Constructing a Model Transport Equation for a Massless Bose Gas and its Analytic Solution

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Abstract

A model kinetic equation is constructed for the transport of a massless Bose gas. This equation is applied to solution of the boundary value problem for the transport of radiation in the half-space occupied by a dispersive medium that is in local thermal equilibrium with the radiation. It is shown that the difference in temperature between the dispersive medium and the incident radiation depends substantially on the character of the scattering properties of the particles of the medium.

Key words: massless Bose gas, boundary value problem.

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Introduction. Statement of the problem and basic equations

In recent decades, transport theory has been substantially developed for Boltzmann gas [1, 2], Boltzmann plasma [3, 4], and Fermi gas [3-6]. Progress in describing radiation transport in dispersive ("turbid") media should also be mentioned [7, 8]. Both approximate and exact analytic methods (for different collision – integral models) have been used in these investigations.

In contrast to the case of Boltzmann and Fermi gases, no general statement of the transport problem is known for Bose gas, although some particular results have been obtained in this case (e.g., see [7, 8]).

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At the same time, transport phenomena are frequently encountered in practice, e.g., heat transport by phonons in a solid and radiative transport by photons in dispersive media. All of these processes have a number of common properties. In what follows, we confine ourselves to the case of massless Bose systems for which the chemical potential vanishes in the thermodynamic equilibrium.

Consider the problem of constructing a transport equation describing the evolution of Bose gas. The collision integrals used in transport theory take into account either the elastic scattering of photons (electromagnetic waves) or their absorption by a dispersive medium. However, if the dispersive medium in which radiative transport takes place is in local equilibrium with the photon gas, then it not only absorbs but also re-radiates the photons. As an example, we mention the case of radiative transport in gaseous dust clouds in the cosmos, where precisely this type of radiative transport from central regions to the outside of the cloud results in cooling and compression of the cloud, which, by–turn, gives rise to the star-formation process [9].

As an example of the construction of a model transport equation, we consider the Bhatnagar — Gross — Krook (BGK) model for the Boltzmann equation. The Boltzmann equation for the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ of the velocity of gas molecules has the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = J[f], \tag{1}$$

where $J[f]$ is the collision integral.

In constructing the model, the left-hand side of Eq. (1) remains unchanged, whereas the collision integral is transformed according to the following scheme. It is assumed that the molecules come to the equilibrium distribution after a single collision, i.e., the molecule dictistribution function after the collision coincides with the equilibrium distribution function

$$f_0 = n_0\Big(\frac{m}{2kT_0}\Big)^{3/2}\exp\Big[-\frac{m}{2kT_0}(\mathbf{v}-\mathbf{u})^2\Big],$$

where n_0, T_0 and $\mathbf{u}$ are defined below.

In this case, the Boltzmann equation becomes

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = \nu(v)(f_0 - f), \tag{2}$$

where $\nu(v)$ is the effective collision frequency.

The Boltzmann equation (1) is compatible with the conservation laws for the number of molecules, momentum, and energy. The model equation (2) must possess similar properties. This requirement leads to the uniquely determined parameters n_0, T_0 and $\mathbf{u}$ (for more detail, see [10]). In the case $\nu(v) = \text{const}$, the parameters n_0, T_0 and $\mathbf{u}$ coincide with the average molecule concentration, temperature, and velocity, respectively.

We consider the case of a massless Bose gas, meaning a photon gas. The case of a phonon gas differs only in some details, which are indicated in the course of the presentation.

The state of a photon gas is described by the distribution function $f_i(\omega, \mathbf{r}, \mathbf{n})$, where ω is the frequency, $\mathbf{n}$ is the direction of photon propagation ($\mathbf{n} = \dfrac{c}{\omega}\mathbf{k}$, $\mathbf{k}$ is the wave vector and c is the velocity of light), and the subscript i ($i = 1, 2$) indexes the polarization direction. For a phonon gas, there are three possible types of polarization and, therefore, the subscript i assumes the values $i = 1, 2, 3$ in this case. The transport equation for the distribution function $f_i(\omega, \mathbf{r}, \mathbf{n})$ has the form

$$\frac{\partial f}{\partial t} + c\mathbf{n}\nabla f = J[f]. \tag{3}$$

Here the functional $J[f]$ describes the absorption and re-radiation process for phonons in the dispersive medium where they propagate. We note that in contrast to photons, the phonons interact with one another. In this case, $J[f]$ means the collision integral for phonons. In Eq. (3), the parameter c is the velocity of light for the photon gas and the velocity of sound in the case of the phonon gas (on the condition that the propagation velocities of the longitudinal and transverse oscillation coincide).

