Long time averaged reflection force and homogenization of oscillating Neumann boundary conditions.

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Abstract. This paper concerns with two issues. The first issue is the existence and the uniqueness of the ergodic type number d which appears in the oblique boundary condition. The second issue is the application of the number for the study of homogenizations of oscillating Neumann boundary conditions.

Résumé. Dans cette article, nous traitons deux problèmes. Le premier est l'existence et l'unicité d'un nombre du type ergodique d qui apparaît dans la condition oblique sur le bord. Le second est l'application de ce nombre pour la recherche des homogénéizationses conditions Neumann sur des bords oscillants.

1 Introduction

First, we are concerned with the existence and uniqueness of the number d in the following problem.

$$F(x,\nabla u,\nabla^2 u)=0 \qquad \text{in} \quad \Omega, \tag{1}$$

$$d+<\nabla u,\gamma(x)>-g(x)=0 \qquad \text{on} \quad \partial\Omega, \tag{2}$$

where Ω is a domain in $\mathbf{R}^n$, F is a fully nonlinear uniformly elliptic Hamilton-Jacobi-Bellman (HJB in short) operator:

$$F(x,\nabla u,\nabla^2 u)=\sup_{\alpha\in \mathrm{A}}\{-\sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}-\sum_{i=1}^{n} b_i^{\alpha}(x)\frac{\partial u}{\partial x_i}\}, \tag{3}$$

satisfying the following conditions. A is a set of controls, and by denoting $n\times n$ matrices $A^{\alpha}=(a_{ij}^{\alpha}(x))_{ij}$ ($\alpha\in \mathrm{A}$), there exist $n\times m$ matrices σ^{α} such that

$$\begin{aligned} &A^{\alpha}(x)=\sigma^{\alpha}(\sigma^{\alpha})^t(x) \qquad \text{any} \quad x\in\Omega, \quad \alpha\in \mathrm{A},\\ &\lambda_1 I\leq A^{\alpha}(x)\leq\Lambda_1 I \qquad \text{any} \quad x\in\Omega, \quad \alpha\in \mathrm{A}, \end{aligned} \tag{4}$$

where $0<\lambda_1\leq\Lambda_1$ positive constants, I the $n\times n$ identity matrix. There exists a positive constant $L>0$ such that

$$\begin{aligned} |a_{ij}^{\alpha}(x)-a_{ij}^{\alpha}(y)| &\leq L|x-y| \qquad \text{any} \quad 1\leq i,j\leq n, \quad x\in\Omega, \quad \alpha\in A,\\ |b_i^{\alpha}(x)-b_i^{\alpha}(y)| &\leq L|x-y| \qquad \text{any} \quad 1\leq i\leq n, \quad x\in\Omega, \quad \alpha\in A. \end{aligned} \tag{5}$$

There also exists a positive constant γ_0, such that for the outward unit normal vector $\mathbf{n}(x)$ ($x\in\partial\Omega$), $\gamma(x)$ satisfies

$$<\gamma(x),\mathbf{n}(x)> \ \geq\gamma_0>0 \qquad \text{any} \quad x\in\partial\Omega. \tag{6}$$

The domain Ω is assumed to be either one of the following:

$$\text{Bounded open domain in} \quad \mathbf{R}^n \qquad \text{with} \quad C^{3,1} \quad \text{boundary}, \tag{7}$$

or

Half space in $\mathbf{R}^n$, periodic in the first $n-1$ variables with $C^{3,1}$ boundary
: $\{(x',x_n)|$ periodic in $x'=(x_1,...,x_{n-1})\in(\mathbf{R}/\mathbf{Z})^{n-1}, \quad x_n\geq f_1(x')\}$,
where $f_1\in C^{3,1}((\mathbf{R}/\mathbf{Z})^{n-1})$. (8)

(In the latter case (8), a supplement boundary condition at $x_n = \infty$ will be added to (1)-(2).)

The following example implies the qualitative meaning of the number d.

Example 1.1. *Let Ω be a domain in (7), and $g(x)$ be a Lipschitz continuous function on $\partial\Omega$. Assume that there exists a number d such that the following problem has a viscosity solution.*

$$-\Delta u = 0 \qquad in \quad \Omega,$$

$$d + < \nabla u, \mathbf{n}(x) > - g(x) = 0 \qquad on \quad \partial\Omega.$$

Then,

$$d = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} g(x) dS.$$

Proof of Example 1.1. In the Green's first identity:

$$\int_{\Omega} \Delta u v dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS,$$

we put $v = 1$, and get $d|\partial\Omega| = \int_{\partial\Omega} g(x) dS$.

