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Average degree in graph powers

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Abstract

The kth power of a simple graph G, denoted G^k, is the graph with vertex set $V(G)$ where two vertices are adjacent if they are within distance k in G. We are interested in finding lower bounds on the average degree of G^k. Here we prove that if G is connected with minimum degree $d \geq 2$ and $|V(G)| \geq \frac{8}{3}d$, then G^4 has average degree at least $\frac{7}{3}d$. We also prove that if G is a connected d-regular graph on n vertices with diameter at least $3k+3$, then the average degree of G^{3k+2} is at least

$$(2k+1)(d+1) - k(k+1)(d+1)^2/n - 1.$$

Both of these results are shown to be essentially best possible; the second is best possible even when n/d is arbitrarily large.

1 Introduction

Throughout this paper we restrict our attention to finite simple connected graphs. This allows us, in particular, to refer to the average degree $a(G)$ of a graph G. The kth power of

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a graph G, denoted G^k, is the graph with vertex set $V(G)$ where two vertices are adjacent if they are within distance k in G, i.e., joined by a path of length at most k in G. It is natural to expect that $a(G^k)$ should generally be large and for this reason we are interested in finding lower bounds on this quantity.

The maximum distance between any pair of vertices in a graph G is called the diameter of G and denoted $diam(G)$. If $diam(G) \leq r$, then G^k is a clique for all $k \geq r$, and powers of G higher than r do not have any additional edges. For this reason, proving good lower bounds on $a(G^k)$ often necessitates a large diameter assumption. Indeed, the problem of counting edges in G^k was first considered by Hegarty [5], who proved that for a connected d-regular graph G with diameter at least 3,

$$a(G^3) \geq (1+c)d,$$

where $c = 0.087$. The constant c was improved to $1/6$ by Pokrovskiy [8], and then to $3/4$ by DeVos and Thomassé [3], who also weakened the assumption to minimum degree $\delta(G)$ at least d. The later authors provided a family of examples proving that $3/4$ is best possible for G^3. In contrast to this result, when $k = 2$ there is no positive constant c for which $a(G^k) > (1+c)a(G)$, even in the case when G is connected, regular, and has a diameter constraint (see [5]).

In this paper we prove the following two new essentially best-possible lower bounds on $a(G^k)$, handling the cases $k = 4$ and $k \equiv 2 \pmod 3$.

Theorem 1.1 *If G is a connected n vertex graph with $\delta(G) \geq d$ and $n \geq \frac{8}{3}d$ for an integer $d \geq 2$, then*

$$a(G^4) \geq \tfrac{7}{3}d.$$

Theorem 1.2 *If G is a connected d-regular graph on n vertices, $k \equiv 2$ (mod 3), and $diam(G) > k$, then*

$$a(G^k) \geq \left(\tfrac{2k-1}{3}\right)(d+1) - \tfrac{(k-2)(k+1)(d+1)^2}{9n} - 1.$$

The proof of Theorem 1.1 comprises Section 3 of this paper and the proof of Theorem 1.2 is the subject of Section 2.

Note that the assumption $n \geq \frac{8}{3}d$ in Theorem 1.1 could be replaced by the more restrictive $diam(G) \geq 6$. This is because a shortest path of length six contains three vertices whose neighbourhoods are completely disjoint, so $n \geq 3d > \frac{8}{3}t$. In addition to proving Theorem 1.1 in Section 3, we also present a family of examples showing the value $8/3$ cannot be further lowered. To see that the coefficient $\frac{7}{3}$ cannot be increased, consider the graph in Figure 1 (here and in later figures, each line segment represents a complete bipartite graph of the appropriate size, and each "-M" indicates the removal of a perfect matching). The graph is

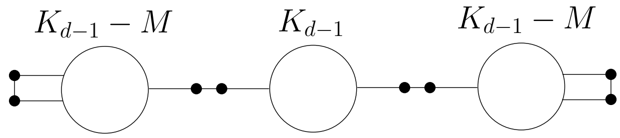

Figure 1: a d-regular graph which is extreme for Theorem 1.1

d-regular for every odd $d > 1$ and a quick calculation reveals that it has $3d + 4$ vertices and its fourth power has degree sum $7d^2 + 19d + 6$.

For both Theorem 1.1 and for the G^3 result of DeVos and Thomassé, we know of no tight examples with arbitrarily large diameter, or with n/d arbitrarily large. So it is possible that as diameter grows to infinity, better bounds could be obtained for both G^3 and G^4. The graph G^5, and in fact all graph powers that are 2 modulo 3, seem easier to understand. Namely, our Theorem 1.2 is best possible even as n/d grows to infinity. To see this, let $d > 1$ be odd and consider the graph H given in Figure 2 — H is d-regular graph and a generalization of Figure 1. Similar graphs have appeared in the papers of Hegarty [5] and Pokrovskiy [8], and a straightforward calculation (which we carry out in an appendix) shows that if $k \equiv 2 \pmod 3$ and $diam(H) \geq k + 1$, then

$$a(H^k) \leq \left(\tfrac{2k-1}{3}\right)(d+1) - \tfrac{(k-2)(k+1)(d+1)^2}{9n} + 3.$$

This implies that Theorem 1.2 cannot be improved by an additive constant greater than 4. On the other hand, it should be noted that the average degree of H^{k-1} is nearly that of H^k, so it seems quite possible that our theorem could be improved by decreasing the exponent of G from k to $k - 1$, and perhaps increasing the constant term slightly.