Let a photon gas be in local equilibrium with a dispersive medium. In this case, the distribution function of photons re-radiated by the medium coincides with the Planck function

$$f_P = \left[\exp\frac{\hbar\omega}{kT} - 1\right]^{-1},$$

where T is a parameter defined below. Equation (3) takes the form

$$\frac{\partial f_i}{\partial t} + c\mathbf{n}\nabla f_i = \nu(\omega)(f_P - f_i), \tag{4}$$

where $\nu(\omega)$ is the effective froquency of the phonon collisions. For photons, this quantity is related to the effective absorption cross-section for the dispersing particles, namely,

$$\nu(\omega) = c\int n(\mathbf{r})\sigma_r(\omega)dr,$$

where $n(\mathbf{r})$ is the radial distribution of the particles (on the condition that they have a spherical form) and $\sigma_r(\omega)$ is the absorption cross-section for a particle of radius r.

Note that in the general case, it is necessary to distinguish between photon scattering and re-radiation. For simplicity, we do not discuss this question in the present paper.

In general, the number of photons and their momenta are not preserved under re-radiation. At the same time, the photon gas energy must be invariant in local thermodynamic equilibrium between the photon gas and the dispersion system. This means that the integral on the right-hand side of Eq. (4) vanishes,

$$\int\sum_{i=1}^{2}\omega^3\nu(\omega)(f_P - f_i)d\mathbf{n}d\omega = 0. \tag{5}$$

Relationship (5) is the one whichn determines the parameter T.

We consider the case where the Bose system is in a nearly equilibrium state. Here the distribution function f_i can be represented in the form

$$f_i = f_P^{(0)} + \psi_i, \tag{6}$$

where ψ_i is a linear correction to the Planck distribution $f_P^{(0)}$.

In what follows, we assume that the radiation is not polarized and the distribution function does not depend on i. Therefore, we omit the subscript i in the distribution functions f_i and ψ_i. The closeness to equilibrium means that T_0 and the parameter T, which has the meaning of the effective temperature, vary weakly in the gas volume, which is equivalent to the relationship

$$|\delta| \ll 1, \tag{7}$$

where $\delta = \dfrac{T - T_0}{T_0}$, and T_0 is the temperature of the Bose gas at some point in the volume.

Under the condition (7), we can linearize the Planck function $f_P(\delta)$ with respect to the parameter δ, $f_P(\delta) = f_P(0) + f_P'(0)\delta$, where

$$f_P(\delta) = \left[\exp\Big(\frac{\hbar\omega}{k(T_0 + \delta T_0)}\Big) - 1\right]^{-1},$$

or, in explicit form,

$$f_P = f_P^{(0)} + \frac{\hbar\omega}{kT_0^2} E\Big(\frac{\hbar\omega}{kT_0}\Big)(T - T_0). \tag{8}$$

Here $E(x)$ is the Einstein function,

$$E(x) = \frac{e^x}{(e^x - 1)^2}.$$

On substituting (6)-(8) into Eq. (4), we derive

$$\frac{\partial\psi}{\partial t} + c\mathbf{n}\nabla\psi = \nu(\omega)\left[\frac{\hbar\omega}{kT_0^2} E\Big(\frac{\hbar\omega}{kT_0}\Big)(T - T_0) - \psi(\omega, \mathbf{r}, \mathbf{n})\right]. \tag{9}$$

Equation (9) is the linearized form of the transport equation (4).