Thus, d is a kind of the averaged quantity on $\partial\Omega$. For general Hamiltonians F, the way to construct the number d and $u(x)$ in (1)-(2) is the following. Here we assume that (7) holds. (The case (8) is more complicated, and will be treated in Section 3 below.) For any $\lambda > 0$, consider

$$F(x, \nabla u_\lambda, \nabla^2 u_\lambda) = 0 \qquad \text{in} \quad \Omega, \tag{9}$$

$$\lambda u_\lambda + < \nabla u, \gamma(x) > - g(x) = 0 \qquad \text{on} \quad \partial\Omega. \tag{10}$$

The regularity of u_λ ($\lambda \in (0, 1)$) which will be shown in Section 2 yields, for any fixed $x_0 \in \Omega$

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d \qquad \text{uniformly in} \quad \overline{\Omega}, \tag{11}$$

and by taking a subsequence $\lambda' \downarrow 0$,

$$\lim_{\lambda' \downarrow 0} (u_{\lambda'}(x) - u_{\lambda'}(x_0)) = u(x) \qquad \text{uniformly in} \quad \overline{\Omega}. \tag{12}$$

The limit number d is unique in the sense that with which (1)-(2) has a viscosity solution. The above limit function $u(x)$ is one of such solutions. (The solution of (1)-(2) is not unique, for $u+C$ (C constant) is also a solution.) We shall show in Section 2 these facts. Now, the meaning of the number d can be stated by using (11). For any fixed measurable function $\alpha(t):[0,\infty)\to A$ (control process), let (X_t^α, A_t^α) be the stochastic process defined by

$$
\begin{aligned}
X_t^\alpha &= x+\int_0^t \sigma^\alpha(X_s^\alpha)dW_s+\int_0^t b^\alpha(X_s^\alpha)ds-\int_0^t \gamma(X_s^\alpha)dA_s \qquad t\geq 0,\\
A_t^\alpha &= \int_0^t 1_{\partial\Omega}(X_s^\alpha)dA_s \quad \text{is continuous, non decreasing in} \quad t\geq 0, \qquad (13)
\end{aligned}
$$

where $b^\alpha=(b_i^\alpha)_i$, $1_{\partial\Omega}(\cdot)$ a characteristic function on $\partial\Omega$, W_t ($t\geq 0$) an $m-$dimensional Brownian motion. The study of the existence and the uniqueness of (X_t^α, A_t^α) is called the Skorokhod problem, and its solvability is known under the preceding assumptions. We refer the readers to P.-L. Lions and A.S. Sznitman [30], P.-L. Lions, J.M. Menaldi and A.S. Sznitman [28], and P.-L. Lions [27]. Let

$$J_\lambda^\alpha(x)=E_x\int_0^\infty e^{-\lambda t}g(X_t^\alpha)1_{\partial\Omega}(X_t^\alpha)dA_t,$$

and define

$$u_\lambda(x)=\inf_{\alpha(\cdot)} J_\lambda^\alpha(x) \qquad \text{in} \quad \Omega, \tag{14}$$

where the infimum is taken over all possible control processes. It is known that u_λ is the unique solution of (9)-(10). (See, P.-L. Lions and N.S. Trudinger [31], and M.I. Freidlin and A.D. Wentzell [21].) Thus,

$$d=\lim_{\lambda\downarrow 0}\inf_{\alpha(\cdot)}\lambda E_x\int_0^\infty e^{-\lambda t}g(X_t^\alpha)1_{\partial\Omega}(X_t^\alpha)dA_t, \tag{15}$$

if the right hand side of (11) exists, which represents the fact that the number d is the long time averaged reflection force on the boundary. (Each time the tragectory reaches to $\partial\Omega$, it gains the force $g(x)$ and is pushed back in the direction of $-\gamma(x)$.) We remark the similarity of the convergence (11) to the so-called ergodic problem for HJB equations. That is, by considering,

$$\lambda u_\lambda(x)+F(x,\nabla u_\lambda,\nabla^2 u_\lambda)=0 \qquad \text{in} \quad \Omega,$$

$$<\nabla u_\lambda(x),\gamma(x)>=0 \qquad \text{on} \quad \partial\Omega,$$

it is known that an unique number d' exists such that

$$\lim_{\lambda\downarrow 0}\lambda u_\lambda(x)=d' \qquad \text{uniformly in} \quad \Omega.$$

We refer the readers to M. Arisawa and P.-L. Lions [6], M. Arisawa [1], [2], A. Bensoussan [10] for the various types (operators and boundary conditions) of ergodic problems. As the above ergodic problem "in the domain", the existence of d in (2) "on the boundary" relates to the ergodicity of the stochastic process (13). Even for some classes of degenerate elliptic operators F, the number d in (2) exists. We remark this in Section 4, below.

Next, we turn our interests to the homogenization. The unique existence of d in (1)-(2) plays an essential role to study the homogenization of oscillating Neumann boundary conditions. The simplest example is as follows.

Example 1.2. *Let c, g, $f_1(x,\xi_1)$ be functions defined in $(x,\xi_1)\in\mathbf{R}^2\times\mathbf{R}\backslash\mathbf{Z}$ (periodic in ξ_1 with period 1). Assume that $f_1\geq 0$, and that there exists a constant $c_0>0$ such that $c>c_0>0$. For any $\varepsilon\geq 0$, let*

$$\Omega_\varepsilon=\{(x_1,x_2)| \quad \varepsilon f_1(x,\frac{x_1}{\varepsilon})\underline{\leq} x_2\underline{\leq} b, \quad |x_1|\underline{\leq} a\},$$

$$\Gamma_\varepsilon=\{(x_1,x_2)| \quad x_2=\varepsilon f_1(x,\frac{x_1}{\varepsilon})\}\cap\partial\Omega_\varepsilon.$$

