Noncommutative circle bundles and new Dirac operators

Ludwik Dąbrowski[1] and Andrzej Sitarz[2,*]

[1] SISSA (Scuola Internazionale Superiore di Studi Avanzati), via Bonomea 265, 34136 Trieste, Italy

[2] Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

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Abstract

We study the spectral triples over the total space of noncommutative principal $U(1)$ bundles. Basing on the classical situation and the abstract algebraic approach we propose a definition of connections and compatibility between the connection and the Dirac operator. We analyse in details the example of the noncommutative three-torus viewed as a $U(1)$ bundle over the noncommutative two-torus and find all connections compatible with an admissible Dirac operator. Conversely, we find a family of new Dirac operators on the noncommutative tori, which arise from the base-space Dirac operator and a suitable connection.

1 Introduction

The principal $U(1)$ bundles are simplest and fundamental examples of fibre bundles, often encountered in mathematics and physics. They are usually equipped with a connection and a metric, which are in principle independent, though an interesting situation arises when they are compatible in some natural way. This reflects in particular on the spectral geometry of $U(1)$ bundles, which in terms of Laplace operator has been studied in [8] whereas the analysis of Dirac operator was presented in [1, 2]. In this note we shall extend part of the latter analysis to the analogue of principal $U(1)$ bundles in noncommutative geometry, encoding their geometric aspects in terms of spectral triples [5, 6].

2 Spin Geometry of U(1)-bundles

We suppose that M is a $n+1$ dimensional ($n+1$ odd) compact manifold which is the total space of $U(1)$-principal bundle over the n-dimensional (n even) base space $N = M/U(1)$. Moreover assume M is equipped with a Riemannian metric $\tilde{g}$ and the $U(1)$ action (free and transitive on fibres) is isometric. The base space N carries a unique metric g such that the projection $\pi : (M, \tilde{g}) \longrightarrow (N, g)$ is a Riemannian submersion.

We can and shall use a suitable local orthonormal frame (basis) of the tangent space TM, $e = (e_0, e_1, \ldots, e_n)$, such that e is $U(1)$ invariant and e_0 is the (normalized) Killing vector field K associated to the $U(1)$-action. For simplicity we assume that the fibres are of constant length $2\pi\ell$.

There exists a unique principal connection 1-form $\omega : TM \to \mathbb{R} \approx u(1)$, such that $\ker \omega$ is orthogonal to the fibres for all $m \in M$ with respect to $\tilde{g}$. Obviously it is given by $\omega = e^0/\ell$, where $(e^0, e^1, \ldots, e^n)$ is the dual frame to e. Conversely, if we are given a principal connection on the principal $U(1)$ bundle and a metric on the base space N then there exists a unique $U(1)$-invariant metric on M, such that the horizontal vectors are orthogonal to the fundamental (Killing) vector field K of length ℓ.

Assume now M is spin and let ΣM be its spinor bundle (which is hermitian, rank $2^{\frac{n}{2}}$ vector bundle). The $U(1)$ action either lifts to the spin structure and then to an action

$$\kappa : U(1) \times \Sigma M \to \Sigma M,$$

or to a projective action (up to a sign), i.e. to the action of a non-trivial double cover of $U(1)$, which happens to be still $U(1)$ as a group.

Assuming the former case, we have a *projectable* spin structure on M. As explained in [2] this induces a spin structure on N. Conversely, any spin structure on N canonically induces a projectable spin structure on M via a pull-back construction.

We recall that the Dirac operator $\tilde{D}$ on M can be constructed as follows. Let γ_j, $j = 0, 1, \ldots, n$, be the antihermitian matrices in $M(2^{\frac{n}{2}}, \mathbb{C})$, which satisfy the relations

$$\gamma_j \gamma_k + \gamma_k \gamma_j = -2\delta_{jk}. \tag{2.1}$$

Then Dirac operator $\tilde{D}$ acting on sections of ΣM can be explicitly written as

$$\tilde{D} = \sum_{i=0}^{n} \gamma_i \partial_{e_i} + \frac{1}{4} \sum_{i,j,k=0}^{n} \tilde{\Gamma}_{ij}^k \gamma_i \gamma_j \gamma_k,$$

where $\tilde{\Gamma}^k_{ij}$ are Christoffel symbols (in the orthonormal basis e) of the Levi-Civita connection on M. In particular

$$\begin{aligned} -\tilde{\Gamma}^0_{ij} = \tilde{\Gamma}^j_{i0} = \tilde{\Gamma}^j_{0i} = \frac{\ell}{2}\, d\omega(e_i, e_j), \\ \tilde{\Gamma}^0_{i0} = \tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^i_{00} = \tilde{\Gamma}^0_{00} = 0. \end{aligned} \tag{2.2}$$

Since the metric on M is completely characterized by the connection 1-form ω, the length ℓ of the fibres and the metric g on N, the Dirac operator $\tilde{D}$ on M can be expressed in terms of ω, and g. Conversely, the metric on N, the connection ω and the length of the fibres can be recovered from $\tilde{D}$.
Following this line, Ammann and Bär [2] achieved to present the Dirac operator $\tilde{D}$ as a sum of two first order differential operators on $L^2(\Sigma M)$ and a zero order term (endomorphism of the spinor bundle).
The first operator, called the *vertical* Dirac operator is

$$D_v := \frac{1}{\ell}\gamma_0\, \partial_K,$$

where

$$\partial_K(\Psi)(m) = \frac{d}{dt}|_{t=0}\, \kappa(e^{-it}, \Psi(m \cdot e^{it}))$$

is the Lie derivative of a spinor Ψ along the U(1) Killing field. Note that $D_v := \gamma_0\, \partial_{e_0}$, where ∂_{e_0} could be interpreted as the Dirac operator associated to the typical fibre $S^1 \simeq U(1)$, whereas γ_0 is the Clifford representation of the (normalized) one-form $e^0 = \ell\omega$.
It follows from (2.2) that the spinor covariant derivative differs from the Lie derivative in the direction of e_0:

$$\nabla_{e_0} = \partial_{e_0} + \frac{\ell}{4} \sum_{j<k} d\omega(e_j, e_k)\gamma_j\gamma_k. \tag{2.3}$$

