THE GIESEKER-PETRI DIVISOR IN $\mathcal{M}_g$ FOR $g \leq 13$

MARGHERITA LELLI–CHIESA

ABSTRACT. The Gieseker-Petri locus GP_g is defined as the locus inside $\mathcal{M}_g$ consisting of curves which violate the Gieseker-Petri Theorem. It is known that GP_g has always some divisorial components and it has been conjectured that it is of pure codimension 1 inside $\mathcal{M}_g$. We prove that this holds true for genus up to 13.

INTRODUCTION

Let $\mathcal{M}_g$ be the coarse moduli space of smooth irreducible projective curves of genus g. Given $[C] \in \mathcal{M}_g$ and a line bundle L on C, we consider the Petri map

$$\mu_{0,L} : H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C).$$

This map has been studied in detail because of its importance in the description of the Brill-Noether varieties $G^r_d(C)$ and $W^r_d(C)$. The most important result in this sense is the Gieseker-Petri Theorem (cf. [Gi], [EH1]), which asserts that for the generic curve and for any line bundle on it the Petri map is injective. This implies that if $[C] \in \mathcal{M}_g$ is general and the Brill-Noether number $\rho(g, r, d) := g-(r+1)(g-d+r)$ is nonnegative, then $G^r_d(C)$ is smooth of dimension $\rho(g, r, d)$ and the natural map $G^r_d(C) \to W^r_d(C)$ is a rational resolution of singularities. The Gieseker-Petri locus is defined as

$$GP_g = \{[C] \in \mathcal{M}_g \,|\, C \text{ does not satisfy the Gieseker-Petri Theorem}\}.$$

It is conjectured that GP_g has pure codimension 1 inside $\mathcal{M}_g$; an explanation why this is plausible is given below. The expectation has been proved in genus up to 8 by Castorena (cf. [Ca1], [Ca3]). Our main result is:

Theorem 0.1. *The locus GP_g has pure codimension 1 inside $\mathcal{M}_g$ for $9 \leq g \leq 13$.*

Our strategy is to look at the different components of GP_g determined by the numerical type of the linear series for which the Gieseker-Petri Theorem fails. For values of g, r, d such that both $r + 1$ and $g - d + r$ are at least 2 we define the Gieseker-Petri locus of type (r, d) as

$$GP^r_{g,d} := \{[C] \in \mathcal{M}_g \,|\, \exists \text{ a base point free } (L, V) \in G^r_d(C) \text{ with } \ker \mu_{0,V} \neq 0\},$$

where $\mu_{0,V}$ denotes the restriction of the Petri map to $V \otimes H^0(C, K_C \otimes L^{-1})$. Clifford's Theorem, along with Riemann-Roch Theorem, restricts the values of g, r, d for which it is necessary to study the component $GP^r_{g,d}$ to the range $0 < 2r \leq d \leq g - 1$. We also recall that, given $[C] \in GP_g$, at least one of the linear series on C for which the Gieseker-Petri Theorem fails is *primitive*, that is, complete and such that both L and $K_C \otimes L^{-1}$ are base point free.

In some cases the codimension of $GP^r_{g,d}$ inside $\mathcal{M}_g$ is known but in general it seems quite difficult to determine the irreducible components of GP_g and control their dimension. When $\rho(g, r, d) < 0$, the Petri map corresponding to any g^r_d on a genus g

curve cannot be injective for dimension reasons and the study of $GP^r_{g,d}$ essentially coincides with that of the Brill-Noether variety

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g \,|\, W^r_d(C) \neq \emptyset\}.$$

In particular, when $\rho(g, r, d) = -1$, the locus $\mathcal{M}^r_{g,d}$, if nonempty, is an irreducible divisor (cf. [EH3], [St]), known as the Brill-Noether divisor. On the other side, if $\rho(g, r, d) < -1$, the codimension of any component Z of $\mathcal{M}^r_{g,d}$ in $\mathcal{M}_g$ is strictly greater than 1. If it is true that GP_g has pure codimension 1 inside $\mathcal{M}_g$, then Z must be contained in some *divisorial*[1] component of GP_g.

When $\rho(g, r, d) \geq 0$, the Gieseker-Petri locus $GP^r_{g,d}$ can be described as the image, under the natural projection $p : \mathcal{G}^r_d \to M_g$, of the degeneracy locus X of a map of vector bundles locally defined on $\mathcal{G}^r_d$ and globalizing the Petri map [2]. Divisoriality of $GP^r_{g,d}$ is suggested by the fact that the expected codimension of X is $\rho(g, r, d) + 1$ and it would imply that the restriction of p to X has finite fibers. Farkas proved that $GP^r_{g,d}$ always has a divisorial component if $\rho(g, r, d) \geq 0$ (cf. [F1], [F2]). However, there are only two cases when $GP^r_{g,d}$ is completely understood. The first one is $GP^1_{g,g-1}$, which can be identified with the locus of curves with a vanishing theta-null and is an irreducible divisor (cf. [Te]). The second case is $GP^1_{g,\frac{g+2}{2}}$, for even genus $g \geq 4$. It has been proved by Eisenbud and Harris (cf. [EH2]), that this is a divisor which can be described as the branch locus of the natural map $H_{g,\frac{g+2}{2}} \to \mathcal{M}_g$ from the Hurwitz scheme $H_{g,\frac{g+2}{2}}$ parametrizing coverings of $\mathbb{P}^1$ of degree $(g + 2)/2$ having as source a smooth curve C of genus g.

We summarize our results. We show that when $g \leq 13$ the components of GP_g whose codimension is either unknown or strictly greater than 1 are contained in some divisorial components. Most of the inclusions easily follow from some basic remarks made in the first section. In particular, the components $GP^1_{g,k}$ with $\rho(g, 1, k) < -1$ are all contained in the Brill-Noether divisor $\mathcal{M}^1_{g,\frac{g+1}{2}}$ if g is odd, and in the locus $GP^1_{g,\frac{g+2}{2}}$ if g is even.

As a matter of notation, let $\overset{\circ}{\mathcal{M}}{}^r_{g,d}$ be the locus of curves having a primitive g^r_d. We define

$$\widetilde{GP}^r_{g,d} := \{[C] \in \mathcal{M}_g \,|\, \exists\, (L, V) \in G^r_d(C) \text{ with } \ker \mu_{0,V} \neq 0\};$$

notice that here we do not require that (L, V) be base point free. If the Brill-Noether number is either 0 or 1, we can prove the inclusion of both $\overset{\circ}{\mathcal{M}}{}^{r+1}_{g,d+1}$ and $\overset{\circ}{\mathcal{M}}{}^r_{g,d-1}$ inside $\widetilde{GP}^r_{g,d}$. We use a very recent result, due to Bruno and Sernesi, according to which for values of g, r, d such that $\rho(g, r, d) \geq 0$ and $\rho(g, r + 1, d) < 0$, the locus $\widetilde{GP}^r_{g,d}$ is divisorial outside its intersection with $\mathcal{M}^{r+1}_{g,d}$ (cf. [BS]). As a corollary we obtain that, in even genus, $\widetilde{GP}^1_{g,\frac{g+2}{2}}$ coincides with the divisor $GP^1_{g,\frac{g+2}{2}}$ studied by Eisenbud and Harris.

In the second paragraph we prove Theorem 0.1 in genera $9, 10, 11$. In addition to the remarks made in the previous section, we use some well known facts about plane curves. The study of the component $\mathcal{M}^3_{10,9}$ requires extra work: we prove that it

[1]By divisorial we will always mean a locus of pure codimension 1.

[2]Here $\mathcal{G}^r_d$ denotes the stack parametrizing pairs (C, l), where $[C]$ is the isomorphism class of a smooth irreducible projective curve of genus g and $l \in G^r_d(C)$; the map p is the projection on the moduli stack M_g.

is contained in $GP^1_{10,6}$ by remarking that any curve of degree 9 and genus 10 in $\mathbb{P}^3$ is either a curve of type $(3,6)$ on a non singular quadric surface or the intersection of two cubic surfaces; linear series on a cubic surface X can be easily written down remembering that X is isomorphic to the blow-up of the projective plane in 6 points.

In the last paragraph we deal with genera 12 and 13. The situation gets more complicated because the methods used before do not enable us to control the codimension of $GP^1_{g,g-2}$. We prove the following theorem:

Theorem 0.2. *Let $[C] \in GP^1_{g,g-2}$ be a non hyperelliptic curve with no vanishing theta-null. Let us assume that for any $L \in G^1_{g-2}(C)$ such that $\mu_{0,L}$ is not injective, L is primitive and $K_C \otimes L^{-1} \in W^2_g(C)$ is big. Then C carries only a finite number of $L \in W^1_{g-2}(C)$ for which* $\ker \mu_{0,L} \neq 0$.

