/M satisfies (P_m) for $m = u_{L/M}$.*

Note that any finite extension of the residue field of K has no Galois extension of degree p if and only if the same holds for M , since the maximal pro- p quotient of

the absolute Galois group of a field of characteristic p is pro- p free ([8, Chapter 2, §2.2, Corollary 1]). Hence Proposition 2.1 allows us to reduce the proof of Theorem 1.1 for L/K to that for L/M .

The proposition is a direct consequence of the following lemma.

Lemma 2.2. *Let K, L, M be as in Proposition 2.1. Let E be an algebraic extension of K . Then the following are equivalent:*

- (1) *There exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{u_{L/K}}$.*
- (2) *The field M is contained in E and there exists an \mathcal{O}_M -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/M}^{u_{L/M}}$.*

Proof. (1) \implies (2). First we show that M is contained in E . The case L/K is unramified is trivial, so we assume L/K is not unramified. By assumption, there exists an \mathcal{O}_K -algebra homomorphism $\eta: \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{u_{L/K}}$. Consider the composite of the inclusion $\mathcal{O}_M \hookrightarrow \mathcal{O}_L$ and η . Since $u_{L/K} > u_{M/K}$, the extension M/K satisfies (P_m) for $m = u_{L/K}$. Hence M is contained in E .

Next we show the existence of an \mathcal{O}_M -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/M}^{u_{L/M}}$. Let α be a generator of \mathcal{O}_L as an \mathcal{O}_K -algebra, let f be the minimal polynomial of α over K , and let β be a lift of $\eta(\alpha) \in \mathcal{O}_E/\mathfrak{a}_{E/K}^{u_{L/K}}$ to \mathcal{O}_E . We have $v_K(f(\beta)) \geq u_{L/K}$ by the well-definedness of η . By [2, the second proposition of §1.3], we have $\tilde{\psi}_{L/K}(v_K(f(\beta))) = v_K(\beta - \sigma(\alpha))$ for some $\sigma \in \text{Gal}(L/K)$. Hence we have $v_K(\beta - \sigma(\alpha)) \geq \tilde{\psi}_{L/K}(u_{L/K}) = i_{L/K}$. Since $\text{Gal}(L/M)_{(i)} = \text{Gal}(L/K)_{(i/e_{M/K})} \cap \text{Gal}(L/M)$ for $i \geq 0$ and $\text{Gal}(L/M) = \text{Gal}(L/K)_{(i_{L/K})}$, we have $i_{L/M} = e_{M/K}i_{L/K}$. Hence $v_M(\beta - \sigma(\alpha)) \geq e_{M/K}i_{L/K} = i_{L/M}$. Thus we have $v_M(g(\beta)) \geq u_{L/M}$ by [2, loc. cit.], where g is the minimal polynomial of $\sigma(\alpha)$ over M . Since $\mathcal{O}_L = \mathcal{O}_K[\sigma(\alpha)] = \mathcal{O}_M[\sigma(\alpha)]$, we can define an \mathcal{O}_M -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{u_{L/M}}$ by sending $\sigma(\alpha)$ to β . This proves (1) \implies (2).

(2) \implies (1). By assumption, there exists an \mathcal{O}_M -algebra homomorphism $\eta: \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/M}^{u_{L/M}}$. Let α be a generator of \mathcal{O}_L as an \mathcal{O}_K -algebra, let g be the minimal polynomial of α over M , and let β be a lift of $\eta(\alpha) \in \mathcal{O}_E/\mathfrak{a}_{E/M}^{u_{L/M}}$ to \mathcal{O}_E . We have $v_M(g(\beta)) \geq u_{L/M}$ by the well-definedness of η . We have $\tilde{\psi}_{L/M}(v_M(g(\beta))) = v_M(\beta - \sigma(\alpha))$ for some $\sigma \in \text{Gal}(L/M)$ by [2, loc. cit.]. Hence $v_M(\beta - \sigma(\alpha)) \geq \tilde{\psi}_{L/M}(u_{L/M}) = i_{L/M}$. Using the equality $i_{L/M} = e_{M/K}i_{L/K}$ obtained above, we have $v_K(\beta - \sigma(\alpha)) \geq i_{L/K}$. Thus we have $v_K(f(\beta)) \geq u_{L/K}$ by [2, loc. cit.], where f is the minimal polynomial of α over K . Thus we can define an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{u_{L/K}}$ by sending α to β . This proves (2) \implies (1). \square

3. REDUCTION TO TOTALLY RAMIFIED E/K

In this section, we reduce the proof of Theorem 1.1 to the case that the extensions E/K appearing in the definition of (P_m) are totally ramified. To be more precise, we consider the following property (P_m^{tr}) :

- (P_m^{tr}) If E is a totally ramified algebraic extension of K and if there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then $L \subset E$.

Proposition 3.1. *Let K be a complete discrete valuation field with perfect residue field k , let L be a finite Galois totally ramified extension of K and let m be a non-negative real number. Then L/K satisfies (P_m) if and only if any finite unramified base change L'/K' of L/K satisfies (P_m^{tr}) .*

Proof. First we assume that any finite unramified base change L'/K' of L/K satisfies (P_m^{tr}) and show that L/K satisfies (P_m) . Suppose E/K is an algebraic extension and there exists an \mathcal{O}_K -algebra homomorphism $\eta: \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$. We may assume E/K is finite. Let $K' = E \cap K^{\text{ur}}$ and set $L' = LK'$. Let $\eta': \mathcal{O}_{L'} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K'}^m$ be the base change of η by $\mathcal{O}_{K'}$. By assumption, we have $L' \subset E$. Hence $L \subset E$. This shows that L/K satisfies (P_m) .