Let $u_\varepsilon(x)$ ($\varepsilon>0$) be the solution of

$$-\Delta u_\varepsilon=0 \qquad in \qquad \Omega_\varepsilon, \tag{16}$$

$$<\nabla u_\varepsilon(x),\mathbf{n}_\varepsilon(x)>+c(x,\frac{x_1}{\varepsilon})u_\varepsilon=g(x,\frac{x_1}{\varepsilon}) \qquad on \quad \Gamma_\varepsilon, \tag{17}$$

$$u_\varepsilon=0 \qquad on \quad \partial\Omega_\varepsilon\backslash\Gamma_\varepsilon, \tag{18}$$

where $\mathbf{n}_\varepsilon(x)$ is the outward unit normal to Γ_ε. Then, as $\varepsilon\downarrow 0$, u_ε converges to a unique functiont $u(x)$ uniformly in $\overline{\Omega_0}$, which is the solution of

$$-\Delta u=0 \qquad in \quad \Omega_0,$$

$$<\nabla u(x),\nu(x)>+\overline{L}(x,u,\nabla u)=0 \qquad on \quad \Gamma_0, \tag{19}$$

$$u=0 \qquad on \quad \partial\Omega_0\backslash\Gamma_0,$$

where ν is the outward unit normal to Γ_0, and $\overline{L}$ is defined as follows.
Let $O(x) = \{(\xi_1, \xi_2)| \quad \xi_2 \geq f_1(x, \xi_1), \quad \xi_1 \in \mathbf{R} \backslash \mathbf{Z}\}$. Then, for any fixed $(x, r, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^2$, there exists a unique number $d(x, r, p)$ such that

$$-\Delta_\xi v \equiv -(\frac{\partial^2 v}{\partial \xi_1^2} + \frac{\partial^2 v}{\partial \xi_2^2}) = 0 \qquad in \quad O(x),$$

$$d(x, r, p) + < \nabla_\xi v, \gamma(\xi) > -(\sqrt{1 + (\frac{\partial f_1}{\partial \xi_1})^2} g - \sqrt{1 + (\frac{\partial f_1}{\partial \xi_1})^2} cr - p_1 \frac{\partial f_1}{\partial \xi_1}) = 0$$

on $\partial O(x)$, where $\gamma(\xi) = (\frac{\partial f_1}{\partial \xi_1}, -1)$ $(\xi \in \partial O(x))$, and

$$\overline{L}(x, r, p) = -d(x, r, p). \tag{20}$$

In A. Friedman, B. Hu, and Y. Liu [22], a similar problem to the above example (linear, three scales case) was treated by the variational approach. (See also [13].) We shall extend the result (including Example 1.2.) to nonlinear problems by using the existence of the long time averaged reflection number d in (1)-(2). As Example 1.2 indicates, the effective limit boundary condition (19) is defined by using the long time averaged number in (20). Our present approach was inspired by the classical method of formal asymptotic expansions of A. Bensoussin, J.L. Lions, and G. Papanicolaou [11]. This approach is closely related to the ergodic problem for HJB equations described in the preceding part of this introduction. For the application of the ergodic problem ([6], [1], [2]) to obtain the effective P.D.E. in the domain, we refer the readers to M. Arisawa [3], [4], M. Arisawa and Y. Giga [5], L.C. Evans [18], [19], and P.-L. Lions, G, Papanicolaou, and S.R.S. Varadhan [29]n. As far as we know, there is no existing reference which treats the homogenization of the oscillating Neumann boundary conditions from the view point of the ergodic problem.

The plan of this paper is the following.
§1. Introduction.
§2. Existence and uniqueness of the number d in the case of the bounded domain.
§3. Existence and uniqueness of the number d in the case of the half space.
§4. Some remarks on the degenerate elliptic operators case.

§5. Homogenization of the oscillating Neumann boundary conditions.

Throughout of this paper, the gradient and the Hesse matrix of $u(x)$ ($x \in \Omega \subset \mathbf{R}^n$) (resp. $v(\xi)$ ($\xi \in \Omega' \subset \mathbf{R}^n$)) are denoted by $\nabla u(x)$, $\nabla^2 u(x)$ (resp. $\nabla_\xi v(\xi)$, $\nabla_\xi^2 v(\xi)$ or $D_\xi^2 v(\xi)$). For $u(x)$ ($x \in \Omega \subset \mathbf{R}^n$), the partial derivatives in x_i, x_j ($1 \leq i, j \leq n$) are denoted by $\frac{\partial u}{\partial x_i} = D_i u$, $\frac{\partial^2 u}{\partial x_i \partial x_j} = D_{ij} u$, etc., and the derivatives in the directions of $y, z \in \mathbf{R}^n$ are denoted by $D_y u = \sum_{i=1}^n y_i \frac{\partial u}{\partial x_i}$, $D_{yz} u = \sum_{i,j=1}^n y_i z_j \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc.. When a function $w(x, \xi)$ depends on both variables of $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$, and when we consider the derivatives $\frac{\partial^2 w(x,\xi)}{\partial x_k \partial \xi_l}$ etc., we denote them by $D_{ij} w(x, \xi)$ ($1 \leq i, j \leq 2n$), etc.. For the twice continuously differentiable function $u(x)$ ($x \in \Omega \subset \mathbf{R}^n$), we denote $|u|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |u|$, $|\nabla u|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \sup_{1 \leq i \leq n} |\frac{\partial u}{\partial x_i}(x)|$, $|\nabla^2 u|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \sup_{1 \leq i, j \leq n} |\frac{\partial^2 u}{\partial x_i \partial x_j}(x)|$,

$$|u|_{\beta;\Omega} = \sup_{(x,y) \in \Omega \times \Omega} \frac{|u(x) - u(y)|}{|x - y|^\beta}, \quad |\nabla u|_{\beta;\Omega} = \sup_{1 \leq i \leq n} \sup_{(x,y) \in \Omega \times \Omega} \frac{|\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y)|}{|x - y|^\beta} \quad 0 < \beta \leq 1,$$

$$|u|_{j,\beta;\Omega} = |\nabla^j u|_{L^\infty(\Omega)} + \sup_{x \neq y \in \Omega} \frac{|\nabla^j u(x) - \nabla^j u(y)|}{|x - y|^\beta} \qquad 0 < \beta \leq 1, \quad j = 1, 2.$$