This generalizes [Ca2], where it is assumed that the plane model Γ of C corresponding to $K_C \otimes L^{-1}$ has only singularities which become nodes after a finite number of blow-ups (in a somewhat oldfashioned way these are called possibly infinitely near nodes). The idea of our proof is to show that we do not need any assumption on the singularities of Γ because the non injectivity of $\mu_{0,L}$ implies that Γ has at least one double point, which cannot be a cusp of any order if $[C] \notin GP^1_{g,g-1}$; then we proceed like in [Ca2]. Theorem 0.2 implies Theorem 0.1 in genus 13 because no g^2_{13} can be composed with an involution. Instead, for a curve $[C] \in GP^1_{12,10}$ it may happen that a g^2_{12}, for which the Petri map is not injective, induces a finite covering of a plane curve of lower genus. We prove that this can be the case only for $[C] \in GP^1_{12,7} \cup GP^1_{12,8}$ (cf. Theorem 3.5).

I would like to thank my Ph.D. advisor Gavril Farkas for all the helpful discussions.

1. Some useful inclusions

In this section we prove some inclusions among different components of GP_g, which enable us to restrict the values of r and d for which the codimension of $GP^r_{g,d}$ must be determined.

We start by stating the following result, due to Sernesi and Bruno, which exhibits some other divisorial components of GP_g:

Theorem 1.1. *Let g, r, d be integers such that $0 < 2r \leq d \leq g-1$, $\rho(g,r,d) \geq 0$ and $\rho(g, r+1, d) < 0$. Then $\widetilde{GP}^r_{g,d} \setminus (\mathcal{M}^{r+1}_{g,d} \cap \widetilde{GP}^r_{g,d})$ has pure codimension 1 inside $\mathcal{M}_g$.*

The proof of Theorem 1.1 has just appeared in [BS] and we briefly recall the idea. The condition $\rho(g, r+1, d) < 0$ assures that on a generic curve of genus g every g^r_d is complete. In this situation we consider $\varphi : \mathcal{C} \to S$ a family of smooth curves of genus g not belonging to $GP^{r+1}_{g,d}$ such that the induced map $S \to \mathcal{M}_g$ is dominant and finite, and the relative scheme $\mathbf{W}^{r,d}_{\mathcal{C}/S} \xrightarrow{\sigma} S$ parametrizing couples (C_s, L_s), with $L_s \in W^r_d(C_s)$ (which in this case implies $h^0(C_s, L_s) = r+1$). The scheme

$$GP^r_{g,d}(\mathcal{C}/S) := \{s \in S \,|\, \varphi^{-1}(s) \in \widetilde{GP}^r_{g,d}\}$$

turns out to be image in S of the degeneracy locus $X_{(r+1)(g-d+r)-1}(\mu)$ of a map of vector bundles $\mu : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{F}$ defined over $\mathbf{W}^{r,d}_{\mathcal{C}/S}$; if $X_{(r+1)(g-d+r)-1}(\mu)$ is nonempty, then its codimension inside $\mathbf{W}^{r,d}_{\mathcal{C}/S}$ is at most $\rho(g,r,d)+1$. The finiteness of the fibers of the restriction of σ to $X_{(r+1)(g-d+r)-1}(\mu)$ follows by a result of Steffen (cf.[St]), which can be applied because σ is projective and dominant and the sheaf $(\mathcal{E}_1 \otimes \mathcal{E}_2)^\vee \otimes \mathcal{F}$ is ample relative to σ, namely it is ample when restricted to any fiber of σ.

Without the condition $\rho(g, r+1, d) < 0$, we could still define the sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ in the same way but $\mathcal{E}_1$ and $\mathcal{E}_2$ would be locally free only when restricted to the open subset $\mathbf{W}^{r,d}_{\mathcal{C}/S} \setminus \mathbf{W}^{r+1,d}_{\mathcal{C}/S}$. Unfortunately, the restriction of σ to $\mathbf{W}^{r,d}_{\mathcal{C}/S} \setminus \mathbf{W}^{r+1,d}_{\mathcal{C}/S}$ is not projective and so Steffen's Theorem cannot be applied in this situation.

We now prove some basic inclusions:

Remark 1. For $\rho(g, r-1, d-1) < 0$ and $r > 1$, we have that:

$$\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1} = \widetilde{GP}^{r-1}_{g,d-1}.$$

Proof. From any g^r_d we can trivially get a g^{r-1}_{d-1} by subtracting a point outside its base locus. □

Next remark concerns the components $GP^1_{g,k}$:

Remark 2. If g is odd, the following sequence of inclusions holds:

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \ldots \subseteq \mathcal{M}^1_{g,\frac{g+1}{2}},$$

and $\mathcal{M}^1_{g,\frac{g+1}{2}}$ is a Brill-Noether divisor.

Similarly when g is even we have that:

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \ldots \subseteq \widetilde{GP}^1_{g,\frac{g+2}{2}}.$$

Proof. Cosider $k < \frac{g+1}{2}$ if g is odd and $k < \frac{g+2}{2}$ if g is even. Let $[C] \in \mathcal{M}^1_{g,k}$ and L be a complete g^1_k on C. By defining $L' := L \otimes \mathcal{O}_C(P)$ with P a point outside the base locus of $K_C \otimes L^{-1}$, one may prove all the inclusions but $\mathcal{M}^1_{g,\frac{g}{2}} \subset \widetilde{GP}^1_{g,\frac{g+2}{2}}$. When L is a complete $g^1_{\frac{g}{2}}$ on C with base locus B (not necessarily empty), the Base Point Free Pencil Trick implies both

$$\dim \ker \mu_{0,L} = h^0(C, K_C \otimes L^{-2} \otimes \mathcal{O}_C(B)) \geq -\rho(g, 1, g/2) = 2$$

and

$$\dim\, \ker \mu_{0,L'} = h^0(C, K_C \otimes L^{-2} \otimes \mathcal{O}_C(B-P)) \geq 1.$$

Thus L' is a $g^1_{\frac{g+2}{2}}$ on C violating the Gieseker-Petri Theorem and $[C] \in \widetilde{GP}^1_{g,\frac{g+2}{2}}$. □

The following result is a corollary of Theorem 1.1. Together with the previous Remark, it implies that all the loci $GP^1_{g,k}$ such that $\rho(g,1,k) < 0$ are contained in a divisorial component of GP_g.

Corollary 1.2. *In even genus the following equality holds:*

$$\widetilde{GP}^1_{g,\frac{g+2}{2}} = GP^1_{g,\frac{g+2}{2}}.$$

Proof. By Remark 2, we have that $\mathcal{M}^1_{g,\frac{g}{2}} \subset \widetilde{GP}^1_{g,\frac{g+2}{2}}$ and so we can write

$$\widetilde{GP}^1_{g,\frac{g+2}{2}} = GP^1_{g,\frac{g+2}{2}} \cup \mathcal{M}^1_{g,\frac{g}{2}},$$

where $GP^1_{g,\frac{g+2}{2}}$ is a divisor on $\mathcal{M}_g$. Furthermore $\mathcal{M}^1_{g,\frac{g}{2}}$ is irreducible and of codimension 2 in $\mathcal{M}_g$ (cf. [Fu]). Our goal is to show that $\mathcal{M}^1_{g,\frac{g}{2}} \subset GP^1_{g,\frac{g+2}{2}}$.

Theorem 1.1 implies that $\widetilde{GP}^1_{g,\frac{g+2}{2}} \setminus \mathcal{M}^2_{g,\frac{g+2}{2}}$ is divisorial, and by Remark 1 we know that $\mathcal{M}^2_{g,\frac{g+2}{2}} \subset \mathcal{M}^1_{g,\frac{g}{2}}$. It follows that

$$\mathcal{M}^1_{g,\frac{g}{2}} \setminus \mathcal{M}^2_{g,\frac{g+2}{2}} \subset GP^1_{g,\frac{g+2}{2}},$$

and the same must be true for its closure. If we show that $\mathcal{M}^1_{g,\frac{g}{2}} \setminus \mathcal{M}^2_{g,\frac{g+2}{2}}$ is open in $\mathcal{M}^1_{g,\frac{g}{2}}$, then the irreducibility of $\mathcal{M}^1_{g,\frac{g}{2}}$ implies that $\mathcal{M}^1_{g,\frac{g}{2}} \subset GP^1_{g,\frac{g+2}{2}}$ and we have finished. To end the proof it is enough to remark that the generic curve in $\mathcal{M}^1_{g,\frac{g}{2}}$ has a unique $g^1_{\frac{g}{2}}$ (cf. [AC2]) while a curve inside $\mathcal{M}^2_{g,\frac{g+2}{2}}$ has at least a 1-dimensional space of $g^1_{\frac{g}{2}}$'s (all obtained from a $g^2_{\frac{g+2}{2}}$ by the subtraction of a point). $\square$

Other useful inclusions come from the following remark:

Remark 3. If $\rho(g,r,d) \in \{0,1\}$, then $\overset{\circ}{\mathcal{M}}{}^{r+1}_{g,d+1} \subset GP^r_{g,d}$ and $\overset{\circ}{\mathcal{M}}{}^{r}_{g,d-1} \subset \widetilde{GP}^r_{g,d}$.

