Comparisons and asymptotics for empty space hazard functions of germ-grain models

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Abstract

We study stochastic properties of the empty space for stationary germ-grain models in $\mathbb{R}^d$, in particular we deal with the inner radius of the empty space with respect to a general structuring element which is allowed to be lower-dimensional. We consider Poisson cluster germ-grain models and Boolean models with grains that are clusters of convex bodies and show that more variable size of clusters results in stochastically greater empty space in terms of the empty space hazard function. We also study impact of clusters being more spread in the space on the value of the empty space hazard. Further we obtain asymptotic behavior of the empty space hazard functions at zero and at infinity.

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1 Introduction

The statistical analysis of a spatial pattern $Z \subset \mathbb{R}^d$ is based on the assumption that Z is a random set in $\mathbb{R}^d$. Distance methods for point patterns usually begin by estimating nonparametric summary functions such as the empty space function, nearest neighbor distance distribution, Ripley's K-function and derived statistics such as the pair correlation and the J-function. For a spatial pattern $Z \subset \mathbb{R}^d$, which is not necessarily a point process, such as germ-grain model a particularly useful functional for estimating properties of Z is the empty space hazard (rate) function. Smaller empty space hazard function in a germ-grain pattern with the same germ intensity intuitively corresponds to a more clustered pattern with more empty space. In this paper we shall give some sufficient conditions for the hazard rate ordering of the empty space distribution in two compared germ-grain

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models using particular classes of such spatial patterns such as Poisson cluster point processes, Poisson cluster germ-grain models, and mixed Poisson germ-grain models. Our sufficient conditions will be given in a language of stochastic orderings which we define for some parameters of these models. Some early results on stochastic ordering of random closed sets can be found in Stoyan and Stoyan [21], and a characterization of the strong stochastic ordering of random closed sets is given by Molchanov [15] in Theorem 4.42. However, apart from Section 3.8 of Hall [6] (dealing with volume fractions) and Last and Holtmann [12] (dealing with the spherical contact distribution of a Gauss-Poisson model) we are not aware of papers comparing functionals of stationary random closed sets. We introduce two stochastic ordering relations which are useful for comparison two distributions with equal expected values in which case both relations imply that a larger variable in these orderings has a bigger variance. It turns out that the impact of a larger variance in a spatial pattern for some component of a germ grain model with fixed expectation is that a larger variance gives more empty space with more clustering. But the situation is more complicated if we compare models with equal intensities of Poisson cluster germ points but at the same time increasing or decreasing intensities of underlying Poisson processes.

To be more precise, for two random variables η, $\tilde{\eta}$, taking values $0, 1, 2, \ldots$ we say that they are ordered in the length biased probability generating functions ordering, and write $\eta <_{l-g} \tilde{\eta}$ if the corresponding length biased variables η_l and $\tilde{\eta}_l$ are ordered in the probability generating ordering, see [19, Section 1.8]. For two nonnegative random variables Λ_1, Λ_2 we say that they are ordered in the first cumulant order and write $\Lambda <_{cum} \tilde{\Lambda}$ if for the corresponding cumulant generating functions C_Λ, $C_{\tilde{\Lambda}}$, taking the first derivatives we have $C'_\Lambda(s) \geq C'_{\tilde{\Lambda}}(s)$, $s \in [-1, 0]$. We show for Neyman-Scott processes with cluster sizes η, $\tilde{\eta}$ that $\eta <_{l-g} \tilde{\eta}$ implies that the corresponding empty space distributions are ordered in the hazard rate ordering. For mixed Poisson germ-grain models we show that if the random intensities are ordered $\Lambda <_{cum} \tilde{\Lambda}$ then the corresponding empty space distributions are ordered in the hazard rate ordering. For both introduced orderings, under the additional assumption that the ordered variables have equal expected values, the greater variable has larger variance. Therefore the mixed Poisson germ-grain model behaves similarly as the Neyman-Scott germ-grain model with respect to variance changes inside the model. We study also behavior of the Gauss-Poisson model. For details, additional properties of the orderings, and examples see Sections 4 and 5. We shall also study there the asymptotic behavior of empty space hazard rates at zero and at infinity.

2 Preliminaries

2.1 Empty space hazard functions

Let us recall the definition of the empty space distribution function F (called also the first contact distribution function). For $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^d$, and $d(x, A) := \inf\{\|x - y\| : y \in A\}$ the distance between $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$, this distribution is given by

$$F(t) = P(d(0, Z) \leq t), \quad t \geq 0,$$

where 0 denotes the zero vector. The value $F(0) = P(0 \in Z)$ is the *volume fraction* of Z. F can also be written in terms of the *capacity functional* T_Z (defined by $T_Z(K) := P(Z \cap K \neq \emptyset), \quad K \subset \mathbb{R}^d$ compact), namely $F(t) = P(Z \cap B(x,t) \neq \emptyset) = T_Z(B(x,t))$, $t \geq 0$, where $B(x,t)$ is the closed ball with center x and radius t. Stationarity of Z ensures that $F(t)$ does not depend on x. Hence, $F(t)$ is the probability that Z hits the ball $B(t) := B(0,t)$. There are many reasons for studying other distances than the Euclidean distance. For example digital image analysers estimate rather polygonal distance than spherical one. To estimate isotropy of point patterns one needs elliptical distance. It is clear that the distribution of Z is not determined by $T_Z(B(x,t))$ for all balls, and a larger class of sets than the class of balls provides a better information on Z.

The usual way of introducing other distances than the spherical one is to fix a *structuring element* (*gauge body*) $B \subset \mathbb{R}^d$. This is a compact convex set having $0 \in B$. Then the B-distance of a point $x \in \mathbb{R}^d$ to a set $K \subset \mathbb{R}^d$ is defined by

$$d_B(x,K) := \inf\{t \geq 0 : (x + tB) \cap K \neq \emptyset\}.$$

It is possible that the set on the right side is empty, e.g. if B is lower-dimensional. In such a case we set $d_B(x,K) := \infty$. Note that we have the translation invariance property $d_B(z + x, z + K) = d_B(x,K)$, for all $z \in \mathbb{R}^d$. Clearly, $d_B(x,K) \leq t$ if and only if x is contained in the *generalized outer parallel set* $K + tB^*$ of K, where B^* denotes the reflected set $\{-x : x \in B\}$. If B is full dimensional (i.e. has a non-empty interior) and centrally symmetric (i.e. $B^* = B$), then $d_B(\cdot,\cdot)$ is a metric on $\mathbb{R}^d$ induced by the norm $d_B(\cdot,0)$, and the pair $(\mathbb{R}^d, d_B)$ is called a *Minkowski space*.

If the B-distance $d_B(x,K)$ of a point $x \notin K$ is attained in a unique point y in the boundary ∂K of K (that means, if $(x + d_B(x,K)B) \cap K = \{y\}$), then we define the *relative metric projection* of x on K by $p_B(K,x) := y$, and the *contact direction vector* $u_B(K,x)$ as the element of ∂B^* given by

$$u_B(K,x) := \frac{x - y}{d_B(x,K)}.$$

The points $x \in \mathbb{R}^d \setminus K$ for which the distance $d_B(x,K)$ is attained in more than one point of K (and for which $u_B(x,K)$ is therefore not defined) form the *exoskeleton* $\mathrm{exo}_B(K)$ of K (see Hug, Last and Weil [10]). In the Euclidean case, and if K is a finite or locally finite set, $\mathrm{exo}_{B(1)}(K)$ is (the boundary of) the *Voronoi tessellation* generated by K.

