DOUBLY-WEIGHTED PSEUDO-ALMOST PERIODIC FUNCTIONS

TOKA DIAGANA

Abstract. We introduce and study a new concept called doubly-weighted pseudo-almost periodicity, which generalizes the notion of weighted pseudo-almost periodicity due to Diagana. Properties of such a new concept such as the stability of the convolution, translation-invariance, existence of a doubly-weighted mean for almost periodic functions, and a composition result will be studied.

1. Introduction

The impetus of this paper comes from one main source, that is, the paper by Diagana [7] in which the concept of weighted pseudo almost periodicity was introduced and studied. Because of the weights involved, the notion of weighted pseudo-almost periodicity is more general and richer than the classical notion of pseudo-almost periodicity, which was introduced in the literature in the early nineties by Zhang [22, 23, 24] as a natural generalization of the classical almost periodicity in the sense of Bohr. Since its introduction in the literature, the notion of weighted pseudo-almost periodicity has generated several developments, see for instance [4], [8], [9], [10], [11], [14], [16], [25], and [26] and the references therein.

Inspired by the weighted Morrey spaces [12], in this paper we introduce and study a new class of functions called doubly-weighted pseudo-almost periodic functions (respectively, doubly-weighted pseudo-almost automorphic functions), which generalizes in a natural fashion weighted pseudo-almost periodic functions (respectively, weighted pseudo-almost automorphy). In addition to the above, we also introduce the class of doubly-weighted pseudo-almost periodic functions of order κ (respectively, doubly-weighted pseudo-almost automorphic functions of order κ), where $\kappa \in (0,1)$. Note that the notion of weighted pseudo-almost automorphy was introduced by Blot *et al.* [2] and is a generalization of both the notions of weighted pseudo-almost periodicity and that of pseudo-almost automorphy due to Liang *et al.* [20, 21]. For recent developments on the notion of weighted pseudo-almost automorphy and related issues, we refer the reader to [2], [5], and [17].

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In order to do all these things, the main idea consists of enlarging the weighted *ergodic* component utilized in Diagana's definition of the weighted pseudo-almost periodicity, with the help of two weighted measures $d\mu(x) = \mu(x)dx$ and $d\nu(x) = \nu(x)dx$, where $\mu, \nu : \mathbb{R} \mapsto (0, \infty)$ are *locally integrable* functions.

In this paper, we take a closer look into properties of these doubly-weighted pseudo-almost periodic functions (respectively, doubly-weighted pseudo-almost automorphic functions) and study their relationship with the notions of weighted pseudo-almost periodicity (respectively, weighted pseudo-almost automorphy). Among other things, properties of these new functions will be discussed including the stability of the convolution operator (Proposition 5.1), translation-invariance (Theorem 5.4), the uniqueness of the decomposition involving these new functions as well as some composition theorems (Theorem 5.8).

In Liang *et al.* [14], the original question which consists of the existence of a weighted mean for almost periodic functions was raised. In particular, Liang *et al.* have shown through an example that there exist weights for which a weighted mean for almost periodic functions may not exist. In this paper we investigate the broader question, which consists of the existence of a doubly-weighted mean for almost periodic functions. Namely, we give some sufficient conditions, which do guarantee the existence of a doubly-weighted mean for almost periodic functions. Moreover, under those conditions, it will be shown that the doubly-weighted mean and the classical (Bohr) mean are proportional (Theorem 4.2).

2. Preliminaries

If $\mathbb{X}, \mathbb{Y}$ are Banach space, we then let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the space of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

2.1. **Properties of Weights.** Let $\mathbb{U}$ denote the collection of functions (weights) $\rho : \mathbb{R} \mapsto (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere.

In the rest of the paper, if $\mu \in \mathbb{U}$ and for $T > 0$, we then set $Q_T := [-T, T]$ and

$$\mu(Q_T) := \int_{Q_T} \mu(x)dx.$$

As in the particular case when $\mu(x) = 1$ for each $x \in \mathbb{R}$, in this setting, we are exclusively interested in those weights, μ, for which, $\lim_{T\to\infty} \mu(Q_T) = \infty$. Consequently, we define the space of weights $\mathbb{U}_\infty$ by

$$\mathbb{U}_\infty := \left\{\mu \in \mathbb{U} : \inf_{x\in\mathbb{R}} \mu(x) = \mu_0 > 0 \text{ and } \lim_{T\to\infty} \mu(Q_T) = \infty\right\}.$$

In addition to the above, we define the set of weights $\mathbb{U}_B$ by

$$\mathbb{U}_B := \left\{\mu \in \mathbb{U}_\infty : \sup_{x\in\mathbb{R}} \mu(x) = \mu_1 < \infty\right\}.$$

We also need the following set of weights, which makes the spaces of weighted pseudo-almost periodic functions translation-invariant,

$$\mathbb{U}_\infty^{\rm Inv} := \left\{\mu \in \mathbb{U}_\infty : \lim_{x\to\infty} \frac{\mu(x+\tau)}{\mu(x)} < \infty \text{ and } \lim_{T\to\infty} \frac{\mu(Q_{T+\tau})}{\mu(Q_T)} < \infty \text{ for all } \tau \in \mathbb{R}\right\}.$$

It can be easily seen that if $\mu \in \mathbb{U}_\infty^{\rm Inv}$, then the corresponding space of weighted pseudo almost periodic functions $PAP(\mathbb{X}, \mu)$ is translation-invariant. In particular, since $\mathbb{U}_B \subset \mathbb{U}_\infty^{\rm Inv}$, it follows that for each $\mu \in \mathbb{U}_B$, then $PAP(\mathbb{X}, \mu)$ is translation-invariant.