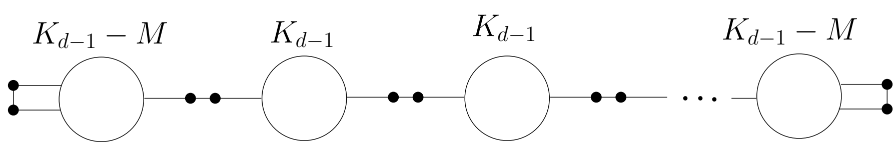

Figure 2: a d-regular graph which is extreme for Theorem 1.2

We can drop the assumption $diam(G) > k$ in Theorem 1.2 for only a small cost. That is, when $diam(G) \leq k$ the graph G^k is complete and we have

$$\begin{aligned} a(G^k) &\geq n - 1 - \left(\sqrt{n} - \tfrac{1}{3\sqrt{n}}(k - \tfrac{1}{2})(d+1)\right)^2 \\ &= \left(\tfrac{2k-1}{3}\right)(d+1) - \tfrac{(k-1/2)^2(d+1)^2}{9n} - 1. \end{aligned}$$

Hence we get the following corollary to Theorem 1.2.

Corollary 1.3 *If G is a connected d-regular graph on n vertices and $k \equiv 2$ (mod 3), then*

$$a(G^k) \geq \left(\tfrac{2k-1}{3}\right)(d+1) - \tfrac{(k-1/2)^2(d+1)^2}{9n} - 1.$$

We can also rewrite Corollary 1.3 using the parameter $diam(G)$ instead of n. To see this, set $t = diam(G)$ and choose $v_0, v_1, \ldots, v_t$ to be the vertex sequence of a geodesic (shortest) path between v_0 and v_t. Now, the neighbourhoods of $v_0, v_3, \ldots v_{3\lfloor \frac{t}{3} \rfloor}$ are pairwise disjoint and it follows that $n \geq (d+1)\frac{diam(G)}{3}$. Hence we get the following.

Corollary 1.4 *If G is a connected d-regular graph and $k \equiv 2$ (mod 3), then*

$$a(G^k) \geq \left(\tfrac{2k-1}{3}\right)(d+1)\left(1 - \tfrac{2k-1}{4\,diam(G)}\right) - 1.$$

For vertex transitive graphs, the bound of Theorem 1.2 may be improved. Let G be a finite d-regular vertex transitive graph and let $k < diam(G)$. Now the degree of a vertex x in G^k will be $|N^1(x) \cup N^2(x) \ldots \cup N^k(x)|$ where $N^i(x) = \{y \in V(G) : dist(x, y) = i\}$. It is immediate that $|N^1(x)| = d$ and since each $N^i(x)$ with $1 \leq i < diam(G)$ is a vertex cut, it follows from a theorem of Mader [7] and Watkins [10] (see page 40 of [4] for a proof) that $|N^i(x)| \geq \frac{2}{3}(d+1)$. This gives us the bound

$$a(G^k) \geq \tfrac{2k+1}{3}(d+1) - 1. \tag{1}$$

This bound is best possible due to a graph which is constructed from a collection of $> k$ disjoint cliques of size $\frac{1}{3}(d+1)$ by placing them in a cyclic order and joining each completely to its neighbours in this ordering. Let us note that (1) is still quite close to the bound of Theorem 1.2. Indeed, in both cases, increasing k by 3 has the effect of improving the bound by $2(d+1)$.

This last result has consequences in additive number theory and group theory by way of Cayley graphs. Let Γ be a finite multiplicative group and let $A \subseteq \Gamma$ be a generating set with $1 \in A$ and with the property that $g \in A \Rightarrow g^{-1} \in A$. If G is the Cayley graph generated by $A \setminus \{1\}$ then G is a regular graph of degree $|A| - 1$ and G^k will be the Cayley graph generated by $A^k \setminus \{1\}$, so it will be regular of degree $|A^k| - 1$. Since Cayley graphs are vertex-transitive we may apply (1), which shows that whenever $A^k \neq \Gamma$

$$|A^k| \geq \tfrac{2k+1}{3}|A|.$$

This bound is traditionally obtained in additive number theory by way of Kneser's addition theorem [6].

2 Proof of Theorem 1.2

Let P_n denote a path on n vertices. Our proof of Theorem 1.2 relies on the number of edges $e(T^k) = |E(T^k)|$ in the kth power of a tree T.

