FROM L-SERIES OF ELLIPTIC CURVES TO MAHLER MEASURES

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Abstract. We prove the conjectural relations between Mahler measures and L-values of elliptic curves of conductors 20 and 24. We also present new hypergeometric expressions for L-values of CM elliptic curves of conductors 27 and 36. Furthermore, we prove a new functional equation for the Mahler measure of the polynomial family $(1+X)(1+Y)(X+Y)-\alpha XY$, $\alpha\in\mathbb{R}$.

1. Introduction

The Mahler measure of a two-variate Laurent polynomial $P(X,Y)$ is defined by

$$\mathrm{m}(P) := \iint_{[0,1]^2} \log|P(e^{2\pi i t}, e^{2\pi i s})|\,\mathrm{d}t\,\mathrm{d}s.$$

In this paper we are mostly concerned with the Mahler measures of three polynomial families,

$$\begin{aligned} m(\alpha) &:= \mathrm{m}\Big(\alpha + X + \frac{1}{X} + Y + \frac{1}{Y}\Big), \\ g(\alpha) &:= \mathrm{m}\big((1+X)(1+Y)(X+Y) - \alpha XY\big), \\ n(\alpha) &:= \mathrm{m}\big(X^3+Y^3+1-\alpha XY\big). \end{aligned}$$

Based on numerical experiments, Boyd observed that these functions can be related to the values of L-series of elliptic curves [7]. For example, he hypothesized that

$$m(8) = 4m(2) = \frac{24}{\pi^2}L(E,2), \tag{1}$$

$$g(4) = \frac{3}{4}n(\sqrt[3]{32}) = \frac{10}{\pi^2}L(E',2), \tag{2}$$

where E and E' are elliptic curves of conductors 24 and 20, respectively. The primary goal of this article is to present rigorous proofs of (1) and (2). In the remainder of the introduction we will briefly describe our method, define notation, review facts about Mahler measures, and present additional theorems.

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The modularity theorem provides the key to proving Boyd's formulas, because it allows one to reformulate his conjectures in more elementary terms. The modularity theorem shows that L-functions of elliptic curves can be equated to Mellin transforms of weight-two modular forms. For a generic elliptic curve E, we can write

$$L(E,2) = -\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q,$$

where $f(e^{2\pi i\tau})$ is a modular form of weight 2 on a congruence subgroup of $SL_2(\mathbb{Z})$. The choice of $f(q)$ is predetermined by the elliptic curve E. For instance, if E has conductor 20, then $f(q) = e_2^2 e_{10}^2$. If E has conductor 24, then $f(q) = e_2 e_4 e_6 e_{12}$. For brevity we use the following notation for the eta product:

$$e_j = e_j(q) := q^{j/24}\prod_{k=1}^{\infty}(1-q^{jk}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{j(6n+1)^2/24}.$$

Our first step is to find modular functions $x(q)$, $y(q)$, and $z(q)$ which depend on $f(q)$, such that

$$-\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q = \int_0^1 x(q)\log y(q)\,\mathrm{d}z(q).$$

Next express x and y as algebraic functions of z. If we write $x(q) = X(z(q))$, and $y(q) = Y(z(q))$, then the substitution reduces $L(E,2)$ to a complicated(!) integral of elementary functions:

$$L(E,2) = \int_{z(0)}^{z(1)} X(z)\log Y(z)\,\mathrm{d}z.$$

The final step is to relate the integral to Mahler measures. In order to accomplish this reduction, it will be necessary to use many properties of hypergeometric functions.

Let us note that $m(\alpha)$, $n(\alpha)$ and $g(\alpha)$ can all be expressed in terms of hypergeometric functions. These formulas provide an efficient way to compute the Mahler measures numerically. It was shown by Rodriguez-Villegas [15] that for every $\alpha\in\mathbb{C}$,

$$m(\alpha) = \operatorname{Re}\left(\log\alpha - \frac{2}{\alpha^2}\,{}_4F_3\left(\begin{matrix}\frac32,\,\frac32,\,1,\,1\\ 2,\,2,\,2\end{matrix}\,\middle|\,\frac{16}{\alpha^2}\right)\right); \tag{3}$$

furthermore [11] if $\alpha\ge 0$, then

$$m(\alpha) = \frac{\alpha}{4}\operatorname{Re}{}_3F_2\left(\begin{matrix}\frac12,\,\frac12,\,\frac12\\ 1,\,1\end{matrix}\,\middle|\,\frac{\alpha^2}{16}\right). \tag{4}$$

More involved hypergeometric expressions are known for $g(\alpha)$ and $n(\alpha)$ [16, Theorem 3.1]; in particular, the formulas

$$\begin{aligned} g(\alpha) = \frac13\operatorname{Re}\Biggl(&\log\frac{(\alpha+4)(\alpha-2)^4}{\alpha^2} - \frac{2\alpha^2}{(\alpha+4)^3}\,{}_4F_3\left(\begin{matrix}\frac43,\,\frac53,\,1,\,1\\ 2,\,2,\,2\end{matrix}\,\middle|\,\frac{27\alpha^2}{(\alpha+4)^3}\right)\\ &- \frac{8\alpha}{(\alpha-2)^3}\,{}_4F_3\left(\begin{matrix}\frac43,\,\frac53,\,1,\,1\\ 2,\,2,\,2\end{matrix}\,\middle|\,\frac{27\alpha}{(\alpha-2)^3}\right)\Biggr) \end{aligned} \tag{5}$$

and

$$n(\alpha) = \mathrm{Re}\left(\log\alpha - \frac{2}{\alpha^3}\,{}_4F_3\left(\begin{matrix}\frac{4}{3},\ \frac{5}{3},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\ \middle|\ \frac{27}{\alpha^3}\right)\right) \tag{6}$$

are valid for $|\alpha|$ sufficiently large. Formula (5) holds on the real line if $\alpha \in \mathbb{R}\backslash[-4,2]$.

It is a subtle but important point that our proofs are essentially elementary. The modularity theorem shows that $L(E,2) = L(f,2)$, however the formulas we prove for $L(f,2)$ are true *unconditionally*. For example, many of Boyd's conjectures can be restated as relations between Mahler measures and the quadruple lattice sum [16]

$$F(b,c) := (b+1)^2(c+1)^2 \times \sum_{\substack{n_i=-\infty\\ i=1,2,3,4}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\left((6n_1+1)^2 + b(6n_2+1)^2 + c(6n_3+1)^2 + bc(6n_4+1)^2\right)^2}, \tag{7}$$

so that the above examples can be written as

$$m(8) = 4m(2) = \frac{24}{\pi^2}F(2,3), \tag{8}$$

$$g(4) = \frac{3}{4}n(\sqrt[3]{32}) = \frac{10}{\pi^2}F(1,5). \tag{9}$$

Formulas (8) and (9) are true even without the modularity theorem. In fact, one significant aspect of Boyd's work, is that it provides a recipe to relate slowly-converging lattice sums to hypergeometric functions. For more details, many other conjectural examples as well as for state-of-art in the area, the reader may consult [7], [15] and [16].

We will prove many additional theorems with the strategy we have described. For instance, we will construct new hypergeometric evaluations

$$F(1,3) = \frac{\Gamma^3(\frac{1}{3})}{27}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\ \frac{1}{3},\ 1\\ \frac{2}{3},\ \frac{4}{3}\end{matrix}\ \middle|\ 1\right) - \frac{\Gamma^3(\frac{2}{3})}{18}\,{}_3F_2\left(\begin{matrix}\frac{2}{3},\ \frac{2}{3},\ 1\\ \frac{4}{3},\ \frac{5}{3}\end{matrix}\ \middle|\ 1\right),$$

$$F(1,1) = -\frac{2\pi^2\log 2}{27} + \frac{\Gamma^3(\frac{1}{3})}{3\cdot 2^{7/3}}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\ \frac{1}{3},\ 1\\ \frac{5}{6},\ \frac{4}{3}\end{matrix}\ \middle|\ -\frac{1}{8}\right) + \frac{\Gamma^3(\frac{2}{3})}{2^{11/3}}\,{}_3F_2\left(\begin{matrix}\frac{2}{3},\ \frac{2}{3},\ 1\\ \frac{7}{6},\ \frac{5}{3}\end{matrix}\ \middle|\ -\frac{1}{8}\right)$$

for the L-series of CM elliptic curves of conductors 27 and 36. We also prove elementary integrals for lattice sums which are not associated to elliptic curves: $F(3,7)$, $F(6,7)$ and $F(3/2,7)$. Finally, we will derive a new functional equation

$$g\big(4p(1+p)\big) + g\left(\frac{4(1+p)}{p^2}\right) = 2g\left(\frac{2(1+p)^2}{p}\right), \qquad \frac{\sqrt{3}-1}{2} \le p \le 1,$$

for the Mahler measure $g(\alpha)$. This last formula resembles some of the functional equations due to Lalín and Rogers [12].

We will conclude the introduction with a word about notation. This paper involves a large number of q-series manipulations, and draws heavily from Berndt's versions of Ramanujan's Notebooks [3, 4, 5], and from Ramanujan's Lost Notebook [1]. For this reason, we have chosen to preserve Ramanujan's theta function notation

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

We will also define the notation for signature-three theta functions in the next section.

2. Conductor 27

In this section we will look at the CM elliptic curves of conductors 27 and 36, as well as at some non-elliptic curve lattice sums. Recall that elliptic curves of conductor 27 are associated to $e_3^2 e_9^2$, and elliptic curves of conductor 36 are associated to e_6^4 [13]. Define

$$H(x) := \int_0^1 \frac{\log q}{q} \, \frac{e_3^3}{e_1} \, \frac{e_x^3}{e_{3x}} \, \mathrm{d}q.$$

The following lemma shows that certain values of (7) are expressed in terms of $H(x)$.

Lemma 1. *The following relations are true:*

$$9F(1,3) = -H(1), \tag{10}$$

$$144F(1,1) = -16H\left(\frac{4}{3}\right) + H\left(\frac{1}{12}\right), \tag{11}$$

$$\frac{27}{16}F(3,7) = \frac{8}{7}H(1) - H(7) - \frac{1}{49}H\left(\frac{1}{7}\right), \tag{12}$$

$$\frac{27}{49}F(6,7) = \frac{1}{49}H\left(\frac{2}{7}\right) + H(14) - \frac{8}{7}H(2), \tag{13}$$

$$\frac{27}{25}F\left(\frac{3}{2},7\right) = \frac{2}{7}H\left(\frac{1}{2}\right) - \frac{1}{4}H\left(\frac{7}{2}\right) - \frac{1}{14^2}H\left(\frac{1}{14}\right). \tag{14}$$

Proof. Equation (10) follows from the definition of $H(x)$. Formula (11) follows from integrating a modular equation equivalent to Somos [19, Entry $t_{36,9,39}$]:

$$\frac{e_1^3 e_{36}^3}{e_3 e_{12}} - \frac{e_4^3 e_9^3}{e_{12} e_3} + e_6^4 = 0.$$

We can recover (12) by integrating a modular equation equivalent to Ramanujan [4, pg. 236, Entry 68]:

$$e_1^2 e_3^2 + 7e_7^2 e_{21}^2 + 3e_1 e_3 e_7 e_{21} = \frac{e_3^3 e_7^3}{e_1 e_{21}} + \frac{e_1^3 e_{21}^3}{e_3 e_7}.$$

Equation (14) follows from a modular equation equivalent to Somos [19, Entry $x_{42,8,56}$]:

$$\frac{e_1^3 e_{42}^3}{e_3 e_{14}} + \frac{e_6^3 e_7^3}{e_2 e_{21}} - \frac{e_1^3 e_6^3}{e_3 e_2} - 7\frac{e_7^3 e_{42}^3}{e_{21} e_{14}} = 3e_2 e_3 e_{14} e_{21},$$

and (13) follows from a modular equation equivalent to Somos [19, Entry $x_{42,8,64}$]:

$$\frac{e_2^3e_{21}^3}{e_6e_7}+\frac{e_3^3e_{14}^3}{e_1e_{42}}-\frac{e_2^3e_3^3}{e_1e_6}-7\frac{e_{14}^3e_{21}^3}{e_{42}e_7}=-3e_1e_6e_7e_{42}.$$

□

In the next proposition we will reduce $H(x)$ to an integral of signature three theta functions. Recall that

$$a(q)=\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2},$$

and

$$b(q)=\frac{1}{2}\big(3a(q^3)-a(q)\big),\qquad c(q)=\frac{1}{2}\big(a(q^{1/3})-a(q)\big).$$

The functions $a(q)$, $b(q)$ and $c(q)$ were studied in great detail by Ramanujan [5]. They form the basis of his theory of signature-three theta functions.