The description of the second differential operator D_h, called a *horizontal* Dirac operator, uses an orthogonal decomposition of the Hilbert space into irreducible representations of $U(1)$:

$$L^2(\Sigma M) = \bigoplus_{k\in\mathbb{Z}} V_k,$$

where V_k are the closures of eigenspaces V_k of the Lie derivative ∂_{e_0} for the eigenvalue ik, $k \in \mathbb{Z}$. This decomposition is preserved by $\tilde{D}$, since it commutes with the (isometric) $U(1)$-action on M.

Next, let $L := M \times_{U(1)} \mathbb{C}$ be the complex line bundle associated to the $U(1)$-bundle $M \to N$. In [2] it is shown that there is a natural homothety of Hilbert spaces (isomorphism if the fibres are of length $\ell = 1$)

$$Q_k : L^2(\Sigma N \otimes L^{-k}) \to V_k,$$

which satisfies

$$Q_k(\gamma_i \Psi) = \gamma_i Q_k(\Psi), \quad i = 1, \ldots, n$$

and

$$\nabla_{e_i} Q_k(\Psi) = Q_k(\nabla_{f_i}\Psi) + \frac{1}{4}\sum_{j=1}^{n}\left(\tilde{\Gamma}_{i0}^{j} - \tilde{\Gamma}_{ij}^{0}\right)\gamma_0\gamma_j Q_k(\Psi), \tag{2.4}$$

where $f = (f_1, f_2, \ldots, f_n)$, $f_i := \pi_*(e_i)$ is a local orthonormal frame on N.
Then $D_h : L^2(\Sigma M) \to L^2(\Sigma M)$ is defined as the unique closed linear operator, such that on each V_k it is:

$$D_h := Q_k \circ D'_k \circ {Q_k}^{-1},$$

where D'_k is the twisted (of charge k) Dirac operator on $\Sigma N \otimes L^{-k}$ acting on sections of $\Sigma M \otimes E$ by using the gamma matrices on the first factor and the tensor product connection $\nabla_N + k\omega$. Here, ∇_N is the covariant spinor derivative on N coming from the Levi-Civita connection on N, whose Christoffel symbols with respect to the projected frame $f = (f_1, \ldots, f_n)$ on N are given by

$$\Gamma_{ij}^{k} = \tilde{\Gamma}_{ij}^{k} \qquad \forall\, i, j, k \in \{1, \ldots, n\}. \tag{2.5}$$

Using the above results the Dirac operator $\tilde{D}$ on M can expressed as a sum

$$\tilde{D} = D_v + D_h + Z,$$

where

$$Z := -(\ell/4)\,\gamma_0 \sum_{j<k} d\omega(e_j, e_k)\gamma_j\gamma_k.$$

Observe that since D_h, γ_0 and Z are $U(1)$-invariant, they commute with ∂_{e_0}. Since for even n, $\gamma_1\gamma_2 \ldots \gamma_n$ anticommutes with any twisted Dirac operator on N and $\gamma_0 \sim \gamma_1\gamma_2 \ldots \gamma_n$ (up to a constant $1, i, -1, -i$ depending on n and the representation of gamma matrices), γ_0 anticommutes with D_h.
Finally, let us observe that the presence of the zero-order term Z is responsible for the torsion-free condition. In other words, omitting Z still provides a Dirac operator of M for the linear connection, which preserves the metric $\tilde{g}$ but has a non-vanishing (in general) torsion. This can be see easily by looking at the Christoffel

symbols defined by (2.5) and by (2.2). If, in the latter formula we put $\tilde{\Gamma}^k_{ij} = 0$ whenever one or more of the indices i, j, k is zero, we get a linear connection, which is still compatible with the metric but the components

$$T^0_{ij} = e^0(\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j]) = de^0(e_i, e_j) = \ell\, d\omega(e_i, e_j) \tag{2.6}$$

of the torsion tensor do not vanish (in general).

3 Noncommutative $U(1)$ principal bundles

We turn now to the noncommutative picture, where the concept of principal bundles is given by the Hopf-Galois theory. Let us shortly recall the basic definitions, for details and examples see [3, 4, 7, 10].

Definition 3.1. Let H be a unital Hopf algebra and $\mathcal{A}$ be a right H-comodule algebra. We denote by $\mathcal{B}$ the subalgebra of invariant elements of $\mathcal{A}$. We say the $\mathcal{B} \hookrightarrow \mathcal{A}$ is a Hopf-Galois extension iff the canonical map χ :

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \ni a' \otimes a \mapsto \chi(a' \otimes a) = a' a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H, \tag{3.1}$$

is an isomorphism.

In the purely algebraic settings the connections are defined as right-colinear maps from the Hopf algebra H to the first order universal differential calculus $\Omega^1_u(\mathcal{A})$ over $\mathcal{A}$.