Observation 2.1 $e(P_n^k) \geq kn - \frac{1}{2}k(k+1)$

Proof: The total degree sum in P_n^k is at least $2kn - 2(1 + 2 + \cdots + k) = 2kn - k(k+1)$ so $e(P_n^k) \geq kn - \frac{1}{2}k(k+1)$.

Lemma 2.2 *If T is a tree on n vertices, then $e(T^k) \geq kn - \frac{1}{2}k(k+1)$*

Proof: Since $e(P_n^k) \geq kn - \frac{1}{2}k(k+1)$, it suffices to prove that $e(T^k) \geq e(P_n^k)$. We prove this by induction on $\sum_{v \in V(G)} \max\{0, deg(v) - 2\}$. As a base, observe that if this sum is 0, then T is isomorphic to P_n and the result is immediate. For the inductive step, we may then assume that there exists a vertex of degree ≥ 3. Fix a root vertex r, and choose a vertex v so that $deg(v) \geq 3$ and subject to this, v has maximum distance from the root. Then T contains a path P between two leaf vertices u, u' so that v is an interior vertex of P, and all other interior vertices of P have degree 2 in T. Let X be the vertex set of this path. Now, we modify our tree T to form a new tree U by deleting an edge of P which is incident with v and then adding the new edge uu'. In the new graph U, the subgraph induced by X is still a path, so the number of edges in U^k with both ends in X is the same as that in T^k. For a vertex $w \in V(G) \setminus X$ the set of neighbours of w in $V(G) \setminus X$ in the two graphs T^k and U^k are identical, and the number of neighbours of w in X in the graph U^k is at most that in T^k. It follows that $e(U^k) \leq e(T^k)$, and now applying the inductive hypothesis to U completes the proof. □

We are now ready to prove Theorem 1.2, save for one very useful definition. For a graph G, a vertex $v \in V(G)$, and a nonnegative integer k, the ball of radius k around v is defined to be $B_k(v) = \{u \in V(G) : dist(u, v) \leq k\}$.

Theorem 2.3 *If G is a connected d-regular graph on n vertices and $diam(G) > 3k+2$, then*

$$a(G^{3k+2}) \geq (2k+1)(d+1) - k(k+1)(d+1)^2/n - 1.$$

Proof: Choose a geodesic path of length $\geq 3k+3$ in G and let $X_0 \subseteq V(G) = V$ consist of every third vertex of this path. Now, we extend X_0 to a set X by the following procedure. At each stage, if there exists a vertex which has distance 3 to X, then we add such a point, and otherwise we stop. Note that $|X| \geq |X_0| \geq k+1$. Now, construct a new graph H with vertex set X by the rule that $u, v \in X$ are adjacent in H if they have distance 3 in G. Observe that by our construction, the graph H must be connected. We set $Z = \bigcup_{w \in X} B_1(w)$, set $Y = V \setminus Z$, and set $z = |Z|$ and $y = |Y|$ and $x = |X|$ (noting that $z = (d+1)x$). We proceed with a sequence of claims. In what follows, $e_{3k+2}(Z, Y)$ denotes the number of edges between Z and Y in G^{3k+2}, and similarly, $e_{3k+2}(Z, Z)$ denotes the number of edges induced on Z in G^{3k+2}.

(1) $e_{3k+2}(Z, Z) \geq (k + \frac{1}{2})(d+1)z - \frac{1}{2}z - \frac{1}{2}k(k+1)(d+1)^2$

First note that if $u \in X$, then $B_1(u)$ induces a clique in G^{3k+2} of size $d+1$. Next, observe that if $u, v \in X$ are adjacent in H^k, then $B_1(u)$ and $B_1(v)$ will be completely joined in the graph G^{3k+2}, so we will have $e_{3k+2}(B_1(u), B_1(v)) = (d+1)^2$. Since H is connected, Lemma 2.2 gives us

$$\begin{aligned} e_{3k+2}(Z, Z) &\geq \tfrac{1}{2}d(d+1)x + e(H^k)(d+1)^2 \\ &\geq \tfrac{1}{2}d(d+1)x + (kx - \tfrac{1}{2}k(k+1))(d+1)^2 \\ &= (k + \tfrac{1}{2})(d+1)z - \tfrac{1}{2}z - \tfrac{1}{2}k(k+1)(d+1)^2 \end{aligned}$$

as desired.

(2) $e_{3k+2}(Z, Y) \geq k(d+1)y$

Let $w \in Y$ and note that by assumption, w must be distance 2 from some point $u \in X$ (were w to have distance ≥ 3 to every point in X, then the set X could have been augmented by adding a new point at distance 3). Now, $|X| \geq k$ and it follows that $deg_{H^{k-1}}(u) \geq k-1$. For every point v which is either equal to u or a neighbour of u in the graph H^{k-1} we have that w will be joined to $B_1(v)$ in the graph G^{3k+2}. It follows from this that $|B_{3k+2}(w) \cap Z| \geq k(d+1)$ and the proof of (2) now follows by summing this over all $w \in Y$.