We consider the solvability of PDEs in the framework of viscosity solutions, and treat the second-order sub and super differentials of upper and lower semi continuous functions $u(x)$ and $v(x)$ ($x \in D \subset \mathbf{R}^n$) at a point $\overline{x}$ in the domain D. We denote them by $J_D^{2,+} u(\overline{x})$ (the second-order superjets of u at $\overline{x}$) and $J_D^{2,-} v(\overline{x})$ (the second-order subjets of v at $\overline{x}$) respectively. (See M.G. Crandall and P.-L. Lions [16], M.G. Crandall, H. Ishii and P.-L. Lions [15], and W.H. Fleming and H.M. Soner n[20].) We use the notation $B(x, r)$ ($x \in \Omega, \quad r > 0$) for the open ball centered at x with radius $r > 0$.

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2 Existence and uniqueness of the long time averaged reflection force in the bounded domain.

In this section, the existence and uniqueness of the number d in (1)-(2) is shown in the case that Ω satisfies (7). The Hamiltonian $F(x,\nabla u,\nabla^2 u)$, given in (3), positively homogeneous in degree one, is assumed to satisfy (4) and (5); the vector field γ on $\partial\Omega$ is assumed to satisfy (6). For the existence, we further assume that

$$|a^\alpha_{ij}, |\nabla a^\alpha_{ij}|, |\nabla^2 a^\alpha_{ij}|, |b^\alpha_i|, |\nabla b^\alpha_i|, |\nabla^2 b^\alpha_i| \leq K \qquad \text{any} \quad x\in\Omega, \quad 1\leq i,j\leq n, \quad \alpha\in A, \tag{21}$$

where $K>0$ is a constant, and that γ, g can be extendable in a neighborhood U of $\partial\Omega$ to twice continuously differentiable functions so that

$$|\nabla\gamma|, |\nabla^2\gamma|, |\nabla^2 g|, |\nabla^2 g| \leq K \qquad \text{in} \quad U, \tag{22}$$

where $K>0$ is the constant in (21). For the existence of d, we approximate (1)-(2) by (9)-(10) ($\lambda\in(0,1)$) and examine the regularity of u_λ, uniformly in λ. In order to have (11)-(12), we need the following estimates.

Theorem 2.1. *Assume that Ω is (7), and that (4), (6), (21) and (22) hold. Then there exists a unique solution $u_\lambda\in C^{1,1}(\overline{\Omega})\cap C^{2,\beta}(\Omega)$ of (9)-(10), where $\beta>0$ depends on n and Λ_1/λ_1. Moreover for any fixed $x_0\in\Omega$, there exists a constant $C>0$ such that the following estimates hold.*

$$|u_\lambda - u_\lambda(x_0)|_{L^\infty(\overline{\Omega})} \leq C \qquad \textit{any} \quad \lambda\in(0,1), \tag{23}$$

$$|\nabla u_\lambda|_{L^\infty(\overline{\Omega})} \leq C \qquad \textit{any} \quad \lambda\in(0,1), \tag{24}$$

$$|\nabla u_\lambda|_{1;\overline{\Omega}} \leq C \qquad \textit{any} \quad \lambda\in(0,1). \tag{25}$$

Remark 2.1 One can replace the conditions (21)-(22) to other conditions ,for example those in [24]n, to have

$$|u_\lambda(x) - u_\lambda(y)| \leq C|x-y|^\theta \qquad \text{any} \quad x,y\in\overline{\Omega}, \quad \lambda\in(0,1),$$

where $C>0$, $\theta\in(0,1)$ are independent on $\lambda>0$. The proof of this inequality can be done in a similar way to [24], but by taking account of the

Neumann type boundary conditions, and also by using the estimate (23). We do not write the proof in this direction here, but shall use the method in a future occassion.

Proof of Theorem 2.1. For each $\lambda > 0$, the existence and uniqueness of $u_\lambda \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$ is established in P.-L. Lions and N.S. Trudinger [31]n. We are to show the uniform (in $\lambda \in (0,1)$) regularity (23)-(25) in the following two steps. In Step 1, (23) will be shown, and in Step 2, (24) and (25) will be shown.