Definition 3.2. We say that a right H-colinear map $\omega : H \to \Omega^1_u(\mathcal{A})$ is *a strong* universal connection if the following conditions hold:

$$\begin{aligned}
&\omega(1) = 0, \\
&\Delta_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}_R, \\
&d_u(a) - a_{(0)} \omega(a_{(1)}) \in \left(\Omega^1_u(\mathcal{B})\right) \mathcal{A}, \quad \forall a \in \mathcal{A}, \\
&(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) \circ \omega = 1 \otimes (\mathrm{id} - \varepsilon).
\end{aligned} \tag{3.2}$$

where we use the natural Sweedler notation for the right coaction of H on $\mathcal{A}$:

$$\Delta_R(a) = a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H.$$

It is possible to extend this definition of connections for nonuniversal differential calculi, however only after requiring certain compatibility conditions between the differential calculus on $\mathcal{A}$ and a given calculus over the Hopf algebra H. Choosing a subbimodule $\mathcal{N} \subset \mathcal{A} \otimes \mathcal{A}$ we have an associated first order differential calculus over $\mathcal{A}$. If the canonical map χ maps $\mathcal{N}$ to $\mathcal{A} \otimes Q$, where $Q \subset \ker \varepsilon \subset H$ is an Ad-invariant vector space then it is possible to use a calculus over H determined by Q using the Woronowicz construction of bicovariant calculi [13]. For details see [3, 10, 11].

As in the case of spectral geometry it will be more convenient to use action of the $U(1)$ group rather then the coaction of the algebra of functions over $U(1)$. Since as the algebra of functions on $U(1)$ we consider the space of polynomials, and effectively we work with homogeneous elements $a \in \mathcal{A}^{(k)} \subset \mathcal{A}$ of a fixed degree k, which are defined as follows:

$$a \in \mathcal{A}^{(k)} \Leftrightarrow \Delta(a) = a \otimes z^k,$$

we can easily reformulate all conditions above using the language of $U(1)$ action, where we have:

$$a \in \mathcal{A}^{(k)} \Leftrightarrow e^{i\phi} \triangleright a = e^{ik\phi} a.$$

Definition 3.3. For a $U(1)$ Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ we say that $\omega : \mathbb{Z} \to \Omega^1_u(\mathcal{A})$ is *a strong* universal connection iff:

$$\begin{aligned}
&\omega(0) = 0, \\
&g \triangleright \omega = \omega, \quad \forall g \in U(1), \\
&d_u(a) - a\omega(k) \in \left(\Omega^1(\mathcal{B})\right) \mathcal{A}, \quad \forall a \in \mathcal{A}^{(k)}, \\
&m \circ (\mathrm{id} \otimes \pi_n)\omega(k) = \delta_{kn} - \delta_{n0}.
\end{aligned} \tag{3.3}$$

Her π_n projects an element on the part of a fixed homogeneity degree n. We shall see in section 5 that the third condition (strongness) will play a significant role in the extension of Dirac operator, we shall also rewrite suitably the last one (fundamental vertical field condition).

4 Spectral triples over $U(1)$ bundles

We assume that there exists a real spectral triple over $\mathcal{A}$ (for details on real spectral triples, notation and basic properties we refer to the textbook [9]), which is $U(1)$ equivariant, that is the action of $U(1)$ extends to the Hilbert space and the representation, the Dirac operator and the reality structure are $U(1)$ equivariant. We denote by π the representation of $\mathcal{A}$ on $\mathcal{H}$, D is the Dirac operator and J the reality structure.

Let δ be the operator on $\mathcal{H}$ which generates the action of $U(1)$ on the Hilbert space. The $U(1)$ equivariance of the reality structure and D means that:

$$J\delta = -\delta J, \quad D\delta = \delta D, \tag{4.1}$$

whereas the equivariance of the representation is:

$$[\delta, \pi(a)] = \pi(\delta(a)), \quad \forall a \in \mathcal{A},$$

where $\delta(a)$ is the derivation of a arising from the $U(1)$ action.
For simplicity, we take the dimension of the spectral triple over $\mathcal{A}$ to be odd, then the dimension of spectral triple over $\mathcal{B}$ is even (in particular, the spectral triple over $\mathcal{B}$ has a $\mathbb{Z}_2$ grading). We shall require that the signs of algebraic relations between J, D and the chirality (in the even case) are not changed when we pass to the quotient. For concreteness we take the top dimension 3 and the dimension of the quotient 2. Our example will be also three-dimensional. Therefore we have:

$$DJ = JD, \quad J^2 = -1.$$

4.1 Projectable spectral triples

We define the space $\mathcal{H}_k \subset \mathcal{H}$, $k \in \mathbb{Z}$, to be a subspace of vectors homogeneous of degree k in $\mathcal{H}$ that is, they are eigenvectors of δ of eigenvalue k. Let us denote by P_k the projection $\mathcal{H} \to \mathcal{H}_k$. The relation (4.1) means that

$$J\mathcal{H}_k = \mathcal{H}_{-k}.$$

In particular the subspace $\mathcal{H}_0$ is J invariant. From the equivariance of D we see that each $\mathcal{H}_k$ is preserved by the action of D:

$$D\mathcal{H}_k \subset \mathcal{H}_k.$$

We start by assuming an additional structure on the spectral triple.