Let $\mathbb{U}_\infty^c$ denote the collection of all continuous functions (weights) $\mu : \mathbb{R} \mapsto (0, \infty)$ such that $\mu > 0$ almost everywhere. Define

$$\mathbb{U}_\infty^s := \left\{\mu \in \mathbb{U}_\infty^c \cap \mathbb{U}_\infty : \lim_{x\to\infty} \frac{\mu(x+\tau)}{\mu(x)} < \infty \text{ for all } \tau \in \mathbb{R}\right\}.$$

The set of weights $\mathbb{U}_\infty^s$ is very rich and contains several types of weights including some polynomials. Moreover, the next lemma shows that elements of $\mathbb{U}_\infty^s$ satisfy the translation-invariance conditions.

Lemma 2.1. *The inclusion* $\mathbb{U}_\infty^s \subset \mathbb{U}_\infty^{\rm Inv}$ *holds.*

Proof. Let $\mu \in \mathbb{U}_\infty^s$. To complete the proof, it suffices to show that

$$\lim_{T\to\infty} \left[\frac{\mu(Q_{T+\tau})}{\mu(Q_T)}\right] < \infty \text{ for all } \tau \in \mathbb{R}.$$

For that, note that by definition $\mu(Q_{T+\tau}) \to \infty$ and $\mu(Q_T) \to \infty$ as $T \to \infty$ for all $\tau \in \mathbb{R}$. Moreover,

$$\frac{d\mu(Q_{T+\tau})}{dT} = \mu(T+\tau) + \mu(-T-\tau) \text{ and } \frac{d\mu(Q_T)}{dT} = \mu(T) + \mu(-T).$$

Setting $\lim_{x\to\infty} \dfrac{\mu(x+\tau)}{\mu(x)} = l$ for all $\tau \in \mathbb{R}$, it follows that

$$\begin{aligned} \lim_{T\to\infty} \frac{\dfrac{d\mu(Q_{T+\tau})}{dT}}{\dfrac{d\mu(Q_T)}{dT}} &= \lim_{T\to\infty} \frac{\mu(T+\tau) + \mu(-T-\tau)}{\mu(T) + \mu(-T)} \\ &= \frac{2l}{1+l} < \infty. \end{aligned}$$

Therefore, using the L'Hôpital's rule from calculus it follows that

$$\begin{aligned} \lim_{T\to\infty}\left[\frac{\mu(Q_{T+\tau})}{\mu(Q_T)}\right] &= \lim_{T\to\infty}\frac{\dfrac{d\mu(Q_{T+\tau})}{dT}}{\dfrac{d\mu(Q_T)}{dT}} \\ &= \lim_{T\to\infty}\frac{\mu(T+\tau)+\mu(-T-\tau)}{\mu(T)+\mu(-T)} \\ &= \frac{2l}{1+l} < \infty \end{aligned}$$

for all $\tau \in \mathbb{R}$, and hence $\mu \in \mathbb{U}_\infty^{\mathrm{Inv}}$. □

Definition 2.2. Let $\mu, \nu \in \mathbb{U}_\infty$. One says that μ is equivalent to ν and denote it $\mu \prec \nu$, if $\dfrac{\mu}{\nu} \in \mathbb{U}_B$.

Let $\mu, \nu, \gamma \in \mathbb{U}_\infty$. It is clear that $\mu \prec \mu$ (reflexivity); if $\mu \prec \nu$, then $\nu \prec \mu$ (symmetry); and if $\mu \prec \nu$ and $\nu \prec \gamma$, then $\mu \prec \gamma$ (transitivity). Therefore, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$.

The proofs of Proposition 2.3 and Proposition 2.4 are not too difficult and hence are omitted.

Proposition 2.3. Let $\mu, \nu \in \mathbb{U}_\infty^{\mathrm{Inv}}$. If $\mu \prec \nu$, then $\sigma = \mu + \nu \in \mathbb{U}_\infty^{\mathrm{Inv}}$.

Proposition 2.4. Let $\mu, \nu \in \mathbb{U}_\infty^s$. Then their product $\pi = \mu\nu \in \mathbb{U}_\infty^s$. Moreover, if $\mu \prec \nu$, then $\sigma := \mu + \nu \in \mathbb{U}_\infty^s$.

In the next theorem, we describe all the nonconstant polynomials belonging to the set of weights $\mathbb{U}_\infty$.

Theorem 2.5. *If $\mu \in \mathbb{U}_\infty$ is a nonconstant polynomial of degree N, then N is necessarily even ($N = 2n'$ for some nonnegative integer n'). More precisely, μ can be written in the following form:*

$$\mu(x) = a\prod_{k=0}^{n}(x^2 + a_k x + b_k)^{m_k}$$

where $a > 0$ is a constant, a_k and b_k are some real numbers satisfying $a_k^2 - 4b_k < 0$, and m_k are nonnegative integers for $k = 0, ..., n$. Furthermore, the weight μ given above belongs to $\mathbb{U}_\infty^s$.

Proof. Let $\mu \in \mathbb{U}_\infty$ be a nonconstant polynomial of degree N. Since $\inf_{t\in\mathbb{R}} \mu(x) = \mu_0 > 0$ it follows that $N \geq 2$ and that μ has no real roots. Namely, all the roots of μ are complex numbers of the form z_k and its conjugate $\overline{z_k}$ whose imaginary parts are nonzero. Consequently, factors of μ, up to constants, are of the form:

$$(x - z_k)(x - \overline{z_k}) = x^2 + a_k x + b_k$$

where $a_k = -2\Re e\, z_k$, $b_k = |z_k|^2 > 0$ with $a_k^2 - 4b_k = 4(\Re e\, z_k)^2 - 4|z_k|^2 < 0$.