(3) Every $w \in V$ satisfies $deg_{G^{3k+2}}(w) \geq (k+1)(d+1) - 1$

If $B_{3k+2}(w) = V$ then V contains $\geq k+1$ disjoint balls of radius 1 which gives the desired bound. Otherwise, we may choose a geodesic path of length $3k$ starting at w, say with vertex sequence $w = w_0, w_1, w_2, \ldots, w_{3k}$. Now we find that $B_{3k+2}(w)$ contains the disjoint sets $B_1(w_0)$, $B_1(w_3), \ldots, B_1(w_{3k})$ which again gives the desired bound.

We are now ready to complete the argument. Below we use (1), (2), and (3) in getting to the third line.

$$\begin{aligned}\sum_{w\in V} deg_{G^{3k+2}}(w) &= \sum_{w\in Z} deg_{G^{3k+2}}(w) + \sum_{w\in Y} deg_{G^{3k+2}}(w)\\ &= 2e_{3k+2}(Z,Z) + e_{3k+2}(Z,Y) + \sum_{w\in Y} deg_{G^{3k+2}}(w)\\ &\geq (2k+1)(d+1)z - z - k(k+1)(d+1)^2 + k(d+1)y + (k+1)(d+1)y - y\\ &= (2k+1)(d+1)n - k(k+1)(d+1)^2 - n.\end{aligned}$$

This completes the proof. □

3 The 4th Power

Before we prove Theorem 1.1, we give an example to show that the value $\frac{8}{3}d$ in the theorem cannot be further lowered. To this end, consider the graph in Figure 3. This is a graph with minimum degree d and $n = d(2+\alpha)+2$ vertices. We claim that if d is large and $n < \frac{8}{3}d$ (and consequently $\alpha < \frac{2}{3}$), then G^4 has fewer than the $\frac{7}{6}nd$ edges expected by Theorem 1.1. Since G^4 is complete except for edges between vertices at distance 5, we get $e(G^4) = \frac{n(n-1)}{2} - \alpha^2 d^2$. Substituting for n, this gives

$$e(G^4) - \tfrac{7}{6}nd = \tfrac{1}{2}d^2\left(\alpha - \tfrac{2}{3}\right)(1-\alpha) + d\left(\tfrac{2}{3} + \tfrac{3}{2}\alpha\right) + 1.$$

This value will indeed be negative when $\alpha < 2/3$, provided d is chosen large enough.

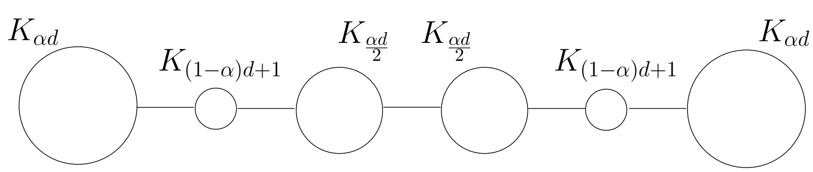

Figure 3: A d-regular graph which is extreme for Theorem 1.1 when $\alpha < 2/3$

The following lemma deals precisely with the boundary case of $\frac{8}{3}d \leq n \leq 3d$.

Lemma 3.1 *If G is a connected n vertex graph with $\delta(G) \geq d$ and $\frac{8}{3}d \leq n \leq 3d$ for an integer $d \geq 2$, then G^4 has average degree $\geq \frac{7}{3}d$.*

Proof: If $diam(G) \leq 4$ then G^4 is complete, so it has average degree $\geq \frac{8}{3}d - 1$. For $d \geq 3$ this is at least $\frac{7}{3}d$, and when $d = 2$ we must have $n \geq 6$ so again G^4 has average degree $\geq \frac{7}{3}d$. In the remaining case, we may choose a geodesic path with vertex sequence $v_1, v_2, \ldots, v_6$. Note that both v_1 and v_6 must have a neighbour not belonging to that path, thus we have $n \geq 8$. Suppose there exists a vertex w with $dist(w, v_3), dist(w, v_4) \geq 3$. Then either $dist(w, v_1) \geq 3$ or $dist(w, v_6) \geq 3$. Hence either the sets $B_1(w)$, $B_1(v_4)$, and $B_1(v_1)$ or the sets $B_1(w)$, $B_1(v_3)$, and $B_1(v_6)$ are disjoint, which is contradictory to $n \leq 3d$. Thus, we may assume $B_2(v_3) \cup B_2(v_4) = V(G) = V$. Now partition V into the following three sets.

$$A = B_2(v_3) \setminus B_2(v_4)$$
$$B = B_2(v_3) \cap B_2(v_4)$$
$$C = B_2(v_4) \setminus B_2(v_3)$$