Definition 4.1. We say that the $U(1)$ equivariant spectral triple $(\mathcal{A}, D, J, \mathcal{H}, \delta)$ is projectable along the fibres if there exists an operator Γ, a $\mathbb{Z}_2$ grading of the Hilbert space $\mathcal{H}$, which satisfies the following conditions:

$$\begin{aligned} &\forall a \in \mathcal{A} : [\Gamma, \pi(a)] = 0, \\ &\Gamma J = -J\Gamma, \quad \Gamma\delta = \delta\Gamma, \quad \Gamma^* = -\Gamma, \end{aligned} \tag{4.2}$$

and the *horizontal Dirac operator*:

$$D_h = \frac{1}{2}\Gamma[D, \Gamma],$$

generates the same bimodule of one-forms over $\mathcal{B}$ as D:

$$[D_h, b] = [D, b], \quad \forall b \in \mathcal{B}. \tag{4.3}$$

The first two lines of conditions will assure that we can use Γ to project the spectral triple to obtain an even triple over $\mathcal{B}$, the second condition is necessary so that the differential calculus over $\mathcal{B}$ does not depend on the choice of projection. Note, that the signs in the definition are adjusted to the case of dimension 3 bundle.
Since $[D_h, \delta] = 0$ we see that D_h preserves the subspaces $\mathcal{H}_k$. We shall denote by D_k its restriction to each subspace $\mathcal{H}_k$. Similarly, by denoting by γ_k the restriction of Γ to $\mathcal{H}_k$ and j_k the restriction of J (as a map $\mathcal{H}_k \to \mathcal{H}_{-k}$).
In what follows, we shall make one additional assumption, which in the classical case amounts to the situation, when we assume that the $U(1)$ fibres are of equal length. What we propose, is a geometric characterization of the Dirac operator, which closely follows the analysis of Amman and Bär [2].
Let D_v denote the vertical part of the Dirac operator:

$$D_v = \frac{1}{\ell}\Gamma\delta.$$

Definition 4.2. We say that the $U(1)$ bundle has fibre of constant length (taken to be $2\pi\ell$) if

$$Z = D - D_h - D_v$$

is an operator of zero order, which commutes with the elements from the commutant:

$$[Z, Ja^* J^{-1}] = 0, \;\; \forall a \in \mathcal{A}.$$

Now we have:

Proposition 4.3. *The data $(\mathcal{B}, \mathcal{H}_0, D_0, \gamma_0, j_0)$ gives a real spectral triple of KR-dimension 2 over $\mathcal{B}$. For $k \neq 0$, $(\mathcal{B}, \mathcal{H}_k, D_k, \gamma_k)$ are twisted spectral triples over $\mathcal{B}$, which are pairwise real:*

$$\begin{aligned} &\gamma_k D_k = -D_k \gamma_k \quad j_k D_k = -D_{-k} j_k, \\ &j_k \gamma_k = -\gamma_{-k} j_k. \end{aligned} \tag{4.4}$$

Proof. Clearly D_h is a selfadjoint operator, which has the same commutation relation with J as D [1]. Therefore, relations (4.4) follow. □

As for the spectral properties, it is not difficult to observe that each of the D_k operators has compact resolvent. Indeed, consider $D - Z$, which is a bounded perturbation of D. Since, it is again $U(1)$ invariant, we can restrict it to $\mathcal{H}_k$. Its eigenvalues are:

$$\pm\sqrt{k^2/\ell^2 + \lambda_{(k)}^2}$$

[1] In the case of dimension other than 3 it is possible to adjust the signs in the definition of Γ and j_k so that the resulting KR-dimension of the projected spectral triple shall be correct.

where $\lambda_{(k)}$ are eigenvalues of D_k. Hence spectral properties of D_k are the same as properties of D restricted to $\mathcal{H}_k$.
Therefore spectral dimension of each D_k can be at most the same as that of D, which does not imply that it is exactly 2 as we know in the classical case.
Actually, taking a pair $\mathcal{H}_k \oplus \mathcal{H}_{-k}$ yields again a full, real spectral triple, which is, however, reducible.

Remark 1. In the classical situation, when one is able to consider the fibres over points of the base space, there is no problem to define the length of a fibre and, consequently, to restrict the considerations to the case when all fibres are of equal length. In the general noncommutative setup, this is no longer possible. Instead, we have proposed in the definition 4.2 above how to replace and extend this property in a way which links the length of fibres to the form of the Dirac operator. There may be, however, some other alternatives. We mention here just one other possible definition, which will be illustrated later on the example of the noncommutative tori.
We can say that the $U(1)$ bundle $\mathcal{A}$ with the equivariant spectral triple and the Dirac D has fibres of length ℓ, if the restriction of D to a $U(1)$ invariant subspace of $\mathcal{H}$ is an operator of spectral dimension $(n-1)$ and for any element $b \in \mathcal{B}$:

$$\fint b|D|^{-n} = \ell \fint b|D_0|^{-n+1},$$

where D_0 is the restriction of D to the invariant Hilbert subspace $\mathcal{H}_0$.
Although in the classical case it is obvious that this definition implies that the fibres are indeed of equal length, in the noncommutative setup it is far from being clear. The advantage of this definition, is that it is not sensitive to the bounded perturbations of the Dirac operator, which do not commute with the algebra elements.

4.2 Twisted spectral triples

In this subsection we shall discuss how to twist real spectral triples by a left-module.
Let M be a finitely generated projective left module over $\mathcal{B}$ and let the data: $(\mathcal{B}, \mathcal{H}_0, \pi, D_0, \gamma_0, j_0)$ define a real spectral triple over $\mathcal{B}$ of KR-dimension 2.

$$D_0 j_0 = j_0 D_0, \quad j_0^2 = -1, \quad j_0 \gamma_0 = -\gamma_0 j_0.$$

We assume that M has a structure of pre-Hilbert module, with $\mathcal{B}$-valued pairing. First, we construct the linear space $V_M = \mathcal{H}_0 \otimes_{\mathcal{B}} M$, using the real structure of the spectral triple for the right $\mathcal{B}$-module structure of $\mathcal{H}_0$:

$$hb = j_0 b^* j_0^{-1} h.$$

We equip V_M with the natural scalar product:

$$(v_1 \otimes_{\mathcal{B}} m_1, v_2 \otimes_{\mathcal{B}} m_2) = (v_1, v_2(m_2, m_1)),$$

so that the completion $\mathcal{H}_k$ of V_M is a Hilbert space.