We define the *directed, B-relative empty space function* F_B of Z by

$$F_B(t,C) := P(d_B(0,Z) \leq t, u_B(Z,0) \in C), \quad t \geq 0,\, C \in \mathcal{B}^d, \tag{2.1}$$

where $\mathcal{B}^d$ is the system of Borel subsets of $\mathbb{R}^d$. Here we use the convention $u_B(Z,0) := u_0$ if $0 \in Z$ or $d_B(0,Z) = \infty$, where $u_0 \in \partial B^*$ is fixed. Definition (2.1) is subject to the assumption that the vector $u_B(Z,0)$ is P-a.s. well-defined on $\{0 < d(0,Z) < \infty\}$. If Z is a random closed set and B is strictly convex, containing 0 in its interior, then this is indeed the case. This follows from the fact that $\mathrm{exo}_B(Z)$ has volume 0 and from stationarity of Z. (More general cases require a suitable assumption on the relative positions of Z and B, see Subsection 2.2). The function F_B determines the joint distribution of the pair $(d_B(0,Z), u_B(Z,0))$, and hence that of the *contact vector* $d_B(0,Z)u_B(Z,0)$. For each

fixed t, the function $F_B(t,\cdot)$ is a measure on $\mathbb{R}^d$ concentrated on ∂B^*. The function $F_B(\cdot) := F_B(\cdot,\mathbb{R}^d)$ is called *(B-relative) empty space function* of Z.

The *Minkowski addition* of two sets $C, D \subset \mathbb{R}^d$ ($C \oplus D := \{x+y : x \in C, y \in D\}$) gives another form for F_B. Stationarity easily implies that

$$F_B(t) := F_B(t,\mathbb{R}^d) = V_d(A)^{-1} E[V_d((Z \oplus tB^*) \cap A)], \quad t \geq 0, \tag{2.2}$$

for each Borel test set A, such that the *volume* $V_d(A)$ of A is positive and finite, see e.g. [11, 7].

Hansen, Baddeley and Gill [7], utilizing Federer's coarea theorem, showed that the empty space function F_B of a random closed set Z is absolutely continuous on $(0,\infty)$ with density

$$f_B(t) = V_d(A)^{-1} E\left[\int_{A \cap \partial(Z \oplus tB^*)} \|\nabla d_B(s,Z)\|^{-1} \mathcal{H}^{d-1}(ds)\right] \tag{2.3}$$

where $\mathcal{H}^i$, $i \in \{0,\ldots,d\}$, denotes i-dimensional Hausdorff measure on $\mathbb{R}^d$, ∂A denotes the boundary of A, and ∇d_B denotes the gradient of the function d_B. In the Euclidean case, this formula reduces to

$$f(t) = V_d(A)^{-1} E[\mathcal{H}^{d-1}(\partial(Z \oplus B(t)) \cap A)], \tag{2.4}$$

In this case the empty space hazard equals the ratio of the expected measure of the boundary $\partial(Z \oplus B(t))$ inside the window A to the volume of of the space not occupied by $Z \oplus B(t)$ inside A.

For general B the empty space hazard is given by

$$r_B(t) = \frac{1}{V_d(A) - E[V_d(Z \oplus tB^*) \cap A)]} E\left[\int_{A \cap \partial(Z \oplus tB^*)} \|\nabla d_B(s,Z)\|^{-1} \mathcal{H}^{d-1}(ds)\right], \tag{2.5}$$

which is intuitive in the sense that the empty space hazard depends on the speed of increase of the distance function d_B along all coordinates. It is possible to rewrite this formula in terms of the support function h_B of the gauge body B (see [7]).

$$r_B(t) = \frac{1}{V_d(A) - E[V_d(Z \oplus tB^*) \cap A)]} E\left[\int_{A \cap \partial(Z \oplus tB^*)} (h_B(u_B(Z,s)))^{-1} \mathcal{H}^{d-1}(ds)\right]. \tag{2.6}$$

The direction dependent (sub)distribution functions, $F_B(\cdot,C)$, as defined in (2.1) are also absolutely continuous on $(0,\infty)$ for any $C \in \mathcal{B}^d$. Letting $f_B(\cdot,C)$ denote its density, we define

$$r_B(t,C) := \frac{f_B(t,C)}{1 - F_B(t)}, \tag{2.7}$$

where $a/0 := 0$ for all $a \in \mathbb{R}$. We call the function $r_B(\cdot,C)$ the directed, *B-relative empty space hazard of Z*.

For fixed B, we shall order two random sets Z, and $\tilde{Z}$ with respect to their B-relative empty space hazard functions, and write

$$Z <_{h-B} \tilde{Z} \tag{2.8}$$

iff for all $t \geq 0$, and $C \in \mathcal{B}^d$

$$r_B(t,C) \geq \tilde{r}_B(t,C), \quad t \geq 0. \tag{2.9}$$

This ordering is stronger than the usual (strong) stochastic ordering of empty space distributions, for further details on such orderings see e.g. Szekli [22, Section 1.4].

2.2 Empty space hazard rates via support measures

By a *germ-grain model* in $\mathbb{R}^d$ we mean a random set of the form

$$Z = \bigcup_{n=1}^{\infty}(X_n + \xi_n) = \bigcup_{n=1}^{\infty}\{x + \xi_n : x \in X_n\},$$

where the random points ξ_n, $n \in \mathbb{N}$, represent the locations of the *germs* and the *primary grains* X_n, $n \in \mathbb{N}$, are assumed to be random non-empty compact subsets of $\mathbb{R}^d$. We assume that the (simple) *point process* $N := \{\xi_n : n \in \mathbb{N}\}$ is *stationary*, that is the distribution of the *shifted* point process $N + x := \{\xi_n + x : n \in \mathbb{N}\}$ does not depend on $x \in \mathbb{R}^d$, and that N is independent of $(X_n)_{n\geq 1}$ which is a sequence of independent and identically distributed random sets. The *intensity* $\lambda := E\,\mathrm{card}\{n \in \mathbb{N} : \xi_n \in [0,1]^d\}$ of N is assumed to be finite. An important special case is the *Boolean model*, where the germs are located according to a homogenous *Poisson process*. The grains X_n as well as the germ-grain model Z itself are measurable mappings from Ω into the set $\mathbf{F}$ of all closed subsets of $\mathbb{R}^d$. Measurability refers to the smallest σ-field of subsets of $\mathbf{F}$, containing the sets $\mathbf{F}_K = \{F : F \cap K \neq \emptyset\}$, for all compact sets $K \subset \mathbb{R}^d$. Stationarity of N entails that also Z is stationary, i.e. that the distribution of $Z + x$ does not depend on x. It is convenient to denote by X_0 a *typical grain* having its distribution equal to that of X_n. We assume that $E[V_d(X_0 + K)]$ is finite for all compact $K \subset \mathbb{R}^d$. We will use later that the capacity functional of a Boolean model is given by

$$P(Z \cap K \neq \emptyset) = 1 - \exp[-\lambda E[V_d(X_0 + K^*)]]. \tag{2.10}$$

In particular, the volume fraction of a Boolean model is given by

$$P(0 \in Z) = 1 - \exp[-\lambda E[V_d(X_0)]]. \tag{2.11}$$

We refer to Stoyan, Kendall and Mecke [20] and Schneider and Weil [18] for a detailed introduction to germ-grain models.

Consider a convex, compact and non-empty set $K \subset \mathbb{R}^d$. We assume that K and B^* are in *general relative position*, which means that K and B^* do not contain parallel segments in parallel and equally oriented support (hyper)planes. This means that K and B have *independent support* sets, see [18, p. 611] for more detail. A sufficient condition is that K or B is strictly convex, This assumption guarantees that $p_B(K,x)$ (and hence $u_B(K,x)$) is defined for all $x \notin K$. Then there are finite measures $C_0(K;B;\cdot), \ldots, C_{d-1}(K;B;\cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which satisfy the *local Steiner formula*

$$V_d(\{x \in \mathbb{R}^d : 0 < d_B(x,K) \leq t, (p_B(K,x), u_B(K,x)) \in A \times C\}) = \sum_{i=0}^{d-1} t^{d-i} b_{d-i} C_i(K;B;A \times C) \tag{2.12}$$

for all $A, C \in \mathcal{B}^d$, where b_i ($i \in \mathbb{N}$) denotes the volume of the unit ball in $\mathbb{R}^i$, and $b_0 := 1$. These *relative support measures* of K are uniquely determined by (2.12). They are concentrated on $\partial K \times \partial B^*$ and in fact on the *relative normal bundle*

$$N_B(K) := \{(p_B(K,x), u_B(K,x)) : x \notin K\}$$

of K. If $B = B(1)$ then the measures $C_i(K;\cdot) := C_i(K;B(1);\cdot)$ are the *generalized curvature measures* of K. The total mass $V_i(K) := C_i(K;\mathbb{R}^d \times \mathbb{R}^d)$ is the ith *intrinsic volume* of K. In particular, $V_d(K)$ is the volume of K, $V_{d-1}(K)$ equals one half of the surface area, $V_{d-2}(K)$ is proportional to the integral mean curvature, $V_1(K)$ is proportional to the mean width of K, and $V_0(K) = 1$. Equation (2.12) implies the classical *Steiner formula*

$$V_d(K \oplus B(t)) = \sum_{i=0}^{d} b_{d-i} t^{d-i} V_i(K). \tag{2.13}$$

In the general case, the total mass $C_i(K;B;\mathbb{R}^d \times \mathbb{R}^d)$ is a special *mixed volume*, namely

$$C_i(K;B;\mathbb{R}^d \times \mathbb{R}^d) = b_{d-i}^{-1} \binom{d}{i} V(K[i], B^*[d-i]), \quad i = 0, \ldots, d-1. \tag{2.14}$$

For $i = 0$ we have

$$C_0(K;B;\mathbb{R}^d \times \mathbb{R}^d) = b_d^{-1} V_d(B^*), \tag{2.15}$$

see [18] for the notation used here and for further details on support and curvature measures.