Proof. Assume $\rho(g,r,d) = 0$. We fix $[C] \in \overset{\circ}{\mathcal{M}}{}^{r+1}_{g,d+1}$ and L a primitive g^{r+1}_{d+1} on C. For any $P \in C$, $L \otimes \mathcal{O}_C(-P)$ is a g^r_d on C and so $G^r_d(C)$ contains

$$C' := \{L \otimes \mathcal{O}_C(-P) \,|\, P \in C\} \cong C.$$

It follows that $\dim T_{L\otimes\mathcal{O}_C(-P)}(G^r_d(C)) \geq \dim_{L\otimes\mathcal{O}_C(-P)} G^r_d(C) \geq 1$. By remembering that

$$\dim T_{L\otimes\mathcal{O}_C(-P)}(G^r_d(C)) = \rho(g,r,d) + \dim \ker \mu_{0,L\otimes\mathcal{O}_C(-P)} = \dim \ker \mu_{0,L\otimes\mathcal{O}_C(-P)},$$

one deduces that $L \otimes \mathcal{O}_C(-P)$ does not satisfy the Gieseker-Petri Theorem. Analogously, given $[C] \in \overset{\circ}{\mathcal{M}}{}^{r}_{g,d-1}$ and L a primitive, complete g^r_{d-1} on C, one defines

$$C'' := \{L \otimes \mathcal{O}_C(P) \,|\, P \in C\} \cong C$$

and, reasoning as above, proves that $[C] \in \widetilde{GP}^r_{g,d}$.

For $\rho(g,r,d) = 1$, we consider $[C] \in \overset{\circ}{\mathcal{M}}{}^{r}_{g,d-1}$ and L a primitive g^r_{d-1} on C. The definition of C'' is the same. Since we can assume that $\dim G^r_d(C) = 1$ (otherwise we could soon conclude that $[C] \in GP^r_{g,d}$), it follows that C'' is an irreducible component of $G^r_d(C)$. As C must have a base point free g^r_d, there exist components of $G^r_d(C)$ different from C''. By the Connectedness Theorem (cf.[ACGH], p. 212), $G^r_d(C)$ is connected. It follows that $G^r_d(C)$ is singular and so $[C] \in \widetilde{GP}^r_{g,d}$. We proceed very similarly if $[C] \in \overset{\circ}{\mathcal{M}}{}^{r+1}_{g,d+1}$. $\square$

2. Proof of Theorem 0.1 in genus 9, 10, 11

In this section we prove that, for genus $g \in \{9,10,11\}$, the Gieseker-Petri locus GP_g is of pure codimension 1 inside $\mathcal{M}_g$.

Let us fix $g = 9$. For $r \in \{4,3\}$ and $2r \leq d \leq 8$ and for $r = 2$ and $4 \leq d \leq 6$, the Brill-Noether number $\rho(g, r-1, d-1)$ is negative and so, by Remark 1, we can restrict our analysis to the components $GP^2_{9,d}$ and $GP^1_{9,k}$ for $d \in \{7,8\}$ and $2 \leq k \leq 8$. Moreover, Remark 2 implies that $\mathcal{M}^1_{9,k}$ is contained in the Brill-Noether divisor $\mathcal{M}^1_{9,5}$ for $k \leq 4$.

Since $\rho(9,2,7)<0$, we now study $\overset{\circ}{\mathcal{M}}{}^2_{9,7}$. Given $[C]\in\overset{\circ}{\mathcal{M}}{}^2_{9,7}$, if we assume that C does not lie in $\mathcal{M}^1_{9,5}$, then any g^2_7 on C is base point free and defines an embedding

$$\phi : C \to \Gamma \subset \mathbb{P}^2,$$

where Γ is a plane curve of degree 7 and genus 9. By the Genus Formula it follows that Γ is singular, which is a contradiction.

Regarding the component $GP^2_{9,8}$, we note that $\rho(9,2,8)=0$ and $\rho(9,3,8)<0$, so Theorem 1.1 implies that $\widetilde{GP}^2_{9,8}\setminus(\mathcal{M}^3_{9,8}\cap\widetilde{GP}^2_{9,8})$ is divisorial. We do not need to study $\mathcal{M}^3_{9,8}\cap GP^2_{9,8}$ separately because, by Remark 1, the inclusion $\mathcal{M}^3_{9,8}\subseteq\mathcal{M}^2_{9,7}$ holds.

Let us consider the components $GP^1_{9,k}$ for $k\in\{6,7,8\}$. For $k\in\{6,7\}$ we have that $\rho(9,1,k)>0$ and $\rho(g,2,k)<0$ and so the locus $\widetilde{GP}^1_{g,k}\setminus(\widetilde{GP}^1_{g,k}\cap\mathcal{M}^2_{g,k})$ is divisorial. As $GP^1_{9,8}$ is the irreducible divisor consisting of curves with a vanishing theta-null, Theorem 0.1 is proved in genus 9.

Before dealing with the case of genus 10, we prefer to treat the case of genus 11, which is very similar to the one we have just studied. As before, by applying Remark 1 and Remark 2 we reduce to considering the components $GP^2_{11,d}$ and $GP^1_{11,k}$ for $8\le d\le 10$ and $7\le k\le 10$.

We can prove that $\overset{\circ}{\mathcal{M}}{}^2_{11,8}$ is contained in the Brill-Noether divisor $\mathcal{M}^1_{11,6}$ simply by remarking that any g^2_8 on a genus 11 curve $[C]\notin\mathcal{M}^1_{11,6}$ is base point free and defines an embedding

$$\phi : C \to \Gamma \subset \mathbb{P}^2.$$

We get a contradiction because Γ is a plane curve of degree 8 and genus 11 and so it must be singular by the Genus Formula.

Concerning the other components, the locus $\mathcal{M}^2_{11,9}$ is a Brill-Noether divisor, while $\widetilde{GP}^2_{11,10}$ is divisorial outside its intersection with $\mathcal{M}^3_{11,10}$ because $\rho(11,2,10)>0$ and $\rho(11,3,10)<0$.

Theorem 1.1 can be applied in order to prove that the locus $\widetilde{GP}^1_{11,k}\setminus(\mathcal{M}^2_{11,k}\cap\widetilde{GP}^1_{11,k})$ is divisorial for $7\le k\le 9$, too. The component $GP^1_{11,10}$ is the irreducible divisor of curves with a vanishing theta-null and so Theorem 0.1 is proved in genus 11.

We now deal with the case of genus 10. As above, by Remarks 1 and 2, the only components of GP_{10} we have to consider are $GP^2_{10,d}$ and $GP^1_{10,k}$ for $7\le d\le 9$ and $7\le k\le 9$.

As $\rho(10,1,6)=0$, Remark 3 implies that $\overset{\circ}{\mathcal{M}}{}^2_{10,7}\subset GP^1_{10,6}$. Moreover, $\rho(10,2,9)=1$ and so Remark 3 implies that $\overset{\circ}{\mathcal{M}}{}^2_{10,8}\subset\widetilde{GP}^2_{10,9}$, too. Since $\rho(10,3,9)<0$, the locus $\widetilde{GP}^2_{10,9}$ is divisorial outside $\mathcal{M}^3_{10,9}$. In this case we have to study the component $\mathcal{M}^3_{10,9}$ separately because our remarks imply only that $\overset{\circ}{\mathcal{M}}{}^3_{10,9}\subseteq\overset{\circ}{\mathcal{M}}{}^2_{10,8}\subseteq GP^2_{10,9}$. We postpone the study of $\overset{\circ}{\mathcal{M}}{}^3_{10,9}$. For $k\in\{7,8\}$, the locus $\widetilde{GP}^1_{10,k}\setminus(\widetilde{GP}^1_{10,k}\cap\mathcal{M}^2_{10,k})$ is divisorial because $\rho(10,2,k)<0$, while $GP^1_{10,9}$ is the irreducible divisor consisting of curves with a vanishing theta-null.