Next we assume that L/K satisfies (P_m) and show that any finite unramified base change L'/K' of L/K satisfies (P_m^{tr}) . Suppose E/K' is a totally ramified algebraic extension and there exists an $\mathcal{O}_{K'}$ -algebra homomorphism $\eta': \mathcal{O}_{L'} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K'}^m$. Consider the composite of the inclusion $\mathcal{O}_L \hookrightarrow \mathcal{O}_{L'}$ and η' . By assumption, (P_m) is true for L/K . Hence $L \subset E$. Thus $L' = LK' \subset E$. This shows that L'/K' satisfies (P_m^{tr}) . \square

4. PRELIMINARY RESULTS ON THE LOCAL CLASS FIELD THEORY OF SERRE AND HAZEWINKEL

In this section, we provide some preliminary results on the local class field theory of Serre and Hazewinkel ([6], [1, Appendice]). We work on the site $(\text{Perf}/k)_{\text{fpqc}}$ of perfect schemes over a perfect field k with the fpqc topology (compare with [4, Chapter III, §0, “Duality for unipotent perfect group schemes”). The category of sheaves of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$ contains both the category of commutative affine pro(-quasi)-algebraic groups over k in the sense of Serre ([5]) and the category of commutative étale group schemes over k as thick abelian full subcategories. For $i \geq 0$, we denote by Ext_k^i the i -th ext functor on the category of sheaves of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$. For a sheaf A of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$ and a non-negative integer i , we define the i -th homotopy group $\pi_i^k(A)$ of A to be the Pontryagin dual of the injective limit of the torsion abelian groups $\text{Ext}_k^i(A, n^{-1}\mathbb{Z}/\mathbb{Z})$ for $n \geq 1$. The system $\{\pi_i^k\}_{i \geq 0}$ is a covariant homological functor from the category of sheaves of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$ to the category of profinite abelian groups.

First we extend the local class field theory of Serre and Hazewinkel in a canonical way for abelian extensions with unramified part (Proposition 4.1 below). Let K be a complete discrete valuation field with perfect residue field k and let \mathcal{O}_K be its ring of integers. We define a sheaf \mathbf{O}_K of rings on $(\text{Perf}/k)_{\text{fpqc}}$ as follows. For each perfect k -algebra R , we set $\mathbf{O}_K(R) = W(R) \otimes_{W(k)} \mathcal{O}_K$ if K has mixed characteristic, and $\mathbf{O}_K(R) = R \hat{\otimes}_k \mathcal{O}_K$ if K has equal characteristic, where W is the sheaf of the rings of Witt vectors of infinite length and $\hat{\otimes}$ denotes the completed tensor product. Let \mathbf{K} be the sheaf of rings on $(\text{Perf}/k)_{\text{fpqc}}$ with $\mathbf{K}(R) = \mathbf{O}_K(R)[(1 \otimes \pi_K)^{-1}]$, where π_K is a prime element of \mathcal{O}_K . This is independent of the choice of π_K . We set $\mathbf{U}_K = \mathbf{O}_K^\times$. For each $n \geq 0$, the sheaf of rings \mathbf{O}_K has a subsheaf of ideals \mathfrak{p}_K^n with $\mathfrak{p}_K^n(R) = \mathbf{O}_K(R) \otimes_{\mathcal{O}_K} \mathfrak{p}_K^n$ for a perfect k -algebra R . The presentation $\mathbf{O}_K = \text{projlim}_{n \rightarrow \infty} \mathbf{O}_K/\mathfrak{p}_K^n$ gives an affine proalgebraic ring structure for \mathbf{O}_K . Likewise, \mathbf{U}_K has a subsheaf of groups $\mathbf{U}_K^n = 1 + \mathfrak{p}_K^n$ for each $n \geq 1$ (for $n = 0$, we set $\mathbf{U}_K^0 = \mathbf{U}_K$). The presentation $\mathbf{U}_K = \text{projlim}_{n \rightarrow \infty} \mathbf{U}_K/\mathbf{U}_K^n$ gives an affine proalgebraic group structure for \mathbf{U}_K . We have a split exact sequence $0 \rightarrow \mathbf{U}_K \rightarrow \mathbf{K}^\times \rightarrow \mathbb{Z} \rightarrow 0$.

For a finite extension L/K with residue extension k'/k , we can define sheaves $\mathbf{O}_{L,k}$, \mathbf{L}_k , $\mathbf{U}_{L,k}$ and $\mathbf{U}_{L,k}^n$ on $(\text{Perf}/k)_{\text{fpqc}}$ and sheaves $\mathbf{O}_{L,k'}$, $\mathbf{L}_{k'}$, $\mathbf{U}_{L,k'}$ and $\mathbf{U}_{L,k'}^n$ on $(\text{Perf}/k')_{\text{fpqc}}$ in similar ways. For example, in the mixed case, we define

$\mathbf{O}_{L,k}(R) = W(R) \otimes_{W(k)} \mathcal{O}_L$ for a perfect k -algebra R and $\mathbf{O}'_{L,k'}(R') = W(R') \otimes_{W(k')} \mathcal{O}_L$ for a perfect k' -algebra R' , etc. When L/K is totally ramified, we can omit the subscript k ($= k'$) without ambiguity.