Consider now a germ-grain model Z with convex, compact grains. Assume that the *reduced second moment measure* of N on $\mathbb{R}^d$, defined by

$$E\left[\sum_{\substack{x,y \in N \\ x \neq y}} \mathbf{1}\{x \in [0,1]^d\}\{x - y \in \cdot\} \right]. \tag{2.16}$$

is absolutely continuous, and assume that the typical grain X_0 and B^* are a.s. in general relative position. It then follows, that $u_B(Z,0)$ is almost surely well-defined on $\{0 < d_B(0,Z) < \infty\}$. This can be proved as Proposition 4.9 in [9].

By the Steiner formula (2.13) our general integrability assumption on X_0 (see Subsection 2.2) is equivalent to the finiteness of the mean intrinsic volumes

$$\bar{V}_i := EV_i(X_0), \quad i = 0, \ldots, d, \tag{2.17}$$

of the typical grain. The Steiner formula (2.13) together with the local Steiner formula (2.12) imply that

$$\bar{V}_{i,B} := EC_i(X_0;B;\mathbb{R}^d \times \mathbb{R}^d), \quad i = 0, \ldots, d-1, \tag{2.18}$$

are finite as well. Therefore the mean relative support measures of the typical grain, defined by

$$\bar{C}_{i,B}(\cdot) := EC_i(X_0;B;\cdot), \quad i = 0, \ldots, d-1,$$

are finite measures on $\mathbb{R}^d \times \mathbb{R}^d$.

We further use the *Palm probability* P_N^0 of P with respect to N (see [4, 20]). We can interpret $P_N^0(A)$ as the conditional probability of the event $A \in \mathcal{F}$ given that 0 is a "randomly chosen point" of N. Let us define $X(x) := X_n$ if $x = \xi_n$ for some n, and

$X(x) := \emptyset$, otherwise. Then under P_N^0, $\{(x, X(x)) : x \in N\}$ is an independently marked point process, i.e. conditionally on N, the grains $\{X(x) : x \in N\}$ are independent and have the same distribution as X_0. For $\bar{V}_{i,B} > 0$, $t \geq 0$, and $C \in \mathcal{B}^d$, let

$$G_{i,B}(t, C) := \bar{V}_{i,B}^{-1} \int \mathbf{1}\{u \in C\} P_N^0(d_B(y + tu, Z^!) \leq t) \bar{C}_{i,B}(d(y, u)), \tag{2.19}$$

where

$$Z^! := \bigcup_{x \in N \setminus \{0\}} (X(x) + x) \tag{2.20}$$

is the union of all grains except for the grain located at the origin. We set $G_{i,B} \equiv 0$ for $\bar{V}_{i,B} = 0$. The function $G_{i,B}(\cdot, \mathbb{R}^d)$ can be interpreted as the distribution function of a random variable ξ, say, which can be constructed as follows. First one selects a point Y of N at random. Then one samples a random element (X, W) according to the distribution $\bar{V}_{i,B}^{-1} \bar{C}_{i,B}$. If Y is not covered by $\bigcup_{x \in N \setminus \{Y\}} (X(x) + x)$, then ξ is the B-distance from $Y + X$ to the exoskeleton $\mathrm{exo}_B(Z)$, in the direction W. Otherwise $\xi = 0$. We shall utilize the following functions

$$J_{i,B}(t, C) := \frac{G_{i,B}(\infty, C) - G_{i,B}(t, C)}{1 - F_B(t)}, \quad i = 0, \ldots, d-1, \tag{2.21}$$

where $i \in \{0, \ldots, d-1\}$, $t \geq 0$, $C \subset \mathbb{R}^d \times \mathbb{R}^d$ is a Borel set, and

$$G_{i,B}(\infty, C) := \lim_{t \to \infty} G_{i,B}(t, C) = \bar{V}_{i,B}^{-1} \bar{C}_{i,B}(\mathbb{R}^d \times C).$$

Special cases of these functions were introduced in [12, Section 5] after the point process case had been treated in [14]. The functions $J_{i,B}(t, C)$ can be used as non-parametric measures for expressing differences between a general germ-grain model and Boolean model with the same values of $\lambda \bar{V}_{i,B}$. Intuitively speaking, such measures detect interactions and clustering effects. In the Euclidean case (and for $C = \mathbb{R}^d$) the following theorem was proved in [13, 12]. The following relative version is implicit in Hug and Last [9], at least in the case of a strictly convex B. The general result can be derived from Theorem 5.1 in Hug, Last and Weil [10].

Theorem 2.1. *Consider a stationary germ-grain model satisfying the assumptions formulated above. Then F_B is absolutely continuous and the B-relative empty space hazard r_B is given by*

$$r_B(t, C) = \sum_{i=0}^{d-1} (d - i) t^{d-i-1} b_{d-i} \lambda \bar{V}_{i,B} J_{i,B}(t, C). \tag{2.22}$$

If N is a Poisson process (i.e. Z is the Boolean model with convex grains), then Slivnyak's theorem (see e.g. Stoyan, Kendall and Mecke [20]) implies that

$$P_N^0(d_B(y + tu, Z^!) > t) = 1 - F_B(t). \tag{2.23}$$

Hence (2.22) simplifies to

$$r_B(t, C) = \sum_{i=0}^{d-1} (d - i) b_{d-i} t^{d-i-1} \lambda \bar{S}_{i,B}(C), \tag{2.24}$$

where $\bar{S}_{i,B}(C) := \bar{C}_{i,B}(\mathbb{R}^d \times C)$. In the case of a strictly convex gauge body B this result can be found in [9]. Note that in the Boolean model $r_B(\cdot, \mathbb{R}^d)$ is determined by the intensity λ and the mean mixed volumes $\bar{V}_{1,B}, \ldots, \bar{V}_{d-1,B}$ of X_0. For $d = 2$ and $B = B(1)$, for instance, the only parameter of X_0 influencing the empty space hazard rate is its mean boundary length $\bar{V}_1$. Note that in the Boolean model with convex grains the empty space hazard rate is increasing, and asymptotically

$$\lim_{t\to\infty} t^{1-d} r_B(t, C) = \lambda d b_d \bar{S}_{0,B}(C) = \lambda d b_d E[C_0(X_0; B; \mathbb{R}^d \times C)].$$

By (2.15) we obtain in particular that

$$\lim_{t\to\infty} t^{1-d} r_B(t, \mathbb{R}^d) = \lambda d b_d E[b_d^{-1} V_d(B^*)] = \lambda d V_d(B^*).$$

3 Results for Poisson cluster point processes

In this section we shall consider germ-grain models where the germ process will be a Poisson cluster point process, and grains will be one point grains attached to germs, hence $Z = N$. Such a model is a special case of a germ-grain model with convex (one point) grains but, alternatively, this model might be seen as a Boolean model (germs form a Poisson process) with non-convex grains (point clusters). Recall from [4] that a Poisson cluster point process can be written as

$$N = \bigcup_{x\in\Pi} L_x + x, \tag{3.1}$$

where Π is a Poisson process with positive and finite intensity λ_Π and the family $\{L_x : x \in \Pi\}$ consists of finite random point processes on $\mathbb{R}^d$. Given Π, the family $\{L_x : x \in \Pi\}$ is i.i.d. with the same distribution as a *typical cluster* L_0. We assume that

$$\gamma := E \operatorname{card} L_0$$

is finite and positive, hence $\lambda = \lambda_\Pi \gamma$.