It follows from (5) that the difference $T-To$ is given by the formula

$$T - T_0 = \frac{kT_0^2}{\hbar}\left[\int \nu(\omega)\omega^4 E\Big(\frac{\hbar\omega}{kT_0}\Big)\,d\omega d\mathbf{n}\right]^{-1}\int \nu(\omega)\omega^3\psi d\omega d\mathbf{n}.$$

Equation (9) shows that it is convenient to introduce a new function φ instead of ψ,

$$\psi(\omega,\mathbf{r},\mathbf{n}) = \omega E\Big(\frac{\hbar\omega}{kT_0}\Big)\varphi(\omega,\mathbf{r},\mathbf{n}),$$

and, then, Eq. (9) can be rewritten as

$$\frac{\partial\varphi}{\partial t} + c\mathbf{n}\nabla\varphi =$$

$$= \frac{\nu(\omega)}{l}\int \nu(\omega')\omega'^4 E\Big(\frac{\hbar\omega'}{kT_0}\Big)\varphi(\omega',\mathbf{r},\mathbf{n})\,d\omega' d\mathbf{n} - \nu(\omega)\varphi(\omega,\mathbf{r},\mathbf{n}), \quad (10)$$

where

$$l = \int \nu(\omega')\omega'^4 E\Big(\frac{\hbar\omega'}{kT_0}\Big)\,d\omega' d\mathbf{n}.$$

We consider the case of a stationary process and assume that the function $\nu(\omega)$ can be approximated by a power function, $\nu(\omega) = \nu_0\omega^\alpha$. Note that the case $\alpha = 2$ corresponds to the absorption of electromagnetic radiation by particles whose size is small in comparison with the wavelength. For large-sized particles, $\alpha = 0$. We introduce the dimensionless variables

$$\omega^* = \frac{\hbar\omega}{kT_0} \quad \text{and} \quad \mathbf{r}^* = \frac{\nu_0}{c}\Big(\frac{kT_0}{\hbar}\Big)^\alpha \mathbf{r}.$$

In this case, Eq. (10) takes the form (here and henceforth, the asterisks in the variables are omitted)

$$\mathbf{n}\nabla\varphi = \frac{\omega^\alpha}{4\pi l_0(\alpha)}\int \omega'^{\alpha+4}E(\omega')\varphi(\omega',\mathbf{r},\mathbf{n})\,d\omega' d\mathbf{n} - \omega^\alpha\varphi(\omega,\mathbf{r},\mathbf{n}), \quad (11)$$

where

$$4\pi l_0(\alpha) = \int \omega^{\alpha+4}E(\omega)d\omega d\mathbf{n} = 4\pi\int_0^\infty \omega^{\alpha+4}E(\omega)d\omega,$$

hence

$$l_0(\alpha) = \int_0^\infty \omega^{\alpha+4} E(\omega) d\omega.$$

We note that it is assumed in the above calculations that $\nu(\omega)$ does not depend on the coordinate, though, generally, this is not the case. This fact can be taken into account by introducing the optical length [8]. We do not dwell on this in what follows.

We consider the one-dimensional problem in which $\varphi = \varphi(x, v, \omega)$, where $v = n_x \text{v}$, i.e., v is the cosine of the angle between the direction of the quantum motion and the x-axis. Here, Eq. (11) can be rewritten as

$$v \frac{\partial}{\partial x} \varphi(x, v, \omega) + \omega^\alpha \varphi(x, v, \omega) =$$

$$= \frac{\omega^\alpha}{2 l_0(\alpha)} \int_0^\infty \omega'^{\alpha+4} E(\omega') d\omega' \int_{-1}^{1} \varphi(x, v', \omega') dv', \tag{12}$$

where

$$l_0 \equiv l_0(\alpha) = \int_0^\infty \omega^{\alpha+4} E(\omega) d\omega = \int_0^\infty \frac{e^\omega \omega^{\alpha+4}}{(e^\omega - 1)^2} d\omega.$$

In the transition to the Eq. (12), we integrated with respect to the azimuthal angle from 0 to 2π because the function φ does not depend on this angle.

We consider a concrete Milne-type problem. Let a dispersive medium occupy the half-space $x > 0$ and let there be a temperature gradient along the x-axis in the medium far from the boundary. We assume that radiation at equilibrium with the temperature T_0 comes from the half-space $x < 0$ into the medium and the total radiation outgoing from the medium is absorbed somewhere and does not come back. The temperature far from the boundary has the form

$$T = T_1 + Kx, \tag{13}$$

where K is the temperature gradient. Our aim is to find the relationship between T_0, T_1, and K. This can be written in the linear approximation as

$$T_1 - T_0 = fK, \tag{14}$$

where f is an unknown coefficient depending on the properties of the medium.