Set $a = |A|$ and $c = |C|$. It is immediate from the assumption $B_2(v_3) \cup B_2(v_4) = V$ that in the graph G^4 the vertices in B are adjacent to every other vertex and that A and C induce complete graphs. Note further that A and $B_1(v_3)$ and $B_1(v_6)$ are disjoint so $n - 2(d+1) \geq a$ and by a similar argument $n - 2(d+1) \geq c$. Using these observations we find

$$\begin{aligned} \sum_{w \in V} deg_{G^4}(w) &\geq n(n-1) - 2ac \\ &\geq n(n-1) - 2(n - 2(d+1))^2 \\ &= \tfrac{7}{3}nd + (n - \tfrac{8}{3}d)(3d + 6 - n) + n - 8 \\ &\geq \tfrac{7}{3}nd \end{aligned}$$

as desired. □

We require one additional lemma before our proof of Theorem 1.1.

Lemma 3.2 *Let G be a connected n vertex graph with $\delta(G) \geq d$ and $n > 3d$. If there exist $u, v \in V(G)$ with $dist(u, v) = 3$ so that $B_4(u) \neq V$, $B_4(v) \neq V(G)$ and $B_2(u) \cup B_2(v) = V(G)$, then G^4 has average degree $\geq \frac{7}{3}d$.*

Proof: We define the following sets for $3 \leq i \leq 5$

$$B = B_2(u) \cap B_2(v)$$
$$A_i = \{w \in V : dist(w, v) = i\}$$
$$C_i = \{w \in V : dist(w, u) = i\}$$

and set $b = |B|$, $a_i = |A_i|$ and $c_i = |C_i|$. Note that by our assumptions, these sets are disjoint and have union equal to V. Since $B_2(u) \cup B_2(v) = V(G) = V$, every point in B is adjacent to every other vertex in G^4. For a vertex $w \in A_3$ we have that $B_4(w)$ contains the disjoint sets $B_1(u)$, $B_1(v)$, and A_5, so it will have degree $\geq 2d + a_5$ in G^4. These two observations plus a similar one for C_3 give us

$$\sum_{w \in A_3 \cup B \cup C_3} deg_{G^4}(w) > 3db + 2d(a_3 + c_3) + a_3 a_5 + c_3 c_5$$

If $w \in A_4$, we may choose $w' \in V$ so that $dist(w, w') = 3$ and $dist(w', v) = 1$. Now $B_4(w)$ contains the disjoint sets $A_5 \cup A_4 \cup A_3$ and $B_1(w')$ so $deg_{G^4}(w) \geq d + a_3 + a_4 + a_5$. If $w \in A_5$ we may choose $w' \in V$ so that $dist(w, w') = 3$ and $dist(w', v) = 2$. Now $B_4(w)$ contains the disjoint sets $A_5 \cup A_4$ and $B_1(w')$, so $deg_{G^4}(w) \geq a_5 + a_4 + d$. This gives us

$$\sum_{w \in A_4 \cup A_5} deg_{G^4}(w) \geq (d + a_4 + a_5)(a_4 + a_5) + a_3 a_4.$$

By a similar argument we get

$$\sum_{w \in C_4 \cup C_5} deg_{G^4}(w) \geq (d + c_4 + c_5)(c_4 + c_5) + c_3 c_4.$$

Now, $B_4(u) \neq V$ so there exists a point $w \in C_5$ and $B_1(w) \subseteq C_4 \cup C_5$. Thus $c_4 + c_5 \geq d$ and $c_3 c_4 + c_3 c_5 \geq c_3 d$. Similarly $a_3 a_4 + a_3 a_5 \geq a_3 d$. Setting $\bar{a} = a_4 + a_5$ and $\bar{c} = c_4 + c_5$ and combining our above inequalities with these observations yields

$$\begin{aligned}
\sum_{w \in V} deg_{G^4}(w) - \tfrac{7}{3}dn &\geq 3d(n - \bar{a} - \bar{c}) + (d + \bar{a})\bar{a} + (d + \bar{c})\bar{c} - \tfrac{7}{3}dn \\
&= \tfrac{2}{3}dn - 2d(\bar{a} + \bar{c}) + \bar{a}^2 + \bar{c}^2 \\
&\geq 2d^2 - 2d(\bar{a} + \bar{c}) + \frac{(\bar{a} + \bar{c})^2}{2} \\
&= \left(\sqrt{2}d - \frac{\bar{a} + \bar{c}}{\sqrt{2}}\right)^2 \\
&\geq 0
\end{aligned}$$

which completes the proof. □

Theorem 3.3 *If $G = (V, E)$ is a connected n vertex graph with $\delta(G) \geq d$ and $n \geq \frac{8}{3}d$ for an integer $d \geq 2$, then G^4 has average degree $\geq \frac{7}{3}d$.*

Proof: Let G be a counterexample with n minimum. Define a vertex v to be *good* if $deg_{G^4}(v) \geq 3d$ and *bad* otherwise. Let $Z \subseteq V$ be the set of good vertices, and set $\gamma = \frac{|Z|}{n}$. We prove the result with a sequence of claims.