Example 3.1. Assume that

$$L_0 = \begin{cases} \emptyset, & \text{if } \eta = 0, \\ \{Y_{i,n} : i = 1, \ldots, n\}, & \text{if } \eta = n \geq 1, \end{cases} \tag{3.2}$$

where the random cluster size $\eta \geq 0$, and the random vectors $Y_{i,n}$, $n \in \mathbb{N}$, $i = 1, \ldots, n$, are independent. Assume also that the $Y_{i,n}$ have the same (cluster point) distribution V, say. Then N is called *Neyman-Scott process*. We always assume that $\gamma = E[\eta]$ is positive and finite.

Example 3.2. Let η be a $\{1, 2\}$-valued random variable and assume that

$$L_0 = \begin{cases} \{0\}, & \text{if } \eta = 1, \\ \{0, Y\}, & \text{if } \eta = 2, \end{cases} \tag{3.3}$$

where Y is a random vector independent of η. Then N is called *Gauss-Poisson process*. In this case a cluster $L_0 + x$ associated with a parent point x say, contains x and, with probability $p := P(\eta = 2)$, also a secondary point x. Note that the mean cluster size is given by $\gamma = 1 + p$.

In this paper we always assume that the *reduced second moment measure*

$$E\left[\sum_{\substack{x,y\in L_0\\ x\neq y}} \mathbf{1}\{x-y\in\cdot\}\right] \tag{3.4}$$

of L_0 is absolutely continuous on $\mathbb{R}^d$. The well-known second order properties of Poisson cluster point processes (see e.g. [4]) easily imply that the measure defined at (2.16) is absolutely continuous as well. For the Neyman-Scott process of Example 3.1 our assumption on L_0 is implied by the absolute continuity of the cluster point distribution V. For a Gauss-Poisson process it is sufficient to assume that the seondary point Y has an absolutely continuous distribution.

In this section we fix a gauge body B that contains a non-empty neighborhood of 0. This way we exclude the trivial case $F_B(t)=0$, $t\geq 0$. Let ν_B be the measure on $\mathbb{R}^d$ given by

$$\nu_B(\cdot) := d\int_{B^*} \mathbf{1}\{x/d_B(0,x)\in\cdot\}dx, \tag{3.5}$$

where $d_B(0,x) := d_B(0,\{x\})$. Using this measure we can express the empty space hazard of N as follows.

Proposition 3.3. *The B-relative empty space hazard of the Poisson cluster point process $Z=N$ is given by*

$$r_B(t,C) = \lambda_\Pi t^{d-1}\int_C E\left[\int_{\mathbb{R}^d} \mathbf{1}\{((L_0-x)\setminus\{0\})\cap(tu+tB)=\emptyset\}L_0(dx)\right]\nu_B(du). \tag{3.6}$$

Proof. Computing the left-hand side of (2.12) for $K=\{0\}$ and $A=\{0\}\times C$ easily shows that

$$db_dC_0(\{0\};B;\{0\}\times\cdot) = \nu_B(\cdot)$$

and that $C_i(\{0\};B;\cdot)=0$ for $i\geq 1$. The result is then a consequence of Proposition 4.1 below. □

If $L_0=\{0\}$ we have $N=\Pi$ and

$$r_B(t,C) = \lambda t^{d-1}\nu_B(C), \tag{3.7}$$

in accordance with (2.24).

If $B=B(1)$ then ν_B is the $(d-1)$-dimensional Hausdorff measure on the unit sphere $S^{d-1} := \partial B(1)$ and

$$r_{B(1)}(t,C) = \lambda_\Pi t^{d-1}\int_C E\left[\int_{\mathbb{R}^d} \mathbf{1}\{((L_0-x)\setminus\{0\})\cap B(tu,t)=\emptyset\}L_0(dx)\right]\mathcal{H}^{d-1}(du). \tag{3.8}$$

Example 3.4. Assume that N is a Neyman-Scott process as defined in Example 3.1. From (3.6) and a straightforward calculation,

$$r_B(t,C) = \lambda_\Pi t^{d-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g'(P(Y_{1,1}-x\notin tu+tB))V(dx)\nu_B(du), \tag{3.9}$$

where g' is the derivative of the probability generating function g of η. This result generalizes formula (30) in [11].

Example 3.5. Assume that N is a Gauss-Poisson process as defined in Example 3.5. Then

$$(\lambda_\Pi)^{-1}t^{1-d}r_B(t,C) =(1-p)\nu_B(C) + p\int_C P(Y \notin tu+tB)\nu_B(du) + p\int_C P(-Y \notin tu+tB)\nu_B(du). \tag{3.10}$$

Let N be a general Poisson cluster point process as above. It is interesting to note that $t^{1-d}r_B(t,C)$ is monotone decreasing in t. This is a direct consequence of (3.6). Then it is instructive to compare r_B with the right side of (3.7). From monotone convergence

$$\lim_{t\to 0} t^{1-d}r_B(t,C) = \lambda d\nu_B(C), \tag{3.11}$$

i.e. for small values of t the empty space hazard of a Poisson cluster point process behaves approximately like the empty space hazard of a Poisson process with the same intensity.

Next we deal with the asymptotics of the empty space hazard for large t. It turns out that it is the same as that of the Poisson process Π thinned at points x where L_x is empty, for which intensity equals $P(L_0 \neq \emptyset)\lambda_\Pi$. This means in a sense, that points in clusters cannot be distinguished from a very far distance, irrespective of any specific assumptions on the typical cluster L_0. This is generalizing equation (22) in [11]. A weaker version of this latter result has been rediscovered by Bordenave and Torrisi [3].

Proposition 3.6. *The B-relative empty space hazard of a Poisson cluster point process satisfies*

$$\lim_{t\to\infty} t^{1-d}r_B(t,C) = P(L_0 \neq \emptyset)\lambda_\Pi\nu_B(C). \tag{3.12}$$

Proof. The *tangential cone* (or *support cone*) $T(B,u)$ of B at $u \in B$ is the closure of $T'(B,u) := \{t(x-u) : t > 0, x \in B\}$, see Schneider [17]. From (3.6) and monotone convergence

$$\lim_{t\to\infty} t^{1-d}r_B(t,C) = \lambda_\Pi \int_C E\left[\int \mathbf{1}\{((L_0-x)\setminus\{0\})\cap T'(B,-u)=\emptyset\}L_0(dx)\right]\nu_B(du). \tag{3.13}$$

Since the measure (3.4) is absolutely continuous it follows that Y_0 a.s. does not intersect the boundary of $T(B,-u)$. Hence

$$\lim_{t\to\infty} t^{1-d}r_B(t,C) = \lambda_\Pi \int_C E\left[\int \mathbf{1}\{((L_0-x)\setminus\{0\})\cap T(B,-u)=\emptyset\}L_0(dx)\right]\nu_B(du). \tag{3.14}$$

Let us now fix for a moment a *regular* boundary point u of B^*. This means that B has a unique supporting hyperplane at $-u$, see Schneider [17]. Then $T(B,-u)$ is the supporting half-space of B at $-u$, see Section 2.2 in Schneider [17]. It now follows as in [11, Section 6.5] that

$$E\left[\int \mathbf{1}\{((L_0-x)\setminus\{0\})\cap T(B,-u)=\emptyset\}L_0(dx)\right] = P(L_0 \neq \emptyset).$$

It remains to note that ν_B-a.a. $u \in \mathbb{R}^d$ are regular boundary points of B^*. This follows from the fact that ν_B is absolutely continuous with respect to $(d-1)$-dimensional Hausdorff measure on ∂B^* and Theorem 2.2.4 in Schneider [17]. □

We shall now consider two Poisson cluster processes N and $\tilde{N}$ having the same intensity, satisfying the technical assumption formulated at (3.4) and with B-relative hazard rates r_B and $\tilde{r}_B$, respectively. We will establish several sets of assumptions implying the hazard rate ordering (2.8). In all cases $\tilde{N}$ will have more clustered patterns in a sense. Therefore one may expect more empty space in the Poisson cluster process N. Still it is somewhat surprising that this happens in the strong sense of (3.17). In Section 4 we will state the corresponding results for Poisson cluster grain models. Our first ordering result, dealing with Neyman Scott processes, is a special case of Proposition 4.9 which will be proved later.