<u>Step 1.</u> We prove (23) by a contradiction argument. Let $x_0 \in \Omega$ be fixed. Assume, as $\lambda > 0$ goes to 0

$$|u_\lambda - u_\lambda(x_0)|_{L^\infty(\overline{\Omega})} \to \infty.$$

Set

$$\varepsilon_\lambda \equiv |u_\lambda - u_\lambda(x_0)|^{-1}_{L^\infty(\overline{\Omega})} \qquad \lambda \in (0,1),$$

and let $v_\lambda \equiv \varepsilon_\lambda(u_\lambda - u_\lambda(x_0))$. Then,

$$|v_\lambda|_{L^\infty(\overline{\Omega})} = 1, \quad v_\lambda(x_0) = 0 \qquad \text{any} \quad \lambda \in (0,1).$$

From (3), v_λ satisfies $F(x, \nabla v_\lambda, \nabla^2 v_\lambda) = 0$ in Ω, and from (4) the Krylov-Safonov inequality (see [12]n for instance) leads: for any compact set $V \subset\subset \Omega$, there exists a constant $M_V > 0$ such that

$$|\nabla v_\lambda|_{L^\infty(\overline{V})} \leq M_V \qquad \text{any} \quad \lambda \in (0,1). \tag{26}$$

We denote

$$v^*(x) = \limsup_{\lambda \downarrow 0, y \to x} v_\lambda(y), \quad v_*(x) = \liminf_{\lambda \downarrow 0, y \to x} v_\lambda(y).$$

Then, since $v_\lambda(x_0) = 0$ ($\forall \lambda \in (0,1)$), from (26) we have

$$v^*(x_0) = v_*(x_0) = 0, \tag{27}$$

$$|v^*|_{L^\infty(\overline{\Omega})} = 1, \quad \text{or} \quad |v_*|_{L^\infty(\overline{\Omega})} = 1. \tag{28}$$

From (2), v_λ satisfies

$$< \nabla v_\lambda, \gamma(x) >= \varepsilon_\lambda g - \lambda(v_\lambda + \varepsilon_\lambda u_\lambda(x_0)),$$

and by the comparison result for (9)-(10)

$$|\lambda u_\lambda(x_0)|_{L^\infty(\overline{\Omega})}\leq C \qquad \text{any} \quad \lambda\in(0,1),$$

where $C>0$ is a constant. By letting $\lambda\downarrow 0$, v^* and v_* are viscosity solutions of

$$<\nabla v^*,\gamma(x)>\leq 0 \qquad \text{on} \quad \partial\Omega, \tag{29}$$

$$<\nabla v_*,\gamma(x)>\geq 0 \qquad \text{on} \quad \partial\Omega, \tag{30}$$

and $v^*(x)$ (resp. $v_*(x)$) ($x\in\Omega$) satisfies

$$F(x,\nabla v^*,\nabla^2 v^*)\leq 0, \qquad (\text{resp.} \quad F(x,\nabla v_*,\nabla^2 v_*)\geq 0) \qquad \text{in} \quad \Omega. \tag{10'}$$

(We refer the readers to [15] and G. Barles and B. Perthame [9]n for this stability result.)

Now we employ the strong maximum principle of M. Bardi and F. Da-Lio [7]. Remark that $F(x,p,R)$ given in (3), satisfying (4) and (21) enjoys the following two properties of (31) and (32).

(Scaling property) For any $x_0\in\Omega$, for any $\eta>0$, there exists a function ϕ: $(0,1)\to(0,\infty)$ such that

$$\overline{F}(x,\xi p,\xi R)\geq\phi(\xi)\overline{F}(x,p,R) \qquad \text{any} \quad \xi\in(0,1), \tag{31}$$

holds for any $x\in B(x_0,\eta)$, $0<|p|\leq\eta$, $|R|\leq\eta$.

(Nondegeneracy property) For any $x_0\in\Omega$, for any small vector $\nu\neq 0$, there exists a positive number r_0 such that

$$\overline{F}(x_0,\nu,I-r\nu\otimes\nu)>0 \qquad \text{any} \quad r>r_0. \tag{32}$$

We cite the following result for our present and later purposes.

Lemma A. (n[7]n) *(Strong maximum priciple) Let $\Omega\subset\mathbf{R}^n$ be an open set and let u be an upper semicontinuous viscosity subsolution of*

$$\overline{F}(x,\nabla u,\nabla^2 u)=0 \qquad \textit{in} \quad \Omega,$$

which attains a maximum in Ω. Assume that $\overline{F}$ satisfies (31), (32), and *for any $x_0 \in \Omega$ there exists $\rho_0 > 0$ such that for any $\nu \in B(0,\rho_0)\backslash\{0\}$, (32)*

holds for some $\quad r_0 > 0.$

$$(33)$$

Then, u is a constant.

We go back to the proof of (23). Assume that $|v^*|_{L^\infty(\overline{\Omega})} = 1$ holds in (28). (The another case of $|v_*|_{L^\infty(\overline{\Omega})} = 1$ can be treated similarly.) Thus from (27), v^* is not constant, and from (10)' and the strong maximum principle (Lemma A), v^* attains its maximum at a point $x_1 \in \partial\Omega$:

$$v^*(x_1) > v^*(x) \qquad \text{any} \quad x \in \Omega.$$

Since $\partial\Omega$ is $C^{3,1}$, the interior sphere condition (see D. Gilbarg and N.S. Trudinger [23]n) is satisfied : there exists $y \in \Omega$ such that for $R = |x_1 - y|$

$$B(y,R) \in \Omega, \qquad x_1 \in \partial B(y,R).$$

Let

$$\phi(x) = e^{-cR^2} - e^{-c|x-y|^2} \qquad x \in \Omega,$$

where $c > 0$ is a constant large enough so that

$$\begin{aligned}
&F(x_1, \nabla\phi(x_1), \nabla^2\phi(x_1)) \\
&= F(x_1, 2c(x_1-y)e^{-c|x_1-y|^2}, 2ce^{-c|x_1-y|^2}(I - 2c(x_1-y)\otimes(x_1-y))) \\
&= 2ce^{-c|x_1-y|^2}F(x_1, x_1-y, I - 2c(x_1-y)\otimes(x_1-y)) > 0
\end{aligned}$$

holds. (Here, we used (3), (32) and (33).) By the lower semicontinuity of F in x, there exists $r \in B(0,R)$ and $C' > 0$ such that

$$F(x, \nabla\phi(x), \nabla^2\phi(x)) \geq C' > 0 \qquad \text{in} \quad B(x_1,r)\cap\overline{\Omega}. \tag{34}$$