In application to the cosmic gaseous dust cloud, this statement of the problem means the following. The radiation at equilibrium with the temperature T_0 close to 3K is incident on the boundary of a gaseous dust cloud. Under cooling of the cloud, energy is transformed and radiated outward. Therefore, there is a temperature gradient in the cloud. We need to find the relationship between this gradient, which is defined by the heat flow, the temperature T_0, and the temperature at the boundary. This relationship is given by formula (14) with an unknown coefficient f that must be determined.

It is easy to show that the following two discrete modes are solutions to Eq. (12) (see Sec. 1 below):

$$\varphi_+(x, v, \omega) = 1, \qquad \varphi_-(x, v, \omega) = x - \frac{v}{\omega^\alpha}.$$

The boundary conditions are as follows:

$$\varphi(0, v, \omega) = 0, \qquad 0 < v < 1, \tag{15}$$

$$\varphi(x, v, \omega) = K_0 + K\Big(x - \frac{v}{\omega^\alpha}\Big) + o(1), \quad x \to +\infty, \quad -1 < v < 0, \tag{16}$$

where K is the temperature gradient (see (13)) and $K_0 = fKT_0^{-1}$ is an unknown coefficient proportional to the relative temperature gradient. Note that a similar question for the Boltzmann gas can be stated as the Smolukhowski problem on the discontinuity of temperature.

1. Eigenfunctions corresponding to continuous and discrete spectra

It can be clearly seen from Eq. (12) and boundary conditions (15), (16) that the desired function φ depends on two variables, x and $\mu = v\omega^{-\alpha}$. Let us perform the change of variables $v = \mu\omega'^{\alpha}$ and $\omega = \omega'$ in Eq. (12). The corresponding Jacobian is equal to ω'^{α}. Consequently, Eq. (12) can be rewritten in the new variables as

$$\mu\frac{\partial\varphi}{\partial x} + \varphi(x,\mu) = \frac{1}{2l_0(\alpha)}\int\limits_0^\infty \omega^{2\alpha+4}E(\omega)d\omega \int\limits_{-1/\omega^\alpha}^{1/\omega^\alpha} \varphi(x,\mu')d\mu'. \qquad (1.1)$$

Accordingly, under the above transformation, the boundary conditions (15), (16) for Eq. (1.1) take the form

$$\varphi(0,\mu) = 0, \qquad 0 < \mu < \infty, \qquad (1.2)$$

$$\varphi(x,\mu) = K_0 + K(x-\mu) + o(1), \quad x \to +\infty, \quad -\infty < \mu < 0. \quad (1.3)$$

On replacing μ' by $\omega^{-\alpha}\mu'$ in Eq. (1.1), we can rewrite it as

$$\mu\frac{\partial}{\partial x}\varphi(x,\mu) + \varphi(x,\mu) =$$

$$= \frac{1}{2l_0(\alpha)}\int\limits_0^\infty \omega^{\alpha+4}E(\omega)d\omega \int\limits_{-1}^{1} \varphi(x,\omega^{-\alpha}\mu')d\mu'. \qquad (1.4)$$

Let us perform the separation of variables in Eq. (1.4) as follows:

$$\varphi_\eta(x,\mu) = \exp(-\frac{x}{\eta})\Phi(\eta,\mu), \qquad \eta \in \mathbb{C}.$$

As a result, we derive the characteristic equation

$$(\eta-\mu)\Phi(\eta,\mu) = \frac{1}{2l_0(\alpha)}\eta\int\limits_0^\infty \omega^{\alpha+4}E(\omega)d\omega \int\limits_{-1}^{1} \Phi(\eta,\omega^{-\alpha}\mu')d\mu'. \quad (1.5)$$

We denote

$$n(\eta) = \int\limits_0^\infty \omega^{\alpha+4}E(\omega)d\omega \int\limits_{-1}^{1} \Phi(\eta,\omega^{-\alpha}\mu')d\mu' \qquad (1.6)$$

and rewrite Eq. (1.5) in the form

$$(\eta-\mu)\Phi(\eta,\mu)=\frac{1}{2l_0(\alpha)}\eta n(\eta). \qquad (1.7)$$

Equation (1.7) implies that the eigenfunctions corresponding to the continuous spectrum are the distributions [11]

$$\Phi(\eta,\mu)=\frac{1}{2l_0(\alpha)}\eta n(\eta)P\frac{1}{\eta-\mu}+g(\eta)\delta(\eta-\mu), \quad \eta,\mu\in(-\infty,+\infty). \qquad (1.8)$$