Proposition 4.1. *There exists a canonical isomorphism $\pi_1^k(\mathbf{K}^\times) \xrightarrow{\sim} G_K^{\text{ab}}$ and a commutative diagram with exact rows*

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \pi_1^k(\mathbf{U}_K) & \longrightarrow & \pi_1^k(\mathbf{K}^\times) & \longrightarrow & \pi_1^k(\mathbb{Z}) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & T(K^{\text{ab}}/K) & \longrightarrow & G_K^{\text{ab}} & \longrightarrow & G_k^{\text{ab}} \longrightarrow 0. \end{array}$$

Here G_K^{ab} (resp. G_k^{ab}) is the Galois group of the maximal abelian extension K^{ab}/K (resp. k^{ab}/k) and T denotes the inertia group. The left vertical isomorphism is the reciprocity map of the local class field theory of Hazewinkel ([1, Appendice, §7.3]). The right vertical isomorphism is dual to the isomorphism $H^1(G_k, \mathbb{Q}/\mathbb{Z}) \cong \text{Ext}_k^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$.

The proof below requires a lemma on calculation of Tate cohomology. Let G be a finite group and let A be a sheaf of G -modules on $(\text{Perf}/k)_{\text{fpqc}}$. For a perfect k -algebra R , let $\{C^i(G, A(R))\}_{i \in \mathbb{Z}}$ be the standard complete resolution of the G -module $A(R)$. The functor $C^i(G, A): R \mapsto C^i(G, A(R))$ is a sheaf of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$, which is isomorphic to a finite product of copies of A . We define the i -th Tate cohomology sheaf of G with values in A , denoted by $\hat{H}^i(G, A)$, to be the i -th cohomology of the complex $\{C^i(G, A)\}_{i \in \mathbb{Z}}$. The sheaf $\hat{H}^i(G, A)$ is associated with the presheaf $R \mapsto \hat{H}^i(G, A(R))$ of Tate cohomology groups.

Let L/K be a finite Galois extension with residue extension k'/k and set $G = \text{Gal}(L/K)$, $\mathfrak{g} = \text{Gal}(k'/k)$ and $T_a = T(L \cap K^{\text{ab}}/K)$. We regard these groups as constant groups over k . Let $\mathbb{Z}[\mathfrak{g}]$ be the group ring regarded as an étale group over k . Let I_G (resp. $I_{\mathfrak{g}}$) be the augmentation ideal of $\mathbb{Z}[G]$ (resp. $\mathbb{Z}[\mathfrak{g}]$). We set $N_G = \sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$. The sheaves $\mathbf{U}_{L,k}$, \mathbf{L}_k^\times are sheaves of G -modules on $(\text{Perf}/k)_{\text{fpqc}}$.

Lemma 4.2. *The Tate cohomology sheaf $\hat{H}^i(G, \mathbf{L}_k^\times)$ vanishes for all $i \in \mathbb{Z}$.*

Proof. First we show that the sheaf $\hat{H}^i(G, \mathbf{L}_k^\times)$ is represented by an affine scheme. The groups $\mathbf{U}_{L,k}$, \mathbf{L}_k^\times and $\mathbb{Z}[\mathfrak{g}]$ are the Weil restrictions of $\mathbf{U}_{L,k'}$, $\mathbf{L}_{k'}^\times$ and \mathbb{Z} , respectively, from k' to k . Thus we have an exact sequence $0 \rightarrow \mathbf{U}_{L,k} \rightarrow \mathbf{L}_k^\times \rightarrow \mathbb{Z}[\mathfrak{g}] \rightarrow 0$. This short exact sequence gives a long exact sequence

$$\cdots \rightarrow \hat{H}^{i-1}(G, \mathbb{Z}[\mathfrak{g}]) \xrightarrow{d_{i-1}} \hat{H}^i(G, \mathbf{U}_{L,k}) \rightarrow \hat{H}^i(G, \mathbf{L}_k^\times) \rightarrow \hat{H}^i(G, \mathbb{Z}[\mathfrak{g}]) \xrightarrow{d_i} \cdots$$

and a short exact sequence

$$0 \rightarrow \text{Coker}(d_{i-1}) \rightarrow \hat{H}^i(G, \mathbf{L}_k^\times) \rightarrow \text{Ker}(d_i) \rightarrow 0,$$

where d_i is the coboundary map. The group $\hat{H}^i(G, \mathbf{U}_{L,k})$ is affine since $\mathbf{U}_{L,k}$ is affine. The group $\hat{H}^i(G, \mathbb{Z}[\mathfrak{g}])$ is étale since $\mathbb{Z}[\mathfrak{g}]$ is étale. Moreover, the étale group $\hat{H}^i(G, \mathbb{Z}[\mathfrak{g}])$ is finite since G is finite and $\mathbb{Z}[\mathfrak{g}]$ is finitely generated over \mathbb{Z} ([7, Chap. VIII, §2, Cor. 2]). Hence $\hat{H}^i(G, \mathbb{Z}[\mathfrak{g}])$ is affine. Thus $\text{Coker}(d_{i-1})$ and $\text{Ker}(d_i)$ are affine. Therefore $\hat{H}^i(G, \mathbf{L}_k^\times)$ is affine.

Thus, to show the vanishing of $\hat{H}^i(G, \mathbf{L}_k^\times)$, it is enough to show the vanishing of the group of its \bar{k} -points. We have $\hat{H}^i(G, \mathbf{L}_k^\times)(\bar{k}) = \hat{H}^i(G, \mathbf{L}_k^\times(\bar{k})) = \hat{H}^i(G, (\hat{K}^{\text{ur}} \otimes_K$

$L)^\times$). By [7, Chap. XIII, §5, Exercise 2, a)], we have $\hat{H}^i(G, (\hat{K}^{\text{ur}} \otimes_K L)^\times) = 0$. This completes the proof of the lemma. \square