Let us also mention that if z_k is a complex root of multiplicity m_k so is its conjugate $\overline{z_k}$. From the previous observations it follows that

$$\mu(x) = a \prod_{k=0}^{n} (x^2 + a_k x + b_k)^{m_k}$$

where $a > 0$ is a constant, and m_k are nonnegative integers for $k = 0, ..., n$. In view of the above it follows that $N \geq 2$ is even. Namely,

$$N = 2 \sum_{k=0}^{n} m_k.$$

Now

$$\lim_{x\to\infty} \frac{\mu(x+\tau)}{\mu(x)} = 1,$$

and hence $\mu \in \mathbb{U}_\infty^s$. □

3. Doubly-Weighted Pseudo-Almost Periodic and Pseudo-Almost Automorphic Functions

Definition 3.1. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \varepsilon \ \text{ for each } \ t \in \mathbb{R}.$$

The number τ above is called an *ε-translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 3.2. A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t+\tau, y) - F(t, y)\| < \varepsilon \ \text{ for each } \ t \in \mathbb{R}, \ \ y \in K.$$

The collection of those functions is denoted by $AP(\mathbb{Y}, \mathbb{X})$.

Definition 3.3. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup-norm $\|\cdot\|_\infty$ turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 3.4. [18, 19] *If $f, f_1, f_2 \in AA(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA(\mathbb{X})$,
(ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,
(iii) $f_\alpha \in AA(\mathbb{X})$ *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,
(iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm,*
(v) *if* $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$, *too.*

In addition to the above-mentioned properties, we have the the following property due to Bugajewski and Diagana [6]:

(vi) If $g \in L^1(\mathbb{R})$ and if $f \in AA(\mathbb{R})$, then $f * g \in AA(\mathbb{R})$, where $f * g$ is the convolution of f with g on $\mathbb{R}$.

Definition 3.5. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \mapsto F(t, x)$ is almost automorphic for all $x \in K$ ($K \subset \mathbb{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$G(t, x) := \lim_{n\to\infty} F(t + s_n, x)$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n\to\infty} G(t - s_n, x) = F(t, x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

To introduce the notion of doubly-weighted pseudo-almost periodicity (respectively, doubly-weighted pseudo-almost automorphy), we need to define the "doubly-weighted ergodic" space $PAP_0(\mathbb{X}, \mu, \nu)$. Doubly-weighted pseudo-almost periodic functions will then appear as perturbations of almost periodic functions by elements of $PAP_0(\mathbb{X}, \mu, \nu)$.

If $\mu, \nu \in \mathbb{U}_\infty$, we then define

$$PAP_0(\mathbb{X}, \mu, \nu) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} \|f(\sigma)\| \, \nu(\sigma)\, d\sigma = 0 \right\}.$$

Similarly, if $\kappa \in (0, 1)$, we define

$$PAP_0^\kappa(\mathbb{X},\mu,\nu) := \left\{ f \in BC(\mathbb{R},\mathbb{X}) : \quad \lim_{T\to\infty} \frac{1}{\left[\mu(Q_T)\right]^\kappa} \int_{Q_T} \|f(\sigma)\| \; \nu(\sigma)\, d\sigma = 0 \right\}.$$

Clearly, when $\mu \prec \nu$, one retrieves the so-called weighted ergodic space introduced by Diagana [7], that is, $PAP_0(\mathbb{X},\mu,\nu) = PAP_0(\mathbb{X},\nu,\mu) = PAP_0(\mathbb{X},\mu)$, where

$$PAP_0(\mathbb{X},\mu) := \left\{ f \in BC(\mathbb{R},\mathbb{X}) : \quad \lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} \|f(\sigma)\| \; \mu(\sigma)\, d\sigma = 0 \right\}.$$

The previous fact suggests that the weighted ergodic spaces $PAP_0(\mathbb{X},\mu,\nu)$ are probably more interesting when both μ and ν are not necessarily equivalent.

Obviously, the spaces $PAP_0(\mathbb{X},\mu,\nu)$ are richer than $PAP_0(\mathbb{X},\mu)$ and give rise to an enlarged space of weighted pseudo-almost periodic functions.

In the same way, we define $PAP_0(\mathbb{Y},\mathbb{X},\mu,\nu)$ as the collection of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ such that $F(\cdot,y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{T\to\infty} \frac{1}{\mu(Q_T)} \left\{ \int_{Q_T} \|F(s,y)\|\, \nu(s)\, ds \right\} = 0$$

uniformly in $y \in \mathbb{Y}$.

Similarly, if $\kappa \in (0,1)$, we then define $PAP_0^\kappa(\mathbb{Y},\mathbb{X},\mu,\nu)$ as the collection of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ such that $F(\cdot,y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{T\to\infty} \frac{1}{\left[\mu(Q_T)\right]^\kappa} \left\{ \int_{Q_T} \|F(s,y)\|\, \nu(s)\, ds \right\} = 0$$

uniformly in $y \in \mathbb{Y}$.

We are now ready to define doubly-weighted pseudo-almost periodic functions.

Definition 3.6. Let $\mu,\nu \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R},\mathbb{X})$ is called doubly-weighted pseudo-almost periodic if it can be expressed as $f = g+\phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X},\mu,\nu)$. The collection of such functions will be denoted by $PAP(\mathbb{X},\mu,\nu)$.