Lemma 4.4. *$V_M = \mathcal{H}_0 \otimes_{\mathcal{B}} M$ carries a canonical left representation π_M of $\mathcal{B}$ given by*

$$\pi_M(b)\,(h \otimes_{\mathcal{B}} m) = \pi(b)h \otimes_{\mathcal{B}} m.$$

Our aim is to construct a twisted Dirac operator over $\mathcal{H}_k$ using connections over M. Let ∇ be any connection on M valued in $\Omega^1_{D_0}(\mathcal{B})$.

$$\nabla : M \to \Omega^1_D(\mathcal{B}) \otimes_{\mathcal{B}} M.$$

Lemma 4.5. *The following defines on a dense domain V_M in $\mathcal{H}_k$ an operator D_M:*

$$D_M(h \otimes_{\mathcal{B}} m) = (D_0 h \otimes_{\mathcal{B}} m) + h\nabla(m),$$

where the last product is defined in the following way:

$$h(\omega \otimes_{\mathcal{B}} m) = (j_0 \omega^* j_0^{-1})h \otimes_{\mathcal{B}} m.$$

Proof. We need to show that the definition is well posed. Clearly, it is well defined for the simple tensor product $h \otimes m$, therefore it remains to check that D_M vanishes on the ideal generated by $hb \otimes m - h \otimes bm$ for any h, b, m.
We use here extensively the right $\mathcal{B}$ module structure on $\mathcal{H}_0$ set by the reality operator j_0 and the order one condition:

$$\forall b, b' \in \mathcal{B} : \left[[D_0, b], j_0(b')^* j_0^{-1}\right] = 0.$$

The latter is needed to show that the action of D_M on $\mathcal{H}_0 \otimes_{\mathcal{B}} M$ is well-defined and to calculate the action of a one-form

$$\begin{aligned} D_M(hb \otimes m - h \otimes bm) =& D_0(hb) \otimes m + hb \otimes \nabla(m) \\ & - D_0(h) \otimes bm - h \otimes \nabla(bm) \\ =& \big([D_0, j_0 b^* j_0^{-1}]h + (D_0 h)b\big) \otimes m + hb \otimes \nabla(m) \\ & - D_0(h) \otimes bm - h([D_0, b]) \otimes m - h \otimes b\nabla(m) \\ =& (j_0[D_0, b]^* j_0^{-1})h \otimes m - h([D_0, b]) \otimes m = 0. \end{aligned}$$

Similarly one proofs that for any $b \in \mathcal{B}$ the commutator is bounded:

$$\begin{aligned} [D_M, b]h \otimes m &= D_0(bh) \otimes m + bh \otimes \nabla(m) - bD_0(h) \otimes m - bh \otimes \nabla(m) \\ &= [D_0, b]h \otimes m. \end{aligned}$$

Notice that if T is a bounded operator on $\mathcal{H}_0$, which extends to an operator on $\mathcal{H}_k$ then the extension is also bounded. □

5 Connections

In order to define a strong $U(1)$ connection over the bundle using the differential calculus given by the Dirac operator, we need to impose certain conditions on the Dirac operator itself.

Definition 5.1. We say that the first order differential calculus over $\mathcal{A}$ given by the Dirac operator D is compatible with the standard de Rham calculus over $U(1)$ if the following holds:

$$\forall p_i, q_i \in \mathcal{A} : \sum_i p_i[D, q_i] = 0 \Rightarrow \sum_i p_i \delta(q_i) = 0. \tag{5.1}$$

We have:

Lemma 5.2. *The image by the canonical map of the ideal defining the first order differential calculus is in* $\mathcal{A} \otimes (\ker \varepsilon)^2$.

Proof. Indeed, assume that $\sum_i p_i \otimes q_i \in \mathcal{N}$, where

$$\mathcal{N} \subset \ker m \subset \mathcal{A} \otimes \mathcal{A},$$

is the subbimodule defined by the relation $\sum_i p_i[D, q_i] = 0$.
Let us decompose q_i as a sum of homogeneous elements:

$$q_i = \sum_n q_i^{(n)}.$$

Then, using the identity (5.1) we have:

$$\sum_{i,n} p_i q_i^{(n)} = 0, \quad \sum_{i,n} n p_i q_i^{(n)} = 0.$$

which we can solve for $p_i q_i^{(0)}$ and $p_i q_i^{(1)}$:

$$\begin{aligned} \sum_i p_i q_i^{(1)} &= -\sum_{i,n\neq 1} n p_i q_i^{(n)}, \\ \sum_i p_i q_i^{(0)} &= -\sum_{i,n\neq 0} p_i q_i^{(n)} = \sum_{i,n\neq 0,1} (n-1) p_i q_i^{(n)}. \end{aligned}$$

Applying canonical map χ to $\sum p_i \otimes q_i$ we have:

$$\chi(\sum_i p_i \otimes q_i) = \sum_{i,n} p_i q_i^{(n)} \otimes z^n$$
$$= \sum_{i,n\neq 0,1} p_i q_i^{(n)} \otimes (z^n - 1 - n(z-1)).$$

Since the second factor on the right-hand side can be written as $(z-1)(z^{n-1} + \cdots + z + 1 - n)$, it is in $(\ker \varepsilon)^2$, which means that the differential calculus over $\mathcal{A}$ is compatible with the standard de Rham calculus over $U(1)$. □

Now, we are ready to define a strong connection for a principal $U(1)$ bundle with a differential calculus set by the Dirac operator.