In order to end the proof of Theorem 0.1 in genus 10, we now study $\mathcal{M}^3_{10,9}$. We consider $[C] \in \mathcal{M}^3_{10,9}$ and L a g^3_9 on C. We can assume $[C] \notin \mathcal{M}^3_{10,8}$ and so L, being base point free, defines a morphism $\phi : C \to \Gamma \subset \mathbb{P}^3$. Furthermore, we can assume that $[C] \notin \mathcal{M}^2_{10,7}$, which forces ϕ to be an embedding. Therefore C can be seen as a curve of genus 10 and degree 9 in $\mathbb{P}^3$. By the classification of curves in $\mathbb{P}^3$, we know that C is either a curve of type $(3,6)$ on a non singular quadric surface S or the intersection of two cubic surfaces (cf. [Ha2] Example 6.4.3. chp.IV). In the first case the lines of type $(0,1)$ on S cut out a g^1_3 on Γ. The second case is treated in the following lemma:

Lemma 2.1. *Let $[C] \in \mathcal{M}_{10}$ be the intersection of two cubic surfaces X, Y in $\mathbb{P}^3$. Then $[C] \in GP^1_{10,6}$.*

Proof. It is classically known that X is isomorphic to the blow-up of $\mathbb{P}^2$ in 6 points $P_1, \ldots, P_6$. We denote by $\pi : X \to \mathbb{P}^2$ the projection and by E_i the exceptional divisors. $\mathrm{Pic}(X) \cong \mathbb{Z}^7$ and it is generated by $l, e_1, e_2, \ldots, e_6$, where l is the class of the strict transform of a line in $\mathbb{P}^2$ and e_i is the class of E_i. The class of the hyperplane section is $h = 3l - \sum e_i$, while

$$K_X \sim -h = -3l + \sum e_i.$$

As C lies on another cubic surface Y, then

$$C \sim 3h = 9l - 3\sum e_i,$$

namely C is the strict transform of a plane curve $\widetilde{C}$ of degree 9 with 6 triple points. The pencil of cubics through $P_1, \ldots, P_6$ with a double point in P_1 cuts out a g^1_6 on $\widetilde{C}$. The strict transforms of these cubics cut out on C the linear series

$$L := (3l - \sum_{i \neq 1} e_i - 2e_1) \otimes \mathcal{O}_C.$$

In order to check that L is a g^1_6 on C, we tensor with $\mathcal{O}_X(3l - \sum_{i\neq 1} e_i - 2e_1)$ the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

getting

$$0 \to \mathcal{O}_X(-6l + 2\sum_{i\neq 1} e_i + e_1) \to \mathcal{O}_X(3l - \sum_{i\neq 1} e_i - 2e_1) \to \mathcal{O}_C(3l - \sum_{i\neq 1} e_i - 2e_1) \to 0.$$

As $6l - 2\sum_{i\neq 1} e_i - e_1$ is ample (cf. [Ha2] Cor.4.13 chap.V), Kodaira Vanishing Theorem implies that $h^i(X, \mathcal{O}_X(-6l + 2\sum_{i\neq 1} e_i + e_1)) = 0$ for $i = 0, 1$. It follows that

$$\begin{array}{rcll} h^0(C, \mathcal{O}_C(3l - \sum_{i\neq 1} e_i - 2e_1)) & = & h^0(X, \mathcal{O}_X(3l - \sum_{i\neq 1} e_i - 2e_1)) & = \\ & = & h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\mathbb{P}^2}(-\sum_{i\neq 1} P_i - 2P_1)) & = \\ & = & 2 & \end{array}$$

and this is enough to conclude that L is a pencil on C; it is trivial to check that its degree is 6.
By the Base Point Free Pencil Trick we have that

$$\ker \mu_{0,L} \cong H^0(C, K_C \otimes L^{-2}) = H^0(C, \mathcal{O}_C(2e_1)).$$

As $\mathcal{O}_C(2e_1)$ is effective, it follows that $[C] \in GP^1_{10,6}$. □

Remark 4. The previous Lemma can also be proved by using the results of [Ma]. Curves of genus 10 which are the complete intersection of two cubic surfaces in $\mathbb{P}^3$ are the only curves of Clifford dimension 3. Martens proved that such curves are 6-gonal and carry a one-dimensional family of g^1_6. Since $\rho(10,1,6)=0$, this is enough to conclude that they lie in $GP^1_{10,6}$.

It is natural to ask whether all curves of Clifford dimension greater than 1 lie in a divisorial component of the Gieseker-Petri locus. Curves of Clifford dimension 2 are smooth plane curves of degree $d\geq 5$. Their gonality is $d-1$ and there is a one-dimensional family of pencils computing it. As $\rho\left(\binom{d-1}{2},1,d-1\right)\leq 0$ for $d\geq 5$, Remark 2 implies that they lie in the Brill-Noether divisor $\mathcal{M}^1_{g,\frac{g+1}{2}}$ when $g=\binom{d-1}{2}$ is odd, and in the irreducible divisor $GP^1_{g,\frac{g+2}{2}}$ when g is even.

It is conjectured in [ELMS] that if C is a curve of Clifford dimension $r>3$, then $g(C)=4r-2$, $\mathrm{gon}(C)=2r$ and there is a one-dimensional family of pencils computing the gonality (this conjecture was proved in [ELMS] for $r\leq 9$). Since $\rho(4r-2,1,2r)=0$, such curves lie in the divisor $GP^1_{g,\frac{g+2}{2}}=GP^1_{4r-2,2r}$.

3. Proof of Theorem 0.1 in genus 12, 13

The situation in genus 12 and 13 is slightly more complicated as there is a component in GP_g which cannot be studied using the methods explained in the previous sections.

In genus 12, by Remarks 1 and 2, we have to analyze only the components $\overset{\circ}{\mathcal{M}}{}^3_{12,11}$, $GP^2_{12,d}$ for $8\leq d\leq 11$ and $GP^1_{12,k}$ for $8\leq k\leq 11$. Since $\rho(12,2,10)=0$, Remark 3 implies that both $\overset{\circ}{\mathcal{M}}{}^3_{12,11}$ and $\overset{\circ}{\mathcal{M}}{}^2_{12,9}$ are contained in $\widetilde{GP}^2_{12,10}$. Remark 3 can also be used in order to show that $\overset{\circ}{\mathcal{M}}{}^2_{12,8}\subset GP^1_{12,7}$; indeed, $\rho(12,1,7)=0$.

As $\rho(12,3,d)<0$ for $d\in\{10,11\}$, the loci $\widetilde{GP}^2_{12,10}$ and $\widetilde{GP}^2_{12,11}$ are divisorial outside their intersection with $\mathcal{M}^3_{12,10}$ and $\mathcal{M}^3_{12,11}$ respectively. We have to study $\mathcal{M}^3_{12,10}$ separately because our remarks only imply that $\overset{\circ}{\mathcal{M}}{}^3_{12,10}\subset\overset{\circ}{\mathcal{M}}{}^2_{12,9}\subset GP^2_{12,10}$.
Given $[C]\in\mathcal{M}^3_{12,10}$, we can suppose that $[C]\notin\mathcal{M}^2_{12,8}$ and so any g^3_{10} on C is base point free and defines an embedding $\phi: C\to\Gamma\subset\mathbb{P}^3$. It can be seen that Γ has ten 4-secant lines (cf. [ACGH], p. 351), each of which corresponds to a g^1_6 on it.

Theorem 1.1 can be applied in order to show that the locus $\widetilde{GP}^1_{12,k}$ is divisorial outside $\mathcal{M}^2_{12,k}$ for $k\in\{8,9\}$. The component $GP^1_{12,11}$ is an irreducible divisor. We postpone the study of $GP^1_{12,10}$ to the end of the section.

The situation in genus 13 is very similar to that in genus 12. By Remarks 1 and 2, we reduce to considering $\overset{\circ}{\mathcal{M}}{}^3_{13,12}$, $GP^2_{13,d}$ for $9\leq d\leq 12$ and $GP^1_{13,k}$ for $8\leq k\leq 12$.

As $\rho(13,2,11)=1$, Remark 3 implies that both $\overset{\circ}{\mathcal{M}}{}^3_{13,12}$ and $\overset{\circ}{\mathcal{M}}{}^2_{13,10}$ are contained in $\widetilde{GP}^2_{13,11}$.

Concerning $\mathcal{M}^2_{13,9}$, any g^2_9 on a genus 13 curve $[C]\notin\mathcal{M}^1_{13,7}$ defines an embedding $\phi: C\to\Gamma\subset\mathbb{P}^2$. We get a contradiction because the Genus Formula forces Γ to be singular.