Proposition 1. *Suppose that $x>0$, then*

$$H(x)=\frac{2\pi}{\sqrt{3}x}\int_0^1 b(q)c(q^3)\log\left(3\frac{c(q^{9x})}{c(q^{3x})}\right)\frac{\mathrm{d}q}{q}. \tag{15}$$

Proof. Begin by setting $q=e^{-2\pi u}$, then

$$H(x)=-(2\pi)^2\int_0^\infty u\,\frac{e_3^3e_x^3}{e_1e_{3x}}\,\mathrm{d}u.$$

We will use the following Eisenstein series expansion [1, pg. 406]:

$$\frac{e_3^3}{e_1}=\sum_{n=1}^{\infty}\chi_{-3}(n)\left(\frac{q^{n/3}}{1-q^{n/3}}-\frac{q^n}{1-q^n}\right)$$

Rearranging the series, and then applying the involution for the eta function, we find that

$$\frac{e_3^3}{e_1}=\sum_{n,k=1}^{\infty}\chi_{-3}(n)(e^{-2\pi nku/3}-e^{-2\pi nku}), \tag{16}$$

$$\frac{e_x^3}{e_{3x}}=\frac{\sqrt{3}}{xu}\sum_{r,s=1}^{\infty}\chi_{-3}(r)(e^{-2\pi rs/(9xu)}-e^{-2\pi rs/(3xu)}). \tag{17}$$

Therefore, the integral becomes

$$H(x)=-\frac{(2\pi)^2\sqrt{3}}{x}\sum_{n,k,r,s\geq 1}\chi_{-3}(nr)\int_0^\infty(e^{-2\pi nku/3}-e^{-2\pi nku})\times\big(e^{-2\pi rs/(9xu)}-e^{-2\pi rs/(3xu)}\big)\,\mathrm{d}u.$$

Use linearity and a u-substitution, to regroup the integral:

$$H(x)=-\frac{(2\pi)^2\sqrt{3}}{x}\sum_{n,k,r,s\geq 1}\chi_{-3}(nr)\int_0^\infty e^{-2\pi nku}\big(e^{-2\pi rs/(3xu)}-4e^{-2\pi rs/(9xu)}+3e^{-2\pi rs/(27xu)}\big)\,\mathrm{d}u.$$

Finally make the u-substitution $u \mapsto ru/k$. This permutes the indices of summation inside the integral and we obtain

$$H(x) = -\frac{(2\pi)^2\sqrt{3}}{x}\sum_{n,k,r,s\geq 1}\frac{r\chi_{-3}(rn)}{k}\int_0^\infty e^{-2\pi nru}\Big(e^{-2\pi ks/(3xu)} - 4e^{-2\pi ks/(9xu)} + 3e^{-2\pi ks/(27xu)}\Big)\,\mathrm{d}u.$$

Simplifying reduces things to

$$H(x) = -\frac{(2\pi)^2\sqrt{3}}{x}\int_0^\infty\left(\sum_{n,r=1}^\infty r\chi_{-3}(rn)e^{-2\pi rnu}\right) \times \log\prod_{s=1}^\infty\frac{(1-e^{-2\pi s/(9xu)})^4}{(1-e^{-2\pi s/(27xu)})^3(1-e^{-2\pi s/(3xu)})}\,\mathrm{d}u.$$

Notice that the product equals a ratio of Dedekind eta functions (all of the $q^{1/24}$ terms have cancelled out). Applying the involution for the eta function, we obtain

$$H(x) = -\frac{(2\pi)^2\sqrt{3}}{x}\int_0^\infty\left(\sum_{n,r=1}^\infty r\chi_{-3}(rn)e^{-2\pi rnu}\right) \times \log\left(\frac{e^{4\pi xu}}{3}\prod_{s=1}^\infty\frac{(1-e^{-2\pi s(9xu)})^4}{(1-e^{-2\pi s(27xu)})^3(1-e^{-2\pi s(3xu)})}\right)\mathrm{d}u.$$

Set $q = e^{-2\pi u}$, and then use the product expansion $c(q) = e_3^3/e_1$ [5, pg. 109], to obtain

$$H(x) = \frac{(2\pi)\sqrt{3}}{x}\int_0^1\left(\sum_{n,r=1}^\infty r\chi_{-3}(rn)q^{rn}\right)\log\left(3\frac{c(q^{9x})}{c(q^{3x})}\right)\frac{\mathrm{d}q}{q}. \tag{18}$$

To simplify the Eisenstein series, notice that

$$\chi_{-3}(n) = \frac{2}{\sqrt{3}}\operatorname{Im}(e^{2\pi in/3}),$$

and therefore

$$\sum_{n,r=1}^\infty r\chi_{-3}(rn)q^{rn} = -\frac{1}{12\sqrt{3}}\operatorname{Im}L(e^{2\pi i/3}q),$$

where

$$L(q) := 1 - 24\sum_{n=1}^\infty\frac{nq^n}{1-q^n}. \tag{19}$$

By Ramanujan's Eisenstein series for $a^2(q)$ [5, pg. 100], we have

$$2a^2(q) = 3L(q^3) - L(q),$$

so it follows that

$$\sum_{n,r=1}^\infty r\chi_{-3}(rn)q^{rn} = \frac{1}{6\sqrt{3}}\operatorname{Im}a^2(e^{2\pi i/3}q).$$

Finally, if we use

$$a(e^{2\pi i/3}q) = b(q) + i\sqrt{3}c(q^3),$$

then

$$\operatorname{Im} a^2(e^{2\pi i/3}q) = 2\sqrt{3}b(q)c(q^3),$$

which implies

$$\sum_{n,r=1}^{\infty} r\chi_{-3}(rn)q^{rn} = \frac{1}{3}b(q)c(q^3). \tag{20}$$

Substituting (20) into (18) concludes the proof of (15). □

In the next proposition, we will pass from an integral involving modular functions, to a purely elementary integral. In order to accomplish this, we will use the inversion formulas for signature-three theta functions.

Proposition 2. *Suppose that $x > 0$, and assume that β has degree $3x$ over α in the theory of signature 3. Then*

$$H(x) = \frac{2\pi}{3\sqrt{3}x}\int_0^1 \frac{(1-\alpha)^{1/3}\big(1-(1-\alpha)^{1/3}\big)}{\alpha(1-\alpha)} \log\frac{1-(1-\beta)^{1/3}}{\beta^{1/3}}\,\mathrm{d}\alpha. \tag{21}$$

Now suppose that β has degree x over α in the theory of signature 3. Then

$$H(x) = \frac{2\pi}{3\sqrt{3}x}\int_0^1 \frac{\alpha^{1/3}(1-\alpha^{1/3})}{\alpha(1-\alpha)} \log\frac{1-(1-\beta)^{1/3}}{\beta^{1/3}}\,\mathrm{d}\alpha. \tag{22}$$

Proof. Let us prove (21) first. By formulas (2.8) and (2.9) in [5, pg. 93–94], we know that

$$3c(q^3) = a(q) - b(q).$$

Therefore (15) reduces to

$$H(x) = \frac{2\pi}{3\sqrt{3}x}\int_0^1 b(q)\big(a(q)-b(q)\big)\log\frac{a(q^{3x})-b(q^{3x})}{c(q^{3x})}\,\frac{\mathrm{d}q}{q}.$$

Now set

$$q = \exp\left(\frac{-2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha)}\right)$$

and notice that

$$a^2(q)\frac{\mathrm{d}q}{q} = \frac{\mathrm{d}\alpha}{\alpha(1-\alpha)}.$$

It is also known [5, pg. 103] that $b(q)/a(q) = (1-\alpha)^{1/3}$ and $c(q)/a(q) = \alpha^{1/3}$. Substituting these relations completes the proof of (21). Equation (22) follows if we first let $q \mapsto q^{1/3}$ in (15), then use

$$b(q^{1/3}) = a(q) - c(q),$$

and finally make the same substitution for q. □

While it is known that algebraic relations exist between α and β for all rational values of x, it is very difficult to apply those relations except in a few cases.

Theorem 1. *We have*

$$F(1,3)=\frac{\Gamma^3(\frac13)}{27}\,{}_3F_2\left(\begin{matrix}\frac13,\ \frac13,\ 1\\ \frac23,\ \frac43\end{matrix}\,\middle|\,1\right)-\frac{\Gamma^3(\frac23)}{18}\,{}_3F_2\left(\begin{matrix}\frac23,\ \frac23,\ 1\\ \frac43,\ \frac53\end{matrix}\,\middle|\,1\right). \tag{23}$$

Proof. If $x=1$ in (22), then $\alpha=\beta$, and we obtain a formula for $F(1,3)$:

$$F(1,3)=-\frac{2\pi}{27\sqrt3}\int_0^1\frac{\alpha^{1/3}(1-\alpha^{1/3})}{\alpha(1-\alpha)}\log\frac{1-(1-\alpha)^{1/3}}{\alpha^{1/3}}\,\mathrm{d}\alpha. \tag{24}$$

It is possible to simplify (24) with `Mathematica`. The easiest method is to make the substitution

$$\log\frac{1-(1-\alpha)^{1/3}}{\alpha^{1/3}}=\sum_{n=1}^{\infty}\frac{(1-\alpha)^n-3(1-\alpha)^{n/3}}{3n},$$

and then perform term-by-term integration using beta integrals. □

The new formula for $F(1,3)$ should be compared to the well-known ${}_4F_3$ evaluation [16, Eq. (43)]:

$$\frac{81}{4\pi^2}F(1,3)=\log 6+\frac{1}{108}\,{}_4F_3\left(\begin{matrix}\frac43,\ \frac53,\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,-\frac18\right). \tag{25}$$

It seems to be a tricky task to demonstrate the equivalence of (23) and (25) by purely hypergeometric techniques.

Note that a similar identity can be derived for $H(1/3)$, by setting $x=1/3$ in (21).

3. Conductor 24

It is known that an elliptic curve E of conductor 24 is associated to the eta product $e_2e_4e_6e_{12}$ [13]. Thus $L(E,2)=F(2,3)$, where $F(b,c)$ is the four-dimensional lattice sum (7). Let us define $G(x)$ as follows:

$$G(x):=\int_0^1\frac{\log q}{q}\,\frac{e_2^2e_6^2}{e_1e_3}\,\frac{e_x^2e_{3x}^2}{e_{2x}e_{6x}}\,\mathrm{d}q.$$

It is easy to see that $G(1)=-4F(2,3)$. It follows that we can solve Boyd's conductor 24 conjectures by reducing $G(1)$ to hypergeometric functions.

Proposition 3. *Let $\omega=e^{2\pi i/3}$. The following formulas hold for $x>0$:*

$$G(x)=\frac{2\pi}{3x}\operatorname{Im}\int_0^1\omega q\psi^4(\omega^2q^2)\log\left(4q^{3x}\frac{\psi^4(q^{12x})}{\psi^4(q^{6x})}\right)\frac{\mathrm{d}q}{q} \tag{26}$$

$$=\frac{\pi}{2\sqrt3 x}\int_0^1(A-B)(A-3B)(A^2-3B^2)\log\left(4q^{3x/2}\frac{\psi^4(q^{6x})}{\psi^4(q^{3x})}\right)\frac{\mathrm{d}q}{q}, \tag{27}$$

where $A=q^{1/8}\psi(q)$ and $B=q^{9/8}\psi(q^9)$.