Proof of Proposition 4.1. First we construct a homomorphism $\pi_1^k(\mathbf{K}^\times) \rightarrow \text{Gal}(L/K)^{\text{ab}}$ for each finite Galois extension L/K . We keep the above notation for the extension L/K . Let $\mathbf{U}_{L, k'/k}$ be the kernel of the composite of the map $\mathbf{L}_k^\times \rightarrow \mathbb{Z}[\mathfrak{g}]$ defined in the proof of Lemma 4.2 and the augmentation map $\mathbb{Z}[\mathfrak{g}] \rightarrow \mathbb{Z}$. We have an exact sequence $0 \rightarrow \mathbf{U}_{L, k'/k} \rightarrow \mathbf{L}_k^\times \rightarrow \mathbb{Z} \rightarrow 0$. Using Lemma 4.2, we have $0 = \hat{H}^{-1}(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^0(G, \mathbf{U}_{L, k'/k}) \cong \mathbf{U}_K / N_G \mathbf{U}_{L, k'/k}$. Hence $N_G: \mathbf{U}_{L, k'/k} \rightarrow \mathbf{U}_K$ is surjective. Similarly we have $G^{\text{ab}} \cong \hat{H}^{-2}(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^{-1}(G, \mathbf{U}_{L, k'/k})$. The group $\hat{H}^{-1}(G, \mathbf{U}_{L, k'/k})$ is the quotient of the kernel of the norm map $N_G: \mathbf{U}_{L, k'/k} \rightarrow \mathbf{U}_K$ by the product $I_G \mathbf{U}_{L, k'/k}$ of the ideal I_G and the G -module $\mathbf{U}_{L, k'/k}$. Thus we have an exact sequence $0 \rightarrow G^{\text{ab}} \rightarrow \mathbf{U}_{L, k'/k} / I_G \mathbf{U}_{L, k'/k} \xrightarrow{N_G} \mathbf{U}_K \rightarrow 0$ and hence an exact sequence $0 \rightarrow G^{\text{ab}} \rightarrow \mathbf{L}_k^\times / I_G \mathbf{U}_{L, k'/k} \xrightarrow{N_G} \mathbf{K}^\times \rightarrow 0$. The resulting long exact sequence of homotopy groups gives a homomorphism $\pi_1^k(\mathbf{K}^\times) \rightarrow \pi_0^k(G^{\text{ab}}) = G^{\text{ab}}$.

Next we show that the homomorphism $\pi_1^k(\mathbf{K}^\times) \rightarrow G^{\text{ab}} = \text{Gal}(L/K)^{\text{ab}}$ and its limit $\pi_1^k(\mathbf{K}^\times) \rightarrow G_K^{\text{ab}}$ satisfy the required properties. The map $\mathbf{L}_k^\times \rightarrow \mathbb{Z}[\mathfrak{g}]$ gives a surjection $\mathbf{U}_{L, k'/k} \twoheadrightarrow I_{\mathfrak{g}}$ and hence a surjection $I_G \mathbf{U}_{L, k'/k} \twoheadrightarrow I_{\mathfrak{g}}^2$. Therefore we have a following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T_a & \longrightarrow & \mathbf{U}_{L, k} / (\mathbf{U}_{L, k} \cap I_G \mathbf{U}_{L, k'/k}) & \longrightarrow & \mathbf{U}_K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (2) \quad 0 & \longrightarrow & G^{\text{ab}} & \longrightarrow & \mathbf{L}_k^\times / I_G \mathbf{U}_{L, k'/k} & \longrightarrow & \mathbf{K}^\times \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathfrak{g}^{\text{ab}} & \longrightarrow & \mathbb{Z}[\mathfrak{g}] / I_{\mathfrak{g}}^2 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

By the resulting long exact sequences of homotopy groups and by passage to the limit, we get the required commutative diagram (1). The map $\pi_1^k(\mathbf{U}_K) \rightarrow T(K^{\text{ab}}/K)$ coincides with the reciprocity map of the local class field theory of Hazewinkel. Hence it is isomorphism. Also the map $\pi_1^k(\mathbb{Z}) \rightarrow G_k^{\text{ab}}$ is an isomorphism. Thus the map $\pi_1^k(\mathbf{K}^\times) \rightarrow G_K^{\text{ab}}$ is an isomorphism as well. \square

In the rest of this section, we prove some lemmas needed for the next section.

Lemma 4.3. *Let E/K be a finite extension with residue extension k'/k and let $N_{E/K}: \mathbf{E}_k^\times \rightarrow \mathbf{K}^\times$ be the norm map. Then we have $\pi_0^k(\text{Ker}(N_{E/K})) \cong \text{Gal}(E/K^{\text{ab}}/K)$.*

Proof. First we reduce to the case where E/K is separable. Let F be the separable closure of K in E . Then $E = F(\pi_F^{1/q})$ for a prime π_F of \mathcal{O}_F and $q = [E:F]$. Since q is a p -power and $N_{E/F}$ is the q -th power map, the map $N_{E/F}$ gives an isomorphism

$\mathbf{E}_k^\times \xrightarrow{\sim} \mathbf{F}_k^\times$ and hence an isomorphism $\text{Ker}(N_{E/K}) \xrightarrow{\sim} \text{Ker}(N_{F/K})$. Thus we may assume E/K is separable.

Next we reduce to the case where E/K is Galois. Take a finite Galois extension \tilde{E} of K containing E . The map $N_{\tilde{E}/E}$ gives an isomorphism $\text{Ker}(N_{\tilde{E}/K})/\text{Ker}(N_{\tilde{E}/E}) \xrightarrow{\sim} \text{Ker}(N_{E/K})$. Hence we have $\pi_0^k(\text{Ker}(N_{\tilde{E}/K}))/\pi_0^k(\text{Ker}(N_{\tilde{E}/E})) \cong \pi_0^k(\text{Ker}(N_{E/K}))$. On the other hand, we have $\text{Gal}(\tilde{E} \cap K^{\text{ab}}/K)/\text{Gal}(\tilde{E} \cap E^{\text{ab}}/E) \cong \text{Gal}(E \cap K^{\text{ab}}/K)$. Comparing them, it follows that we may assume E/K is Galois.