(1) $n > 3d$

This follows from Lemma 3.1.

(2) Every $v \in V$ satisfies $deg_{G^4}(v) \geq 2d$.

If $B_2(v) = V$ then the above inequality follows from (1). Otherwise there exists $u \in V$ with $dist(u, v) = 3$ and now $B_4(v)$ contains the disjoint sets $B_1(u)$ and $B_1(v)$ so $deg_{G^4}(v) \geq 2d$.

(3) G^4 has average degree $\geq (2 + \gamma)d$.

This is a consequence of the following calculation.

$$\begin{aligned}\sum_{v \in V} deg_{G^4}(v) &= \sum_{v \in Z} deg_{G^4}(v) + \sum_{v \in V \setminus Z} deg_{G^4}(v) \\ &\geq (\gamma n)(3d) + (1 - \gamma)n(2d) \\ &= (2 + \gamma)dn\end{aligned}$$

(4) If $u, v, v' \in V$ satisfy $dist(v, v') \geq 3 = dist(u, v) = dist(u, v')$ then u is good.

The sets $B_1(u)$, $B_1(v)$, and $B_1(v')$ are pairwise disjoint and are all contained in $B_4(u)$ so $deg_{G^4}(u) \geq 3d$ and u is good.

(5) There do not exist bad vertices u_1, u_2 with $dist(u_1, u_2) = 3$.

If u_1, u_2 are bad, then since $n \geq 3d$, it follows from Lemma 3.2 that we may assume that $V \setminus (B_2(u_1) \cup B_2(u_2)) \neq \emptyset$ and it follows that there exists a vertex w so that $\min\{dist(u_1, w), dist(u_2, w)\} = 3$. But then this a contradiction as (4) implies that one of u_1, u_2 is good.

Let $X_1, \ldots, X_k$ be the vertex sets of the components of $G - Z$.

(6) Every X_i induces a clique in G^2.

Let $v \in X_i$ and suppose (for a contradiction) that $X_i \not\subseteq B_2(v)$. In this case, we may choose a vertex $u \in X_i \setminus B_2(v)$ which is adjacent to a point in $B_2(v)$. This gives us $dist(u, v) = 3$ contradicting (5).

We now define a relation on $\{X_1, \ldots, X_k\}$ by the rule that $X_i \sim X_j$ if $N(X_i) \cap N(X_j) \neq \emptyset$.

(7) If $X_i \sim X_j$ then $X_i \cup X_j$ is a clique in G^2.

Let $v \in X_i$ satisfy $N(v) \cap N(X_j) \neq \emptyset$ and suppose (for a contradiction) that $X_j \not\subseteq B_2(v)$. Then we may choose a vertex $u \in X_j \setminus B_2(v)$ which is adjacent to a point in $B_2(v)$. But then u and v have distance 3 contradicting (5). It follows that every point in X_j is distance 2 from v and then by a similar argument has distance 2 to any point in X_i.

(8) $\sim$ is an equivalence relation.

It is immediate from the definitions that $\sim$ is both reflexive and symmetric. To see that it is transitive, we suppose that $X_i \sim X_j \sim X_k$. If every point in G is distance ≤ 2 to $X_i \cup X_j \cup X_k$ then it follows from (6) and (7) that in the graph G^4 every point in X_j is adjacent to every other vertex, but this contradicts (1) and the assumption that these vertices are bad. It follows that we may choose a vertex $w \in V$ so that $dist(w, X_i \cup X_j \cup X_k) = 3$. First suppose that there exists $v \in X_j$ so that $dist(w, v) = 3$. Now, choose $u \in X_i$ and $u' \in X_k$. It follows from (7) that $B_4(v)$ contains $B_1(u) \cup B_1(u') \cup B_1(w)$ and since v is bad this implies that $B_1(u) \cap B_1(u') \neq \emptyset$, so $X_i \sim X_k$. Thus, we may assume without loss that $dist(w, v) = 3$ for some $v \in X_i$ and that $dist(w, X_j) \geq 4$. Now choose a vertex $u \in N(X_j) \cap N(X_k)$. We must have $dist(u, w) \geq 3$ (otherwise $dist(w, X_j) \leq 3$) and $dist(v, u) \leq 3$ by (7). It follows that $B_4(v)$ contains $B_1(v) \cup B_1(w) \cup B_1(u)$. Since v is bad it must be that $dist(u, v) \leq 2$. But then we have that $dist(v, X_k) \leq 3$. If there is a point in X_k which is distance 3 from v we get a contradiction to (5), so we must have $dist(v, X_k) = 2$ which implies $X_i \sim X_k$ as desired.

We now define $\{Y_1, \ldots, Y_\ell\}$ to be the unions of the equivalence classes of $\sim$. Note that by (6) and (7) every Y_i induces a clique in G^2.