Let η and $\tilde{\eta}$ be two counting variables (i.e. taking values in $\{0, 1, 2, \ldots\}$) and let η_l be the (shifted) *length-biased* version of η. This means that η_l has distribution $E[\eta]^{-1}E[\eta \mathbf{1}\{\eta - 1 \in \cdot\}]$. Denoting by $\tilde{\eta}_l$ the length-biased version of $\tilde{\eta}$, we define the *length biased probability generating functions ordering*

$$\eta <_{l-g} \tilde{\eta} \tag{3.15}$$

by

$$E[s^{\eta_l}] \geq E[s^{\tilde{\eta}_l}], \quad s \in [0, 1].$$

This means that η_l is smaller than $\tilde{\eta}_l$ in the the generating function order (see Stoyan [19], Section 1.8.) Note that $\eta <_{l-g} \tilde{\eta}$ is equivalent to

$$E[\eta]^{-1}E[\mathbf{1}\{\eta \geq 1\}\eta a^{\eta-1}] \geq E[\tilde{\eta}]^{-1}E[\mathbf{1}\{\tilde{\eta} \geq 1\}\tilde{\eta} a^{\tilde{\eta}-1}], \quad a \in [0, 1].$$

Another way of expressing this relation is by the generating functions of η and $\tilde{\eta}$. Denote by

$$g_\eta(s) := E[s^\eta], \quad s \in [0, 1],$$

the probability generating function of η. Then (3.15) means that

$$E[\eta]^{-1} g'_\eta(s) \geq E[\tilde{\eta}]^{-1} g'_{\tilde{\eta}}(s), \quad s \in [0, 1]. \tag{3.16}$$

The relation (3.15) does not imply that the corresponding expected values $E[\eta]$,$E[\tilde{\eta}]$ are ordered. For example if $\eta \equiv 1$, $\eta_1 \equiv 2$, and η_2 equals 0 with probability 1/2 and 1 with probability 1/2 then $\eta <_{l-g} \eta_1$, and $\eta <_{l-g} \eta_2$, but $E[\eta] = 1 < E[\eta_1] = 2$, and $E[\eta] = 1 > E[\eta_2] = 1/2$. However, if $E[\eta] = E[\tilde{\eta}]$ then (3.15) implies that $\mathbb{V}\mathrm{ar}[\eta] \leq \mathbb{V}\mathrm{ar}[\tilde{\eta}]$, therefore this relation is a variability ordering.

Proposition 3.7. *Assume that N and $\tilde{N}$ are Neyman-Scott processes with cluster sizes η and $\tilde{\eta}$, respectively, having the same distribution V of cluster points. If N and $\tilde{N}$ have the same intensity and*

$$\eta <_{l-g} \tilde{\eta}$$

then

$$Z <_{h-B} \tilde{Z} \tag{3.17}$$

.

Example 3.8. Let N and $\tilde{N}$ be as in Proposition 3.7 and assume that $\eta \equiv 1$. That is N is again a Poisson process with intensity 1 since the points of the original Poisson process Π are independently shifted. Let $a \in [0,1]$. The length biased variable $\eta_l \equiv 0$, and its generating function $E[s^{\eta_l}] \equiv 1$, therefore for each $\tilde{\eta}$ we have $1 \equiv \eta <_{l-g} \tilde{\eta}$, and $Z <_{h-B} \tilde{Z}$. This example shows that within the class of Neyman-Scott processes with fixed intensity the stochastically smallest empty space appears for pure Poisson germ processes.

Example 3.9. Let N and $\tilde{N}$ be as in Proposition 3.7 and assume that η and $\tilde{\eta}$ are Poisson distributed with parameter c and $\tilde{c}$, respectively. (Then the clusters are finite Poisson processes.) We have for any $a \in [0,1]$ that

$$E[\eta]^{-1} E[\mathbf{1}\{\eta \geq 1\}\eta a^{\eta-1}] = e^{-c(1-a)}.$$

Therefore (3.15) (and hence $Z <_{h-B} \tilde{Z}$) holds iff $c \leq \tilde{c}$. Note that for Poisson distributed η the length-biased η_l have the same (Poisson) distribution. Proportionally more Poisson cluster points in a Neyman-Scott process indeed generate more clustering and lead to a stochastically larger empty space.

Example 3.10. Assume that η and $\tilde{\eta}$ are binomial distributed with parameter (n,p) and $(\tilde{n},\tilde{p})$, respectively. Since $g_\eta(s) = ((1-p)+ps)^n$, it follows that (3.15) is equivalent to

$$((1-p)+ps)^{n-1} \geq ((1-\tilde{p})+\tilde{p}s)^{\tilde{n}-1}, \quad s \in [0,1]. \tag{3.18}$$

If, for instance $n = \tilde{n}$, then this inequality is implied by $p \leq \tilde{p}$.

Example 3.11. Assume that η and $\tilde{\eta}$ are negative binomial distributed with parameter (p,r) and $(\tilde{p},\tilde{r})$, respectively. The corresponding length-biased variables η_l , $\tilde{\eta}_l$ are again negative binomial distributed with parameters $(p,r+1)$ and $(\tilde{p},\tilde{r}+1)$, respectively. For $p = \tilde{p}$, if $r \leq \tilde{r}$ then $\eta <_{l-g} \tilde{\eta}$, and for $r = \tilde{r}$, if $p \geq \tilde{p}$ then $\eta <_{l-g} \tilde{\eta}$. This is a special case of a more general setting. If $\eta = \sum_{i=1}^{\kappa} \vartheta_i$, and $\tilde{\eta} = \sum_{i=1}^{\tilde{\kappa}} \tilde{\vartheta}_i$, for iid variables $\{\vartheta_i\}_{i\geq 1}$, and independent of κ (all variables taking on natural values) then $\vartheta_i <_{l-g} \tilde{\vartheta}_i$, and $\kappa <_{l-g} \tilde{\kappa}$ implies $\eta <_{l-g} \tilde{\eta}$.

The next result is a special case of Proposition 4.10.

Proposition 3.12. *Assume that N and $\tilde{N}$ are Gauss-Poisson processes having the same distribution V of the secondary point. Assume that N and $\tilde{N}$ have the same intensity and that the probabilities p and $\tilde{p}$ for having a secondary point satisfy $p \leq \tilde{p}$. Then $Z <_{h-B} \tilde{Z}$.*

In our next result we will multiply each point of the typical cluster L_0 of N with a random variable $W \leq 1$, i.e. we assume that the typical cluster $\tilde{L}_0$ of $\tilde{N}$ has the distribution of $WL_0 = \{Wx : x \in L_0\}$. Compared with $\tilde{L}_0$, the points of L_0 are more spread out. Note that we allow any sort of dependence between L_0 and D.

Proposition 3.13. *Consider two Poisson cluster processes N and $\tilde{N}$ based on the same Poisson process Π and typical clusters L_0 and $\tilde{L}_0$, respectively. Assume that $\tilde{L}_0$ is distributed as WL_0 for some random variable $W \leq 1$. Then $Z <_{h-B} \tilde{Z}$.*

Proof. Take $x_1, \dots, x_n \in \mathbb{R}^d$ and set $\psi := \{x_j : j = 1, \dots, n\}$. Let $w \le 1$ and define $\tilde{\psi} := \{wx_j : j = 1, \dots, n\}$. Let $u \in \partial B^*$ and $t \ge 0$. In view of (3.6) it is sufficient to show that

$$\mathbf{1}\{((\tilde{\psi} - wx_i) \setminus \{0\}) \cap (tu + tB) = \emptyset\} \le \mathbf{1}\{((\psi - x_i) \setminus \{0\}) \cap (tu + tB) = \emptyset\} \qquad (3.19)$$

holds for any $i \in \{1, \dots, n\}$. Assume that the left-hand side of (3.19) equals 1. This is equivalent to $w(x_j - x_i) \notin tu + tB$ for all $j \ne i$. Since $u \in \partial B^*$ we have that $0 \in \partial(tu + tB)$. Therefore the convexity of $tu + tB$ and $w \le 1$ imply that also $x_j - x_i \notin tu + tB$ for all $j \ne i$. This shows (3.19) and hence the proposition. □

Recall that a random variable $\tilde{W}$ is *stochastically smaller* than another r.v. W, if $P(\tilde{W} > t) \le P(W > t)$ for all $t \ge 0$.