We show the lemma for Galois E/K . The short exact sequence $0 \rightarrow \text{Ker}(N_{E/K}) \rightarrow \mathbf{E}_k^\times \rightarrow \mathbf{K}^\times \rightarrow 0$ gives a long exact sequence

$$\cdots \rightarrow \pi_1^k(\mathbf{E}_k^\times) \rightarrow \pi_1^k(\mathbf{K}^\times) \rightarrow \pi_0^k(\text{Ker}(N_{E/K})) \rightarrow \pi_0^k(\mathbf{E}_k^\times) \rightarrow \pi_0^k(\mathbf{K}^\times) \rightarrow 0.$$

Since \mathbf{E}_k^\times is the Weil restriction of $\mathbf{E}_{k'}^\times$ from k' to k , we know that $\pi_i^k(\mathbf{E}_k^\times) \cong \pi_i^{k'}(\mathbf{E}_{k'}^\times)$ for $i \geq 0$. The map $\pi_1^k(\mathbf{E}_k^\times) \rightarrow \pi_1^k(\mathbf{K}^\times)$ can be translated into the restriction map $G_E^{\text{ab}} \rightarrow G_K^{\text{ab}}$ by Proposition 4.1. The map $\pi_0^k(\mathbf{E}_k^\times) \rightarrow \pi_0^k(\mathbf{K}^\times)$ can be translated into the identity map $\mathbb{Z} \rightarrow \mathbb{Z}$. Hence we have $\pi_0^k(\text{Ker}(N_{E/K})) \cong \text{Gal}(E \cap K^{\text{ab}}/K)$ as required. \square

For the next two lemmas, let L/K be a finite wild Galois extension with a unique jump in the ramification filtration of its Galois group. Then L/K is a totally ramified abelian extension whose Galois group is killed by p . Set $m = u_{L/K} > 1$ and $G = \text{Gal}(L/K)$. Then m is an integer by the Hasse-Arf theorem. We write $\psi = \psi_{L/K}$.

Lemma 4.4. *Let L/K , m , G , ψ be as above. Let $N_{L/K}: \mathbf{U}_L \rightarrow \mathbf{U}_K$ be the norm map and $\overline{N_{L/K}}: \mathbf{U}_L/\mathbf{U}_L^{\psi(m-1)+1} \rightarrow \mathbf{U}_K/\mathbf{U}_K^m$ be its quotient. Then we have $\pi_0^k(\overline{N_{L/K}}) \cong G$.*

Proof. We have $N_{L/K}(\mathbf{U}_L^{\psi(m-1)+1}) = \mathbf{U}_K^m$ by [7, Chap. V, §6, Cor. 3]. Thus we have $\text{Ker}(\overline{N_{L/K}}) = \text{Ker}(N_{L/K})/\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1}$. Hence $\pi_0^k(\text{Ker}(\overline{N_{L/K}})) = \pi_0^k(\text{Ker}(N_{L/K}))/\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1})$ by the right exactness of π_0^k . For the numerator, we have $\pi_0^k(\text{Ker}(N_{L/K})) \cong G$ by Lemma 4.3. For the denominator, we show that the image of $\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1})$ in G is zero. We have

$$\begin{aligned} \frac{\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)}}{\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1}} &\cong \text{Ker} \left(N_{L/K}: \frac{\mathbf{U}_L^{\psi(m-1)}}{\mathbf{U}_L^{\psi(m-1)+1}} \rightarrow \frac{\mathbf{U}_K^{m-1}}{\mathbf{U}_K^m} \right) \\ &\cong G^{m-1}/G^m \end{aligned}$$

by [7, Chap. V, §6, Prop. 9]. From our assumption on L/K , we have $G^{m-1} = G$ and $G^m = 0$. Taking π_0^k , we have

$$\frac{\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)})}{\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1})} \cong G.$$

But since $\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)}) \cong G$ by [6, §3.5, Prop. 8, (ii)], the image of $\pi_0^k(\text{Ker}(N_{L/K}) \cap \mathbf{U}_L^{\psi(m-1)+1})$ in G is zero. Thus we have $\pi_0^k(\text{Ker}(\overline{N_{L/K}})) \cong G$, as desired. \square

Lemma 4.5. *Let L/K , m , G and ψ be as above. Then we have canonical isomorphisms*

$$K^\times/N_{L/K}L^\times \cong U_K^{m-1}/U_K^m N_{L/K}U_L^{\psi(m-1)} \cong \mathrm{Hom}(G_k, G).$$

Moreover, if k' is a finite extension of k and K' (resp. L') is the corresponding unramified base change of K (resp. L), then the following diagram commutes:

$$\begin{array}{ccccc} K^\times/N_{L/K}L^\times & \xlongequal{\quad} & U_K^{m-1}/U_K^m N_{L/K}U_L^{\psi(m-1)} & \xlongequal{\quad} & \mathrm{Hom}(G_k, G) \\ \downarrow & & \downarrow & & \downarrow \\ K'^\times/N_{L'/K'}L'^\times & \xlongequal{\quad} & U_{K'}^{m-1}/U_{K'}^m N_{L'/K'}U_{L'}^{\psi(m-1)} & \xlongequal{\quad} & \mathrm{Hom}(G_{k'}, G), \end{array}$$

where the vertical arrows are induced by the inclusions $K^\times \hookrightarrow K'^\times$, $U_K^{m-1} \hookrightarrow U_{K'}^{m-1}$ and $G_{k'} \hookrightarrow G_k$.