Definition 5.3. We say that $\omega \in \Omega^1_D(\mathcal{A})$ is a strong connection for the $U(1)$ bundle $\mathcal{B} \hookrightarrow \mathcal{A}$ if the following conditions hold:

$$[\delta, \omega] = 0, \quad \textit{(}U(1)\textit{ invariance of }\omega\textit{)}$$
$$\text{if } \omega = \sum_i p_i [D, q_i] \text{ then} \sum_i p_i \delta(q_i) = 1, \quad \textit{(vertical field condition)},$$
$$\forall a \in \mathcal{A} : [D, a] - \delta(a)\omega \in \Omega^1_D(\mathcal{B})\mathcal{A}, \quad \textit{(strongness)}$$

Observe that the second condition (which in the classical case corresponds to the value of ω on fundamental vertical vector field) makes sense due to assumption (5.1).

5.1 Lifting the Dirac operator through connection

Let $\mathcal{A}^{(k)}$ be the left module of elements in $\mathcal{A}$, which are homogeneous of degree k with respect to the action of $U(1)$.

Proposition 5.4. *The map:*

$$\nabla_\omega : \mathcal{A}^{(k)} \ni a \mapsto [D, a] - na\omega \in \Omega^1_D(\mathcal{B}) \otimes_{\mathcal{B}} \mathcal{A}^{(k)},$$

defines a $\Omega^1_D(\mathcal{B})$-valued connection (covariant derivative) over $\mathcal{A}^{(k)}$.

Indeed, observe first that by the strongness condition and the fact that ω is $U(1)$-invariant the element $\nabla_\omega(a)$ is indeed in $\Omega^1_D(\mathcal{B}) \otimes_{\mathcal{B}} \mathcal{A}^{(k)}$. Since we have assumed that the differential calculus over $\mathcal{B}$ are isomorphic for D and D_0, we easily see that ∇_ω indeed satisfies the condition required for a connection (connection should be $\Omega^1_{D_0}(\mathcal{B})$ - valued and we used the commutator with D in the definition).

Using the construction from the previous section we can now construct Dirac operators, which arise from the connection ∇_ω for $\mathcal{H}_0 \otimes \mathcal{A}^{(k)}$ for any k. Since we have

$$\mathcal{A} = \bigoplus_k \mathcal{A}^{(k)},$$

and $\mathcal{H} \cdot \mathcal{A}^{(k)}$ is a dense linear subset of $\mathcal{H}_k$, we can define D_ω on a dense linear subspace of $\mathcal{H}$ an operator associated to the connection ω:

Definition 5.5. For each $k \in \mathbb{Z}$ we use the construction of the twisted spectral triple to define the Dirac operator $D_\omega^{(k)}$ on a dense subset of $\mathcal{H}_k$ identified as $\mathcal{H}_0 \otimes_{\mathcal{B}} \mathcal{A}^{(k)}$. Taking D_ω to be the closure of the operator given by the collection of $D_\omega^{(k)}$, we obtain a Dirac operator D_ω on $\mathcal{H}$, twisted by the connection ω.

Note that from the construction it is not entirely obvious that the Dirac operator D_ω has bounded commutators with the elements from the algebra $\mathcal{A}$. We have:

Proposition 5.6. *The twisted Dirac operator D_ω is selfadjoint if ω is an antiselfadjoint one form and has bounded commutators with all elements of $\mathcal{A}$.*

Proof. To see it let us calculate D_ω on an element hp, with $h \in \mathcal{H}_0$ and $p \in \mathcal{A}^{(n)}$:

$$\begin{aligned} D_\omega(hp) =& (D_0 h)p + h[D,p] - nhp\omega \\ =& Jp^* J^{-1} D_0 h + [D, Jp^* J^{-1}]h - J\omega^* J^{-1} nhp \\ =& D(hp) + \left((D_0 - D)h\right) p - J\omega^* J^{-1} \delta(hp) \\ =& \left(D - J\omega^* J^{-1}\delta - Z\right)(hp), \end{aligned}$$

where we have used the decomposition of D into the horizontal part, which restricted to $\mathcal{H}_0$ is D_0, the vertical part (which vanishes on $\mathcal{H}_0$) and the bounded perturbation Z. If the bounded perturbation commutes with the multiplication by the elements of the commutant then we have that $(Zh)p = Z(hp)$ and we obtain the formula above.
Now, calculating the commutator with an arbitrary element $a \in \mathcal{A}$ is an easy task. D has bounded commutators and since ω was a one-form, from the order one condition the commutator of the second term with a is $J\omega^* J^{-1}\delta(a)$ and hence it is bounded. The third term, as a commutator of two bounded elements remains bounded. Hence, $[D_\omega, a]$ is bounded for any a.
Next, observe that since D and Z are selfadjoint it suffices to have $J\omega^* J^{-1}\delta$ to be selfadjoint. But since δ commutes with ω and is antiselfadjoint, so must be ω. □

It is interesting to see whether the operator we obtain is related to the Dirac we started with. We propose:

Definition 5.7. We say that the connection ω is compatible with the Dirac operator D if both D_ω and D_h coincide on a dense subset of $\mathcal{H}$.

It is not difficult to see building on the results of [2] that in the classical case the Dirac operator, which gives the metric compatible with a given connection is indeed compatible with this connection in the sense of the above definition.

6 The noncommutative torus

To see how the definitions work in a noncommutative case, we study in detail the case of the 3-dimensional noncommutative torus as a $U(1)$ bundle over the 2-dimensional noncommutative torus.