The components $\widetilde{GP}^2_{13,11}$ and $\widetilde{GP}^2_{13,12}$ are divisorial outside $\mathcal{M}^3_{13,11}$ and $\mathcal{M}^3_{13,12}$ respectively. As before we have to study $\mathcal{M}^3_{13,11}$ separately. Given $[C] \in \mathcal{M}^3_{13,11}$ such that $[C] \notin \mathcal{M}^2_{13,9}$, by taking the 4-secant lines to the space model of C corresponding to any $l \in G^3_{11}(C)$, one shows that C has a g^1_7.

Regarding the other components, the locus $\widetilde{GP}^1_{13,k}$ is divisorial outside its intersection with $\mathcal{M}^2_{13,k}$ for $k \in \{8, 9, 10\}$, while $GP^1_{13,12}$ is an irreducible divisor. Therefore Theorem 0.1 is proved also in genus 13 if we are able to verify that the component $GP^1_{g,g-2}$ is divisorial. In order to show this, we generalize a result of Castorena (cf. [Ca2]) as follows.

We consider curves $[C] \in GP^1_{g,g-2}$ such that for any $L \in G^1_{g-2}(C)$ with $\ker \mu_{0,L} \neq 0$ the following are satisfied:

(1) L is primitive.
(2) The morphism $\phi := \phi_{K_C \otimes L^{-1}}$ is birational.

We remark that the first condition is satisfied if $[C] \notin GP^1_{g,g-3} \cup GP^2_{g,g-2} \cup GP^2_{g,g-1}$, because if L were not complete (respectively not base point free), this would imply $[C] \in GP^2_{g,g-2}$ (resp. $[C] \in GP^1_{g,g-3}$). Similarly, if $K_C \otimes L^{-1}$ is not base base point free, then $[C] \in GP^2_{g,g-1}$. We prove the following result:

Proposition 3.1. *Let us consider $Z_g \subset GP^1_{g,g-2}$ the locus consisting of curves $[C] \in GP^1_{g,g-2}$ such that if $L \in G^1_{g-2}(C)$ satisfies $ker\mu_{0,L} \neq 0$, then L is primitive and $K_C \otimes L^{-1}$ is big. The scheme Z_g has pure codimension 1 in $\mathcal{M}_g$ outside its intersection with the hyperelliptic locus and the divisor $GP^1_{g,g-1}$.*

It is clear that Z_g is open in $GP^1_{g,g-2}$. In order to prove Proposition 3.1 we need the following:

Lemma 3.2. *If $[C] \in Z_g$ and L is a g^1_{g-2} on C such that $\ker \mu_{0,L} \neq 0$, then L is the pullback to C of the pencil cut out on $\Gamma := \phi_{K_C \otimes L^{-1}}(C)$ by the lines through a singular point x. In particular, x is a double point of Γ and $K_C \otimes L^{-2} = \frac{1}{k}\phi^*\mathcal{O}_\Gamma(x)$, where k is the number of blow-ups necessary to desingularize Γ in x (e.g., if x is a tacnode, then $k = 2$).*

Proof. The Base Point Free Pencil Trick implies that $K_C \otimes L^{-2} = \phi^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes L^{-1}$ is effective. Hence L is the pullback to C of the pencil cut out on Γ by the lines through a singular point x, which must be a double point because L is base point free. Furthermore, $K_C \otimes L^{-2}$ is linearly equivalent to $\frac{1}{k}\phi^*\mathcal{O}_\Gamma(x)$. □

We can now prove the following fact:

Lemma 3.3. *If $[C] \in Z_g$, $[C] \notin GP^1_{g,g-1}$ and C is not hyperelliptic, then there exists only a finite number of $L \in W^1_{g-2}(C)$ such that $\mu_{0,L}$ is not injective.*

Proof. We recall and adapt the proof of Castorena, referring to [Ca2] for further details. Given L a g^1_{g-2} on C with $\ker \mu_{0,L} \neq 0$, we have that

$$K_C \otimes L^{-2} = \frac{1}{k}\phi^*\mathcal{O}_\Gamma(x) = \mathcal{O}_C(P + Q),$$

and we can assume $P \neq Q$ because otherwise $L \otimes \mathcal{O}_C(P)$ would be a theta characteristic with a 2-dimensional space of sections, thus contradicting $[C] \notin GP^1_{g,g-1}$. We remark that asking that $P \neq Q$ is equivalent to requiring that x be not a cusp of any order. As C is not hyperelliptic, $h^0(C, \mathcal{O}_C(P + Q)) = 1$ and $h^0(C, K_C \otimes \mathcal{O}_C(-P - Q)) =$

$g-2$. It follows that L^2 lies in the intersection of the following two subvarieties of $\mathrm{Pic}^{2g-4}(C)$:

$$\begin{aligned} X_1 &:= \{L^2 \mid L \in W^1_{g-2}(C)\}, \\ X_2 &:= \{K_C \otimes \mathcal{O}_C(-P-Q) \mid P, Q \in C\} \subset W^{g-3}_{2g-4}(C). \end{aligned}$$

In order to show that $X_1 \cap X_2$ is a finite set, it is enough to prove that the intersection $T_{L^2}(X_1) \cap T_{L^2}(X_2) = \{0\}$ in $H^1(C, \mathcal{O}_C) = T_{L^2}(\mathrm{Pic}^{2g-4}(C))$, or equivalently, that $T_{L^2}(X_1)^\perp + T_{L^2}(X_2)^\perp$ generates the whole $H^0(C, K_C) = T_{L^2}(\mathrm{Pic}^{2g-4}(C))^\perp$. Indeed, it is trivial to see that

$$\dim\, T_{L^2}(X_1) = \dim\, T_L(W^1_{g-2}(C)) = \rho(g, 1, g-2) + \dim\, \ker \mu_{0,L} = g-5$$

while $\dim\, T_{L^2}(X_2) = 2$, because μ_{0,L^2} is injective.
We recall that $T_{L^2}(X_1)^\perp \simeq \mathrm{Im}\, \mu_{0,L}$ and

$$T_{L^2}(X_2)^\perp \simeq \mathrm{Im}\, \mu_{0,L^2} \simeq H^0(C, K_C \otimes \mathcal{O}_C(-P-Q)).$$

We should prove that $\dim\, T_{L^2}(X_1)^\perp \cap T_{L^2(}X_2)^\perp = 3$, that is, P and Q impose independent conditions on the $5-$dimensional space $\mathcal{L} := \mathrm{Im}\, \mu_{0,L}$. As explained in [Ca2], it is enough to show that P and Q impose independent conditions on $D + |L| \subset \mathcal{L}$, where $D \in |K_C \otimes L^{-1}|$ is a divisor not containing $P+Q$. Indeed, let us consider the finite sequence of blow-ups

$$X_k \to X_{k-1} \to \ldots \to X_0 = \mathbb{P}^2$$

necessary to desingularize Γ in x. We denote by C_h the strict transform of Γ in X_h and by $\phi_h : X_h \to \mathbb{P}^2$ the projection; we note that C coincides with C_k and the normalization map $\phi_k|_{C_k}$ is $\phi_{K_C \otimes L^{-1}}$. The curve C_{k-1} has a node in the point x_{k-1} which maps to x via $\phi_{k-1}|_{C_{k-1}}$. The strict transform of a line through x intersects C along a divisor of the form $E_l \otimes \mathcal{O}_C(P+Q)$, with $E_l \in |L|$. As C_{k-1} has a node in x_{k-1}, there exist two lines l_1 and l_2 through x whose strict transforms in X_{k-1} are the two tangent lines to C_{k-1} in x_{k-1}. It follows that the strict transforms of l_1 and l_2 in X_k intersect C in $E_1 \otimes \mathcal{O}_C(2P+Q)$ and $E_2 \otimes \mathcal{O}_C(P+2Q)$ respectively, where $E_1 \otimes \mathcal{O}_C(P) \in |L|$ does not contain Q and $E_2 \otimes \mathcal{O}_C(Q) \in |L|$ does not contain P. It follows that P and Q impose independent conditions on $D + |L|$. $\square$

Proof of Proposition 3.1. Let $[C] \in Z_g$ be a non hyperelliptic curve with no vanishing theta-null. One may find a neighborhood $U \subset \mathcal{M}_g$ of C, intersecting neither the hyperelliptic locus nor the divisor $GP^1_{g,g-1}$, such that there exists a finite ramified covering $\pi : \widetilde{U} \to U$, a universal curve $\varphi : \Gamma_{\widetilde{U}} \to \widetilde{U}$ and a variety $\mathcal{G}^1_{g-2} \xrightarrow{\xi} \widetilde{U}$ proper over $\widetilde{U}$ which parametrizes pairs $(C, (V, L))$ with $[C] \in \widetilde{U}$ and (V, L) a g^1_{g-2} on $\varphi^{-1}(C)$. Up to restricting U, we can also assume that $U \cap GP^1_{g,g-2} \subset Z_g$. The scheme $\mathcal{G}^1_{g-2}$ is smooth of dimension $\rho(g, 1, g-2) + \dim\, \mathcal{M}_g$ (cf. [AC1]). We define the following subvariety of $\mathcal{G}^1_{g-2}$:

$$\widetilde{Z}_g := \{(C, L) \in \mathcal{G}^1_{g-2} \mid [C] \in \pi^{-1}(Z_g \cap U), \ker \mu_{0,L} \neq 0\}.$$