We claim that

$$v^*(x) - v^*(x_1) - \phi(x) \leq 0 \qquad \text{in} \quad B(x_1,r)\cap\overline{\Omega}. \tag{35}$$

In fact, if $x \in B(y,R)^c$, $\phi(x) \geq 0$ and (35) holds. Assume that for $x' \in B(x_1,r)\cap B(y,R)$ (35) does not hold, and

$$v^*(x') - v^*(x_1) - \phi(x') = \max_{B(x_1,r)\cap B(y,R)} v^*(x) - v^*(x_1) - \phi(x).$$

Then by the definition of the viscosity solution,

$$F(x', \nabla\phi(x'), \nabla^2\phi(x')) \leq 0,$$

which contradicts to (34). Therefore, (35) holds. By remarking that $\phi(x_1) = 0$, (35) indicates that $v^* - \phi$ takes its maximum at $x_1 \in \partial\Omega$. Since v^* satisfies (29) in the sense of viscosity solutions, either

$$< \phi(x_1), \gamma(x_1) > \leq 0,$$

or

$$F(x_1, \nabla\phi(x_1), \nabla^2\phi(x_1)) \leq 0$$

must be satisfied. However from the definition of ϕ, (6) and (34), both of the above are not satisfied. We got a contradiction, and proved (23).

Step 2. To obtain (24) and (25), we appply (23) in the argument of [31]n. First, we regularlize the Hamiltonian F. Let ρ be a mollifier on $\mathbf{R}^n$ ($\rho \geq 0$, $\rho \in C_0^\infty(\mathbf{R}^n)$, $\int \rho = 1$). For any $\delta > 0$, set

$$h_\delta(y) = \delta^{-n} \int_{\mathbf{R}^N} \rho(\frac{y-z}{\delta})(\inf_{1 \leq k \leq N} z_k) dz,$$

$$F_\delta^N[u] \equiv h_\delta(L^{\alpha_1}u, ..., L^{\alpha_N}u),$$

where

$$L^{\alpha_l}u = -\sum_{i,j=1}^n a_{ij}^{\alpha_l} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_i^\alpha b_i^{\alpha_l} \frac{\partial u}{\partial x_i} \qquad 1 \leq l \leq N.$$

Remark that for any $\delta \in (0,1)$, the operator $F_\delta^N(x,p,R)$ satisfies

$$\lambda_1 I \leq (\frac{\partial F_\delta^N}{\partial r_{ij}}(x,p,R))_{1 \leq i,j \leq n} \leq \Lambda_1 I \qquad x \in \Omega, \quad R \in \mathbf{S}^n, \tag{36}$$

$$F_\delta^N(x,p,R) \leq \mu_0(1 + |p| + |R|) \qquad x \in \Omega, \quad R \in \mathbf{S}^n, \tag{37}$$

$$|\frac{\partial F_\delta^N}{\partial x}|, |\frac{\partial F_\delta^N}{\partial p}|, |\frac{\partial F_\delta^N}{\partial R}| \leq \mu_1\{(1+|p|+|R|)|x| + |p| + |R|\} \qquad x \in \Omega, \quad R \in \mathbf{S}^n, \tag{38}$$

$$|\frac{\partial^2 F_\delta^N}{\partial x^2}|, |\frac{\partial^2 F_\delta^N}{\partial x \partial p}|, |\frac{\partial^2 F_\delta^N}{\partial x \partial R}| \leq \mu_2\{(1+|p|+|R|)|x|+|p|+|R|\} \times |x| \qquad x \in \Omega, \quad R \in \mathbf{S}^n, \tag{39}$$

where μ_i $(i = 0, 1, 2)$ are positive constants, and $|p| = \max_{1 \leq i \leq n} |p_i|$ $(p = (p_i)_{1 \leq i \leq n})$, $|R| = \max_{1 \leq i,j \leq n} |r_{ij}|$ $(R = (r_{ij})_{1 \leq i,j \leq n})$.

We need the following a priori estimates.

Lemma 2.2. *Let $u^{\delta}_{\lambda,N} \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of*

$$F^N_{\delta}(x, \nabla u^{\delta}_{\lambda,N}, \nabla^2 u^{\delta}_{\lambda,N}) = 0 \qquad in \quad \Omega, \tag{40}$$

$$\lambda u^{\delta}_{\lambda,N} + < \nabla u^{\delta}_{\lambda,N}, \gamma(x) > -g(x) = 0 \qquad on \quad \partial\Omega. \tag{41}$$

Then, there exists $C > 0$ such that

$$|\nabla u^{\delta}_{\lambda,N}|_{L^{\infty}(\overline{\Omega})}, \quad |\nabla^2 u^{\delta}_{\lambda,N}|_{L^{\infty}(\overline{\Omega})} \leq C \qquad any \quad \delta, \quad \lambda \in (0,1), \quad N \in \mathbf{N}, \tag{42}$$

where $C > 0$ depends on n, λ_1, Λ_1, μ_i ($i = 0, 1, 2$), Ω, and K.