Here P symbolizes the principal value of the integral, $\delta(x)$ is the Dirac delta function, and $g(\eta)$ can be found from the normalization condition for (1.6). Substituting (1.8) into (1.6), we obtain

$$n(\eta)\lambda(\eta)=g(\eta)\xi_\alpha(\eta), \qquad (1.9)$$

where

$$\xi_\alpha(\eta)=\int\limits_0^{1/\eta^\alpha}\omega^{2\alpha+4}E(\omega)d\omega,$$

and the dispersion function

$$\lambda(z)=1+\frac{z}{2l_0(\alpha)}\int\limits_0^\infty\omega^{2\alpha+4}E(\omega)d\omega\int\limits_{-1}^{1}\frac{d\mu}{\mu-\omega^\alpha z}$$

of the problem can be expressed via the Case dispersion function [12]

$$\lambda_C(z)=1+\frac{1}{2}z\int\limits_{-1}^{1}\frac{d\mu}{\mu-\omega^\alpha z}$$

in the following way

$$\lambda(z)=\frac{1}{l_0(\alpha)}\int\limits_0^\infty\omega^{2\alpha+4}E(\omega)\lambda_C(\omega^\alpha z)d\omega. \qquad (1.10)$$

The boundary values of the function $\lambda_C(\omega^\alpha z)$ on the edges of the slit

$$\Delta_\alpha=\Big(-\frac{1}{\omega^\alpha},\quad\frac{1}{\omega^{-\alpha}}\Big)$$

are determined by the Sokhotski formulas,

$$\lambda_C^{\pm}(\omega^\alpha\mu) = \begin{cases} \lambda_C(\omega^\alpha\mu) \pm i\dfrac{\pi}{2}\omega^\alpha\mu, & \mu \in \Delta_\alpha, \\ \lambda_C(\omega^\alpha\mu), & \mu \notin \Delta_\alpha. \end{cases} \qquad (1.11)$$

In view of (1.10) and (1.11), the boundary values of $\lambda(z)$ are related by the formula

$$\lambda^{\pm}(\mu) = \lambda(\mu) \pm i\frac{\pi}{2}\frac{\xi_\alpha(\mu)}{l_0(\alpha)}, \qquad \mu \in \mathbb{R}.$$

With the help of expression (1.9), we bring formula (1.8) for the eigenfunctions to the form $\Phi(\eta,\mu) = \tilde{\Phi}(\eta,\mu)n(\eta)$, where

$$\tilde{\Phi}(\eta,\mu) = \frac{1}{2l_0(\alpha)}\eta P\frac{1}{\eta-\mu} + \frac{\lambda(\eta)}{\xi_\alpha(\eta)}\delta(\eta-\mu)$$

is the eigenfunction corresponding to the normalization condition

$$n(\eta) \equiv 1.$$

It can be shown that $\lambda(z)$ has a second–order zero at $z = \infty$ with which two solutions to the original equation (1.4) are associated, namely,

$$\varphi_+(x,\mu) = 1 \quad \text{and} \quad \varphi_-(x,\mu) = x - \mu.$$

2. Eigenfunction expansion for the solution

Theorem. *The boundary value problem (1.2)-(1.4) has a unique solution, whose expansion in terms of the eigenfunctions of the characteristic equation has the form*

$$\varphi(x,\mu) = K_0 + K(x-\mu) + \int\limits_0^\infty e^{-x/\eta}\tilde{\Phi}(\eta,\mu)n(\eta)d\eta. \qquad (2.1)$$

Here the unknowns are the coefficient K_0 associated with the discrete spectrum and the function $n(\eta)$, which is the coefficient corresponding to the continuous spectrum. The expansion (2.1) is interpreted in the classical sense, namely,

$$\varphi(x,\mu) = K_0 + K(x-\mu) +$$

$$+\frac{1}{2l_0(\alpha)}\int\limits_0^{\infty} e^{-x/\eta}\frac{\eta n(\eta)}{\eta-\mu}d\eta+\frac{\lambda(\mu)}{\xi_\alpha(\mu)}e^{-x/\mu}\theta_+(\mu), \tag{2.2}$$

where $\theta_+(\mu)$ is the characteristic function of the positive half-line, i.e., $\theta_+(\mu)=1$ if $\mu\in(0,+\infty)$ and $\theta_+(\mu)=0$ if $\mu\notin(0,+\infty)$.