Lemma 3.3 implies that the fiber of the projection from $\widetilde{Z}_g$ on $Z_g \cap U$ is finite. For any $(C, L) \in \widetilde{Z}_g$, the curve C is not hyperelliptic and so $\dim \mathrm{Im}\mu_{0,L} = 5$. Locally the Petri map defines a homomorphism μ of vector bundles on $\mathcal{G}^1_{g-2}$ and $\widetilde{Z}_g$ can be identified with the fifth degeneracy locus of μ. By the fact that each irreducible component of $\widetilde{Z}_g$ has codimension $\leq \rho(g, 1, g-2) + 1$ in $\mathcal{G}^1_{g-2}$ and by the finiteness of the fibers of

$\pi \circ \xi$ over the points of $\pi \circ \xi(\widetilde{Z}_g)$, we can deduce that each component of $Z_g \cap U$ has codimension at most 1 in U. It must be 1 because of the Gieseker-Petri Theorem. $\square$

As a consequence we gain the following:

Corollary 3.4. *The locus GP_{13} has pure codimension 1 in $\mathcal{M}_{13}$.*

Proof. By the above discussion we should only study the component $GP^1_{13,11}$. Given $[C] \in GP^1_{13,11}$, we assume that $[C]$ does not lie in $GP^1_{13,10} \cup GP^2_{13,11} \cup GP^2_{13,12}$. In particular, condition (1) is satisfied for any $L \in G^1_{13}(C)$ for which the Gieseker-Petri Theorem fails. Moreover, $K_C \otimes L^{-1}$ cannot be composed with any involution and so condition (2) is satisfied, too. It follows that $[C] \in Z_g$ and so Proposition 3.1 is enough to conclude. $\square$

Next we turn to the case of genus 12. Given $[C] \in GP^1_{12,10}$ such that condition (1) is satisfied for any $L \in G^1_{10}(C)$ with $\ker \mu_{0,L} \neq 0$, it could still happen that some of the above $L \in W^1_{10}(C)$ violate condition (2), that is, $K_C \otimes L^{-1}$ is not big. We prove the following:

Theorem 3.5. *Let $[C] \in GP^1_{12,10}$ and let us assume that condition (1) is satisfied for any $L \in G^1_{10}(C)$ such that $\ker \mu_{0,L} \neq 0$. If for one of such $L \in W^1_{10}(C)$ the morphism $K_C \otimes L^{-1}$ defines a finite covering of a plane curve Γ of degree strictly less than 12, then $[C]$ lies in $GP^1_{12,7} \cup GP^1_{12,8}$.*

Proof. Let $[C] \in GP^1_{12,10}$ be as in the hypothesis and L be a g^1_{10} on C for which the Gieseker-Petri Theorem fails. If $\phi := \phi_{K_C \otimes L^{-1}} : C \to \Gamma \subset \mathbb{P}^2$ is not birational, then it is a finite covering of degree 6, 4, 3 or 2. We analyze these cases.

(I): $\deg \phi_{K_C \otimes L^{-1}} = 6$. In this case Γ is rational and so C has a g^1_6.

(II): $\deg \phi_{K_C \otimes L^{-1}} = 3$. Then Γ has degree 4 and genus at most 3. If $g(\Gamma) < 3$, then Γ has at least one singular point and by taking the lines through it one sees that Γ is hyperelliptic and so C has a g^1_6.

Let us consider $g(\Gamma) = 3$. As the triple cover is induced by $K_C \otimes L^{-1}$, it follows that $K_C \otimes L^{-1} = \phi^*\mathcal{O}_\Gamma(1) = \phi^* K_\Gamma$ and so $L = \mathcal{O}_C(R)$, where R is the ramification locus. The Base Point Free Pencil Trick thus implies that

$$\ker \mu_{0,L} \simeq H^0(C, K_C \otimes \mathcal{O}_C(-2R)) \simeq H^0(C, \phi^*\mathcal{O}_\Gamma(1) \otimes \mathcal{O}_C(-R)).$$

If this were not zero, then there would exist a divisor D on Γ, $\mathcal{O}_\Gamma(D) = \mathcal{O}_\Gamma(1)$, such that $\phi^*D - R \ge 0$. This would imply that D contains the branch locus B but this is impossible because $\deg B \ge \frac{1}{2} \deg R = 5$ while $\deg D = 4$.

(III): $\deg \phi_{K_C \otimes L^{-1}} = 4$. The curve Γ has degree 3 and so it is either a rational curve or a smooth elliptic curve. In the first case C has a g^1_4.

If Γ is elliptic , then we have that $K_C \otimes L^{-1} = \phi^*\mathcal{O}_\Gamma(1)$ and

$$L = \phi^*(K_\Gamma \otimes \mathcal{O}_\Gamma(-1)) \otimes \mathcal{O}_C(R) = \phi^*\mathcal{O}_\Gamma(-1) \otimes \mathcal{O}_C(R).$$

It follows that

$$\ker \mu_{0,L} \simeq H^0(C, \mathcal{O}_C(R) \otimes (\mathcal{O}_C(R) \otimes \phi^*\mathcal{O}_\Gamma(-1))^{-2}) = H^0(\phi^*\mathcal{O}_\Gamma(2) \otimes \mathcal{O}_C(-R)).$$

This is nonzero whenever there exists a divisor D on Γ such that $\mathcal{O}(D) = \mathcal{O}_\Gamma(2)$ and $\phi^*D - R \ge 0$. This never happens because D has degree 6 and it should contain the base locus B, whose degree is at least $\frac{1}{3} \deg R > 7$. It follows that there exixts no g^1_{10}

on C which does not satisfy the Gieseker-Petri Theorem and whose residual induces a map $4:1$ on an elliptic curve.

(IV): $\deg \phi_{K_C \otimes L^{-1}} = 2$. The degree of Γ is 6 and by the Riemann-Hurwitz Formula it follows that $g(\Gamma) \leq 6$. We can assume that Γ has only double points as singularities because otherwise Γ has a g^1_k for some $k \leq 3$ and, by Remark 2, the curve $[C] \in GP^1_{12,7}$. If $g(\Gamma) \leq 4$, it is easy to check that Γ has always a g^1_3 and so $[C] \in GP^1_{12,7}$. As a consequence the only two cases that require a more detailed analysis are $g(\Gamma) = 5$ and $g(\Gamma) = 6$.

Let us consider the case when Γ is a plane sextic of genus 5. We can assume that the singularities of Γ are 5 double points $P_1, \ldots, P_5$. Some of the P_i's may coincide; indeed, if we need k blow-ups in order to desingularize Γ in P_i, then this point appears k times in the list. We denote by x_i, y_i the counterimage of P_i under the normalization map $p : Y \to \Gamma$. Denoting by B and R the branch locus and the ramification locus respectively, the Riemann-Hurwitz Formula implies that both B and R have degree 6. The double covering $f : C \to Y$ induced by ϕ is given by means of a divisor η on Y of degree -3 such that $2\eta = -B$ and $f_*\mathcal{O}_C = \mathcal{O}_Y \oplus \mathcal{O}_Y(\eta)$. As $\mathrm{Pic}^{-3}(Y) = Y - Y_4$, we can write $\eta = x - D_4$.