Remark 2.2. In the estimates of [31], Theorem 2.1n, the above constant C depends also on $\lambda \in (0, 1)$.

By delaying the proof of Lemma 2.2, we shall show how (42) leads (24) and (25). By the method of continuity, for each $\delta > 0$ the a priori estimate (42) yields the existence of $u^{\delta}_{\lambda,N} \in C^3(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ of (40)-(41). Put $w^{\delta}_{\lambda,N} = u^{\delta}_{\lambda,N} - u^{\delta}_{\lambda,N}(x_0)$. The same argument as in Step 1 works for $w^{\delta}_{\lambda,N}$, and

$$|w^{\delta}_{\lambda,N}|_{L^{\infty}(\overline{\Omega})} \leq C \qquad \text{any} \quad \delta, \quad \lambda \in (0,1), \quad N \in \mathbf{N}.$$

From (42), by extracting a subsequence of $\delta' \downarrow 0$, there exists $w_{\lambda,N} \in C^{1,1}(\overline{\Omega})$ such that

$$\lim_{\delta' \downarrow 0} w^{\delta}_{\lambda,N} = w_{\lambda,N} \qquad \text{uniformly in} \quad \overline{\Omega},$$

$$\lim_{\delta' \downarrow 0} \nabla w^{\delta}_{\lambda,N} = \nabla w_{\lambda,N} \qquad \text{uniformly in} \quad \overline{\Omega},$$

and

$$|w_{\lambda,N}|_{L^{\infty}(\overline{\Omega})}, \quad |\nabla w_{\lambda,N}|_{L^{\infty}(\overline{\Omega})}, \quad |\nabla w_{\lambda,N}|_{1;\overline{\Omega}} \leq C \qquad \text{any} \quad \lambda \in (0,1), \quad N > 0.$$

On the other hand, from (36) and the Evans-Krylov interior estimate (see, e.g. L.C. Evans [17], X. Cabre and L.A. Caffarelli [12], N.V. Krylov [25], [26], and [31]n,) leads for any $\Omega' \subset\subset \Omega$

$$|\nabla^2 w^{\delta}_{\lambda,N}|_{\alpha;\Omega'} \leq C \qquad \text{any} \quad \delta \in (0,1),$$

where $C > 0$ depends on Ω' and $\alpha \in (0,1)$. Thus, we obtain $w_{\lambda,N} \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$ of

$$\max_{1 \le l \le N} \{L^{\alpha_l} w_{\lambda,N}\} = 0 \qquad \text{in} \quad \Omega,$$

$$\lambda w_{\lambda,N} + < \nabla w_{\lambda,N}, \gamma(x) > - g(x) = 0 \qquad \text{on} \quad \partial\Omega.$$

Letting $N \to \infty$, we obtain (24) and (25) from the preceding estimates.

In the following, we shall prove Lemma 2.2.

Proof of Lemma 2.2. Set

$$v^{\delta}_{\lambda,N} \equiv \frac{u^{\delta}_{\lambda,N} - u^{\delta}_{\lambda,N}(x_0)}{|\nabla(u^{\delta}_{\lambda,N} - u^{\delta}_{\lambda,N}(x_0))|_{L^\infty(\overline{\Omega})}}. \tag{43}$$

From (23), there exists a constant $M_1 > 0$ such that

$$|v^{\delta}_{\lambda,N}|_{L^\infty(\overline{\Omega})}, \quad |\nabla v^{\delta}_{\lambda,N}|_{L^\infty(\overline{\Omega})} \le M_1 \qquad \text{any} \quad \delta, \quad \lambda \in (0,1), \quad N \in \mathbf{N}. \tag{44}$$

It is clear that

$$F^N_\delta(x, \nabla v^{\delta}_{\lambda,N}, \nabla^2 v^{\delta}_{\lambda,N}) = 0 \qquad \text{in} \quad \Omega, \tag{45}$$

$$\lambda v^{\delta}_{\lambda,N} + < \nabla v^{\delta}_{\lambda,N}, \gamma(x) > - \overline{g} = 0 \qquad \text{on} \quad \partial\Omega, \tag{46}$$

where

$$\overline{g} = \frac{g - \lambda u^{\delta}_{\lambda,N}(x_0)}{|\nabla(u^{\delta}_{\lambda,N} - u^{\delta}_{\lambda,N}(x_0))|_{L^\infty(\overline{\Omega})}}.$$

We need the following Lemma.

Lemma 2.3. *Let $v^{\delta}_{\lambda,N}$ be defined in (43). Then, there exists $C > 0$ such that*

$$|\nabla^2 v^{\delta}_{\lambda,N}|_{L^\infty(\overline{\Omega})} \le C \qquad \textit{any} \quad \delta, \quad \lambda \in (0,1), \quad N \in \mathbf{N}. \tag{47}$$

Lemma 2.3 will lead our present goal (42) in Lemma 2.2. In fact, from (43), (47), we have

$$\sup_{\overline{\Omega}} |\nabla^2 u^{\delta}_{\lambda,N}| \le C(1 + \sup_{\overline{\Omega}} |\nabla u^{\delta}_{\lambda,N}|). \tag{48}$$

We use the following interpolation inequality in the above.