Proposition 3.14. *Let N and $\tilde{N}$ be as in Proposition 3.13. Let L be a point process and assume that L_0 is distributed as WL for a random variable $W > 0$ independent of L. Assume that $\tilde{L}_0$ is distributed as $\tilde{W}L$ for a random variable $\tilde{W} \ge 0$ independent of L. If $\tilde{W}$ is stochastically smaller than W then $Z <_{h-B} \tilde{Z}$.*

Proof. By inverse coupling based on an uniformly distributed random variable that it independent of L we can assume that $(W, \tilde{W})$ is independent of L and that $\tilde{W} \le W$ everywhere on the underlying probability space. Since $\tilde{L}_0 = \tilde{W}/W L_0$, we can apply Proposition 3.13 with W replaced by $\tilde{W}/W \le 1$. □

4 Results for Poisson cluster germ-grain models

We now return to a general case of germ-grain models where N is a Poisson cluster point processes as in the previous section, and grains X_n, located at all points of N, are compact and convex. In the next two sections we fix a structuring element B such that our general assumption made after (2.16) holds, that is X_1 and B^* are a.s. in general relative position. Alternatively, we shall treat such a process as a Boolean model with non-convex grains which are

$$\bigcup_{y \in L_x} (X(y) + y), \quad x \in \Pi.$$

For a given finite point pattern $\psi = \{x_n : n = 1, \dots, m\}$, denote by $\Gamma(\psi, \cdot)$ the distribution of the random closed set $\cup_{n=1}^m (X_n + x_n)$, where $X_1, \dots, X_m$ are independent with distribution of X_0. We also set $\Gamma(\emptyset, \cdot) := \delta_\emptyset$. Let Y_0 be a random closed set with distribution

$$\mu(\cdot) := \frac{1}{\gamma} E\left[\sum_{y \in L_0} \int \mathbf{1}\{A \in \cdot\} \Gamma((L_0 - y) \setminus \{0\}, dA)\right]. \qquad (4.1)$$

This probability measure is describing the distribution of the germ-grain model associated with a typical cluster as seen from a randomly chosen cluster point, after removing the grain around the chosen point. We assume that Y_0 and the typical grain X_0 are independent and define for $i \in \{0, \dots, d-1\}$, $t \ge 0$, and a Borel set $C \subset \mathbb{R}^d$

$$K_{i,B}(t, C) := \frac{1}{\overline{V}_{i,B}} E\left[\int \mathbf{1}\{d_B(x + tu, Y_0) > t\} \mathbf{1}\{u \in C\} C_i(X_0; B; d(x, u))\right]. \qquad (4.2)$$

The following proposition yields again an explicit formula for the empty space hazard of a Poisson cluster germ-grain model, this time in terms of X_0 and Y_0 describing locally a cluster.

Proposition 4.1. *The B-relative empty space hazard of a Poisson cluster germ-grain model with compact, convex grains is given by*

$$r_B(t,C) = \sum_{i=0}^{d-1} (d-i) t^{d-i-1} b_{d-i} \lambda \bar{V}_{i,B} K_{i,B}(t,C). \tag{4.3}$$

Proof. Our aim is to use (2.22). To do so we recall that the Palm probability measure P_N^0 of a Poisson cluster process N satisfies

$$P_N^0(N \in \cdot) = E\left[\int \mathbf{1}\{N \cup \psi \in \cdot\} Q_{L_0}^0(d\psi)\right], \tag{4.4}$$

where

$$Q_{L_0}^0(\cdot) := \gamma^{-1} E\left[\sum_{y \in L_0} \mathbf{1}\{L_0 - y \in \cdot\}\right]$$

is the *Palm distribution* of the typical cluster L_0, see e.g. Stoyan, Kendall and Mecke [20]. As in Last and Holtmann [12], this implies that

$$P_N^0(d_B(x+tu, Z^!) > t) = (1 - F_B(t)) \iint \mathbf{1}\{d_B(x+tu, A) > t\} \Gamma(\psi \setminus \{0\}, dA) Q_{L_0}^0(d\psi).$$

By definition (2.19) and definition of Y_0 this means that

$$\frac{G_{i,B}(\infty, C) - G_{i,B}(t,C)}{1 - F_B(t)} = \frac{1}{\bar{V}_{i,B}} E\left[\int \mathbf{1}\{d_B(x+tu, Y_0) > t\} \mathbf{1}\{u \in C\} \bar{C}_{i,B}(d(x,u))\right].$$

By definition (4.2), the above right-hand side equals $K_{i,B}(t,C)$. Inserting this into (2.22) gives the asserted equation (4.3). □

Our next proposition deals with the asymptotic behavior of $r_B(t,C)$ as $t \to 0$ or $t \to \infty$, respectively. Note that $t^{1-d} r_B(t,C)$ is monotone decreasing in t. This is a direct consequence of (4.2). Recall the definition of the tangential cone given in the proof of Proposition 3.6.

Proposition 4.2. *The B-relative empty space hazard $r_B(t,C)$ of a Poisson cluster germ-grain model with compact, convex grains satisfies*

$$\lim_{t \to 0} r_B(t,C) = 2\lambda_\Pi E\left[\int_{\mathbb{R}^d \times C} \mathbf{1}\{x \notin Y_0\} C_{d-1}(X_0; B; d(x,u))\right] \tag{4.5}$$

$$\lim_{t \to \infty} t^{1-d} r_B(t,C) = d b_d \lambda_\Pi E\left[\int_{\mathbb{R}^d \times C} \mathbf{1}\{(x + T(B,-u)) \cap Y_0 = \emptyset\} C_0(X_0; B; d(x,u))\right]. \tag{4.6}$$

Proof. The proof is similar to the proof of (3.11) and (3.14). □

Remark 4.3. One might wonder whether the right-hand side of (4.6) is positive for $C = \mathbb{R}^d$. A simple sufficient condition is that $P(\operatorname{card} L_0 = 1) > 0$, because in this case Y_0 is with positive probability empty. Another sufficient condition is to assume that B is smooth (any boundary point has a unique supporting hyperplane) and that the diameter of the typical grain can take arbitrary small positive values with positive probability. These assumptions would allow to apply the method of [11, Section 6.5] on a set of positive probability. We do not go into further details.

Remark 4.4. The second assertion of the previous proposition shows in particular that F_B is *light-tailed*, i.e. has a finite exponential moment.

Now we shall compare the B-relative empty space hazard r_B of Z with that of another Poisson cluster germ-grain model $\tilde{Z}$ with the same intensity λ of germs and the same typical grain X_0. Both underlying germ processes are assumed to satisfy the technical assumption formulated at (3.4). We denote the characteristics of $\tilde{Z}$ by $\tilde{\Pi}$, $\tilde{L}_0$, $\tilde{K}_{i,B}$, $\tilde{r}_B$ etc. We begin with a direct consequence of Theorem 4.1.

Proposition 4.5. *Let $C \in \mathcal{B}^d$ and $t \geq 0$ such that*

$$K_{i,B}(t, C) \geq \tilde{K}_{i,B}(t, C). \tag{4.7}$$

Then (2.9) *holds.*

An immediate consequence of the above proposition is that the relative empty space hazard of a Boolean model is always greater than that of a Poisson cluster germ-grain model having the same germ intensity and the same typical grain:

Corollary 4.6. *Assume that Z is a Boolean model with typical convex, compact grains distributed as X_0, and $\tilde{Z}$ a Poisson cluster germ-grain model with equal intensity, and also typical grains distributed as X_0. Then $Z <_{h-B} \tilde{Z}$.*

Proof. Since Z is a Boolean model, we have $L_0 = \{0\}$ and $\gamma = 1$. Hence, if $\bar{V}_{i,B} > 0$,

$$K_{i,B}(t, C) = \bar{V}_{i,B}^{-1} \bar{C}_{i,B}(\mathbb{R}^d \times C) \geq \tilde{K}_{i,B}(t, C). \qquad \square$$

While a Boolean model has stochastically smaller empty space than a related Poisson cluster germ-grain model, it has a greater volume fraction. Under a different set of assumptions (more specific Poisson-cluster processes and deterministic but possibly non-convex grains) the result was proved in Section 3.8 of [6].