Proof. Using boundary condition (1.2), we pass from expansion (2.2) to the following singular integral equation with the Cauchy kernel [13]:

$$K_0-K_1\mu+\frac{1}{2l_0(\alpha)}\int\limits_0^{\infty}\frac{\eta n(\eta)d\eta}{\eta-\mu}+\frac{\lambda(\mu)}{\xi_\alpha(\mu)}n(\mu)=0, \quad \mu>0. \tag{2.3}$$

Let us introduce the auxiliary function

$$N(z)=\int\limits_0^{\infty}\frac{\eta n(\eta)d\eta}{\eta-z}. \tag{2.4}$$

Multiplying both sides of Eq. (2.3) by

$$\frac{2}{l_0(\alpha)}[\lambda^+(\mu)-\lambda^-(\mu)]=2\pi i\mu\xi_\alpha(\mu)$$

(2.4) and using the boundary values $N^{\pm}(\mu)$ and $\lambda^{\pm}(\mu)$, we reduce Eq. (2.3) to the Riemann boundary problem

$$\lambda^+(\mu)[N^+(\mu)+2l_0(\alpha)(K_0-K_1\mu)]=$$

$$=\lambda^-(\mu)[N^-(\mu)+2l_0(\alpha)(K_0-K_1\mu)], \quad \mu>0. \tag{2.5}$$

The coefficient in the boundary problem (2.5) is the function

$$G(\mu)=\frac{\lambda^-(\mu)}{\lambda^+(\mu)}.$$

Consider the following factorization problem for the coefficient $G(\mu)$

$$\frac{X^+(\mu)}{X^-(\mu)}=\frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \qquad \mu>0. \tag{2.6}$$

To solve this problem, we calculate the index $\varkappa$ of the coefficient $G(\mu)$,

$$\varkappa = \frac{1}{2\pi}\Big[\arg G(\mu)\Big]_{(0,+\infty)}.$$

where the expression $[\arg G(\mu)]_{(0,+\infty)}$ denotes the increment of the function in the brackets under variation of the argument from 0 to $+\infty$.

Let $\theta_\alpha(\mu) = \arg \lambda^+(\mu)$ be the principal value of the argument of $\lambda^+(\mu)$,

$$\theta_\alpha(\mu) = \arctan \frac{2\int\limits_0^\infty \omega^{\alpha+4}E(\omega)\lambda_C(\omega^\alpha\tau)d\tau}{\pi\tau\int\limits_0^\infty \omega^{2\alpha+4}E(\omega)d\omega}.$$

Since $\overline{\lambda^+(\mu)} = \lambda^-(\mu)$ (where the bar symbolizes complex conjugation), we have $G(\mu) = \exp(-2_\alpha(\mu))$, and, therefore,

$$\varkappa = -\frac{1}{\pi}[\theta_\alpha(\mu)]_{(0,+\infty)} = -\frac{1}{\pi}[\theta_\alpha(+\infty) - \theta_\alpha(0)] = -1.$$

As a solution to problem (2.6), we take the function

$$X(z) = \frac{1}{z}\exp V(z),$$

which does not vanish at zero, where

$$V(z) = \frac{1}{\pi}\int\limits_0^\infty \frac{\theta_\alpha(\tau) - \pi}{\tau - z}d\tau.$$

We now return to the problem (2.5) and apply (2.6) to transform it into the problem of determining an analytic function from the following condition

$$X^+(\mu)[N^+(\mu) + 2l_0(\alpha)(K_0 - K_1\mu)] =$$

$$+X^-(\mu)[N^-(\mu) + 2l_0(\alpha)(K_0 - K_1\mu)], \qquad \mu > 0. \qquad (2.7)$$

Taking into account the behavior of the functions entering (2.7), we derive the general solution to this problem in the form

$$N(z) = -2l_0(\alpha)(K_0 - Kx) + \frac{C_0}{X(z)}, \tag{2.8}$$

where C_0 is an arbitrary constant.