Proof. Begin by setting $q=e^{-2\pi u}$; then the integral becomes

$$G(x)=-(2\pi)^2\int_0^\infty u\,\frac{e_2^2e_6^2}{e_1e_3}\,\frac{e_x^2e_{3x}^2}{e_{2x}e_{6x}}\,\mathrm{d}u.$$

Now consider a Lambert series due to Ramanujan [3, pg. 223, Entry 3.1]:

$$\frac{e_2^2 e_6^2}{e_1 e_3} = \sum_{n=1}^{\infty} \frac{\chi(n) q^{n/2}}{1-q^n},$$

where $\chi(n)$ has conductor 6, with $\chi(5) = -1$. Rearranging Ramanujan's result, and then using the involution for the eta function, we have

$$\frac{e_2^2 e_6^2}{e_1 e_3} = \sum_{n,k=1}^{\infty} \chi(n)(e^{-\pi nku} - e^{-2\pi nku}), \tag{28}$$

$$\frac{e_x^2 e_{3x}^2}{e_{2x} e_{6x}} = \frac{2}{\sqrt{3}xu} \sum_{r,s=1}^{\infty} \chi(r)(e^{-2\pi rs/(12xu)} - e^{-2\pi rs/(6xu)}). \tag{29}$$

Noting that $\chi(n)$ is totally multiplicative, the integral becomes

$$\begin{aligned} G(x) &= -\frac{8\pi^2}{\sqrt{3}x} \sum_{n,k,r,s\geq 1} \chi(rn) \int_0^\infty (e^{-\pi nku} - e^{-2\pi nku})(e^{-2\pi rs/(12xu)} - e^{-2\pi rs/(6xu)})\, \mathrm{d}u \\ &= -\frac{8\pi^2}{\sqrt{3}x} \sum_{n,k,r,s\geq 1} \chi(rn) \int_0^\infty e^{-2\pi nku}\big(2e^{-2\pi rs/(24xu)} \\ &\qquad\qquad - 3e^{-2\pi rs/(12xu)} + e^{-2\pi rs/(6xu)}\big)\, \mathrm{d}u. \end{aligned}$$

Now make the substitution $u \mapsto ru/k$. This step is crucially important, because it groups the r and n indices together:

$$\begin{aligned} G(x) = -\frac{8\pi^2}{\sqrt{3}x} \sum_{n,k,r,s\geq 1} \frac{r\chi(rn)}{k} \int_0^\infty e^{-2\pi rnu}\big(2e^{-2\pi ks/(24xu)} \\ - 3e^{-2\pi ks/(12xu)} + e^{-2\pi ks/(6xu)}\big)\, \mathrm{d}u. \end{aligned}$$

Simplifying the k and s sums, brings the integral to

$$\begin{aligned} G(x) = -\frac{8\pi^2}{\sqrt{3}x} \int_0^\infty \left(\sum_{n,r\geq 1} r\chi(rn) e^{-2\pi rnu} \right) \\ \times \log \prod_{s=1}^{\infty} \frac{(1-e^{-2\pi s/(12xu)})^3}{(1-e^{-2\pi s/(24xu)})^2(1-e^{-2\pi s/(6xu)})}\, \mathrm{d}u. \end{aligned}$$

The product equals a ratio of eta functions (the $q^{1/24}$ terms have cancelled out). Applying the involution again, we have

$$\begin{aligned} G(x) = -\frac{8\pi^2}{\sqrt{3}x} \int_0^\infty \left(\sum_{n,r\geq 1} r\chi(rn) e^{-2\pi rnu} \right) \\ \times \log\left(\frac{e^{-3\pi ux/2}}{\sqrt{2}} \prod_{s=1}^{\infty} \frac{(1-e^{-24\pi sxu})^3}{(1-e^{-48\pi sxu})^2(1-e^{-12\pi sxu})} \right) \mathrm{d}u. \end{aligned}$$

Now use the product expansion $q^{1/8}\psi(q)=e_2^2/e_1$, and simplify:

$$\begin{aligned} G(x) &= -\frac{4\pi}{\sqrt{3}x}\int_0^1\left(\sum_{n,r\ge1} r\chi(rn)q^{rn}\right)\log\left(\frac{q^{-3x/4}}{\sqrt{2}}\frac{\psi(q^{6x})}{\psi(q^{12x})}\right)\frac{\mathrm{d}q}{q}\\ &= \frac{\pi}{\sqrt{3}x}\int_0^1\left(\sum_{n,r\ge1} r\chi(rn)q^{rn}\right)\log\left(4q^{3x}\frac{\psi^4(q^{12x})}{\psi^4(q^{6x})}\right)\frac{\mathrm{d}q}{q}. \end{aligned} \tag{30}$$

The calculation is nearly complete. To simplify the Eisenstein series, we will use

$$\chi(n)=\frac{1}{\sqrt{3}}\operatorname{Im}(e^{2\pi in/3}-(-1)^n e^{2\pi in/3}),$$

and therefore

$$\sum_{n,r\ge1} r\chi(rn)q^{rn} = -\frac{1}{24\sqrt{3}}\operatorname{Im}\big(L(e^{2\pi i/3}q)-L(-e^{2\pi i/3}q)\big),$$

where $L(q)$ is the Eisenstein series (19). Ramanujan proved [3, pg. 114, Entry 8.2] that

$$3\varphi^4(q)=4L(q^4)-L(q),$$

hence

$$\sum_{n,r\ge1} r\chi(rn)q^{rn} = \frac{1}{8\sqrt{3}}\operatorname{Im}\big(\varphi^4(e^{2\pi i/3}q)-\varphi^4(-e^{2\pi i/3}q)\big);$$

finally by [3, pg. 40], we have

$$\sum_{n,r\ge1} r\chi(rn)q^{rn} = \frac{2}{\sqrt{3}}\operatorname{Im}\big(e^{2\pi i/3}q\psi^4(e^{4\pi i/3}q^2)\big). \tag{31}$$

Substituting (31) into (30) completes the proof of (26). To reduce (26) to (27), we can substitute the following identity into (26):

$$2\psi(\omega^2q^2)=2\psi(q^2)-3q^2\psi(q^{18})-i\sqrt{3}q^2\psi(q^{18}).$$

□

Lemma 2. *We have*

$$-4F(2,3)=G(1)=\frac{\pi}{12}\int_0^{1/2}\frac{\sqrt{(1-2p)(2-p)}\log\dfrac{p^3(2-p)}{1-2p}}{(1-p^2)\sqrt{p}}\,\mathrm{d}p. \tag{32}$$

Proof. Set $x=1$ and then manipulate (26), to obtain

$$G(1)=\frac{\pi}{6}\operatorname{Im}\int_0^1\omega q^{1/2}\psi^4(\omega^2q)\log\left(16q^3\frac{\psi^8(q^6)}{\psi^8(q^3)}\right)\frac{\mathrm{d}q}{q}.$$

Now apply complex conjugation, then use $\omega^2=-e^{\pi i/3}=-\omega^{1/2}$, and let $\omega q\mapsto q$, to arrive at

$$G(1)=\frac{\pi}{6}\operatorname{Im}\int_0^{\omega} q^{1/2}\psi^4(q)\log\left(16q^3\frac{\psi^8(q^6)}{\psi^8(q^3)}\right)\frac{\mathrm{d}q}{q}.$$

Now set $\alpha(q) := 1 - \varphi^4(-q)/\varphi^4(q)$, and $z(q) := \varphi^2(q)$. Then by formula [3, pg. 123, Entry 11.1] and [3, pg. 120, Entry 9.1],

$$q^{1/2}\psi^4(q) = \frac{\sqrt{\alpha(q)}}{4} z^2(q),$$
$$\frac{\mathrm{d}\alpha(q)}{\mathrm{d}q} = \frac{\alpha(q)(1-\alpha(q))z^2(q)}{q}.$$

By formulas [3, pg. 123, Entry 11.1] and [3, pg 123, Entry 11.3] we also have

$$16q^3\frac{\psi^8(q^6)}{\psi^8(q^3)} = \alpha(q^3).$$

Thus,

$$\begin{aligned} G(1) &= \frac{\pi}{24}\operatorname{Im}\int_0^{\omega} \frac{\log\alpha(q^3)}{\sqrt{\alpha(q)}(1-\alpha(q))}\,\mathrm{d}\alpha(q) \\ &= \frac{\pi}{24}\operatorname{Im}\int_0^{1} \frac{\log\alpha(q^3)}{\sqrt{\alpha(\omega q)}(1-\alpha(\omega q))}\,\mathrm{d}\alpha(\omega q). \end{aligned}$$

Note that both $\alpha(\omega q)$ and $\alpha(q^3)$ vary from 0 to 1 as q changes in the range from 0 to 1, and that the path for the latter is purely real.

The functions $\alpha(q)$ and $\beta(q) = \alpha(q^3)$ are related by the modular polynomial

$$(\alpha^2+\beta^2+6\alpha\beta)^2 - 16\alpha\beta\big(4(1+\alpha\beta)-3(\alpha+\beta)\big)^2 = 0$$

and admit the rational parametrization

$$\alpha = \frac{p(2+p)^3}{(1+2p)^3}, \quad \beta = \frac{p^3(2+p)}{1+2p} \tag{33}$$

with p ranging from 0 to 1 as as q changes in the range. The same modular relation and parametrization, of course, remain true when we take ωq for q, except that in this case the parameter p ranges along the complex curve

$$\mathcal{P} = \left\{p : 0 < \frac{p^3(2+p)}{1+2p} < 1\right\}$$

in the upper half-plane $\operatorname{Im} p > 0$ joining the points 0 and -1. This gives rise to writing $G(1)$ as

$$G(1) = \frac{\pi}{24}\operatorname{Im}\int_{\mathcal{P}} \frac{\log\beta}{\sqrt{\alpha}(1-\alpha)}\,\mathrm{d}\alpha.$$

First note that the integrand, as function of p, is analytic in the half-plane $\operatorname{Im} p > 0$, so that we can change the path of integration to the straight interval from 0 to -1 understood as the interval along the upper cut of the real axis:

$$G(1) = \frac{\pi}{24}\operatorname{Im}\int_0^{-1} \frac{\log\beta}{\sqrt{\alpha}(1-\alpha)}\,\mathrm{d}\alpha = \frac{\pi}{24}\int_0^{-1}\operatorname{Im}\left(\frac{\log\beta}{\sqrt{\alpha}(1-\alpha)}\right)\mathrm{d}\alpha.$$

Secondly, along the interval $-1 < p < -1/2$ the integrand is purely *real*, so that

$$G(1) = \frac{\pi}{24}\int_0^{-1/2}\operatorname{Im}\left(\frac{\log\beta}{\sqrt{\alpha}(1-\alpha)}\right)\mathrm{d}\alpha.$$

Developing now the substitution (33), computing the imaginary part and putting $-p$ for p, we thus arrive at (32). □

Remark. A similar recipe expresses $G(1/2)$ in the form

$$G(1/2) = \frac{\pi}{24}\,\mathrm{Im}\int_{\mathcal{P}} \frac{\log\beta}{1-\alpha}\,\mathrm{d}\alpha \tag{34}$$

for the path $\mathcal{P}$ given above. The substitution (33) produces an expression whose anti-derivative could be expressed in terms of the logarithmic and dilogarithmic functions, and we finally arrive at

$$G(1/2) = -\frac{\pi^2\log 2}{3}.$$

3.1. **The hypergeometric reduction.** In (32), $G(1)$ splits into two integrals of the form

$$F_1(\lambda) = \int_0^{1/\lambda} \frac{\sqrt{(1-\lambda p)(\lambda-p)}\,\log(1/p)}{(1-p^2)\sqrt{p}}\,\mathrm{d}p$$

and

$$F_2(\lambda) = \int_0^{1/\lambda} \frac{\sqrt{(1-\lambda p)(\lambda-p)}\,\log\dfrac{\lambda-p}{1-\lambda p}}{(1-p^2)\sqrt{p}}\,\mathrm{d}p,$$

where $\lambda = 2$.