Proof. The inclusion induces an isomorphism from $U_K^{m-1}/U_K^m N_{L/K}U_L^{\psi(m-1)}$ to $K^\times/N_{L/K}L^\times$ by [7, Chap. V, §6, Cor. 2] with a similar argument to [7, Chap. V, §3, Cor. 7]. By the assumption on L/K , the sequence

$$0 \longrightarrow G \longrightarrow \mathbf{U}_L^{\psi(m-1)}/\mathbf{U}_L^{\psi(m-1)+1} \xrightarrow{N_{L/K}} \mathbf{U}_K^{m-1}/\mathbf{U}_K^m \longrightarrow 0$$

is an exact sequence of sheaves on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ by [6, §3.4, Prop. 6]. Since $\mathbf{U}_L^{\psi(m-1)}/\mathbf{U}_L^{\psi(m-1)+1}$ is isomorphic to the additive group, we have $H^1(G_k, \mathbf{U}_L^{\psi(m-1)}/\mathbf{U}_L^{\psi(m-1)+1}) = 0$. Thus the resulting long exact sequence of Galois cohomology of k gives an isomorphism from $U_K^{m-1}/U_K^m N_{L/K}U_L^{\psi(m-1)}$ to $\mathrm{Hom}(G_k, G)$. The commutativity of the diagram is obvious. \square

5. THE REDUCED CASE

Throughout this section, let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$ and let L be a wild Galois extension of K with a unique jump in the ramification filtration of its Galois group. Then the extension L/K is a totally ramified abelian extension whose Galois group is killed by p . We put $G := \mathrm{Gal}(L/K)$ and $m := u_{L/K} > 1$. By the Hasse-Arf theorem, m is an integer.

The following proposition, combined with Propositions 2.1 and 3.1, proves Theorem 1.1.

Proposition 5.1. *Let L/K , G , m be as above.*

- (1) *If L/K satisfies $(\mathbf{P}_m^{\mathrm{tr}})$, then k has no Galois extension of degree p .*
- (2) *If E is a totally ramified algebraic extension of K and if there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then there exists an unramified Galois extension K'/K with Galois group isomorphic to a subgroup of G such that $L \subset EK'$.*

Proof. (1). Assume L/K satisfies $(\mathbf{P}_m^{\mathrm{tr}})$. Since $G = \mathrm{Gal}(L/K)$ is a non-trivial abelian group killed by p , to show that k has no Galois extension of degree p , it is enough to prove that $\mathrm{Hom}(G_k, G) = 0$, which is equivalent to showing that $N_{L/K}U_L^{\psi_{L/K}(m-1)} = U_K^{m-1}$ by Lemma 4.5. Take $u \in U_K^{m-1}$. Let $f(x) = x^e + a_{e-1}x^{e-1} + \cdots + a_1x + a_0$ be the minimal polynomial of a prime element π_L of \mathcal{O}_L . Then the polynomial $x^e + a_{e-1}x^{e-1} + \cdots + a_1x + ua_0 \in \mathcal{O}_K[x]$ is an Eisenstein polynomial. Let π_E be a root of this polynomial such that the function $v_K(\pi_E -$

$\sigma\pi_L$) for $\sigma \in G$ takes the maximum at $\sigma = 1$. Set $E = K(\pi_E)$. Then we have a well-defined \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ that sends π_L to π_E , since $v_K(f(\pi_E)) = v_K((u-1)a_0) \geq m$. Since L/K satisfies (P_m^{tr}) by assumption, we have $L \subset E$, which is actually an equality: $L = E$. Since π_L and π_E are both prime in $\mathcal{O}_L = \mathcal{O}_E$, their ratio $u' := \pi_E/\pi_L$ is in $U_L = U_E$. By [2, the second proposition of §1.3], we have $\tilde{\psi}_{L/K}(v_K(f(\pi_E))) = v_K(\pi_E - \pi_L) = (v_L(u' - 1) + 1)/e_{L/K}$. Thus we have $v_L(u' - 1) \geq e_{L/K}\tilde{\psi}_{L/K}(m) - 1 = \psi_{L/K}(m - 1)$. Hence $u' \in U_L^{\psi_{L/K}(m-1)}$. Also we have $N_{L/K}u' = N_{L/K}\pi_E/N_{L/K}\pi_L = ua_0/a_0 = u$. Hence $N_{L/K}U_L^{\psi_{L/K}(m-1)} = U_K^{m-1}$. This proves (1).