Definition 3.7. Let $\mu,\nu \in \mathbb{U}_\infty$. A function $F \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called doubly-weighted pseudo-almost periodic if it can be expressed as $F = G + \Phi$, where $G \in AP(\mathbb{Y},\mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y},\mathbb{X},\mu,\nu)$. The collection of such functions will be denoted by $PAP(\mathbb{Y},\mathbb{X},\mu,\nu)$.

Definition 3.8. Let $\mu,\nu \in \mathbb{U}_\infty$ and let $\kappa \in (0,1)$. A function $f \in C(\mathbb{R},\mathbb{X})$ is called doubly-weighted pseudo-almost periodic of order κ if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0^\kappa(\mathbb{X},\mu,\nu)$. The collection of such functions will be denoted by $PAP^\kappa(\mathbb{X},\mu,\nu)$.

Definition 3.9. Let $\mu,\nu \in \mathbb{U}_\infty$ and let $\kappa \in (0,1)$. A function $F \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called doubly-weighted pseudo-almost periodic of order κ if it can be expressed as $F = G + \Phi$, where $G \in AP(\mathbb{Y}, \mathbb{X})$ and $\Phi \in PAP_0^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$. The collection of such functions will be denoted by $PAP^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$.

We are also ready to define doubly-weighted pseudo-almost automorphic functions.

Definition 3.10. Let $\mu \in \mathbb{U}_\infty$ and $\nu \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called doubly-weighted pseudo-almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \mu, \nu)$. The collection of such functions will be denoted by $PAA(\mathbb{X}, \mu, \nu)$.

Definition 3.11. Let $\mu, \nu \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called doubly-weighted pseudo-almost automorphic if it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{Y}, \mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \mu, \nu)$. The collection of such functions will be denoted by $PAA(\mathbb{Y}, \mathbb{X}, \mu, \nu)$.

Definition 3.12. Let $\mu \in \mathbb{U}_\infty$ and $\nu \in \mathbb{U}_\infty$ and let $\kappa \in (0, 1)$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called doubly-weighted pseudo-almost automorphic of order κ if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0^\kappa(\mathbb{X}, \mu, \nu)$. The collection of such functions will be denoted by $PAA^\kappa(\mathbb{X}, \mu, \nu)$.

Definition 3.13. Let $\mu, \nu \in \mathbb{U}_\infty$ and let $\kappa \in (0, 1)$. A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called doubly-weighted pseudo-almost automorphic of order κ if it can be expressed as $F = G+\Phi$, where $G \in AA(\mathbb{Y}, \mathbb{X})$ and $\Phi \in PAP_0^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$. The collection of such functions will be denoted by $PAA^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$.

4. Existence of a Doubly-Weighted Mean for Almost Periodic Functions

Let $\mu, \nu \in \mathbb{U}_\infty$. If $f : \mathbb{R} \mapsto \mathbb{X}$ is a bounded continuous function, we define its *doubly-weighted mean*, if the limit exists, by

$$\mathcal{M}(f, \mu, \nu) := \lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} f(t)\nu(t)dt.$$

It is well-known that if $f \in AP(\mathbb{X})$, then its mean defined by

$$\mathcal{M}(f) := \lim_{T\to\infty} \frac{1}{2T} \int_{Q_T} f(t)dt$$

exists [3]. Consequently, for every $\lambda \in \mathbb{R}$, the following limit

$$a(f, \lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{Q_T} f(t)e^{-i\lambda t}dt$$

exists and is called the Bohr transform of f.

It is well-known that $a(f, \lambda)$ is nonzero at most at countably many points [3]. The set defined by

$$\sigma_b(f) := \left\{\lambda \in \mathbb{R} : a(f, \lambda) \neq 0\right\}$$

is called the Bohr spectrum of f [15].

Theorem 4.1. (Approximation Theorem) [13, 15] *Let $f \in AP(\mathbb{X})$. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial*

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $a_k \in \mathbb{X}$ and $\lambda_k \in \sigma_b(f)$ such that $\|f(t) - P_\varepsilon(t)\| < \varepsilon$ for all $t \in \mathbb{R}$.

Our result on the existence of a doubly-weighted mean for almost periodic functions can be formulated as follows:

Theorem 4.2. *Let $\mu, \nu \in \mathbb{U}_\infty$ and suppose that $\lim_{T\to\infty} \dfrac{\nu(Q_T)}{\mu(Q_T)} = \theta_{\mu\nu}$. If $f : \mathbb{R} \mapsto \mathbb{X}$ is an almost periodic function such that*

$$\lim_{T\to\infty} \left| \frac{1}{\mu(Q_T)} \int_{Q_T} e^{i\lambda t} \nu(t) dt \right| = 0 \tag{4.1}$$

for all $0 \neq \lambda \in \sigma_b(f)$, then the doubly-weighted mean of f,

$$\mathcal{M}(f, \mu, \nu) = \lim_{T\to\infty} \frac{1}{\mu(Q_T)} \int_{Q_T} f(t)\nu(t)dt$$

exists. Furthermore, $\mathcal{M}(f, \mu, \nu) = \theta_{\mu\nu} \mathcal{M}(f)$.