In order to use the function (2.8) as the auxiliary function $N(z)$ introduced by the formula (2.4), we remove the pole of the solution (2.8) at $z = \infty$ and make the limit of (2.8) at the point $z = \infty$ equal to zero by setting

$$C_0 = -2l_0(\alpha)K, \qquad K_0 = V_1(\alpha)K, \tag{2.9}$$

$$V_1(\alpha) = -\frac{1}{\pi}\int\limits_0^\infty [\theta_\alpha(\mu) - \pi]\, d\mu. \tag{2.10}$$

The coefficient $n(\eta)$ for the continuous spectrum can be found from Eq. (2.8), namely,

$$2\pi i\eta n(\eta) = -2l_0(\alpha)K\left(\frac{1}{X^+(\eta)} - \frac{1}{X^-(\eta)}\right).$$

Thus, the coefficients in (2.1) have been uniquely defined. The uniqueness of the expansion (2.1) is proved by the fact that there can be no nontrivial expansion of zero with respect to the eigenfunctions of the characteristic equation. Obviously, expansion (2.1) automatically satisfies to the boundary condition (1.3) and (1.2) holds by construction. Direct verification shows that the expansion (2.1) satisfies to the Eq. (1.4). The theorem is proved.

3. Numerical results. Discussion and conclusion

First, we note that we can find the exact value of the integral (2.10) for $\alpha = 0$. Indeed, if $\alpha = 0$, then it can be seen from the formula (1.10) that the dispersion function $\lambda(z)$ coincides with the Case dispersion function $\lambda_C(z)$. In this case,

$$V_1^\circ = -\frac{1}{\pi}\int\limits_0^\infty \operatorname{arccot}\frac{\pi\tau}{2\lambda_C(\tau)}d\tau = 0.71045. \tag{3.1}$$

Let us find an approximate value of the integral (2.10) for $\alpha > 0$. Applying the saddle-point method, we replace $\lambda(z)$ by an approximating function $\lambda(z)$. The expression (1.10) implies that the saddle point ω_0 is determined by the equation

$$e^{\omega_0} = \frac{\alpha + 4 + \omega_0}{\alpha + 4 - \omega_0}. \tag{3.2}$$

The approximate solution $\tilde{\omega}_0$ to Eq. (3.2) for $\alpha + 4 > 1$ can be calculated by the formula

$$\tilde{\omega}_0 = (\alpha + 4)(1 - 2e^{-\alpha-4}). \tag{3.3}$$

Note that formula (3.3) provides a satisfactory approximation even for $a = 0$. Indeed, in this case $\omega_0 = 3.83002$ and $\tilde{\omega}_0$=3.85347. For $\alpha = 2$, we have ω_0=5.96941 and $\tilde{\omega}_0 = 5.97025$, i.e., formula (3.3) gives a nearly exact value of the saddle point. According to the saddle-point method, we have

$$\tilde{\lambda}(z) = \lambda_C(\omega_0^\alpha z)$$

and, therefore,

$$\tilde{V}_1 = -\frac{1}{\pi}\int\limits_0^{1/\omega_0^\alpha} [\tilde{\theta}_\alpha(\tau) - \pi]d\tau = -\frac{1}{\pi}\int\limits_0^{1/\omega_0^\alpha} \operatorname{arccot}\frac{\pi\omega_0^\alpha\tau}{2\lambda_C(\omega_0^\alpha\tau)},$$

whence

$$\tilde{V}_1 = \omega_0^{-\alpha} V_1^\circ. \tag{3.4}$$

Note that formula (3.4) with $A = 0$ exactly implies the formula (3.1) and for $\alpha = 2$, we have $\tilde{V}_1 = 0.01994$. The decrease in V_1 with increasing α relates to the fact that the contribution from high frequencies (large values of ω) is reduced during the transport process in the dispersive medium due to the faster growth of the collision frequency $\nu(\omega)$) with increasing α. At the same time, the radiation from the boundary surface is not limited by this factor nor does the high-frequency contribution depend on α.

The result is that the relative outflow of energy from the surface increases together with α (as compared to the volumetric outflow). As a result, the coefficient V_1, which characterizes the temperature drop between the boundary of the exterior medium and the incident radiation, decreases.

In conclusion, we note that in the present paper, a generalization of the Case — Zweifel classical theory [12] to the transport equation for a massless Bose gas was considered for the first time. It turns out that the suggested method permits transport equations with a double integral on the right-hand side to be solved analytically.

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