Lemma 3. *The identity*

$$F_1(\lambda) - F_2(\lambda) = \pi\cdot {}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12\\ \frac32, 1\end{matrix}\,\middle|\,\frac{1}{\lambda^2}\right)$$

is true for all $\lambda \ge 1$.

Proof. Making the change $\hat p = (1-\lambda p)/(\lambda - p)$ in the integral defining $F_2(\lambda)$ we obtain $p = (1-\lambda\hat p)/(\lambda-\hat p)$ and

$$F_2(\lambda) = (1-\lambda^2)\int_0^{1/\lambda} \frac{\sqrt{\hat p}\,\log(1/\hat p)}{(1-\hat p^2)\sqrt{(1-\lambda\hat p)(\lambda-\hat p)}}\,\mathrm{d}\hat p,$$

Then we set $z = 1/\lambda^2$ and perform the changes $p = t\sqrt{z}$ and $\hat p = t\sqrt{z}$, so that the required identity becomes equivalent to

$$(1-z)\int_0^1 \frac{\sqrt{t}\,\log(t\sqrt{z})}{(1-zt^2)\sqrt{(1-t)(1-zt)}}\,\mathrm{d}t - \int_0^1 \frac{\sqrt{(1-t)(1-zt)}\,\log(t\sqrt{z})}{(1-zt^2)\sqrt{t}}\,\mathrm{d}t$$
$$= \pi\cdot {}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12\\ \frac32, 1\end{matrix}\,\middle|\,z\right) \tag{35}$$

for $0 \le z \le 1$. The left-hand side here is

$$\int_0^1 \frac{\big((1-z)t-(1-t)(1-zt)\big)\log(t\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t = \int_0^1 \frac{\big((1-zt^2)-2(1-t)\big)\log(t\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t$$
$$= \int_0^1 \frac{\log(t\sqrt{z})}{\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t - 2\int_0^1 \frac{\sqrt{1-t}\,\log(t\sqrt{z})}{(1-zt^2)\sqrt{t(1-zt)}}\,\mathrm{d}t. \tag{36}$$

Our strategy is to write the series expansions of

$$G_\varepsilon(z)=\int_0^1\frac{t^\varepsilon\,\mathrm{d}t}{\sqrt{t(1-t)(1-zt)}}=\frac{\Gamma(\frac12)\Gamma(\frac12+\varepsilon)}{\Gamma(1+\varepsilon)}\,{}_2F_1\left(\begin{matrix}\frac12,\ \frac12+\varepsilon\\ 1+\varepsilon\end{matrix}\,\middle|\,z\right)$$
$$=\sum_{n=0}^\infty\frac{\Gamma(n+\frac12)\Gamma(n+\frac12+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)}z^n$$

and

$$\tilde G_\varepsilon(z)=\sum_{n=0}^\infty g_nz^n=\int_0^1\frac{t^\varepsilon\sqrt{1-t}}{(1-zt^2)\sqrt{t(1-zt)}}\,\mathrm{d}t.$$

Because

$$\frac{1}{\sqrt{1-zt}}=\sum_{k=0}^\infty\frac{(\frac12)_k}{k!}t^kz^k,\qquad \frac{1}{1-zt^2}=\sum_{m=0}^\infty t^{2m}z^m,$$

we have

$$g_n=\sum_{k=0}^n\frac{(\frac12)_k}{k!}\int_0^1t^{2n-k-1/2+\varepsilon}(1-t)^{1/2}\,\mathrm{d}t=\sum_{k=0}^n\frac{(\frac12)_k}{k!}\,\frac{\Gamma(2n-k+\frac12+\varepsilon)\Gamma(\frac32)}{\Gamma(2n-k+2+\varepsilon)}$$
$$=\frac{\Gamma(\frac32)\Gamma(2n+\frac12+\varepsilon)}{\Gamma(2n+2+\varepsilon)}\sum_{k=0}^n\frac{(\frac12)_k}{k!}\,\frac{(-2n-1-\varepsilon)_k}{(-2n+\frac12-\varepsilon)_k}$$

(we apply [18, (2.6.3)] to the partial sum of the ${}_2F_1$ series to $n+1$ terms)

$$=\frac{\Gamma(\frac32)\Gamma(2n+\frac12+\varepsilon)}{\Gamma(2n+2+\varepsilon)}\,\frac{\Gamma(n+\frac32)\Gamma(-n-\varepsilon)}{\Gamma(n+1)\Gamma(-n+\frac12-\varepsilon)}$$
$$\times\,{}_3F_2\left(\begin{matrix}\frac12,\ -2n-1-\varepsilon,\ -n+\frac12-\varepsilon\\ -n+\frac12-\varepsilon,\ -2n+\frac12-\varepsilon\end{matrix}\,\middle|\,1\right)$$
$$=\frac{\Gamma(\frac32)\Gamma(2n+\frac12+\varepsilon)}{\Gamma(2n+2+\varepsilon)}\,\frac{\Gamma(n+\frac32)\Gamma(-n-\varepsilon)}{\Gamma(n+1)\Gamma(-n+\frac12-\varepsilon)}\,{}_2F_1\left(\begin{matrix}\frac12,\ -2n-1-\varepsilon\\ -2n+\frac12-\varepsilon\end{matrix}\,\middle|\,1\right)$$

(we apply the Gauss summation to the ${}_2F_1$ series)

$$=\frac{\Gamma(\frac32)\Gamma(2n+\frac12+\varepsilon)}{\Gamma(2n+2+\varepsilon)}\,\frac{\Gamma(n+\frac32)\Gamma(-n-\varepsilon)}{\Gamma(n+1)\Gamma(-n+\frac12-\varepsilon)}\,\frac{\Gamma(-2n+\frac12-\varepsilon)\Gamma(1)}{\Gamma(-2n-\varepsilon)\Gamma(\frac32)}$$

(finally we use the functional equations for the Gamma function)

$$=\frac{\Gamma(n+\frac32)\Gamma(n+\frac12+\varepsilon)}{(2n+1+\varepsilon)\Gamma(n+1)\Gamma(n+1+\varepsilon)}.$$

Therefore,

$$\begin{aligned}
G_\varepsilon(z)-2\tilde{G}_\varepsilon(z) &= \sum_{n=0}^{\infty}\biggl(\frac{\Gamma(n+\frac12)\Gamma(n+\frac12+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)} \\
&\qquad - \frac{\Gamma(n+\frac32)\Gamma(n+\frac12+\varepsilon)}{(n+\frac12+\frac12\varepsilon)\Gamma(n+1)\Gamma(n+1+\varepsilon)}\biggr)z^n \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac12)\Gamma(n+\frac12+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)}\biggl(1-\frac{n+\frac12}{n+\frac12+\frac12\varepsilon}\biggr)z^n \\
&= \varepsilon\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac12)\Gamma(n+\frac12+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)(2n+1+\varepsilon)}z^n.
\end{aligned}$$

This implies for (36) that

$$\int_0^1 \frac{\bigl((1-zt^2)-2(1-t)\bigr)\log(\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t = \log\sqrt{z}\cdot\bigl(G_\varepsilon(z)-2\tilde{G}_\varepsilon(z)\bigr)\big|_{\varepsilon=0}=0$$

and

$$\begin{aligned}
\int_0^1 \frac{\bigl((1-zt^2)-2(1-t)\bigr)\log t}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigl(G_\varepsilon(z)-2\tilde{G}_\varepsilon(z)\bigr)\bigg|_{\varepsilon=0} \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac12)^2}{\Gamma(n+1)^2(2n+1)}z^n \\
&= \Gamma(\tfrac12)^2\sum_{n=0}^{\infty}\frac{(\frac12)_n^2}{n!^2(2n+1)}z^n \\
&= \pi\cdot{}_3F_2\biggl(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,1\end{matrix}\biggm| z\biggr),
\end{aligned}$$

thus establishing the required identity (35). □

The method also allows us to give closed forms individually for $F_1(\lambda)$ and $F_2(\lambda)$.

Lemma 4. *For $\lambda\ge 1$,*

$$F_1(\lambda)=\frac{\pi}{2}\log(4\lambda)+\frac{\pi}{2}\cdot{}_3F_2\biggl(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,1\end{matrix}\biggm|\frac{1}{\lambda^2}\biggr)-\frac{\pi}{16\lambda^2}\cdot{}_4F_3\biggl(\begin{matrix}\frac32,\frac32,1,1\\ 2,2,2\end{matrix}\biggm|\frac{1}{\lambda^2}\biggr). \tag{37}$$

Proof. As we have shown in the proof of Lemma 3

$$\begin{aligned}
F_1(1/\sqrt{z}) &= -\int_0^1\frac{(1-t)(1-zt)\log(t\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t \\
&= -\int_0^1\frac{\bigl((1-t)-zt(1-t)\bigr)\log(t\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t,
\end{aligned}$$

and this integral can be computed by examining the constant and linear terms in the ε-expansion of

$$\begin{aligned}\tilde{G}_\varepsilon(z) - z\tilde{G}_{1+\varepsilon}(z) &= \sum_{n=0}^\infty \frac{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{1}{2}+\varepsilon)}{(2n+1+\varepsilon)\Gamma(n+1)\Gamma(n+1+\varepsilon)} z^n \\ &\quad - \sum_{n=0}^\infty \frac{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2}+\varepsilon)}{(2n+2+\varepsilon)\Gamma(n+1)\Gamma(n+2+\varepsilon)} z^{n+1} \\ &= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\varepsilon)}{\Gamma(2+\varepsilon)} + \sum_{n=1}^\infty \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2}+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)} \left(\frac{n+\frac{1}{2}}{2n+1+\varepsilon} - \frac{n}{2n+\varepsilon} \right) z^n \\ &= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\varepsilon)}{\Gamma(2+\varepsilon)} + \frac{\varepsilon}{2} \sum_{n=1}^\infty \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2}+\varepsilon)}{\Gamma(n+1)\Gamma(n+1+\varepsilon)(2n+\varepsilon)(2n+1+\varepsilon)} z^n.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 \frac{(1-t)(1-zt)\log(\sqrt{z})}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t &= \log\sqrt{z} \cdot \left(\tilde{G}_\varepsilon(z) - z\tilde{G}_{1+\varepsilon}(z) \right)\Big|_{\varepsilon=0} \\ &= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} \log\sqrt{z} = \frac{\pi\log\sqrt{z}}{2}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \frac{(1-t)(1-zt)\log t}{(1-zt^2)\sqrt{t(1-t)(1-zt)}}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\tilde{G}_\varepsilon(z) - z\tilde{G}_{1+\varepsilon}(z) \right)\bigg|_{\varepsilon=0} \\ &= \frac{\pi}{2}\left(\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - 1 + \gamma \right) + \frac{1}{2}\sum_{n=1}^\infty \frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n+1)^2(2n)(2n+1)} z^n \\ &= \frac{\pi}{2}(-2\log 2 - 1) + \frac{\pi}{2}\sum_{n=1}^\infty \frac{(\frac{1}{2})_n^2}{n!^2}\left(\frac{1}{2n} - \frac{1}{2n+1} \right) z^n \\ &= -\pi\log 2 + \frac{\pi z}{16} \cdot {}_4F_3\left(\left. \begin{matrix} \frac{3}{2},\ \frac{3}{2},\ 1,\ 1 \\ 2,\ 2,\ 2 \end{matrix} \,\right|\, z \right) - \frac{\pi}{2} \cdot {}_3F_2\left(\left. \begin{matrix} \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \\ \frac{3}{2},\ 1 \end{matrix} \,\right|\, z \right).\end{aligned}$$

Joining the latter results we obtain (37). ∎

Using Lemmas 2, 3, 4, the equality $m(8) = 4m(2)$ as well as the hypergeometric evaluations (3) and (4) of $m(8)$ and $m(2)$ we finally arrive at

Theorem 2. *The following evaluation is true:*

$$F(2,3) = -\frac{1}{4}G(1) = \frac{\pi^2}{6}m(2).$$

3.2. **The elliptic reduction.** In this subsection we give an alternative derivation of Theorem 2. In order to accomplish this, we will use properties of the Jacobian elliptic functions. Recall that $\operatorname{sn} u$ depends implicitly on α, and that it is doubly periodic, with periods $4K$ and $2iK'$, where

$$K = \frac{\pi}{2}\,{}_2F_1\left(\left. \begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{matrix} \,\right|\, \alpha \right), \qquad K' = \frac{\pi}{2}\,{}_2F_1\left(\left. \begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{matrix} \,\right|\, 1-\alpha \right).$$

We will also take the usual definition of the elliptic nome, namely

$$q=\exp\left(-\pi\frac{K'}{K}\right)=\exp\left(-\pi\frac{{}_2F_1(\frac12,\frac12;1;1-\alpha)}{{}_2F_1(\frac12,\frac12;1;\alpha)}\right)$$

In the first lemma we give a Fourier series expansion for a ratio of Jacobian elliptic functions. Formula (38) is absent from most references, however it can be derived from results in [21].