(9) If $1 \leq i < j \leq \ell$ then $N(Y_i)$ and $N(Y_j)$ are disjoint, and there are no edges between them.

It is immediate that $N(Y_i)$ and $N(Y_j)$ are disjoint. Were there to be an edge between $u \in N(Y_i)$ and $v \in N(Y_j)$ then $u, v \in Z$ and we may choose $u' \in Y_i$ and $v' \in Y_j$ so that $uu', vv' \in E$. But then we have that u' and v' have distance 3 which contradicts (5).

Let $Z^* = \{u \in Z : N(u) \subseteq Z\}$.

(10) Every $v \in Z$ satisfies $|B_4(v) \cap Z| \geq d + 1$.

We claim that $B_3(v) \cap Z^* \neq \emptyset$ which immediately yields (10). To show this claim, let us suppose (for a contradiction) that it is false and choose $1 \leq i \leq \ell$ so that v has a neighbour, say u, in Y_i. It now follows from (9) that $B_3(v) \subseteq Y_i \cup N(Y_i)$. However, $Y_i \subseteq B_2(u)$ so $Y_i \cup N(Y_i) \subseteq B_3(u)$ giving us $B_3(v) \subseteq B_3(u)$. But this contradicts the assumptions that u is bad but v is good.

For every $1 \leq i \leq \ell$ and positive integer t let $Y_i^t = \{v \in Y_i : dist(v, Z^*) = t\}$.

(11) For $1 \leq i \leq \ell$ we have $Y_i^t = \emptyset$ whenever $t = 1$ or $t > 4$.

It is immediate from the definitions that $Y_i^1 = \emptyset$. It follows from (9) that there exists a vertex $u \in N(Y_i)$ with a neighbour in Z^*. Choose $v \in Y_i$ adjacent to u. Since Y_i induces a clique in G^2 every point in Y_i must have distance at most four to Z^* as desired.

(12) Every $v \in Y_i^2$ satisfies $deg_{G^4}(v) \geq 2d + |Y_i^4|$.

Choose a vertex $u \in Z^*$ with $dist(u,v) = 2$. There must be a vertex $u' \in B_2(u)$ with $dist(u',v) = 3$ (otherwise $B_2(u) \subseteq B_2(v)$, contradicting that u is good and v is bad). Now $B_4(v)$ contains $B_1(u') \cup B_1(v) \cup Y_i^4$, and we claim that these sets are disjoint (which obviously yields (12)). It is immediate that $B_1(u') \cap B_1(v) = \emptyset$. No point in Y_i^4 could be adjacent to v or u' since v and u' are distance ≤ 2 from Z^*, so $B_1(v) \cap Y_i^4 = B_1(u') \cap Y_i^4 = \emptyset$.

(13) For every $1 \leq i \leq \ell$ and $v \in Y_i \setminus Y_i^4$ we have $|B_4(v) \cap Z| \geq d$.

By our definitions v must have distance ≤ 3 to some vertex $u \in Z^*$ but then $B_1(u) \subseteq B_4(v)$ and $B_1(u)$ is a subset of Z with size $\geq d$.

(14) For every $1 \leq i \leq \ell$ and $v \in Y_i^4$ we have $|B_4(v) \cap Z| \geq d - |Y_i^2|$.

By our definitions v is distance 3 to a point $u \in Z$ which has a neighbour in Z^*. For $j \neq i$ we have $B_1(u) \cap Y_j = \emptyset$, otherwise (9) would imply that a vertex u' satisfying $dist(u,u') = 1$ and $dist(u',v) = 2$ belongs to Z^*, contradicting the fact that v is distance 4 from Z^*. It follows that $B_1(u) \subseteq Z \cup Y_i^2$ and thus $B_1(u) \cap Z$ is a set of size $\geq d - |Y_i^2|$ which is contained in $B_4(v) \cap Z$.

Let $Y = V \setminus Z$, and for any pair of disjoint sets $S, T \subseteq V$ and positive integer k we let $e_k(S,T)$ denote the number of edges between the sets S and T in the graph G^k.

(15) The average degree of G^4 is at least $(3 - 2\gamma)d$.

This is a consequence of the following equation (here we use (2), (10) and (12) in getting to the third line and (13) and (14) in getting to the fifth line).

$$\begin{aligned}
\sum_{v \in V} deg_{G^4}(v) &= \sum_{v \in Z} deg_{G^4}(v) + \sum_{u \in Y} deg_{G^4}(u) \\
&= \sum_{v \in Z} e_4(v, Z \setminus \{v\}) + e_4(Z,Y) + \sum_{u \in Y} deg_{G^4}(u) \\
&\geq d|Z| + e_4(Z,Y) + \sum_{i=1}^{\ell} |Y_i^2||Y_i^4| + 2d|Y| \\
&= d\gamma n + \sum_{v \in Y} |B_4(v) \cap Z| + \sum_{i=1}^{\ell} |Y_i^2||Y_i^4| + 2(1-\gamma)dn \\
&\geq d\gamma n + \sum_{i=1}^{\ell} |Y_i \setminus Y_i^4| d + \sum_{i=1}^{\ell} |Y_i^4|(d - |Y_i^2|) + \sum_{i=1}^{\ell} |Y_i^2||Y_i^4| + 2(1-\gamma)dn \\
&= d\gamma n + 3(1-\gamma)dn \\
&= (3 - 2\gamma)dn
\end{aligned}$$