We choose the generators of the $\mathbb{T}^3_\theta$ to be U_i, $i = 1, 2, 3$, with the relations $U_iU_j = e^{2\pi\theta_{ij}}U_jU_j$, where θ_{ij} is an antisymmetric real matrix. We assume that neither of its entries is rational. An element of the algebra of smooth functions on the noncommutative torus is of the form:

$$a = \sum_{k,l,m\in\mathbb{Z}} \alpha_{klm} U_1^k U_2^l U_3^m,$$

where $\alpha_{k,l,m}$ is a rapidly decreasing sequence. The canonical trace on the algebra is:

$$\tau(a) = \alpha_{000}.$$

We start with the canonical Hilbert space $\mathcal{H}_0$ of the GNS construction with respect to the trace on $\mathbb{T}^3_\theta$, and π the associated faithful representation. With $e_{k,l,m}$ the orthonormal basis of $\mathcal{H}_0$ we have:

$$\begin{aligned} U_1 e_{k,l,m} &= e_{(k+1),l,m}, \\ U_2 e_{k,l,m} &= e^{2\pi k\theta_{21}} e_{k,(l+1),m}, \\ U_3 e_{k,l,m} &= e^{2\pi(k\theta_{31}+l\theta_{32})} e_{k,l,(m+1)}, \end{aligned}$$

where k, l, m are in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on the choice of the spin structure. The projectable spin structures must have the trivial spin structure on the fibre, so $m \in \mathbb{Z}$, which we assume from now on.

We double the Hilbert space taking $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$, with the diagonal representation of the algebra. The canonical equivariant spectral triple over $\mathbb{T}^3_\theta$ is given by the Dirac operator D and the reality structure J of the form:

$$D = \sum_{j=1}^{3} i\sigma^j \delta_j, \qquad J = i\sigma^2 \circ J_0, \tag{6.1}$$

where σ^i are Pauli matrices [9]. J_0 here is the canonical Tomita-Takesaki antilinear map on the Hilbert space $\mathcal{H}_0$.

$$J_0 e_{k,l,m} = e_{-k,-l,-m},$$

and δ_i are the derivations acting diagonally on the basis:

$$\delta_1 e_{k,l,m} = k e_{k,l,m}, \quad \delta_2 e_{k,l,m} = l e_{k,l,m}, \quad \delta_3 e_{k,l,m} = m e_{k,l,m}.$$

Notice that $JD = DJ$ as required in the spectral triple of dimension 3.
We choose the following $U(1)$ action on the torus $\mathbb{T}^3_\theta$, defined through the action on the generators:

$$e^{i\phi} \triangleright U_1 = U_1, \;\; e^{i\phi} \triangleright U_2 = U_2, \;\; e^{i\phi} \triangleright U_3 = e^{i\phi} U_3, \tag{6.2}$$

and the induced diagonal action on the Hilbert space:

$$e^{i\phi} \triangleright e_{k,l,m} = e^{im\phi} e_{k,l,m},$$

The $U(1)$ invariant subalgebra is generated by U_1 and U_2, and can be identified with the 2-dimensional noncommutative torus $\mathbb{T}^2_\theta$, where the indices of θ_{ij} run over $1, 2$. It is straightforward to check that $\mathbb{T}^3_\theta$ is a Hopf-Galois extension of $\mathbb{T}^2_\theta$.
The chosen Dirac operator is one which is fully equivariant, that is invariant under three $U(1)$ symmetries. This is not necessary in our case, as we need only one $U(1)$ invariance, hence we shall allow for the fluctuations of D.

Remark 2. The space of possible perturbations of the Dirac operator of the gauge connection type (by one-forms) is given by:

$$D_A = D + \sigma^i A_i + J(\sigma^i A_i) J^{-1},$$

where $A_i \in \mathbb{T}^3_\theta$, $i = 1, 2, 3$, satisfy: $A_i = A_i^*$.

If we further require that the Dirac operator D_A is $U(1)$-invariant then we must restrict A_i to belong to the invariant subalgebra $\mathbb{T}^2_\theta$.
Before we proceed, observe that any one-form in $\Omega^1_{D_A}$ is (trivially) a one-form in Ω^1_D. This provides us with a more convenient description of the bimodule of one-forms.
We begin with:

Lemma 6.1. *The differential calculus generated by D_A satisfies the compatibility condition from the definition 5.1 if $A_3 = 0$.*

Proof. Take p_i, q_i such that $\sum_i p_i[D, q_i] = 0$. Since σ^i are linearly independent, we have for σ^3:

$$\sum_i p_i \left(\delta_3(q_i) + [A_3, q_i]\right).$$

If $A_3 = 0$ then the condition follows. □

We shall see that this is in agreement with the existence of Γ.

Lemma 6.2. *If $A_3 = 0$ then there exists a unique operator Γ, which satisfies the conditions of the definition 4.1, making the spectral geometry over $\mathbb{T}^3_\theta$ projectable with constant length fibres:*

$$\Gamma = \sigma^3.$$

Proof. It is easy to see that indeed $\Gamma = \sigma^3$ does satisfy all conditions. Conversely, any operator Γ, which commutes with the algebra, is $U(1)$-invariant, anticommutes with J and is a $\mathbb{Z}_2$ grading must be a linear combination of Pauli matrices. Then, the requirement 4.1 that D has equal length fibres fixes Γ to be σ^3. □

Lemma 6.3. *Note that in the case of the chosen general Dirac operator with $A_3 = 0$, $\Gamma = \sigma^3$ satisfies the following orthogonality condition:*

$$⨍ \Gamma\rho|D_A|^{-3} = 0, \quad \forall \rho \in \Omega^1_D(\mathbb{T}^2_\theta). \tag{6.3}$$

In fact, $\Gamma = \sigma^3$ is a unique (up to sign) $U(1)$-invariant $\mathbb{Z}_2$-grading of the Hilbert space, which commutes with the algebra and its commutant and satisfies orthogonality condition (6.3).