Lemma 5. *The following identity is true:*

$$\begin{aligned}\frac{\operatorname{cn}^2 u\,\operatorname{dn}^2 u}{1-\alpha\operatorname{sn}^4 u}&=\frac{\pi}{4K}+\frac{\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\frac{2\pi nu}{K}\\&\quad+\frac{\pi}{\sqrt{\alpha}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\frac{\pi(2n+1)u}{K}.\end{aligned}\tag{38}$$

Proof. Equation (39) is a superposition of results in [21]. Let us begin by decomposing the function using partial fractions

$$\begin{aligned}\frac{\operatorname{cn}^2 u\,\operatorname{dn}^2 u}{1-\alpha\operatorname{sn}^4 u}&=\frac{(1-\operatorname{sn}^2 u)(1-\alpha\operatorname{sn}^2 u)}{1-\alpha\operatorname{sn}^4 u}\\&=-1-\frac{(1-\sqrt{\alpha})^2}{2\sqrt{\alpha}}\frac{1}{1-\sqrt{\alpha}\operatorname{sn}^2 u}\\&\quad+\frac{(1+\sqrt{\alpha})^2}{2\sqrt{\alpha}}\frac{1}{1+\sqrt{\alpha}\operatorname{sn}^2 u}.\end{aligned}$$

By equation (1.1) in [21, pg. 543], we can show that

$$\frac{1}{1-\sqrt{\alpha}\operatorname{sn}^2 u}=\frac{\Pi(\sqrt{\alpha},\alpha)}{K}+\frac{\pi}{(1-\sqrt{\alpha})K}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n/2}}{1+q^n}\cos\frac{\pi nu}{K},\tag{39}$$

$$\frac{1}{1+\sqrt{\alpha}\operatorname{sn}^2 u}=\frac{\Pi(-\sqrt{\alpha},\alpha)}{K}+\frac{\pi}{(1+\sqrt{\alpha})K}\sum_{n=1}^{\infty}\frac{q^{n/2}}{1+q^n}\cos\frac{\pi nu}{K},\tag{40}$$

where $\Pi(\alpha,\beta)$ is the complete elliptic integral of the third kind. Substituting (39) and (40), we obtain

$$\begin{aligned}\frac{\operatorname{cn}^2 u\,\operatorname{dn}^2 u}{1-\alpha\operatorname{sn}^4 u}&=\frac{h(\alpha)}{K}+\frac{\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\frac{2\pi nu}{K}\\&\quad+\frac{\pi}{\sqrt{\alpha}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\frac{\pi(2n+1)u}{K},\end{aligned}$$

where

$$h(\alpha):=-K-\frac{(1-\sqrt{\alpha})^2}{2\sqrt{\alpha}}\Pi(\sqrt{\alpha},\alpha)+\frac{(1+\sqrt{\alpha})^2}{2\sqrt{\alpha}}\Pi(-\sqrt{\alpha},\alpha).$$

Finally, we are grateful to James Wan for pointing out that a more general formula for $\Pi(m,n)$ implies that $h(\alpha)=\pi/4$ (see [22]). We will leave this final calculation as an exercise for the reader. □

Proposition 4. *Suppose that $0 \le \alpha \le 1$. The following identities are true:*

$$-\frac{8}{\pi}\int_0^1 \frac{\sqrt{(1-v^2)(1-\alpha v^2)}}{1-\alpha v^4}\log v\,\mathrm{d}v = m\left(\frac{4}{\sqrt{\alpha}}\right)+\frac{1}{\sqrt{\alpha}}m(4\sqrt{\alpha}) + \log\sqrt{\alpha}, \tag{41}$$

$$-\frac{8}{\pi}\int_0^1 \frac{\sqrt{(1-v^2)(1-\alpha v^2)}}{1-\alpha v^4}\log(1-v^2)\,\mathrm{d}v = 2m\left(\frac{4}{\sqrt{\alpha}}\right)+\log\frac{\alpha}{1-\alpha} + \frac{1}{\sqrt{\alpha}}\log\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}, \tag{42}$$

$$-\frac{8}{\pi}\int_0^1 \frac{\sqrt{(1-v^2)(1-\alpha v^2)}}{1-\alpha v^4}\log(1-\alpha v^2)\,\mathrm{d}v = \frac{2}{\sqrt{\alpha}}m(4\sqrt{\alpha})-\log(1-\alpha) + \frac{1}{\sqrt{\alpha}}\log\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}. \tag{43}$$

Proof. First notice that if we set $v=\operatorname{sn} u$, then (41) becomes

$$\int_0^1 \frac{\sqrt{(1-v^2)(1-\alpha v^2)}}{1-\alpha v^4}\log v\,\mathrm{d}v = \int_0^K \frac{\operatorname{cn}^2 u\,\operatorname{dn}^2 u}{1-\alpha\operatorname{sn}^4 u}\log\operatorname{sn} u\,\mathrm{d}u.$$

We will now substitute Fourier expansions for Jacobian elliptic functions. The following series holds for $u\in(0,K)$ [9, pg. 917]:

$$\log\operatorname{sn} u = \log\frac{2K}{\pi}+\log\sin\frac{\pi u}{2K}-2\sum_{n=1}^\infty \frac{1}{n}\frac{q^n}{1+q^n}\left(1-\cos\frac{\pi n u}{K}\right). \tag{44}$$

Substitute (38) and (44) into the integral, and then integrate term-by-term. It is necessary to use the following formula several times:

$$\int_0^K \cos\frac{\pi n u}{K}\log\sin\frac{\pi u}{2K}\,\mathrm{d}u = \begin{cases} -K\log 2 & \text{if } n=0,\\ -K/(2n) & \text{if } n\ge 1.\end{cases}$$

A substantial amount of work reduces the integral to

$$\begin{aligned}\int_0^K \frac{\operatorname{cn}^2 u\,\operatorname{dn}^2 u}{1-\alpha\operatorname{sn}^4 u}\log\operatorname{sn} u\,\mathrm{d}u &= \frac{\pi}{4}\left(\log\frac{K}{\pi}-2\sum_{n=1}^\infty\frac{1}{n}\frac{q^n}{1+q^n}\right)\\ &\quad+\frac{\pi}{2}\sum_{n=1}^\infty \frac{1}{n}\frac{q^n}{1+q^{2n}}\left(\frac{q^{2n}}{1+q^{2n}}-\frac{1}{2}\right)\\ &\quad+\frac{\pi}{\sqrt{\alpha}}\sum_{n=0}^\infty\frac{1}{2n+1}\frac{q^{n+1/2}}{1+q^{2n+1}}\left(\frac{q^{2n+1}}{1+q^{2n+1}}-\frac{1}{2}\right).\end{aligned}$$

Now substitute the geometric series

$$\frac{x}{1+x^2}\left(\frac{x^2}{1+x^2}-\frac{1}{2}\right) = -\frac{1}{2}\sum_{k=1}^\infty k\chi_{-4}(k)x^k,$$

and then swap the order of summation, to obtain

$$\int_0^K \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u}{1-\alpha \operatorname{sn}^4 u} \log \operatorname{sn} u \, du = -\frac{\pi}{16} \log\left(\frac{\pi^4}{K^4} \cdot q \prod_{k=1}^{\infty} \frac{(1-q^{2k})^{16}}{(1-q^k)^8}\right) + \frac{\pi}{8}\left(\frac{\log q}{2} + 2\sum_{k=1}^{\infty} k\chi_{-4}(k) \log(1-q^k)\right) - \frac{\pi}{4\sqrt{\alpha}} \sum_{k=1}^{\infty} k\chi_{-4}(k) \log \frac{1-q^k}{(1-q^{k/2})^2}.$$

Finally, by the q-series expansion for $m(4/\sqrt{\alpha})$ [12, Entry (2-9)], and by [3, pg. 124, Entries 12.2 and 12.3], this becomes

$$\int_0^K \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u}{1-\alpha \operatorname{sn}^4 u} \log \operatorname{sn} u \, du = -\frac{\pi}{16} \log \alpha - \frac{\pi}{8} m\left(\frac{4}{\sqrt{\alpha}}\right) + \frac{\pi}{8\sqrt{\alpha}}\left(m\left(\frac{4}{\sqrt{\alpha}}\right) - 2m\left(\frac{4}{\sqrt{\alpha'}}\right)\right),$$

where α has degree 2 over α'. By the second degree modular equation of Ramanujan [3, pg. 215], we know that

$$\alpha' = \frac{4\sqrt{\alpha}}{(1+\sqrt{\alpha})^2}.$$

Since $0 \le \alpha \le 1$, we can apply a functional equation of Kurokawa and Ochiai [11], to obtain

$$2m\left(\frac{4}{\sqrt{\alpha'}}\right) = 2m\big(2(\alpha^{1/4} + \alpha^{-1/4})\big) = m(4\sqrt{\alpha}) + m\left(\frac{4}{\sqrt{\alpha}}\right).$$

This last observation completes the proof of (41). Formulas (42) and (43) can be proved with an identical method, except that they require Fourier expansions for $\log \operatorname{cn} u$ and $\log \operatorname{dn} u$, respectively, [9, pg. 917]. □

Alternative proof of Theorem 2. If we let $p \mapsto v^2/2$ in (32), then

$$G(1) = \frac{\pi}{6} \int_0^1 \frac{\sqrt{(1-v^2)(1-\frac{1}{4}v^2)}}{(1-\frac{1}{4}v^4)} \log \frac{v^6(1-\frac{1}{4}v^2)}{4(1-v^2)} \, dv.$$

Theorem 2 follows immediately from combining an elementary result

$$\frac{\pi}{4} = \int_0^1 \frac{\sqrt{(1-v^2)(1-\alpha v^2)}}{(1-\alpha v^4)} \, dv$$

(consider a Taylor series in α), with all three formulas in Proposition 4, and the known identity $m(8) = 4m(2)$ [12]. □

4. Conductor 20

In this section we will prove Boyd's conjectures for elliptic curves of conductor 20. Recall [13] that such curves are associated to the modular form $e_2^2 e_{10}^2$, so it follows that $L(E,2) = F(1,5)$. The first step is to use Ramanujan's modular equations to relate $F(1,5)$ to an elementary integral. The elementary integral can then be reduced to Mahler measures by substituting doubly-periodic elliptic functions, or by using hypergeometric functions. Define $S(x)$ as follows:

$$S(x) := \int_0^1 \frac{\log q}{q}\, q^{(1+x)/4}\psi^2(q^x)\big(\psi^2(q) - 5q\psi^2(q^5)\big)\,\mathrm{d}q.$$

We will begin by expressing $F(1,5)$ in terms of $S(x)$.

Lemma 6. *The following relation is true:*

$$-4F(1,5) = S(1) - S(5). \tag{45}$$

Proof. First notice that

$$S(1) - S(5) = \int_0^1 \frac{\log q}{q} q^{1/2}\big(\psi^2(q) - q\psi^2(q^5)\big)\big(\psi^2(q) - 5q\psi^2(q^5)\big)\,\mathrm{d}q.$$

Ramanujan showed [1, pg. 28] that

$$e_1^2 e_5^2 = q^{1/2}\big(\psi^2(q) - q\psi^2(q^5)\big)\big(\psi^2(q) - 5q\psi^2(q^5)\big),$$

which implies

$$S(1) - S(5) = \int_0^1 \frac{\log q}{q}\, e_1^2 e_5^2\,\mathrm{d}q = -4F(1,5).$$

□

Next we will apply our trick to obtain a transformation for $S(x)$.