(2). Let E be a totally ramified algebraic extension of K such that there exists an \mathcal{O}_K -algebra homomorphism $\eta: \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$. We may assume E/K is finite. We regard $\mathcal{O}_E/\mathfrak{a}_{E/K}^m$ as an \mathcal{O}_L -module via η . By Lemma 5.2 below, we know that $\mathcal{O}_E/\mathfrak{a}_{E/K}^m$ is finite free over $\mathcal{O}_L/\mathfrak{a}_{L/K}^m$. Hence we get the corresponding norm map N_η from $(\mathcal{O}_E/\mathfrak{a}_{E/K}^m)^\times = U_E/U_E^{me_{E/K}}$ to $(\mathcal{O}_L/\mathfrak{a}_{L/K}^m)^\times = U_L/U_L^{me_{L/K}}$. The map N_η can be extended to a morphism $\mathbf{U}_E/\mathbf{U}_E^{me_{E/K}} \rightarrow \mathbf{U}_L/\mathbf{U}_L^{me_{L/K}}$ of sheaves of abelian groups on $(\text{Perf}/k)_{\text{fpqc}}$. By the transitivity of norm, we have $\overline{N_{L/K}} \circ N_\eta = \overline{N_{E/K}}$, where $\overline{N_{L/K}}: \mathbf{U}_L/\mathbf{U}_L^{me_{L/K}} \rightarrow \mathbf{U}_K/\mathbf{U}_K^m$ (resp. $\overline{N_{E/K}}: \mathbf{U}_E/\mathbf{U}_E^{me_{E/K}} \rightarrow \mathbf{U}_K/\mathbf{U}_K^m$) is the map induced by the norm map $N_{L/K}$ for the finite extension L/K (resp. $N_{E/K}$ for E/K). Since $me_{L/K} \geq \psi_{L/K}(m-1) + 1$, the map $\overline{N_{L/K}}$ factors through the canonical projection $\mathbf{U}_L/\mathbf{U}_L^{me_{L/K}} \rightarrow \mathbf{U}_L/\mathbf{U}_L^{\psi_{L/K}(m-1)+1}$. By abuse of notation, we write $\overline{N_{L/K}}$ for the map $\mathbf{U}_L/\mathbf{U}_L^{\psi_{L/K}(m-1)+1} \rightarrow \mathbf{U}_K/\mathbf{U}_K^m$.

We show that there exists an unramified Galois extension K'/K with Galois group isomorphic to a subgroup of G such that the following holds: Let k' be the residue field of K' and let $L' = LK'$, $E' = EK'$ and $N_{\eta'}$ be the unramified base changes by K'/K . Then we can extend $N_{\eta'}$ to an morphism $\tilde{N}_{\eta'}$ so as to obtain the following commutative diagram with exact rows of sheaves of abelian groups on $(\text{Perf}/k')_{\text{fpqc}}$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ker}(N_{E'/K'}) & \longrightarrow & \mathbf{E}'^\times & \xrightarrow{N_{E'/K'}} & \mathbf{K}'^\times & \longrightarrow & 0 \\
 & & \downarrow \tilde{N}_{\eta'} & & \downarrow \tilde{N}_{\eta'} & & \downarrow & & \\
 0 & \longrightarrow & \text{Ker}(\overline{N_{L'/K'}}) & \longrightarrow & \mathbf{L}'^\times / \mathbf{U}_{L'}^{\psi_{L'/K'}(m-1)+1} & \xrightarrow{\overline{N_{L'/K'}}} & \mathbf{K}'^\times / \mathbf{U}_{K'}^m & \longrightarrow & 0.
 \end{array}$$

The construction of $\tilde{N}_{\eta'}$ is as follows. Take a prime element π_E of \mathcal{O}_E . By the isomorphism of Lemma 4.5, $N_{E/K}\pi_E$ defines a homomorphism $\chi: G_k \rightarrow G$. Let k' correspond to $\text{Ker}(\chi)$ via Galois theory. Let $K', L', E', N_{\eta'}$ be the corresponding unramified base changes. The restriction $\chi|_{G_{k'}}: G_{k'} \rightarrow G$ is a trivial character. Hence the image of $N_{E/K}\pi_E$ in $K'^\times/N_{L'/K'}L'^\times$ is trivial by Lemma 4.5. Thus $N_{E/K}\pi_E$ can be written as $N_{L'/K'}\pi_{L'}$ for some prime $\pi_{L'}$ of $\mathcal{O}_{L'}$. We define $\tilde{N}_{\eta'}\pi_E := \pi_{L'}$. Then the commutativity of the right square follows from the equality $\overline{N_{L'/K'}} \circ N_{\eta'} = \overline{N_{E/K}}$ as a map from \mathbf{U}_E to $\mathbf{U}_K/\mathbf{U}_K^m$ and the equality $N_{E'/K'}\pi_E = N_{L'/K'}\pi_{L'}$.

The above diagram and the resulting long exact sequences of homotopy groups give the following commutative diagram:

$$\begin{array}{ccc} \pi_1^{k'}(\mathbf{K}'^\times) & \longrightarrow & \pi_0^{k'}(\mathrm{Ker}(N_{E'/K'})) \\ \downarrow & & \downarrow \tilde{N}_{\eta'} \\ \pi_1^{k'}(\mathbf{K}'^\times/\mathbf{U}_{K'}^m) & \longrightarrow & \pi_0^{k'}(\mathrm{Ker}(\overline{N_{L'/K'}})). \end{array}$$

By Proposition 4.1, Lemma 4.3 and Lemma 4.4, we can translate this diagram into

$$\begin{array}{ccc} G_{K'}^{\mathrm{ab}} & \longrightarrow & \mathrm{Gal}(E' \cap K'^{\mathrm{ab}}/K') \\ \downarrow & & \downarrow \\ G_{K'}^{\mathrm{ab}}/(G_{K'}^{\mathrm{ab}})^m & \longrightarrow & G = \mathrm{Gal}(L'/K'). \end{array}$$

The horizontal arrows are restriction maps. The commutativity of this diagram shows that $L' \subset E' \cap K'^{\mathrm{ab}}$. Thus $L \subset E' = EK'$. This proves (2). \square

Lemma 5.2. *The \mathcal{O}_L -module $\mathcal{O}_E/\mathfrak{a}_{E/K}^m$ is killed by $\mathfrak{a}_{L/K}^m$ and is finite free over $\mathcal{O}_L/\mathfrak{a}_{L/K}^m$.*