We can now complete the proof. By taking a convex combination of the bounds in (3) and (15) we have that the average degree of G^4 must be at least $\frac{2}{3}(2+\gamma)d + \frac{1}{3}(3-2\gamma)d = \frac{7}{3}d$ thus giving us a final contradiction. $\square$

Appendix

Here we carry out the calculation claimed in the introduction giving an upper bound on the average degree of the kth power (when $k \equiv 2 \pmod 3$) of the graphs appearing in Figure 2. For every odd integer $d > 1$ and every positive integer t we shall define such a graph H_t (where $t+1$ is the number of large circles in the picture). The vertex set has a partition as $\{X_{-1}, X_0, X_1, \ldots, X_{3t+1}\}$. The edges are defined as follows. For $-1 \le i \le 3t$ there is a complete bipartite graph between X_i and X_{i+1}. The sets X_{-1} and X_{3t+1} induce K_2, the sets X_0 and X_{3t} induce K_{d-1} minus a perfect matching, the set X_{3i} induces K_{d-1} for $1 \le i \le t-1$, and X_i is a single point when 3 does not divide i and $0 \le i \le 3t$.

For $0 \le i \le t$ let Y_i be the union of X_{3i} together with one vertex from X_{3i-1} and one vertex from X_{3i+1}, set $Y = \bigcup_{i=0}^{t} Y_i$ and set $\{u, u'\} = V(H_t) \setminus Y$. Now, let k be an integer with $k \equiv 2 \pmod 3$ and $k < 3t$. To assist in counting the edges in H_t^k we construct the alternate graph H_t' with vertex partition $\{Y_0, Y_1, \ldots, Y_t\}$ and with edges given by the rule that each Y_i induces a clique, and Y_i and Y_j are completely joined for $0 \le i < j \le t$ if $j - i \le \frac{k-2}{3}$. We then find

$$e(H_t') = (t+1)\tfrac{d(d+1)}{2} + \left(\frac{k-2}{3}(t+1) - \frac{1}{2}\frac{k-2}{3}\frac{k+1}{3}\right)(d+1)^2.$$

A vertex in X_{3i} will have degree at most 2 larger in H_t^k than in H_t' while a vertex in $X_i \setminus \{u, u'\}$ with i not a multiple of 3 will have degree at most $d+1$ larger in H_t^k than in H_t'. Using the fact that u and u' are vertices of minimum degree in H_t^k this gives us

$$\begin{aligned}
a(H_t^k) &\le \frac{1}{(t+1)(d+1)} \sum_{v \in Y} deg_{H_t^k}(v) \\
&\le \frac{1}{(t+1)(d+1)} \left(\sum_{v \in Y} deg_{H_t'}(v) + (t+1)(d-1)2 + 2(t+1)(d+1) \right) \\
&\le \tfrac{2}{(t+1)(d+1)} e(H_t') + 4 \\
&= (\tfrac{2k-1}{3})(d+1) - \tfrac{(k-2)(k+1)(d+1)}{9(t+1)} + 3 \\
&\le (\tfrac{2k-1}{3})(d+1) - \tfrac{(k-2)(k+1)(d+1)^2}{9n} + 3
\end{aligned}$$

as claimed in our introduction.

References

[1] A. L. Cauchy. Recherches sur les nombres, J. École polytech. 9 (1813) 99-116.

[2] H. Davenport. On the addition of residue classes, J. London Math. Soc. 10 (1935) 30-32.

[3] M. DeVos and S. Thomassé. Edge Growth in Graph Cubes, preprint, 2010.

[4] G. Godsil and G. Royle. Algebraic graph theory, Springer, New York, 2001.

[5] P. Hegarty. A Cauchy-Davenport type result for arbitrary regular graphs, preprint, 2009.

[6] M. Kneser. Abschtzung der asymptotischen Dichte von Summenmengen, Math. Z. 58, (1953). 459484.

[7] W. Mader. Über den Zusammenhang symmetrischer Graphen, Arch. Math 22 (1971) 333-336.

[8] A. Pokrovskiy. Growth of graph powers, preprint, 2010.

[9] B. D. Sullivan. A summary of results and problems related to the Caccetta-Häggkvist Conjecture, AIM Preprint 2006-13, 2006.

[10] M.E. Watkins. Connectivity of transitive graphs, J. Comb. Theory 8 (1970) 23-29.