Proposition 5. *Suppose that $x > 0$. Then*

$$S(x) = -\pi \int_0^1 q^{x/2}\psi^4(-q^x)\log\left(5\frac{\varphi^2(q^5)}{\varphi^2(q)}\right)\frac{\mathrm{d}q}{q}. \tag{46}$$

Proof. Begin by setting $q = e^{-2\pi u}$, then

$$S(x) = -(2\pi)^2 \int_0^\infty u e^{-\pi x u/2}\psi^2(e^{-2\pi x u})\big(e^{-\pi u/2}\psi^2(e^{-2\pi u}) - 5e^{-5\pi u/2}\psi^2(e^{-10\pi u})\big)\,\mathrm{d}u.$$

We will use the following Lambert series expansion (which follows from [3, pg. 139, Example 4]):

$$e^{-\pi x u/2}\psi^2(e^{-2\pi x u}) = \sum_{n,k=1}^\infty \chi_{-4}(n)(e^{-\pi n k x u/2} - e^{-\pi n k x u}).$$

By the involution for the psi function and by [3, pg. 114, Entry 8.1], we have

$$\begin{aligned} e^{-\pi u/2}\psi^2(e^{-2\pi u}) - 5e^{-5\pi u/2}\psi^2(e^{-10\pi u}) &= \frac{1}{4u}\big(\varphi^2(-e^{-\pi/u}) - \varphi^2(-e^{-\pi/(5u)})\big) \\ &= \frac{1}{u}\sum_{r,s=1}^\infty (-1)^s \chi_{-4}(r)(e^{-\pi r s/u} - e^{-\pi r s/(5u)}). \end{aligned}$$

Therefore, the integral becomes

$$S(x) = -(2\pi)^2 \sum_{n,k,r,s\geq 1} (-1)^s \chi_{-4}(nr) \int_0^\infty (e^{-\pi nkxu/2} - e^{-\pi nkxu}) \times (e^{-\pi rs/(u)} - e^{-\pi rs/(5u)})\,\mathrm{d}u.$$

Use linearity and a u-substitution, to regroup the integral:

$$S(x) = -(2\pi)^2 \sum_{n,k,r,s\geq 1} (-1)^s \chi_{-4}(nr) \int_0^\infty e^{-\pi nkxu} (2e^{-\pi rs/(2u)} - 2e^{-\pi rs/(10u)} - e^{-\pi rs/u} + e^{-\pi rs/(5u)})\,\mathrm{d}u.$$

Finally make the u-substitution $u \mapsto ru/k$. This permutes the indices of summation inside the integral. We have

$$S(x) = -(2\pi)^2 \sum_{n,k,r,s\geq 1} r\chi_{-4}(nr) \frac{(-1)^s}{k} \int_0^\infty e^{-\pi nrxu} (2e^{-\pi ks/(2u)} - 2e^{-\pi ks/(10u)} - e^{-\pi ks/u} + e^{-\pi ks/(5u)})\,\mathrm{d}u.$$

Simplify the k and s sums, then use the involution for the Dedekind eta function and the product expansion $\varphi(q) = e_2^5/(e_1^2 e_4^2)$, to reduce the integral to

$$S(x) = -2\pi^2 \int_0^\infty \left(\sum_{n,r=1}^\infty r\chi_{-4}(rn) e^{-\pi rnxu} \right) \log\left(5 \frac{\varphi^2(e^{-10\pi u})}{\varphi^2(e^{-2\pi u})} \right) \mathrm{d}u. \tag{47}$$

Finally, the nested sum is easy to simplify. By [3, pg. 139, Example 3],

$$\sum_{n,r=1}^\infty r\chi_{-4}(rn) q^{rn} = \sum_{r=1}^\infty \frac{r\chi_{-4}(r) q^r}{1+q^{2r}} = q\psi^4(-q^2).$$

Substituting this last result into (47) completes the proof of (46). $\square$

Now we can derive an elementary integral for $F(1,5)$. In order to accomplish the reduction, we will need several additional modular equations.

Lemma 7. *We have*

$$F(1,5) = -\frac{\pi}{20} \int_0^1 \frac{(1-6t)\log(1+4t)}{\sqrt{t(1-t)(1+4t^2)}}\,\mathrm{d}t. \tag{48}$$

Proof. By formulas (46) and (45), we find that

$$F(1,5) = \frac{\pi}{4} \int_0^1 \frac{q^{1/2}\psi^4(-q) - q^{5/2}\psi^4(-q^5)}{q} \log\left(5 \frac{\varphi^2(q^5)}{\varphi^2(q)} \right) \mathrm{d}q.$$

Now set $m = \varphi^2(q)/\varphi^2(q^5)$. Then by [1, pg. 26, formula (1.6.4)], we obtain

$$1 - q^2 \frac{\psi^4(-q^5)}{\psi^4(-q)} = \frac{8(3-m)}{(5-m)^2}.$$

Therefore, the integral becomes

$$F(1,5) = 2\pi \int_0^1 \frac{3-m}{(5-m)^2} \log\left(\frac{5}{m}\right) \frac{q^{1/2}\psi^4(-q)}{q}\, \mathrm{d}q.$$

Now set $\alpha = 1 - \varphi^4(-q)/\varphi^4(q)$; then it is known that

$$\begin{aligned}\frac{q^{1/2}\psi^4(-q)}{q} &= \frac{1}{2\sqrt{4\alpha(1-\alpha)}} \frac{\mathrm{d}\alpha}{\mathrm{d}q} \\ &= \frac{1}{8\sqrt{4\alpha(1-\alpha)(1-4\alpha(1-\alpha))}} \frac{\mathrm{d}}{\mathrm{d}q}\big(4\alpha(1-\alpha)\big).\end{aligned}$$

Finally, we have the following relation between α and m:

$$4\alpha(1-\alpha) = \frac{(m-1)(5-m)^5}{64m^5}.$$

This relation between α and m follows from [3, pg. 288, Entry 14]. Notice that the entry holds for $|q| < 1$ by the principle of analytic continuation. Eliminating α, and exercising caution about the square root, reduces the integral to

$$\begin{aligned}F(1,5) &= \frac{\pi}{4} \int_0^1 \log\left(\frac{5}{m}\right) \frac{3-m}{m\sqrt{(5-m)(m-1)(5-2m+m^2)}} \frac{\mathrm{d}m}{\mathrm{d}q}\, \mathrm{d}q \\ &= \frac{\pi}{4} \int_1^5 \log\left(\frac{5}{m}\right) \frac{3-m}{m\sqrt{(5-m)(m-1)(5-2m+m^2)}}\, \mathrm{d}m.\end{aligned}$$

The change of variables from q to m is justified because m ranges monotonically between $m = 1$ and $m = 5$ when $q \in [0,1]$. Finally, the substitution $m \mapsto 5/(1+4t)$ completes the proof of (48). □

Below we use two methods to reduce (48) to Mahler measures. The first method is to substitute doubly-periodic elliptic functions into the integral. The main drawback to this method is that we will first have to construct non-standard elliptic functions. The second method is to prove the identity directly via hypergeometric manipulations. In both approaches we will investigate an integral which generalizes (48). Notice that

$$J(y) := \frac{1}{2\pi} \int_0^1 \frac{(2-y+3yt)\log(1+yt)}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\, \mathrm{d}t \tag{49}$$

reduces to the integral in (48) when $y = 4$.

4.1. **The elliptic reduction.** Throughout this subsection we assume that $k > 4/3$. Notice that when $y = 2k/(k-1)$, we have

$$J\left(\frac{2k}{k-1}\right) = -\frac{1}{\pi} \int_0^1 \frac{(1-3kt)\log\left(1-\frac{2kt}{1-k}\right)}{\sqrt{4t((1-k)^2 - t(1-kt)^2)}}\, \mathrm{d}t. \tag{50}$$

The overarching goal of the following discussion is to obtain formula (57). To prove that identity, it is necessary to use Fourier series expansions for elliptic functions which parameterize the curve

$$F_k : y^2 = 4x\big((1-k)^2 - x(1-kx)^2\big).$$

Since we (regrettably) could not find such formulas in the literature, we will first prove Proposition 6 and Lemma 8.

Notice that F_k is a genus-one curve, with non-zero discriminant when $k > 4/3$. Therefore, F_k can be parameterized by doubly-periodic functions. Suppose that $w(x)$ satisfies the differential equation:

$$\big(w'(x)\big)^2 = 4w(x)\big((1-k)^2 - w(x)(1-kw(x))^2\big). \tag{51}$$

In order to explicitly identify $w(x)$ we can map F_k to $Y^2 = 4X^3 - g_2X - g_3$. It follows easily that

$$w(x) = \frac{3(1-k)^2}{1+3\wp(x)}, \tag{52}$$

where $\wp(x) := \wp(x, \{g_2, g_3\})$ is the Weierstrass function, and

$$g_2 = -\frac{4}{3}(6k^3 - 12k^2 + 6k - 1),$$
$$g_3 = \frac{4}{27}(2 - 6k + 3k^2)(1 - 6k + 12k^2 - 18k^3 + 9k^4).$$

This identification is quite useful for computational purposes.

Proposition 6. *Let $2K$ and $2K'$ denote the real and purely imaginary periods of $w(x)$. Then $w(x)$ has the following values:*

x	$w(x)$	*order of zero/pole*	*residue*
0	0	2	$-$
K	1	$-$	$-$
K'	1	$-$	$-$
$K+K'$	0	2	$-$
$K'/3$	$(1-k)/(2k)$	$-$	$-$
$2K'/3$	∞	1	$i/(2k)$
$4K'/3$	∞	1	$-i/(2k)$
$5K'/3$	$(1-k)/(2k)$	$-$	$-$
$K+K'/3$	∞	1	$-i/(2k)$
$K+2K'/3$	$(1-k)/(2k)$	$-$	$-$
$K+4K'/3$	$(1-k)/(2k)$	$-$	$-$
$K+5K'/3$	∞	1	$i/(2k)$

Proof. It is well known that $\wp(x)$ has a second-order pole at $x=0$, so $w(x)$ has a second-order zero at that point. Since $\wp(x)$ is even, $w(2aK + 2bK' - x) = w(x)$ for all $(a,b) \in \mathbb{Z}^2$. Therefore we only need to evaluate $w(x)$ for $x \in \{K, K', K + K', K'/3, 2K'/3, K + 4K'/3, K + 5K'/3\}$.