Proposition 4.7. *Under the assumptions of Corollary 4.6*

$$P(0 \in Z) \geq P(0 \in \tilde{Z}). \tag{4.8}$$

Proof. The volume fraction of Z is given by (2.11). On the other hand, $\tilde{Z}$ is also a Boolean model, but based on the Poisson process $\tilde{\Pi}$ and with typical (possibly non-convex) grain

$$\tilde{X}_0 := \bigcup_{x \in \tilde{L}_0} X(x) + x,$$

where $\tilde{L}_0$ is the typical cluster associated with $\tilde{Z}$ and, given $\tilde{L}_0$, the family $\{X(x) : x \in \tilde{L}_0\}$ consists of independent random closed sets with the same distribution as X_0. Therefore we obtain from (2.11)

$$P(0 \in \tilde{Z}) = 1 - \exp[-\lambda_{\tilde{\Pi}} E V_d(\tilde{X}_0)]. \tag{4.9}$$

We have

$$E V_d(\tilde{X}_0) \le E \sum_{x \in \tilde{L}_0} V_d(X(x)) = E \sum_{x \in \tilde{L}_0} E[V_d(X(x))|\tilde{L}_0] = \gamma \bar{V}_d.$$

Inserting this into (4.9) and comparing with (2.11), yields the assertion. □

Remark 4.8. Consider the hypothesis of Proposition 4.7. The proof of this proposition shows that we have equality in (4.8) iff

$$V_d(X(x) \cap X(y)) = 0 \quad x, y \in \tilde{L}_0, x \ne y \quad P - \text{a.s.} \tag{4.10}$$

This is, for instance, the case if the cluster points have minimal distance $2t_0$ from each other for some $t_0 > 0$, and X_0 is a.s. contained in the ball $B(t_0)$.

Next we prove a more general germ-grain version of Proposition 3.7.

Proposition 4.9. *Consider two germ-grain models with the same typical grains X_0 such that N and $\tilde{N}$ are Neyman-Scott processes with cluster sizes η and $\tilde{\eta}$, respectively, and the same cluster point distribution V. If N and $\tilde{N}$ have the same intensity and*

$$\eta <_{l-g} \tilde{\eta} \tag{4.11}$$

then $Z <_{h-B} \tilde{Z}$.

Proof. By Proposition 4.5 it suffices to show

$$P(d_B(x + tu, Y_0) > t) \ge P(d_B(x + tu, \tilde{Y}_0) > t) \tag{4.12}$$

for all $x, u \in \mathbb{R}^d$ and all $t \ge 0$. Letting $B' := tB + x + tu$ and using the definition (4.1) of the distribution of Y_0, we obtain that

$$\begin{aligned} E[\eta] P(Y_0 \cap B' = \emptyset) &= E\left[\sum_{y \in L_0} \mathbf{1}\{A \cap B' = \emptyset\} \Gamma((L_0 - y) \setminus \{0\}, dA)\right] \\ &= P(\eta = 1) + \sum_{n=2}^{\infty} P(\eta = n) n \int f(y)^{n-1} V(dy), \end{aligned}$$

where $f(y) := \int P((X_0 + z - y) \cap B' = \emptyset) V(dz)$ and where the second identity comes from a straightforward calculation using the definition of the typical cluster L_0 of a Neyman-Scott process. By Fubini's theorem this means that

$$E[\eta] P(Y_0 \cap B' = \emptyset) = \int E[\eta f(y)^{\eta - 1}] V(dy).$$

We can now use our assumption (3.15) to derive

$$P(Y_0 \cap B' = \emptyset) \ge E[\eta]^{-1} \int E[\tilde{\eta} f(y)^{\tilde{\eta} - 1}] V(dy).$$

Reversing the above steps, we get (4.12) and hence the asserted result. □

Our next result generalizes Theorem 5.4 in [12].

Proposition 4.10. *Consider two germ-grain models with the same typical grains X_0 based on Gauss-Poisson processes N and $\tilde{N}$. Assume that N and $\tilde{N}$ have the same intensity and that the probabilities p and $\tilde{p}$ for having a secondary point satisfy $p \leq \tilde{p}$. Then $Z <_{h-B} \tilde{Z}$.*

Proof. Fix $t \geq 0$ and $C \in \mathcal{B}^d$. From the defining properties of a Gauss-Poisson process (see Example 3.5) we have

$$\begin{aligned}\bar{V}_{i,B}K_{i,B}(t,C) =& \frac{1-p}{1+p}\bar{C}_{i,B}(\mathbb{R}^d \times C) + \frac{p}{1+p}\int_{\mathbb{R}^d\times C} a(x,u)\bar{C}_{i,B}(d(x,u)) \\ &+ \frac{p}{1+p}\int_{\mathbb{R}^d\times C} b(x,u)\bar{C}_{i,B}(d(x,u)),\end{aligned}$$

where

$$a(x,u) := P(d_B(x+tu, Y+X_0) > t), \qquad b(x,u) := P(d_B(x+tu, -Y+X_0) > t),$$

where Y and X_0 are independent. Therefore, by Proposition 4.5, it suffices to show that

$$\frac{1-p}{1+p} + \frac{pa}{1+p} + \frac{pb}{1+p} \geq \frac{1-\tilde{p}}{1+\tilde{p}} + \frac{\tilde{p}a}{1+\tilde{p}} + \frac{\tilde{p}b}{1+\tilde{p}},$$

for all $a, b \in [0,1]$. Simple algebra shows that this inequality is equivalent to

$$2\tilde{p} - a\tilde{p} - b\tilde{p} \geq 2p - ap - bp.$$

The latter is implied by our assumption $p \leq \tilde{p}$. □

5 Results for Mixed Poisson germ-grain models

In this section we consider a germ-grain model Z based on a *mixed Poisson process* N. This means that there is a random variable $\Lambda \geq 0$ such that the conditional distribution of N given Λ is that of a stationary Poisson process with intensity Λ. We assume that $E[\Lambda]$ (the intensity of N) is positive and finite.

It is convenient to use the notation

$$H_B(t) = E[V_d(X_0 + tB^*)], \quad t \geq 0.$$

Proposition 5.1. *The B-relative empty space hazard of a mixed Poisson germ-grain model with compact, convex grains is given by*

$$r_B(t,C) = \sum_{i=0}^{d-1}(d-i)t^{d-i-1}b_{d-i}E[\exp[-\Lambda H_B(t)]]^{-1}E[\Lambda\exp[-\Lambda H_B(t)]]\bar{S}_{i,B}(t,C). \quad (5.1)$$

Proof. Again we will use (2.22). To do so, we note that the Palm probability measure P_N^0 of a mixed Poisson process N satisfies

$$P_N^0((\Lambda, N) \in \cdot) = E[\Lambda]^{-1}E[\Lambda \mathbf{1}\{(\Lambda, N \cup \{0\}) \in \cdot\}]. \quad (5.2)$$

This formula can be derived by conditioning and using the properties of a Poisson process. Since, moreover, the conditional distribution $P_N^0(Z^! \in \cdot|\Lambda)$ (cf. (2.20) for the definition of

the random set $Z^!$) is that of a Boolean model with germ intensity Λ, we obtain for all $(x,u) \in \mathbb{R}^d \times \mathbb{R}^d$ that

$$\begin{aligned} P_N^0(d_B(x+tu, Z^!) > t) &= E[\Lambda]^{-1} E[\Lambda P(d_B(x+tu, Z) > t \mid \Lambda)] \\ &= E[\Lambda]^{-1} E[\Lambda P(d_B(0, Z) > t \mid \Lambda)] \\ &= E[\Lambda]^{-1} E[\Lambda \exp[-\Lambda H_B(t)]], \end{aligned}$$

where we have used (2.10) to obtain the last identity. Again by conditioning and (2.10) we have that $1 - F_B(t) = E[\exp[-\Lambda H_B(t)]]$. Inserting our findings into the general formula (2.22) yields the assertion (5.1). □