We consider the divisor $f^*(D_4) \in \mathrm{Pic}^8(C)$. We can assume that

$$h^0(C, \mathcal{O}_C(f^*D_4))) = h^0(Y, \mathcal{O}_Y(D_4)) + h^0(Y, \mathcal{O}_Y(D_4 + \eta)) = 2,$$

because otherwise we can conclude that $[C] \in \mathcal{M}^2_{12,8} \subset GP^1_{12,7}$. We would like to prove that $\ker \mu_{0,\mathcal{O}_C(f^*D_4)} \neq 0$, which implies $[C] \in GP^1_{12,8}$. By the Base Point Free Pencil Trick we know that $\ker \mu_{0,\mathcal{O}_C(f^*D_4)} \cong H^0(C, K_C \otimes \mathcal{O}_C(f^*D_4)^{-2})$, and this has dimension equal to

$$h^0(C, f^*(K_Y \otimes \mathcal{O}_Y(-\eta - 2D_4))) = h^0(Y, K_Y \otimes \mathcal{O}_Y(-\eta - 2D_4)) + h^0(Y, K_Y \otimes \mathcal{O}_Y(-2D_4));$$

here we have used that $K_C = f^*(K_Y \otimes \mathcal{O}_Y(-\eta))$.
Since $h^0(Y, K_Y \otimes \mathcal{O}_Y(-2D_4)) \neq 0$ whenever D_4 is a theta characteristic on Y, our goal is to show that $h^0(Y, K_Y \otimes \mathcal{O}_Y(-\eta - 2D_4)) > 0$. As

$$K_Y \otimes \mathcal{O}_Y(-\eta - 2D_4) = \mathcal{O}_Y(3)(-x_1 - y_1 - \ldots - x_5 - y_5 - D_4 - x),$$

we need to prove the existence of a plane cubic passing through the points $P_1, P_2, P_3, P_4, P_5, p(x), p(z_1), p(z_2), p(z_3), p(z_4)$, where $D_4 = z_1 + \ldots + z_4$.
We can assume that every g^2_6 on Y is base point free and not composed with an involution and that every plane model of Y as a sextic has only double points as singularities (otherwise Y would have a g^1_3 and C a g^1_6); the same is true for all the curves in a neighborhood U of Y in $\mathcal{M}_5$. Up to shrinking U, we can assume the existence of a proper morphism $\xi : \mathcal{G}^2_6 \to U$, where $\mathcal{G}^2_6$ parametrizes couples (C, l), with $[C] \in U$ and l a g^2_6 on C. We denote by $V_{5,6}$ the variety of irreducible plane curves of degree 6 and genus 5 and by $m : V_{5,6} \to \mathcal{M}_5$ the natural morphism. The locus $m^{-1}(U)$ can be seen as a $PGL(2)$-bundle on $\mathcal{G}^2_6$ parametrizing couples $((C, l), \mathcal{B})$ with $(C, l) \in \mathcal{G}^2_6$ and $\mathcal{B}$ a frame of l. Indeed, giving l and $\mathcal{B}$ is equivalent to fixing a plane model of C. We denote by $p_1 : m^{-1}(U) \to \mathcal{G}^2_6$ the natural morphism. The restriction $m_U : m^{-1}(U) \to U$ coincides with the composition $\xi \circ p_1$ and it is proper because both ξ and p_1 are. Denoting by $\pi : \mathcal{M}_{5,5} \to \mathcal{M}_5$ the forgetful map, the morphism

$$m_1 : m^{-1}(U) \times_U \pi^{-1}(U) \to \pi^{-1}(U)$$

is proper because of the invariance of properness under base extension. A point of $m^{-1}(U) \times_U \pi^{-1}(U)$ is of the form $(\Gamma, (C, z_1, \ldots z_5))$, where C is the normalization of Γ.

We remark that $m^{-1}(U) \times_U \pi^{-1}(U)$ has dimension equal to

$$\dim\, \pi^{-1}(U) + \rho(5,2,6) + \dim\, PGL(2) = \dim\, \pi^{-1}(U) + 10.$$

Let

$$\mathcal{E} := H^0(\mathcal{O}_{\mathbb{P}^2}(3)) \times (m^{-1}(U) \times_U \pi^{-1}(U))$$

be the trivial bundle on $m^{-1}(U) \times_U \pi^{-1}(U)$ and let us define $\mathcal{F}$ to be the bundle on $m^{-1}(U) \times_U \pi^{-1}(U)$ with fiber over $(\Gamma, (C, z_1 \ldots, z_5))$ being the space

$$H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\Delta_\Gamma}) \oplus \bigoplus_{i=1}^{5} H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\phi(z_i)}),$$

where Δ_Γ is the scheme of all singular points of Γ. If Γ is generic this space is

$$H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{P_1}) \oplus \ldots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{P_5}) \oplus \bigoplus_{i=1}^{5} H^0(\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\phi(z_i)}),$$

where $P_1, \ldots, P_5$ are the nodes of Γ. We consider the evaluation map $F : \mathcal{E} \to \mathcal{F}$. Both $\mathcal{E}$ and $\mathcal{F}$ have rank 10 and so the degeneracy locus $X(F)$, if nonempty, has codimension 1 in $m^{-1}(U) \times_U \pi^{-1}(U)$.

In order to show that $X(F) \neq \emptyset$ we observe that, given a cubic $\Gamma_3 \subset \mathbb{P}^2$ and $P_1, \ldots, P_{10}$ ten points on it, one can always find a sextic $\Gamma_6 \subset \mathbb{P}^2$ passing through $P_6, \ldots, P_{10}$ and having nodes in $P_1 \ldots, P_5$ (because there exists a $\mathbb{P}^{27}$ of plane sextics). Denoting by $\tilde{\phi} : \tilde{C} \to \Gamma_6$ the normalization map, the point $(\Gamma_6, (\tilde{C}, \tilde{\phi}^*(P_6), \ldots, \tilde{\phi}^*(P_{10})))$ lies in $X(F)$. Thus we have that

$$\dim\, X(F) = \dim\, m^{-1}(U) \times_U \pi^{-1}(U) - 1 = \dim\, \pi^{-1}(U) + 9.$$

As m_1 is proper, it follows that $m_1(X(F))$ is closed inside $\pi^{-1}(U)$. Moreover,

$$\dim\, m_1(X(F)) = \dim\, X(F) - \dim\, X_e = \dim\, \pi^{-1}(U) + 9 - \dim\, X_e,$$

where X_e is the generic fiber of $m_1|_{X(F)}$. Therefore $\dim\, m_1(X(F)) < \dim\, \pi^{-1}(U)$ if and only if $\dim\, X_e = 10$, that is, the generic fiber of $m_1|_{X(F)}$ coincides with the generic fiber of m_1. If we prove that this cannot happen, then $m_1|_{X(F)}$ is surjective and in particular $(Y, p(x), p(z_1), \ldots, p(z_4)) \in m_1(X(F))$, which implies the existence of a plane model $\widetilde{\Gamma}$ of Y and of a cubic passing through the singular points of $\widetilde{\Gamma}$ and through the images in $\widetilde{\Gamma}$ of $x, z_1, \ldots, z_4$. Therefore it survives only to prove that $\dim\, X_e \neq 10$.

Given a general $[C] \in U$, we have to find general points $z_1, \ldots, z_5 \in C$, a g^2_6 on C, together with a frame $\mathcal{B}$ corresponding to a rational map $\phi : C \to \Gamma \subset \mathbb{P}^2$, such that Γ has 5 nodes $P_1, \ldots, P_5$ and there does not exist a cubic through $P_1, \ldots, P_5, \phi(z_1), \ldots, \phi(z_5)$. We remark that any complete g^2_6 on C is of the form

$$L = K_C \otimes \mathcal{O}_C(-a-b), \;\; a, b \in C.$$

Having chosen a frame for $H^0(C, L)$ and denoted by $\phi : C \to \Gamma \subset \mathbb{P}^2$ the corresponding morphism, this is equivalent to saying that

$$\phi^*\mathcal{O}_\Gamma(1) = \phi^*(\mathcal{O}_\Gamma(3)(-\Delta_\Gamma)) \otimes \mathcal{O}_C(-a-b),$$

that is, every cubic in $\mathbb{P}^2$ passing through the singular points of Γ and the points $\phi(a), \phi(b)$, intersects Γ in other points which are collinear. Choose $\mathcal{B}$ any frame of $K_C(-z_1-z_2)$; it is enough to take z_3, z_4, z_5 such that $\phi(z_3), \phi(z_4), \phi(z_5)$ are not collinear in the plane model of C corresponding to $(K_C \otimes \mathcal{O}_C(-z_1-z_2), \mathcal{B})$.

Now we consider the case when $g(\Gamma) = 6$, namely Γ is a plane sextic with 4 double points $P_1, \ldots, P_4$. Using the notation introduced above, we now have that B has degree 2 and so $\eta \in \mathrm{Pic}^{-1}(Y)$. Choose a point $P \in Y$. Since $\mathrm{Pic}^{-2}(Y) = Y_2 - Y_4$, we can always write $\eta - P = D_2 - D_4$; it follows that $\eta = D_3 - D_4$ with P a point of D_3. As in the previous case, we can assume that

$$h^0(C, \mathcal{O}_C(f^*D_4)) = h^0(Y, \mathcal{O}_Y(D_4)) + h^0(Y, \mathcal{O}_Y(D_3)) = 2,$$

and so $f^*(D_4)$ defines a g^1_8 on C. In trying to prove that it does not satisfy the Gieseker-Petri Theorem, the above method is unsuccessful. Indeed, we should prove the existence of a plane cubic passing through $P_1, \ldots, P_4$, $p(z_1), \ldots, p(z_6)$, $p(P)$, where $D_4 = z_1 + \ldots, z_4$, $D_3 = z_5 + z_6 + P$. As $P \in Y$ is arbitrarily chosen, actually it would be enough to prove the existence of a cubic through $P_1, \ldots, P_4, z_1, \ldots, z_6$ and this is a divisorial condition in $(\mathbb{P}^2)^{10}$. Since $\rho(6,2,6) = 0$, in this case we do not have any degree of freedom in the choice of a g^2_6 on Y, namely in the choice of $P_1, \ldots, P_4$.