We will require additional properties of the Weierstrass $\wp$-function. If $4X^3 - g_2X - g_3 = 4(x-r_1)(x-r_2)(x-r_3)$, then the half-periods of $\wp(x)$ are given by

$$\omega = \int_{\infty}^{r_1} \frac{1}{\sqrt{4y^3 - g_2y - g_3}}\,\mathrm{d}y, \qquad \omega' = \int_{r_1}^{r_2} \frac{1}{\sqrt{4y^3 - g_2y - g_3}}\,\mathrm{d}y.$$

Select $r_1 = (1-k)^2 - \frac{1}{3}$ to be the real zero of $4X^3 - g_2X - g_3 = 0$, and r_2 to be the imaginary zero which lies in the upper half plane. Now set $K := \omega$, and $K' := 2\omega - \omega'$. While it is possible to show $\operatorname{Re} K' = 0$, we will not pursue that calculation here. It follows that $K + K'$ is a period of $w(x)$, so we have the following identities:

$$\begin{aligned} w(K+K') &= w(0) = 0, \\ w(K) &= w(2K+K') = w(K'), \\ w(K'/3) &= w(K+4K'/3), \\ w(2K'/3) &= w(K+5K'/3). \end{aligned}$$

Since $\wp(\omega) = r_1 = (1-k)^2 - \frac{1}{3}$, we can use (52) to conclude that $w(K) = 1$. The values of $w(K'/3)$ and $w(2K'/3)$ can be verified from a polynomial relation between $w(x)$ and $w(3x)$, which follows from the Weierstrass addition formula. Now we will calculate the values of the residues. Since $w(2K'/3) = \infty$, it follows that $\wp\left(\frac{4\omega'-2\omega}{3}\right) = -\frac{1}{3}$, thus $\left(\wp'\left(\frac{4\omega'-2\omega}{3}\right)\right)^2 = -4k^2(1-k)^4$. Extracting a square root we obtain $\wp'\left(\frac{4\omega'-2\omega}{3}\right) = -2ik(1-k)^2$. The choice of square root can be justified by checking the formula numerically at $k = 2$, and then appealing to the fact that ω, ω', and $\wp'\left(\frac{4\omega'-2\omega}{3}\right)$ are analytic functions of k for $k > 4/3$. Finally, by formula (52)

$$\operatorname{Res}_{x=2K'/3} w(x) = \frac{(1-k)^2}{\wp'\left(\frac{4\omega'-2\omega}{3}\right)} = \frac{i}{2k}.$$

The other residues can be verified in a similar fashion. □

Notice that we can integrate (51), and use $w(K) = 1$, to obtain a second formula for K:

$$K = \int_0^1 \frac{\mathrm{d}t}{\sqrt{4t((1-k)^2 - t(1-kt)^2)}}.$$

We will need two Fourier series expansions to finish the elliptic reduction.

Lemma 8. *Suppose that $x > 0$. Then*

$$w(x) = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}q^n}{1 + (-1)^n q^n + q^{2n}} \sin^2 \frac{\pi n x}{2K} \tag{53}$$

and

$$\log\left(1 - \frac{2k}{1-k} w(x)\right) = 8 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \frac{q^n - q^{2n}}{1 + q^{3n}} \sin^2 \frac{\pi n x}{2K}, \tag{54}$$

where $q=e^{2\pi iK'/6K}$. *An alternative formula for* q *is given by*

$$q=\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac13,\frac23;1;1-\alpha)}{{}_2F_1(\frac13,\frac23;1;\alpha)}\right), \tag{55}$$

where

$$\alpha=\frac{27p(1+p)^4}{2(1+4p+p^2)^3}, \qquad p=\frac{-1+\sqrt{(3k-1)/(k-1)}}{2}.$$

Proof. The proof of (53) is an exercise in the theory of elliptic functions. The poles of $w(x)$ inside the fundamental parallelogram are $2K'/3$, $4K'/3$, $K+K'/3$, and $K+5K'/3$. The function has residues $i/(2k)$, $-i/(2k)$, $-i/(2k)$, and $i/(2k)$ at each of the poles. We also know that $w(0)=w'(0)=0$. By [2, formula (27)], we have

$$\begin{aligned}w(x)=\frac{3i}{2k}\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\chi_{-3}(n)\biggl(&\frac{1}{3x-(6mK+2nK')}\\&+\frac{1}{(6mK+2nK')}+\frac{3x}{(6mK+2nK')^2}\biggr)\\-\frac{3i}{2k}\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\chi(n)\biggl(&\frac{1}{3x-((6m+3)K+nK')}\\&+\frac{1}{((6m+3)K+nK')}+\frac{3x}{((6m+3)K+nK')^2}\biggr),\end{aligned}$$

where $\chi(n)$ is the Legendre symbol mod 6. It is a lengthy exercise to reduce this last expression to (53). The fastest (if least rigorous) method for finishing the calculation, is to differentiate the entire expression twice, and then substitute the following Fourier series:

$$\sum_{n=-\infty}^{\infty}\frac{1}{(x+\tau+n)^3}+\frac{1}{(-x+\tau+n)^3}=i(2\pi)^3\sum_{n=1}^{\infty}n^2e^{2\pi in\tau}\cos(2\pi nx),$$

which holds for $\mathrm{Im}(\tau)>0$. Thus one obtains a formula for $w''(x)$, which can be integrated to recover (53).

Proposition 6 shows that $(1-k)/(2k)=w(K'/3)$. It follows that

$$1-\frac{2k}{1-k}w(x)=1-\frac{w(x)}{w(K'/3)}.$$

This function has simple zeros at $K'/3$, $5K'/3$, $K+2K'/3$ and $K+4K'/3$, and simple poles at $2K'/3$, $4K'/3$, $K+K'/3$ and $K+5K'/3$. Since any two elliptic functions with the same zeros and poles are constant multiples, it is easy to obtain an infinite product. We have

$$1-\frac{2k}{1-k}w(x)=C\frac{\theta(x,K'/3)\theta(x,5K'/3)\theta(x,K+2K'/3)\theta(x,K+4K'/3)}{\theta(x,2K'/3)\theta(x,4K'/3)\theta(x,K+K'/3)\theta(x,K+5K'/3)}, \tag{56}$$

where

$$\theta(x,\rho) = \left(1 - e^{2\pi i(x-\rho)/(2K)}\right)\prod_{n=1}^{\infty}\left(1 - e^{2\pi i(x-\rho+2nK')/(2K)}\right)\left(1 - e^{2\pi i(-x+\rho+2nK')/(2K)}\right).$$

The right-hand side of (56) is doubly periodic because $\theta(x,\rho)$ has period $2K$, and satisfies the quasi-periodicity relation

$$\theta(x+2K',\rho) = -e^{2\pi i(\rho-x)/(2K)}\theta(x,\rho).$$

The right-hand side also has the correct zeros and poles, since $\theta(x,\rho)$ vanishes at ρ. The constant C can be determined by using the fact that $w(0)=0$. Finally, (54) follows from taking logarithms of (56), and then using the Taylor series for the logarithm.

We will conclude the proof by simplifying the expression for q. Since $w(K)=1$, we can use (54) to obtain

$$\begin{aligned}\log\frac{3k-1}{k-1} &= 8\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{q^{2n+1}-q^{4n+2}}{1+q^{6n+3}}\\ &= 4\log\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5(1-q^{3n})^2(1-q^{12n})^2}{(1-q^{n})^2(1-q^{4n})^2(1-q^{6n})^5}\\ &= 2\log\frac{\varphi^2(q)}{\varphi^2(q^3)}.\end{aligned}$$

If we let $1+2p = \varphi^2(q)/\varphi^2(q^3)$, then it follows easily that

$$p = \frac{-1+\sqrt{(3k-1)/(k-1)}}{2}.$$

Finally, formula (55) is a consequence of standard inversion formulas in the theory of signature 3. □

Theorem 3. *Suppose that $k \geq 4/3$, and let $p = \frac{1}{2}(-1+\sqrt{(3k-1)/(k-1)})$. The following formula is true:*

$$J\left(\frac{2k}{k-1}\right) = 2g\left(\frac{2(1+p)^2}{p}\right) - g\left(\frac{4(1+p)}{p^2}\right). \tag{57}$$

Proof. First assume that $k > 4/3$. If we set $t = w(x)$, then (50) becomes

$$J\left(\frac{2k}{k-1}\right) = -\frac{1}{\pi}\int_0^K (1-3kw(x))\log\left(1-\frac{2k}{1-k}w(x)\right)\mathrm{d}x.$$

Substituting (54) and (53) reduces the integral to

$$\begin{aligned}J\left(\frac{2k}{k-1}\right) = &-\left(\frac{4K}{\pi} + 12\sum_{j=1}^{\infty}\frac{(-q)^j}{1+(-q)^j+q^{2j}}\right)\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}\frac{q^n-q^{2n}}{1+q^{3n}}\\ &+ 6\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}\frac{q^{2n}(1-q^{2n})}{(1+q^{3n})^2}.\end{aligned}$$

Now substitute

$$3\frac{q^2(1-q^2)}{(1+q^3)^2}=-\frac{q(1-q)}{(1+q^3)}-\sum_{j=1}^{\infty}(-1)^j j\chi_{-3}(j)q^j,$$

to obtain

$$J\left(\frac{2k}{k-1}\right)=-\left(\frac{4K}{\pi}+2a(-q)\right)\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{1}{n}\frac{q^n-q^{2n}}{1+q^{3n}}$$
$$-\sum_{j=1}^{\infty}(-1)^j j\chi_{-3}(j)\log\frac{1+q^j}{1-q^j}.$$

Notice that we have used a Lambert series for $a(-q)$, which follows from [5, pg. 100, Theorem 2.12]. Now we claim that $2K=-\pi a(-q)$. If one substitutes the hypergeometric representation for $a(-q)$, then this statement is equivalent to Lemma 9 in the next subsection. It is also possible to prove the equality directly by setting $x=K$ in (53) and (54), and then performing q-series manipulations. The q-series for $J(2k/(k-1))$ reduces to

$$J\left(\frac{2k}{k-1}\right)=-\sum_{j=1}^{\infty}(-1)^j j\chi_{-3}(j)\log\frac{1+q^j}{1-q^j}.$$

We can now substitute Stienstra's q-series for $g(k)$ [20]: applying formula (2-11) in [12] completes the proof of (57) if $k>4/3$. Finally, notice that both sides of (57) are continuous at $k=4/3$, hence the formula remains true for the boundary value as well. ☐

4.2. **The hypergeometric reduction.** In this part, we show the coincidence of the derivatives of $J(y)$ and $g(y)$ on the interval $2<y<8$ and conclude with the identity $J(y)=g(y)$ for $2\le y\le 8$ by appealing to the equality at $y=8$ deduced in Theorem 3.

Lemma 9. *For $2\le y<8$, we have*

$$\frac{1}{2\pi}\int_0^1\frac{\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}=\frac{1}{y+4}\,{}_2F_1\left(\begin{matrix}\frac13,\ \frac23\\ 1\end{matrix}\,\middle|\,\frac{27y^2}{(y+4)^3}\right). \tag{58}$$

Proof. We apply the transformation [5, p. 112, Theorem 5.6],

$$\frac{1}{1+p+p^2}\,{}_2F_1\left(\begin{matrix}\frac13,\ \frac23\\ 1\end{matrix}\,\middle|\,\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right)=\frac{1}{\sqrt{1+2p}}\,{}_2F_1\left(\begin{matrix}\frac12,\ \frac12\\ 1\end{matrix}\,\middle|\,\frac{p^3(2+p)}{1+2p}\right) \tag{59}$$

with the choice $p=(\sqrt{1+y}-1)/2$ (ranging in $(\sqrt{3}-1)/2\le p<1$), so that $y=4p(1+p)$ and the left-hand side in (59) assumes the form

$$\frac{1}{1+p+p^2}\,{}_2F_1\left(\begin{matrix}\frac13,\ \frac23\\ 1\end{matrix}\,\middle|\,\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right)=\frac{4}{y+4}\,{}_2F_1\left(\begin{matrix}\frac13,\ \frac23\\ 1\end{matrix}\,\middle|\,\frac{27y^2}{(y+4)^3}\right). \tag{60}$$

On the other hand, the substitution $y = 4p(1+p)$ and the change of variable

$$t = \frac{1-u}{1+2pu-p^3(2+p)u^2}$$

in the original integral results in

$$\begin{aligned}\frac{1}{\pi}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}} &= \frac{1}{2\pi}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u(1-u)(1+2p-p^3(2+p)u)}} \\ &= \frac{1}{2\sqrt{1+2p}}\,{}_2F_1\biggl(\begin{matrix}\frac12,\ \frac12\\ 1\end{matrix}\biggm|\frac{p^3(2+p)}{1+2p}\biggr), \qquad (61)\end{aligned}$$

where on the last step we apply the Euler–Pochhammer integral representation of the hypergeometric series [18, equation (1.6.6)]. Combining (59)–(61) we arrive at the desired claim (58). □

Note the range of the argument of the hypergeometric series in (58):

$$\frac12 \le \frac{27y^2}{(y+4)^3} < 1 \quad \text{for } 2 \le y < 8.$$

Lemma 10. *For $2 < y < 8$,*

$$\frac{\mathrm{d}J}{\mathrm{d}y} = \frac{1}{y+4}\,{}_2F_1\biggl(\begin{matrix}\frac13,\ \frac23\\ 1\end{matrix}\biggm|\frac{27y^2}{(y+4)^3}\biggr).$$