Proof. If f is a trigonometric polynomial, say, $f(t) = \sum_{k=0}^{n} a_k e^{i\lambda_k t}$ where $a_k \in \mathbb{X} - \{0\}$ and $\lambda_k \in \mathbb{R}$ for $k = 1, 2, ..., n$, then $\sigma_b(f) = \{\lambda_k : \ k = 1, 2, ..., n\}$. Moreover,

$$\begin{aligned} \frac{1}{\mu(Q_T)} \int_{Q_T} f(t)\nu(t)dt &= a_0 \frac{\nu(Q_T)}{\mu(Q_T)} + \frac{1}{\mu(Q_T)} \int_{Q_T} \Big[\sum_{k=1}^{n} a_k e^{i\lambda_k t} \Big] \nu(t) dt \\ &= a_0 \frac{\nu(Q_T)}{\mu(Q_T)} + \sum_{k=1}^{n} a_k \Big[\frac{1}{\mu(Q_T)} \int_{Q_T} e^{i\lambda_k t} \nu(t) dt \Big] \end{aligned}$$

and hence

$$\left\| \frac{1}{\mu(Q_T)} \int_{Q_T} f(t)\nu(t)dt - a_0 \frac{\nu(Q_T)}{\mu(Q_T)} \right\| \leq \sum_{k=1}^{n} \|a_k\| \left| \frac{1}{\mu(Q_T)} \int_{Q_T} e^{i\lambda_k t} \nu(t) dt \right|$$

which by Eq. (4.1) yields

$$\left\| \frac{1}{\mu(Q_T)} \int_{Q_T} f(t)\nu(t)dt - a_0\theta_{\mu\nu} \right\| \to 0 \ \text{ as } \ T \to \infty$$

and therefore $\mathcal{M}(f, \mu, \nu) = a_0\theta_{\mu\nu} = \theta_{\mu\nu} M(f)$.

If in the finite sequence of λ_k there exist $\lambda_{n_k} = 0$ for $k = 1, 2, ...l$ with $a_m \in \mathbb{X} - \{0\}$ for all $m \neq n_k$ $(k = 1, 2, ..., l)$, it can be easily shown that

$$\mathcal{M}(f, \mu, \nu) = \theta_{\mu\nu} \sum_{k=1}^{l} a_{n_k} = \theta_{\mu\nu} M(f).$$

Now if $f : \mathbb{R} \mapsto \mathbb{X}$ is an arbitrary almost periodic function, then for every $\varepsilon > 0$ there exists a trigonometric polynomial (Theorem 4.1) P_ε defined by

$$P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

where $a_k \in \mathbb{X}$ and $\lambda_k \in \sigma_b(f)$ such that

$$\|f(t) - P_\varepsilon(t)\| < \varepsilon \tag{4.2}$$

for all $t \in \mathbb{R}$.

Proceeding as in Bohr [3] it follows that there exists T_0 such that for all $T_1, T_2 > T_0$,

$$\Big\|\frac{1}{\mu(Q_{T_1})} \int_{Q_{T_1}} P_\varepsilon(t)\nu(t)dt - \frac{1}{\mu(Q_{T_2})} \int_{Q_{T_2}} P_\varepsilon(t)\nu(t)dt\Big\| = \theta_{\mu\nu}\Big\|M(P_\varepsilon) - M(P_\varepsilon)\Big\| = 0 < \varepsilon.$$

In view of the above it follows that for all $T_1, T_2 > T_0$,

$$\begin{aligned}
\Big\|\frac{1}{\mu(Q_{T_1})} \int_{Q_{T_1}} f(t)\nu(t)dt - \frac{1}{\mu(Q_{T_2})} \int_{Q_{T_2}} f(t)\nu(t)dt\Big\| \quad &\leq \quad \frac{1}{\mu(Q_{T_1})} \int_{Q_{T_1}} \|f(t) - P_\varepsilon(t)\|\nu(t)dt \\
+\Big\|\frac{1}{\mu(Q_{T_1})} \int_{Q_{T_1}} P_\varepsilon(t)\nu(t)dt - \frac{1}{\mu(Q_{T_2})} \int_{Q_{T_2}} P_\varepsilon(t)\nu(t)dt\Big\| & \\
+\frac{1}{\mu(Q_{T_2})} \int_{Q_{T_2}} \|f(t) - P_\varepsilon(t)\|\nu(t)dt < 3\varepsilon. &
\end{aligned}$$

□

Example 4.3. Let $\mu(t) = e^{|t|}$ and $\nu(t) = 1 + |t|$ for all $t \in \mathbb{R}$, which yields $\theta_{\mu\nu} = 0$. If $\varphi : \mathbb{R} \mapsto \mathbb{X}$ is a (nonconstant) almost periodic function, then according to the previous theorem, its doubly-weighted mean $\mathcal{M}(\varphi, \mu, \nu)$ exists. Moreover,

$$\lim_{T\to\infty} \frac{1}{2(e^T - 1)} \int_{Q_T} f(t)(1 + |t|)dt = 0. \lim_{T\to\infty} \frac{1}{2T} \int_{Q_T} f(t)dt = 0.$$

5. Properties of Doubly-Weighted Pseudo Almost-Periodic and Doubly-Weighted Pseudo-Almost Automorphic Functions

This section is mainly devoted to properties of doubly-weighted pseudo-almost periodic functions (respectively, doubly-weighted pseudo-almost automorphic functions). These include, the convolution of a doubly-weighted pseudo-almost periodic function with a function which is integrable over $\mathbb{R}$, the translation-invariance of the weighted spaces, the uniqueness of the decomposition of the weighted spaces and well as their compositions.