Thus we proceed in a slightly different way. We have that $\rho(6,2,7) = 3$ and, given l a base point free g^2_7 on Y, we can assume that it defines a birational morphism

$$\varphi : Y \to \Lambda \subset \mathbb{P}^2,$$

where Λ is a plane septic of genus 6; indeed, l cannot be composed with any involution. We expect Λ to have only nodes as singularities but in this case we cannot exclude the possibility that Λ has some triple points. As Y is the normalization of Λ, we have that

$$K_Y = \varphi^*(\mathcal{O}_\Lambda(4)(-\Delta_\Lambda)) \text{ with } \Delta_\Lambda = \sum_{P \in \mathrm{Sing}\Lambda} (r_P - 1)P,$$

where r_P is the multiplicity of Λ in P. Of course for Λ generic, the singular locus Δ_Λ is the sum of the nine nodes $P_1, \ldots, P_9$ and the condition $\ker \mu_{0,\mathcal{O}_C(f^*D_4)} \neq 0$ is equivalent to the existence of a plane quartic through $P_1, \ldots, P_9$, $\varphi(z_1), \ldots, \varphi(z_6)$. In the non generic case we have a different condition equivalent to $\ker \mu_{0,\mathcal{O}_C(f^*D_4)} \neq 0$ (for instance, when Λ has a triple point Q and six double points $P_1, \ldots, P_6$, then we require that the plane quartic has a double point in Q and passes through $P_1, \ldots, P_6$). However, the number of independent conditions imposed on the plane quartics is the same.

As before, we consider a neighborhood U of Y in $\mathcal{M}_6$ such that there exists a proper morphism $\xi : \mathcal{G}^2_7 \to U$, where $\mathcal{G}^2_7$ parametrizes couples (C, l), with $[C] \in U$ and l a g^2_7 on C. We can assume that, given $[C] \in U$, the generic g^2_7 on C is base point free and not composed with an involution but in this case the models of C as a plane septic can have also some triple points. Denoting by $m : V_{6,7} \to \mathcal{M}_6$ the natural morphism, the restriction $m_U : m^{-1}(U) \to U$ is proper. If $\pi : \mathcal{M}_{6,6} \to \mathcal{M}_6$ is the forgetful map, then the induced morphism $m_1 : m^{-1}(U) \times_U \pi^{-1}(U) \to \pi^{-1}(U)$ is proper, too. We have that

$$\begin{array}{rcl} \dim\, m^{-1}(U) \times_U \pi^{-1}(U) & = & \dim\, \pi^{-1}(U) + \rho(6,2,7) + \dim\, PGL(2) \;= \\ & = & \dim \pi^{-1}(U) + 11 \qquad\qquad\qquad\qquad\quad . \end{array}$$

As in the previous case, we define

$$\mathcal{E} := H^0(\mathcal{O}_{\mathbb{P}^2}(4)) \times (m^{-1}(U) \times \pi^{-1}(U))$$

and $\mathcal{F}$ being the bundle over $m^{-1}(U) \times \pi^{-1}(U)$ whose fiber over $(\Lambda, (C, z_1, \ldots, z_6))$ is

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_\Lambda}) \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\varphi(z_1)}) \oplus \ldots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\varphi(z_6)}).$$

For $\Lambda \in V_{6,7}$ generic we have that

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_\Lambda}) = H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P_1}) \oplus \ldots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P_9}),$$

where $P_1, \ldots, P_9$ are the nodes of Λ. Instead, if for instance Λ has one triple point Q and 6 nodes $P_1, \ldots, P_6$, then the following equality holds:

$$H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\Delta_\Lambda}) = H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{2Q}) \oplus \ldots \oplus H^0(\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{P_6}).$$

We define $F : \mathcal{E} \to \mathcal{F}$ to be the evaluation map. As both $\mathcal{E}$ and $\mathcal{F}$ have rank 15, the situation is analogous to the one already treated. Therefore, in order to prove that the image under m_1 of the degeneracy locus $X(F)$ is the whole $\pi^{-1}(U)$, it is enough to show that the generic fiber X_e of $m_1|_{X(F)}$ is nonempty and that it does not coincide with the generic fiber of m_1. The fact that $X_e \neq \emptyset$ follows easily by observing that, given 15 points on a quartic $\Lambda_4 \subset \mathbb{P}^2$, there always exists a plane septic Λ_7 passing through them and having nodes in the first nine. On the other hand, it can be shown that $\dim X_e \neq 15$ by proceeding like in the case of genus 5 because on a curve C of genus 6 any complete g^2_7 is of the form $l = K_C \otimes \mathcal{O}_C(-a-b-c)$, with $a, b, c \in C$. □

Finally, we obtain that:

Corollary 3.6. *The locus GP_{12} has pure codimension 1 in $\mathcal{M}_{12}$.*

Proof. By the remarks at the beginning of the section we have to study only the component $GP^1_{12,10}$. Given $[C] \in GP^1_{12,10}$ we can assume that $[C] \notin GP^1_{12,9} \cup GP^2_{12,10} \cup GP^2_{12,11}$, which forces any $l \in G^1_{10}(C)$ for which the Gieseker-Petri Theorem fails to verify condition (1). Theorem 3.5 implies that if $[C] \notin GP^1_{12,7} \cup GP^1_{12,8}$, then condition (2) is satisfied, too. We can thus apply Proposition 3.1. □

References

[AC1] E. Arbarello, M. Cornalba, *A few remarks about the variety of irreducible plane curves of given degree and genus*, Ann. Scient. Éc. Norm. Sup. 4 sreie **16** (1983), 467-488.

[AC2] E. Arbarello, M. Cornalba, *Footnotes to a Paper of Beniamino Segre*, Math. Ann. **256** (1981), 341-362.

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves*, Grundl. Math. Wiss. **267**, Springer Verlag, 1985.

[BS] A. Bruno, E. Sernesi, *A note on the Petri loci*, `ArXiv`:1012.0856v2.

[Ca1] A. Castorena, *Curves of genus seven that do not satisfy the Gieseker-Petri Theorem*, Bollettino U.M.I. **8-B** (2005), 697-706.

[Ca2] A. Castorena, *A family of plane curves with moduli* $3g-4$, Glasgow Math. J. **49** (2007), 417-422.

[Ca3] A. Castorena, *Remarks on the Gieseker-Petri divisor in genus eight*, Rend. Circ. Mat. Palermo **59** (2010), 143-150.

[EH1] D. Eisenbud, J. Harris, *A simple proof of the Gieseker-Petri theorem on special divisors*, Invent. Math. **74** (1983), 269-280

[EH2] D. Eisenbud, J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), 359-387.

[EH3] D. Eisenbud, J. Harris, *Irreducibility of some families of linear series with Brill-Noether Number* -1, Ann. Scient. Éc. Norm. Sup. 4e série **22** (1989), 33-53.

[ELMS] D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer, *The Clifford dimension of a projective curve*, Compos. Math. **72** (1989), 173-204.

[F1] G. Farkas, *Gaussian maps, Gieseker-Petri loci and large theta-characteristics*, J. Reine Angew. Math **581** (2005), 151-173.

[F2] G. Farkas, *Rational maps between moduli space of curves and Gieseker-Petri divisor*, J. Algebraic Geom. **19** (2010), 243-284.

[Fu] W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. Math. **90** (1969), 542-575.

[Gi] D. Gieseker, *Stable curves and special divisors*, Invent. Math. **66** (1982), 251-275.

[Ha1] R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75-110.
[Ha2] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg-Berlin, 1977.
[Ma] G. Martens, *Über den Clifford-Index algebraischer Kurven*, J. Reine Angew. Math. **36** (1982), 83-90.
[St] F. Steffen, *A generalized principal ideal theorem with an application to Brill-Noether theory*, Invent. Math. **132** (1998), 73-89.
[Te] M. Teixidor, *The divisor of curves with a vanishing theta-null*, Compos. Math. **66** (1988), 15-22.

HUMBOLDT UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, 10099 BERLIN
E-mail address: lelli@math.hu-berlin.de