Proof. (For another proof, see [3, Lem. 2.1].) Put $n = [L : K] = e_{L/K}$ and $T = \mathcal{O}_E/\mathfrak{a}_{E/K}^m$. Let π_K be a prime element of \mathcal{O}_K , let π_L be a prime element of \mathcal{O}_L and let $\beta \in \mathcal{O}_E$ be a lift of $\eta(\pi_L) \in \mathcal{O}_E/\mathfrak{a}_{E/K}^m$. Since $u = \pi_L^n/\pi_K$ is a unit, $\eta(u)$ is a unit in $\mathcal{O}_E/\mathfrak{a}_{E/K}^m$. Since $\beta^n \equiv \eta(u)\pi_K$ in $\mathcal{O}_E/\mathfrak{a}_{E/K}^m$ and $m > 1$, we have $v_K(\beta^n) = v_K(\pi_K) = 1$. Hence $v_K(\beta) = 1/n$. Thus π_L^{nm} kills $T = \mathcal{O}_L/\mathfrak{a}_{L/K}^m$. For an arbitrary element $\gamma \in \mathcal{O}_E$ that is a lift of an element of the π_L -torsion part $T[\pi_L]$ of T , we have $v_K(\beta\gamma) \geq m$. Hence $v_K(\gamma) \geq m - 1/n = v_K(\beta^{nm-1})$. Hence $\gamma \in \beta^{nm-1}\mathcal{O}_E$. Thus $T[\pi_L] = \pi_L^{nm-1}T$. Applying the structure theorem of modules over a principal ideal domain, we get the result. \square

Remark 5.3. One can use Propositions 2.1 and 3.1 to get a generalization of assertion (2) of Proposition 5.1 for a finite wild Galois extension L/K . The result is the following. If E is an algebraic extension of K and if there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ for $m = u_{L/K}$, then the $\mathrm{Gal}(L/K)^{(m)}$ -fixed subfield M of L is contained in E , and there exists a finite unramified subextension M'/M of E/M and an unramified Galois extension M''/M' with $\mathrm{Gal}(M''/M')$ isomorphic to a subgroup of $\mathrm{Gal}(L/K)^{(m)}$ such that $L \subset EM''$.

Remark 5.4. The local class field theory of Serre and Hazewinkel is useful not only for the reduced case but for more general abelian cases. We illustrate it with a simplified proof of the following part of Theorem 1.1: if K is a complete discrete valuation field with algebraically closed residue field k and L is a finite wild abelian extension of K , then L/K satisfies (P_m) for $m = u_{L/K}$.

The proof is given as follows. Let E be a finite extension of K and suppose there exists an \mathcal{O}_K -algebra homomorphism $\eta: \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$. Lemma 5.2 is still valid for this situation. Thus we have a norm map $N_\eta: \mathbf{U}_E/\mathbf{U}_E^{me_{E/K}} \rightarrow \mathbf{U}_L/\mathbf{U}_L^{me_{L/K}}$ relative to η . We have the following commutative diagram with exact rows of

proalgebraic groups over k :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathrm{Ker}(N_{E/K}) & \longrightarrow & \mathbf{U}_E & \xrightarrow{N_{E/K}} & \mathbf{U}_K \longrightarrow 0 \\
 & & \downarrow N_\eta & & \downarrow N_\eta & & \downarrow \\
 0 & \longrightarrow & \mathrm{Ker}(\overline{N_{L/K}}) & \longrightarrow & \mathbf{U}_L/\mathbf{U}_L^{\psi_{L/K}(m-1)+1} & \xrightarrow{\overline{N_{L/K}}} & \mathbf{U}_K/\mathbf{U}_K^m \longrightarrow 0.
 \end{array}$$

The resulting long exact sequences of homotopy groups gives a commutative diagram

$$\begin{array}{ccc}
 \pi_1^k(\mathbf{U}_K) \cong G_K^{\mathrm{ab}} & \xrightarrow{\mathrm{Res}} & \mathrm{Gal}(E \cap K^{\mathrm{ab}}/K) \cong \pi_0^k(\mathrm{Ker}(N_{E/K})) \\
 \downarrow & & \downarrow N_\eta \\
 \pi_1^k(\mathbf{U}_K/\mathbf{U}_K^m) \cong G_K^{\mathrm{ab}}/(G_K^{\mathrm{ab}})^m & \xrightarrow{\mathrm{Res}} & \mathrm{Gal}(L/K) \cong \pi_0^k(\mathrm{Ker}(\overline{N_{L/K}})).
 \end{array}$$

Here we used Lemmas 4.3 and 4.4, which are still valid for this situation. Therefore $L \subset E \cap K^{\mathrm{ab}} \subset E$. Hence L/K satisfies (P_m) for $m = u_{L/K}$.

REFERENCES

- [1] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris, 1970. (Appendix: M. Hazewinkel, Corps de classes local).
- [2] Jean-Marc Fontaine. Il n'y a pas de variété abélienne sur \mathbf{Z} . *Invent. Math.*, 81(3):515–538, 1985.
- [3] Toshiro Hiranouchi and Yuichiro Taguchi. Extensions of truncated discrete valuation rings. *Pure Appl. Math. Q.*, 4(4, part 1):1205–1214, 2008.
- [4] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.
- [5] Jean-Pierre Serre. Groupes proalgébriques. *Inst. Hautes Études Sci. Publ. Math.*, 7, 1960.
- [6] Jean-Pierre Serre. Sur les corps locaux à corps résiduel algébriquement clos. *Bull. Soc. Math. France*, 89:105–154, 1961.
- [7] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.
- [8] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, English edition, 2002. Translated from the French by Patrick Ion and revised by the author.
- [9] Manabu Yoshida. Ramification of local fields and Fontaine's property (P_m) . preprint 2009, [arXiv.org:0905.1171](https://arxiv.org/abs/0905.1171).

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