Proposition 5.1. Let $\mu \in \mathbb{U}_\infty$ and let $\nu \in \mathbb{U}_\infty^{\rm Inv}$ such that

$$\sup_{T>0}\left[\frac{\nu(Q_T)}{\mu(Q_T)}\right] < \infty. \tag{5.1}$$

Let $f \in PAP_0(\mathbb{R},\mu,\nu)$ (respectively, $PAP_0^\kappa(\mathbb{R},\mu,\nu)$) and let $g \in L^1(\mathbb{R})$. Suppose

$$\lim_{T\to\infty}\left[\frac{\mu(Q_{T+|\tau|})}{\mu(Q_T)}\right] < \infty \ \text{ for all } \ \tau \in \mathbb{R}. \tag{5.2}$$

(respectively,

$$\lim_{T\to\infty}\left[\frac{\left(\mu(Q_{T+|\tau|})\right)^\kappa}{\mu(Q_T)}\right] < \infty \ \text{ for all } \ \tau \in \mathbb{R}.) \tag{5.3}$$

Then $f * g$, the convolution of f and g on $\mathbb{R}$, belongs to $PAP_0(\mathbb{R},\mu,\nu)$ (respectively, $PAP_0^\kappa(\mathbb{R},\mu,\nu)$).

Proof. It is clear that if $f \in PAP_0(\mathbb{R},\mu,\nu)$ and $g \in L^1(\mathbb{R})$, then their convolution $f * g \in BC(\mathbb{R},\mathbb{R})$. Now setting

$$J(T,\mu,\nu) := \frac{1}{\mu(Q_T)}\int_{Q_T}\int_{-\infty}^{+\infty} |f(t-s)|\,|g(s)|\nu(t)\,dsdt$$

it follows that

$$\begin{aligned}\frac{1}{\mu(Q_T)}\int_{Q_T}|(f*g)(t)|\nu(t)dt &\leq J(T,\mu,\nu)\\ &= \int_{-\infty}^{+\infty}|g(s)|\left(\frac{1}{\mu(Q_T)}\int_{Q_T}|f(t-s)|\nu(t)dt\right)ds\\ &= \int_{-\infty}^{+\infty}|g(s)|\phi_T(s)ds,\end{aligned}$$

where

$$\begin{aligned}\phi_T(s) &= \frac{1}{\mu(Q_T)}\int_{Q_T}|f(t-s)|\nu(t)dt\\ &= \frac{\mu(Q_{T+|s|})}{\mu(Q_T)}\cdot\frac{1}{\mu(Q_{T+|s|})}\int_{Q_T}|f(t-s)|\nu(t)dt\\ &\leq \frac{\mu(Q_{T+|s|})}{\mu(Q_T)}\cdot\frac{1}{\mu(Q_{T+|s|})}\int_{Q_{T+|s|}}|f(t)|\nu(t+s)dt.\end{aligned}$$

Using the fact that $\nu \in \mathbb{U}_\infty^{\text{Inv}}$ and Eq. (5.2), one can easily see that $\phi_T(s) \mapsto 0$ as $T \mapsto \infty$ for all $s \in \mathbb{R}$. Next, since ϕ_T is bounded, i.e.,

$$|\phi_T(s)| \leq \|f\|_\infty \,.\, \sup_{T>0}\frac{\nu(Q_T)}{\mu(Q_T)} < \infty$$

and $g \in L^1(\mathbb{R})$, using the Lebesgue dominated convergence theorem it follows that

$$\lim_{T\to\infty}\left\{\int_{-\infty}^{+\infty}|g(s)|\phi_T(s)ds\right\} = 0,$$

and hence $f * g \in PAP_0(\mathbb{R},\mu,\nu)$. The proof for $PAP_0^\kappa(\mathbb{R},\mu,\nu)$ is similar to that of $PAP_0(\mathbb{R},\mu,\nu)$ and hence omitted. □

It is well-known that if $h \in AP(\mathbb{R})$ (respectively $h \in AA(\mathbb{R})$) and $\psi \in L^1(\mathbb{R})$, then the convolution $h * \psi \in AP(\mathbb{R})$ (respectively, $h * \psi \in AA(\mathbb{R})$). Using these facts, we obtain the following:

Corollary 5.2. Fix $\kappa \in (0,1)$. Let $\mu \in \mathbb{U}_\infty$ and let $\nu \in \mathbb{U}_\infty^{\text{Inv}}$ such that Eq. (5.1) holds. Let $f \in PAP(\mathbb{R},\mu,\nu)$ (respectively, $PAP^\kappa(\mathbb{R},\mu,\nu)$) and let $g \in L^1(\mathbb{R})$. Suppose Eq. (5.2) holds (respectively, Eq. (5.3)). Then $f * g$ belongs to $PAP(\mathbb{R},\mu,\nu)$ (respectively, $PAP^\kappa(\mathbb{R},\mu,\nu)$).

and

Corollary 5.3. Fix $\kappa \in (0,1)$. Let $\mu \in \mathbb{U}_\infty$ and let $\nu \in \mathbb{U}_\infty^{\text{Inv}}$ and suppose that Eq. (5.1) holds. Let $f \in PAA(\mathbb{R},\mu,\nu)$ (respectively, $PAA^\kappa(\mathbb{X},\mu,\nu)$) and let $g \in L^1(\mathbb{R})$. Suppose Eq. (5.2) holds (respectively, Eq. (5.3)). Then $f * g$ belongs to $PAA(\mathbb{R},\mu,\nu)$ (respectively, $PAA^\kappa(\mathbb{R},\mu,\nu)$).

Theorem 5.4. *Fix $\kappa \in (0,1)$. Let $\mu \in \mathbb{U}_\infty$ and let $\nu \in \mathbb{U}_\infty^{\text{Inv}}$. Suppose Eq. (5.2) holds. Then $PAP(\mathbb{X},\mu,\nu)$ and $PAA(\mathbb{X},\mu,\nu)$ are translation-invariant.*

Proof. Let $f \in PAP_0(\mathbb{X},\mu,\nu)$. We will show that $t \mapsto f(t+s)$ belongs to $PAP_0(\mathbb{X},\mu,\nu)$ for each $s \in \mathbb{R}$.