Proof. Take an arbitrary one form $\Gamma = \sum \sigma^i \rho_i$. From the orthogonality requirement (6.3), taking as $\eta = b\sigma^1$ and $b\sigma^2$ respectively, for $b \in \mathbb{T}^2_\theta$ we obtain:

$$⨍ \rho_1 b|D|^{-3} = ⨍ \rho_2 b|D|^{-3} = 0,$$

as in the noncommutative integral we can use D instead of D_A. As this holds for any b then $\rho_1 = \rho_2 = 0$. Therefore, the only nonvanishing coefficient is ρ_3 and since $\rho_3^2 = 1$ we recover $\Gamma = \sigma^3$ (up to the sign, of course). □

Furthermore, we have:

Lemma 6.4. *If $A_3 = 0$ and $\Gamma = \sigma^3$, then the projection of D onto $\mathbb{T}^2_\theta$ gives a real spectral triple over the two-dimensional torus and the differential calculi over $\mathbb{T}^2_\theta$ have the property (4.3).*

Proof. It is easy to recognize the general (perturbed) Dirac operators over the two-torus. To see that for every $b \in \mathbb{T}^2_\theta$ we have:

$$[D, b] = [D_h, b],$$

it is sufficient to see that $\delta(b) = 0$ and since $A_3 = 0$, only the horizontal part has non-trivial commutators. ☐

Note that, so far, there is a remarkable consistency in all the conditions that we are imposing on the Dirac operator and the spectral geometry of the noncommutative three-torus, viewed as a $U(1)$ bundle over the noncommutative two-torus. As a last note, observe:

Remark 3. The Dirac operator D_A, with $A_3 = 0$, satisfies the condition of equal length fibres proposed in the remark 1.

Indeed, using the results of the explicit calculations of the spectral action, note that the noncommutative integral of the perturbed Dirac operator $|D_A|^{-3}$ does not depend on A. Hence, we can work with D alone and its restriction to $\mathcal{H}_0$. Then, the proof is reduced to a simple exercise. In fact, an easy check suggests that we could have taken remark 1 as a defining condition for Γ (at least in that case).

6.1 Compatible strong connections over $\mathbb{T}^3_\theta$

Next, we turn to the space of possible connections ω. For simplicity, we consider the usual unperturbed Dirac operator (6.1) over $\mathbb{T}^3_\theta$.
Here is the full characterization of the possible connections, according to the definition (5.3):

Lemma 6.5. *A $U(1)$ connection over $\mathbb{T}^3_\theta$ with the $U(1)$ action defined as in (6.2) is a one-form:*

$$\omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1, \tag{6.4}$$

where $\omega^1, \omega^2 \in \mathbb{T}^2_\theta$ are $U(1)$ invariant elements of the algebra $\mathbb{T}^3_\theta$. Every such connection is strong.

Proof. Any $U(1)$-invariant one-form could be written as $\sum_i \sigma^i \omega_i$, where all ω_i are from $\mathbb{T}^2_\theta$. Since $\sigma^3 = U_3^{-1}[D, U_3]$, from the condition related to δ we obtain $\omega_3 = 1$. ☐

Finally, let us calculate the Dirac operators associated to an arbitrary antiselfadjoint connection ω on the noncommutative three-torus, $\omega^i = (\omega^i)^*$.

Lemma 6.6. *For any antiselfadjoint connection ω the associated Dirac operator D_ω has the form:*

$$D_\omega = D - (\sigma^2 J\omega_2 J^{-1} + \sigma^1 J\omega_1 J^{-1})\delta_3. \tag{6.5}$$

The proof follows straight from direct calculation based on the proof of proposition 5.6. We have an immediate corollary:

Corollary 6.7. *The only connection, compatible with the fully ($U(1)^3$) equivariant Dirac operator (6.1) on the noncommutative three-torus $\mathbb{T}^3_\theta$ is: $\omega = \sigma^3$.*

Finally, observe that the new family of Dirac operators gives a class of new, spectral geometries over the noncommutative torus. Although they are not real, as D_ω is not compatible with the real structure, we needed a *real spectral triple* as a a background geometry providing us with necessary tools (in particular, the differential calculus).
The properties of the new class of Dirac operators are yet to be investigated, however we expect that they will correspond to some locally non-flat geometries.

7 Conclusions

We have attempted to reconstruct the compatibility conditions between the metric geometry (as given by the Dirac operator) and connections (as defined in the algebraic setup for the Hopf-Galois extensions) on noncommutative $U(1)$ bundles. Although this project has still to be further developed, we have encountered several new and interesting phenomena. First of all, we observe that the existence of real spectral triples is necessary to provide a kind of background geometry. Furthermore, there are many compatibility conditions, which are necessary to impose on the spectral triple, apart from the simple requirement of $U(1)$-equivariance. We have demonstrated that in the case, where all assumptions are met it is possible to consistently extend the algebraic definition of strong connections to the case of differential calculi given by Dirac operators.
The requirement of compatibility condition between the connection and the metric (given implicitly by the Dirac operator) has led us to the discovery of a new family of Dirac operators. This indicates that these are not objects introduced *ad hoc* but have a deeper geometrical meaning. We postpone the study of their properties to future work.

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