Proof. Note that for real values of y in the interval $2 \le y \le 8$ we have

$$\biggl|\sqrt{t}\Bigl(1-\frac{y}{2}(1-t)\Bigr)\biggr| \le 1 \quad \text{for } 0 \le t \le 1,$$

so that the real-valued function

$$v(t) = v(t;y) = 2\arcsin\biggl(\sqrt{t}\Bigl(1-\frac{y}{2}(1-t)\Bigr)\biggr)$$

is well defined on the interval $0 < t < 1$. Because

$$\frac{\partial v}{\partial t} = \frac{2-y+3yt}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}, \qquad (62)$$

we can write the integral (49) as

$$J(y) = \frac{1}{2\pi}\int_0^1 \log(1+yt)\frac{\partial v}{\partial t}\,\mathrm{d}t. \qquad (63)$$

Denote $u(t) = u(t;y) = \log(1+yt)$ and use, besides (62),

$$\frac{\partial u}{\partial t} = \frac{y}{1+yt}, \quad \frac{\partial u}{\partial y} = \frac{t}{1+yt}, \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{2\sqrt{t(1-t)}}{\sqrt{4+(4-y)yt+y^2t^2}}.$$

It follows from (63) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}J(y)&=\frac{1}{2\pi}\int_0^1\frac{\partial}{\partial y}\biggl(u\frac{\partial v}{\partial t}\biggr)\,\mathrm{d}t=\frac{1}{2\pi}\int_0^1\biggl(\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial t}+u\frac{\partial^2 v}{\partial y\,\partial t}\biggr)\,\mathrm{d}t\\&=\frac{1}{2\pi}\int_0^1\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial t}\,\mathrm{d}t+\frac{1}{2\pi}\int_0^1 u\,\mathrm{d}\biggl(\frac{\partial v}{\partial y}\biggr)\end{aligned}$$

(integrating the second integral by parts)

$$\begin{aligned}&=\frac{1}{2\pi}\int_0^1\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial t}\,\mathrm{d}t+\frac{1}{2\pi}\,u\frac{\partial v}{\partial y}\bigg|_{t=0}^{t=1}-\frac{1}{2\pi}\int_0^1\frac{\partial v}{\partial y}\,\frac{\partial u}{\partial t}\,\mathrm{d}t\\&=\frac{1}{2\pi}\int_0^1\biggl(\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial t}-\frac{\partial v}{\partial y}\,\frac{\partial u}{\partial t}\biggr)\,\mathrm{d}t\\&=\frac{1}{2\pi}\int_0^1\frac{t(2+y+yt)}{(1+yt)\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\,\mathrm{d}t\\&=\frac{1}{2\pi}\int_0^1\frac{t\,\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\\&\qquad+\frac{1}{2\pi}\int_0^1\frac{(1+y)t\,\mathrm{d}t}{(1+yt)\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\end{aligned}$$

(applying the change $t\mapsto(1-t)/(1+yt)$ in the second integral)

$$\begin{aligned}&=\frac{1}{2\pi}\int_0^1\frac{t\,\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\\&\qquad+\frac{1}{2\pi}\int_0^1\frac{(1-t)\,\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}\\&=\frac{1}{2\pi}\int_0^1\frac{\mathrm{d}t}{\sqrt{t(1-t)(4+(4-y)yt+y^2t^2)}}.\end{aligned}$$

It remains to apply Lemma 9 to the resulting integral. □

Theorem 4. *For $2\le y\le 8$, the equality*

$$J(y)=g(y)\tag{64}$$

holds.

Proof. For $2<y<8$, the hypergeometric evaluation (5) of $g(y)$ can be stated in the form

$$g(y)=\frac{1}{3}f\biggl(\frac{y^2}{(y+4)^3}\biggr)+\frac{4}{3}\operatorname{Re}f\biggl(\frac{y}{(y-2)^3}\biggr)$$

where the function

$$f(z)=-\frac{\log z}{3}-2z\,{}_4F_3\biggl(\begin{matrix}\frac43,\ \frac53,\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\biggm|27z\biggr)\tag{65}$$

satisfies the equation

$$\frac{\mathrm{d}f}{\mathrm{d}z} = -\frac{1}{3z}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, 27z\right).$$

Therefore,

$$\frac{\mathrm{d}g}{\mathrm{d}y} = \frac{y-8}{9y(y+4)}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, \frac{27y^2}{(y+4)^3}\right) + \frac{8(y+1)}{9y(y-2)}\,\mathrm{Re}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, \frac{27y}{(y-2)^3}\right),$$

and application of the cubic transformation

$$\mathrm{Re}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, \frac{27y}{(y-2)^3}\right) = \frac{y-2}{y+4}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, \frac{27y^2}{(y+4)^3}\right)$$

result in

$$\frac{\mathrm{d}g}{\mathrm{d}y} = \frac{1}{y+4}\,{}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, \frac{27y^2}{(y+4)^3}\right). \tag{66}$$

Comparing this evaluation with the one from Lemma 10 we conclude that $g(y)$ and $J(y)$ differ on the interval $2 < y < 8$ by a constant; because both $g(y)$ and $J(y)$ are continuous at the end-points, the relation $J(y) - g(y) = C$, a real constant, is true for $2 \le y \le 8$. To determine the constant, take $y = 8$ and apply Theorem 3 with the choice $p = 1$; it follows that

$$g(8) + C = J(8) = 2g(8) - g(8),$$

hence $C = 0$. □

Remark. The derivative (66) can be alternatively obtained by differentiating Stienstra's q-series for $g(y)$ [20, Example #6], [12, formula (2-11)].

4.3. **Culmination.** We conclude this section by listing the major consequences of Theorems 3 and 4.

Theorem 5. *The following formulas are true:*

$$\frac{10}{\pi^2}F(1,5) = 2g(4+2\sqrt{5}) - g(8+4\sqrt{5}) \tag{67}$$

$$= g(4) \tag{68}$$

$$= \frac{3}{4}n(\sqrt[3]{32}). \tag{69}$$

Proof. Equation (67) follows from setting $k = 2$ in (57) and then comparing it to (48), while (68) follows from taking $y = 4$ in (64).

Using the hypergeometric evaluations (5) and (6) in the form

$$g(k) = \frac13 f\left(\frac{k^2}{(k+4)^3}\right) + \frac43 f\left(\frac{k}{(k-2)^3}\right), \qquad n(k) = f\left(\frac{1}{k^3}\right),$$

whenever the arguments lie between 0 and 1/27, with the hypergeometric function $f(z)$ defined in (65), we have

$$\begin{aligned} &2g(4+2\sqrt{5})-g(8+4\sqrt{5}) \\ &\quad=\frac{2}{3}f\biggl(\frac{4}{(7-\sqrt{5})^3}\biggr)+\frac{8}{3}f\biggl(\frac{1}{32}\biggr)-\frac{1}{3}f\biggl(\frac{1}{32}\biggr)-\frac{4}{3}f\biggl(\frac{4}{(7+\sqrt{5})^3}\biggr) \\ &\quad=\frac{7}{3}n(\sqrt[3]{32})+\frac{2}{3}n\biggl(\frac{7-\sqrt{5}}{\sqrt[3]{4}}\biggr)-\frac{4}{3}n\biggl(\frac{7+\sqrt{5}}{\sqrt[3]{4}}\biggr). \end{aligned}$$

Finally, Bertin's "exotic" relation [10, Theorem 6]

$$16n\biggl(\frac{7+\sqrt{5}}{\sqrt[3]{4}}\biggr)-8n\biggl(\frac{7-\sqrt{5}}{\sqrt[3]{4}}\biggr)=19n(\sqrt[3]{32}) \tag{70}$$

reduces the latter sum to $\frac{3}{4}n(\sqrt[3]{32})$. □

Corollary. *The following Boyd's conjectural evaluations are true:*

$$n(\sqrt[3]{2})=\frac{25}{6\pi^2}F(1,5), \qquad g(-2)=\frac{15}{\pi^2}F(1,5).$$

Proof. These readily follow from [12, formula (2-26)],

$$3g(-2)=n(2^{1/3})+4n(2^{5/3}), \qquad 3g(4)=4n(2^{1/3})+n(2^{5/3}),$$

and Theorem 5. □

Theorem 6. *For $(\sqrt{3}-1)/2 \le p \le 1$, the Mahler measure $g(\,\cdot\,)$ satisfies the functional equation*

$$g\bigl(4p(1+p)\bigr)+g\biggl(\frac{4(1+p)}{p^2}\biggr)=2g\biggl(\frac{2(1+p)^2}{p}\biggr).$$

Proof. The result follows by comparing the two different evaluations obtained in Theorems 3 and 4. □

Remark. In view of the proof of Theorem 5, our Theorem 6 may be thought of as a generalization of Bertin's "exotic" relation (70).

Theorem 7. *We have*

$$F(1,1)=-\frac{2\pi^2\log 2}{27}+\frac{\Gamma^3(\frac13)}{3\cdot 2^{7/3}}\,{}_3F_2\biggl(\begin{matrix}\frac13,\frac13,1\\ \frac56,\frac43\end{matrix}\biggm| -\frac18\biggr)+\frac{\Gamma^3(\frac23)}{2^{11/3}}\,{}_3F_2\biggl(\begin{matrix}\frac23,\frac23,1\\ \frac76,\frac53\end{matrix}\biggm| -\frac18\biggr).$$

Proof. Rodriguez-Villegas [15] showed that

$$F(1,1)=\frac{2\pi^2}{9}g(2).$$

Making the change $t^3=u$ in the integral

$$F(1,1)=\frac{2\pi^2}{9}J(2)=\frac{\pi}{3}\int_0^1\frac{\sqrt{t}\log(1+2t)}{\sqrt{1-t^3}}\,\mathrm{d}t$$

and writing the interior logarithm as hypergeometric series, we arrive at the claim. Note that both `Maple` and `Mathematica` produce the evaluation without human assistance. □

5. Concluding remarks

We will conclude by mentioning the fact that this paper settles the conjectures of Bloch and Grayson for elliptic curves of conductor 20 [6]. Since there is a very simple method to translate Mahler measures into elliptic dilogarithms [10], the main results in this paper are equivalent to relations between $L(E,2)$ and values of the elliptic dilogarithm. For instance, given the conductor 20 elliptic curve $E : y^2 = 4x^3 - 432x + 1188$, and the torsion point $P = (-6, 54)$, by formula (24) in [10] we have

$$D^E(2P) = \frac{2\pi}{9} n(2^{5/3}) = \frac{80}{27\pi} L(E,2).$$

The equality to $L(E,2)$ follows immediately from (69). Although Bloch and Grayson did not examine any conductor 24 curves, we can prove similar relations for those cases, by combining Theorem 5 with formulas (30) and (31) in [10].

Finally, there are many additional problems which need to be addressed. The most obvious direction is to try to prove more of Boyd's conjectures. There are still hundreds of outstanding conjectures in Boyd's tables [7]. It would also be interesting to understand what overlap (if any) exists between our techniques, and those of Brunault [8] and Mellit [14]. They proved Boyd's conjectures for elliptic curves of conductors 11 and 14 by using Beilinson's theorem. Rodriguez-Villegas was the first to advocate this K-theoretic approach [15]; he originally suggested that the conductor 24 cases could be proved with Beilinson's theorem.

It would also be interesting to reduce more values of $F(b,c)$ to hypergeometric functions. An easy corollary to the $F(1,5)$ formula of Theorem 5, is a formula for $F(5,9)$. By [17, formula (3.26)], we have

$$\frac{18}{5\pi^2} F(5,9) = g(-4) - 2g(4). \tag{71}$$

Notice that Lemma 1 and Proposition 2 reduce $F(3,7)$, $F(6,7)$, and $F(3/2,7)$ to complicated elementary integrals. We expect these lattice sums to also equal values of hypergeometric functions, although we currently see no way to prove it.

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