Indeed,

$$\begin{aligned}\frac{1}{\mu(Q_T)}\int_{Q_T}\|f(t+s)\|\nu(t)dt &= \frac{\mu(Q_{T+|s|})}{\mu(Q_T)}\cdot\frac{1}{\mu(Q_{T+|s|})}\int_{Q_T}\|f(t+s)\|\nu(t)dt\\ &\le \frac{\mu(Q_{T+|s|})}{\mu(Q_T)}\cdot\frac{1}{\mu(Q_{T+|s|})}\int_{Q_{T+|s|}}\|f(t)\|\nu(t-s)dt.\end{aligned}$$

Using the fact that $\nu \in \mathbb{U}_\infty^{\rm Inv}$ and Eq. (5.2), it follows that

$$\lim_{T\to\infty}\frac{1}{\mu(Q_T)}\int_{Q_T}\|f(t+s)\|\nu(t)dt = 0.$$

Therefore, $PAP_0(\mathbb{R},\mu,\nu)$ is translation invariant. □

Similarly,

Theorem 5.5. *Fix $\kappa \in (0,1)$. Let $\mu \in \mathbb{U}_\infty$ and let $\nu \in \mathbb{U}_\infty^{\rm Inv}$. Suppose Eq. (5.3) holds. Then $PAP^\kappa(\mathbb{X},\mu,\nu)$ and $PAA^\kappa(\mathbb{X},\mu,\nu)$ are translation-invariant.*

In a recent paper by Liang *et. al* [14], it was shown that the uniqueness of the decomposition of weighted pseudo-almost periodic functions (respectively, weighted pseudo-almost automorphic functions) depends upon the translation-invariance of those spaces. Using similar ideas as in [14, Proof of Proposition 3.2], one can easily show the following theorems:

Theorem 5.6. *If $\mu,\nu \in \mathbb{U}_\infty$ such that the space $PAP_0(\mathbb{X},\mu,\nu)$ is translation-invariant and if*

$$\inf_{T>0}\left[\frac{\nu(Q_T)}{\mu(Q_T)}\right] = \delta_0 > 0, \tag{5.4}$$

then the decomposition of doubly-weighted pseudo-almost periodic functions (respectively, doubly-weighted pseudo-almost automorphic functions) is unique.

Similarly,

Theorem 5.7. *If $\mu,\nu \in \mathbb{U}_\infty$ such that the space $PAP_0^\kappa(\mathbb{X},\mu,\nu)$ ($\kappa \in (0,1)$) is translation-invariant and if*

$$\inf_{T>0}\left[\frac{\nu(Q_T)}{\big(\mu(Q_T)\big)^\kappa}\right] = \gamma_0 > 0, \tag{5.5}$$

then the decomposition of doubly-weighted pseudo-almost periodic functions of order κ (respectively, doubly-weighted pseudo-almost automorphic functions of order κ) is unique.

The next composition theorem generalizes existing composition theorems of pseudo-almost periodic functions involving Lipschitz condition especially those given in [1, 10].

Theorem 5.8. *Let $\mu, \nu \in \mathbb{U}_\infty$ and let $f \in PAP(\mathbb{Y}, \mathbb{X}, \mu, \nu)$ (respectively, $PAP^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$) satisfying the Lipschitz condition*

$$\|f(t,u) - f(t,v)\| \le L \,.\, \|u - v\|_{\mathbb{Y}} \ \textit{for all}\ \ u, v \in \mathbb{Y},\ t \in \mathbb{R}.$$

If $h \in PAP(\mathbb{Y}, \mu, \nu)$ (respectively, $PAP^\kappa(\mathbb{Y}, \mu, \nu)$) , then $f(\cdot, h(\cdot)) \in PAP(\mathbb{X}, \mu, \nu)$ (respectively, $PAP^\kappa(\mathbb{X}, \mu, \nu)$).

Proof. The proof will follow along the same lines as that of the composition result given in Diagana [10]. Let $f = g + \phi$ where $g \in AP(\mathbb{Y}, \mathbb{X})$ and $\phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \mu, \nu)$. Similarly, let $h = h_1 + h_2$, where $h_1 \in AP(\mathbb{Y})$ and $h_2 \in PAP_0(\mathbb{Y}, \mu, \nu)$. Clearly, $f(\cdot, h(\cdot)) \in C(\mathbb{R}, \mathbb{X})$. Next, decompose f as follows

$$f(\cdot, h(\cdot)) = g(\cdot, h_1(\cdot)) + f(\cdot, h(\cdot)) - f(\cdot, h_1(\cdot)) + \phi(\cdot, h_1(\cdot)).$$

Using the theorem of composition of almost periodic functions, one can easily see that $g(\cdot, h_1(\cdot)) \in AP(\mathbb{X})$. Now, set $F(\cdot) = f(\cdot, h(\cdot)) - f(\cdot, h_1(\cdot))$. Clearly, $F \in PAP_0(\mathbb{X}, \mu, \nu)$. Indeed, for $T > 0$,

$$\begin{aligned}\frac{1}{\mu(Q_T)}\int_{Q_T}\|F(s)\|\nu(s)ds &= \frac{1}{\mu(Q_T)}\int_{Q_T}\|f(s,h(s)) - f(s,h_1(s))\|\nu(s)ds \\ &\le \frac{L}{\mu(Q_T)}\int_{Q_T}\|h(s) - h_1(s)\|\nu(s)ds \\ &\le \frac{L}{\mu(Q_T)}\int_{Q_T}\|h_2(s)\|\nu(s)ds,\end{aligned}$$

and hence

$$\lim_{T\to\infty}\frac{1}{\mu(Q_T)}\int_{Q_T}\|F(s)\|\nu(s)ds = 0.$$

To complete the proof we have to show that

$$\lim_{T\to\infty}\frac{1}{\mu(Q_T)}\int_{Q_T}\|\phi(s, h_1(s))\|\nu(s)ds = 0.$$