In order to state some stochastic ordering consequences of Proposition 5.1 we introduce a stochastic order using cumulants. For two nonnegative random variables Λ, $\tilde{\Lambda}$ we say that they are ordered in the first cumulant order and write $\Lambda <_{cum} \tilde{\Lambda}$ if for the corresponding cumulant generating functions C_Λ, $C_{\tilde{\Lambda}}$, taking the first derivatives we have $C'_\Lambda(s) \geq C'_{\tilde{\Lambda}}(s),\ s \in [-1, 0]$. Note that $\Lambda <_{cum} \tilde{\Lambda}$ is equivalent to

$$E[\exp[-\Lambda s]]^{-1} E[\Lambda \exp[-\Lambda s]] \geq E[\exp[-\tilde{\Lambda} s]]^{-1} E[\tilde{\Lambda} \exp[-\tilde{\Lambda} s]], \quad s \geq 0. \tag{5.3}$$

The left-hand side of (5.3) is the logarithmic derivative of the Laplace transform $s \mapsto E[\exp[-\Lambda s]]$. It is also the hazard rate of the distribution function G_Λ, defined by

$$G_\Lambda(s) := 1 - E[\exp[-\Lambda s]], \quad s \geq 0.$$

This is a mixture of exponential distributions. Equation (5.3) then means that the corresponding variables are ordered in the hazard rate order, i.e. $G_\Lambda <_h G_{\tilde{\Lambda}}$. Note that for Λ, $\tilde{\Lambda}$ with equal expected values $\Lambda <_{cum} \tilde{\Lambda}$ implies that $\mathbb{V}\mathrm{ar}[\Lambda] \leq \mathbb{V}\mathrm{ar}[\tilde{\Lambda}]$, therefore, similarly to the relation $<_{l-g}$, the relation $<_{cum}$ is a variability ordering in the case of a fixed expectation.

An immediate consequence of Proposition 5.1 is the following counterpart of Propositions 4.9 and 4.10. We use similar notation. Again, intuitively speaking, more variability in the mixed Poisson model results in a stochastically greater empty space (a stochastically larger clustering).

Proposition 5.2. *Consider two germ-grain models with the same typical grains X_0 based on mixed Poisson processes N and $\tilde{N}$ with random intensities Λ and $\tilde{\Lambda}$, respectively. Assume that $\Lambda <_{cum} \tilde{\Lambda}$ then $Z <_{h-B} \tilde{Z}$.*

Example 5.3. Let N and $\tilde{N}$ be as in Proposition 5.1 and assume that Λ is Gamma distributed with shape and scale parameter $\alpha > 0$ and $\beta > 0$, respectively. This means that Λ has density $\beta^\alpha \Gamma(\alpha)^{-1} x^{\alpha-1} \exp[-\beta x]$. The Laplace transform of Λ can be computed as

$$E[\exp[-s\Lambda]] = \frac{\beta^\alpha}{(\beta+s)^\alpha},$$

while an equally easy calculation gives

$$E[\Lambda \exp[-s\Lambda]] = \frac{\alpha}{\beta+s} \frac{\beta^\alpha}{(\beta+s)^\alpha}.$$

Assume now that $\tilde{\Lambda}$ is Gamma distributed with parameters $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. Then assumption (5.3) means that $\alpha/(\beta+s) \geq \tilde{\alpha}/(\tilde{\beta}+s)$ holds for all $s \geq 0$. This is equivalent to

$$\alpha \geq \tilde{\alpha}, \quad \frac{\alpha}{\beta} \geq \frac{\tilde{\alpha}}{\tilde{\beta}}. \tag{5.4}$$

Depending on whether or not $\beta \geq \tilde{\beta}$, only one of these equations is relevant. By Proposition 5.2, (5.4) implies the empty space hazard ordering (3.17). Assume for instance that $\tilde{\Lambda}$ is exponentially distributed with mean 1, i.e. $\tilde{\alpha} = \tilde{\beta} = 1$ and assume furthermore that Λ has also mean 1, that is $\alpha = \beta$. Then (5.4) is equivalent to $\alpha \geq \tilde{\alpha}$. Note that the variance of Λ satisfies $\mathbb{V}\mathrm{ar}[\Lambda] = \alpha/\beta^2 = 1/\beta \leq \mathbb{V}\mathrm{ar}[\tilde{\Lambda}] = 1$ if $\alpha \geq 1$.

As in the Poisson cluster case it follows that the relative empty space hazard of a Boolean model is greater than that of a mixed Poisson germ-grain model with the same germ intensity.

Corollary 5.4. *Assume that Z is a Boolean model with typical convex, compact grains distributed as X_0, and $\tilde{Z}$ a mixed Poisson germ-grain model with equal intensity, and also typical grains distributed as X_0. Then $Z <_{h-B} \tilde{Z}$.*

Proof. Let λ denote the germ intensity of the Boolean model Z and let $\tilde{\Lambda}$ be the random intensity of the mixed Poisson process underlying $\tilde{Z}$. It is assumed that $E[\tilde{\Lambda}] = \lambda$. We check that condition (5.3) holds with $\Lambda \equiv \lambda$. This condition means that

$$\lambda E[\exp[-\tilde{\Lambda}s]] \geq E[\tilde{\Lambda}\exp[-\tilde{\Lambda}s]], \quad s \geq 0.$$

In other words: the covariance between $\tilde{\Lambda}$ and $-\exp[-\tilde{\Lambda}s]$ has to be non-negative. This fact follows from a very well-known statement that a single random variable is associated, see Esary et al. [5]. □

For completeness we provide the mixed Poisson analogue of Corollary 4.7. The result can be found in Section 3.8 of [6] for the more general case of stationary Cox processes with an absolutely continuos intensity measure. Although our proof below can be extended to arbitrary stationary Cox processes we stick to the mixed Poisson case.

Proposition 5.5. *Under the assumptions of Corollary 5.4*

$$P(0 \in Z) \geq P(0 \in \tilde{Z}). \tag{5.5}$$

Proof. By conditioning and (2.11),

$$1 - P(0 \in \tilde{Z}) = E[\exp[-\Lambda E[V_d(X_0)]]].$$

By Jensen's inequality this is bounded from below by $\exp[-\lambda E[V_d(X_0)]]$. This lower bound is just $1 - P(0 \in Z)$. □

6 Concluding remarks

We have derived several variability properties of the empty space function of Poisson cluster and mixed Poisson germ-grain models. It would be worthwhile to study also other classes of germ processes. Another interesting task is to find a good notion of *spread out* for a finite point process (with respect to the origin). Proposition 3.12 and Proposition 3.13 should be both special cases of the same principle. The first proposition is generalized by Proposition 4.10. We believe that also Proposition 3.13 has a germ-grain counterpart.

In this paper we have always fixed the distribution of the typical grain. However, it would be quite interesting to study the variability of empty space in germ-grain models for a fixed germ-process but variable grain distribution. For instance, one might compare models with equal expected volumes of the typical grains. To illustrate this task we give one simple example that is closely related to some of the results in [21].

Example 6.1. Let X_0 be a random convex body such that $E[V_d(X_0+K)]$ is finite for all compact $K \subset \mathbb{R}^d$. Let R and $\tilde{R}$ be positive random variables with a finite d-th moment and assume that X_0 and R (resp. X_0 and $\tilde{R}$) are independent. Consider two Boolean model Z and $\tilde{Z}$ based on the same Poisson process N and typical grains RX_0 and $\tilde{R}X_0$, respectively. If

$$E[R^i] \geq E[\tilde{R}^i], \quad i = 1, \ldots, d-1,$$

then (2.8) holds for all structuring elements B such that X_0 and B^* are a.s. in general relative position, and all Borel sets $C \subset \mathbb{R}^d$. This follows from (2.24) and the scaling property

$$C_i(aX_0; B; \mathbb{R}^d \times C) = a^i C_i(X_0; B; \mathbb{R}^d \times C), \quad a > 0,$$

see e.g. [17] for the Euclidean case $B = B(1)$.

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