As $h_1 \in AP(\mathbb{Y})$, $h_1(\mathbb{R})$ is relatively compact. Thus for each $\varepsilon > 0$ there exists a finite number of open balls $B_k = B(x_k, \dfrac{\varepsilon}{3L})$, centered at $x_k \in h_1(\mathbb{R})$ with radius for instance $\dfrac{\varepsilon}{3L}$ with $h_1(\mathbb{R}) \subset \bigcup_{k=1}^{m} B_k$. Therefore, for $1 \le k \le m$, the set $U_k = \{t \in \mathbb{R} : h_1(t) \subset B_k\}$ is open and $\mathbb{R} = \bigcup_{k=1}^{m} U_k$. Now, set $V_k = U_k - \bigcup_{i=1}^{k-1} U_i$ and $V_1 = U_1$. clearly, $V_i \bigcap V_j = \emptyset$ for all $i \ne j$.

Since $\phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \mu, \nu)$ there exists $T_0 > 0$ such

$$\lim_{T\to\infty}\frac{1}{\mu(Q_T)}\int_{Q_T}\|\phi(s, x_k)\|\nu(s)ds < \frac{\varepsilon}{3m} \ \text{ for }\ T \ge T_0 \tag{5.6}$$

and $k \in \{1, 2, ..., m\}$.

Moreover, since g ($g \in AP(\mathbb{Y}, \mathbb{X})$) is uniformly continuous in $\mathbb{R} \times \overline{h_1(\mathbb{R})}$, one has

$$\|g(t, x_k) - g(t, x)\| < \frac{\varepsilon}{3} \text{ for } x \in B_k,\ k = 1, 2, .., m. \tag{5.7}$$

Using above and the following the decompositions

$$\phi(\cdot, h_1(\cdot)) = f(\cdot, h_1(\cdot)) - g(\cdot, h_1(\cdot))$$

and

$$\phi(t, x_k) = f(t, x_k) - g(t, x_k)$$

it follows that

$$\begin{aligned}
&\frac{1}{\mu(Q_T)} \int_{Q_T} \|\phi(s, h_1(s))\|\nu(s)ds \\
&= \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|\phi(s, h_1(s))\|\nu(s)ds \\
&\le \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|\phi(s, h_1(s)) - \phi(s, x_k)\|\nu(s)ds \\
&+ \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|\phi(s, x_k)\|\nu(s)ds \\
&\le \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|f(s, h_1(s)) - f(s, x_k)\|\nu(s)ds \\
&+ \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|\phi(s, x_k)\|\nu(s)ds \\
&+ \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|g(s, h_1(s)) - g(s, x_k)\|\nu(s)ds \\
&+ \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} L\|h_1(s) - x_k\|_{\mathbb{Y}}\nu(s)ds \\
&+ \frac{1}{\mu(Q_T)} \sum_{k=1}^{m} \int_{V_k \bigcap Q_T} \|g(s, h_1(s)) - g(s, x_k)\|\nu(s)ds \\
&+ \sum_{k=1}^{m} \frac{1}{\mu(Q_T)} \int_{V_k \bigcap Q_T} \|\phi(s, x_k)\|\nu(s)ds.
\end{aligned}$$

For each $s \in V_k \cap Q_T$, $h_1(s) \in B_k$ in the sense that $\|h_1(s) - x_k\|_{\mathbb{Y}} < \dfrac{\varepsilon}{3L}$ for $1 \leq k \leq m$. Clearly, from Eqs. (5.6)-(5.7) it easily follows that

$$\frac{1}{\mu(Q_T)} \int_{Q_T} \|\phi(s, h_1(s))\| \nu(s) ds \leq \varepsilon$$

for $T \geq T_0$, and hence

$$\lim_{T \to \infty} \frac{1}{\mu(Q_T)} \int_{Q_T} \|\phi(s, h_1(s))\| \nu(s) ds = 0.$$

The proof is similar in the case when the order κ is involved. □

Similarly, we have the following composition result for doubly-weighted pseudo-almost automorphic functions.

Theorem 5.9. *Let $\mu, \nu \in \mathbb{U}_\infty$ and let $f \in PAA(\mathbb{Y}, \mathbb{X}, \mu, \nu)$ (respectively, $PAA^\kappa(\mathbb{Y}, \mathbb{X}, \mu, \nu)$) satisfying the Lipschitz condition*

$$\|f(t, u) - f(t, v)\| \leq L \,.\, \|u - v\|_{\mathbb{Y}} \text{ for all } u, v \in \mathbb{Y},\ t \in \mathbb{R}.$$

If $h \in PAA(\mathbb{Y}, \mu, \nu)$ (respectively, $PAA^\kappa(\mathbb{Y}, \mu, \nu)$), then $f(\cdot, h(\cdot)) \in PAA(\mathbb{X}, \mu, \nu)$ (respectively, $PAA^\kappa(\mathbb{X}, \mu, \nu)$).

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Department of Mathematics, Howard University, 2441 6th Street N.W., Washington, D.C. 20059, USA
E-mail address: tdiagana@howard.edu