Lemma B. ([23], Lemma 6.35) *Suppose $j+\beta<k+\alpha$, where $j=0,1,2,...$; $k=1,2,...$, and $0\leq\alpha,\beta\leq 1$. Let D be a $C^{k,\alpha}$ domain in $\mathbf{R}^n$, and assume $u\in C^{k,\alpha}(\overline{D})$. Then, for any $\varepsilon>0$ and some constant $C=C(\varepsilon,j,k,D)$ we have*

$$|u|_{j,\beta;D}\leq C|u|_{L^\infty(D)}+\varepsilon|u|_{k,\alpha;D}.$$

By putting $j=1$, $k=2$, $\alpha=\beta=0$ in Lemma B, (48) leads (42) in Lemma 2.2. Finally, we are to prove Lemma 2.3.

Proof of Lemma 2.3. For simplicity, write $F=F_\delta$, $v=v^\delta_{\lambda,N}$. First, we examine the regularity of v on $\partial\Omega$. By differentiating (45) twice with respect to a vector $\xi\in\mathbf{R}^n$, $|\xi|=1$,

$$\sum_{i,j=1}^n\frac{\partial F}{\partial r_{ij}}\frac{\partial^2}{\partial x_i\partial x_j}D_\xi v+\sum_{i=1}^n\frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i}D_\xi v+\frac{\partial F}{\partial\xi}=0,$$

$$\sum_{i,j=1}^n\frac{\partial F}{\partial r_{ij}}\frac{\partial^2}{\partial x_i\partial x_j}D_{\xi\xi} v+\sum_{i=1}^n\frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i}D_{\xi\xi} v+F_{\overline{XX}}=0,$$

where $F_{\overline{XX}}$ is the derivarives of F with respect to $\overline{X}=(\xi,\nabla(D_\xi v),\nabla^2(D_\xi v))$. Using the structure conditions (36)-(39), we obtain from above inequalities

$$|\sum_{i,j=1}^n\frac{\partial F}{\partial r_{ij}}\frac{\partial^2}{\partial x_i\partial x_j}D_\xi v|\leq C(1+|\nabla^2 v|), \tag{49}$$

$$\sum_{i,j=1}^n\frac{\partial F}{\partial r_{ij}}\frac{\partial^2}{\partial x_i\partial x_j}D_{\xi\xi} v\leq C(1+|\nabla^2 v|+|\nabla^2 D_\xi v|), \tag{50}$$

where $C>0$ depends on n, M_1, μ_1 and μ_2. By the usual argument of flattening the boundary, we may assume that $\partial\Omega=\{(x',x_n)|x_n\geq 0\}$ in a neighborhood of $x=0\in\partial\Omega$. Although by the change of variables, (45)-(46) is transformed into $\overline{F}=0$ ($\overline{F}$ is the new Hamiltonian) etc., we keep to denote $\overline{F}=F$, etc., for simplicity. Denote $B_r^+=\{x\in B(0,r)|x_n>0\}$, and for $\xi=(\xi_1,...,\xi_{n-1},0)\in\mathbf{R}^{n-1}$, $|\xi|\leq 1$, consider

$$w(x,\xi)\equiv\eta^2(x,\xi)(z(x,\xi)+Av'), \tag{51}$$

where η is a smooth cut-off function to be precised in below, A a constant,

$$z(x,\xi) \equiv D_{\xi\xi} v(x) = \sum_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} \xi_i \xi_j, \qquad v' \equiv \sum_{i=1}^{n-1} |\frac{\partial v}{\partial x_i}|^2.$$

By introducing (36), (37), (44) and (45) into (49), we obtain

$$\sum_{i,j=1}^{n} (\frac{\partial F}{\partial r_{ij}} \frac{\partial^2 z}{\partial x_i \partial x_j} + C_{ij} \frac{\partial^2}{\partial x_i \partial x_j} D_\xi v) \leq C(1+|\nabla^2 v|')$$

where the coefficients C_{ij} are such that $C_{in}=0$, $|C_{ij}| \leq C$ depending on n, λ_1, μ_i $(i=0,1,2)$, M_1, and $|\nabla^2 v|' = (\sum_{i+j<2n} |\frac{\partial^2 v}{\partial x_i \partial x_j}|^2)^{\frac{1}{2}}$. Using the relations

$$\frac{\partial}{\partial x_i} D_{\xi_j} z = 2 \frac{\partial^2}{\partial x_i \partial x_j} D_\xi v, \quad D_{\xi_i \xi_j} z = 2 \frac{\partial^2 v}{\partial x_i \partial x_j},$$

we can take constants C_0 and C such that the following $(2n-1)\times(2n-1)$ matrix $(F'_{ij})_{ij}$:

$$\sum_{i,j=1}^{2n-1} F'_{ij} D_{ij} z \equiv \sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 z}{\partial x_i \partial x_j} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} C_{ij} \frac{\partial}{\partial x_i} D_{\xi_j} z + C_0 \sum_{j=1}^{n-1} D_{\xi_j \xi_j} z$$
$$\leq C(1+|\nabla^2 v|')$$

is uniformly elliptic with minimum eigenvalue $\lambda' \geq \frac{\lambda_1}{2}$. From (49),

$$\sum_{k=1}^{n-1} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 v}{\partial x_i \partial x_k} \frac{\partial^2 v}{\partial x_j \partial x_k} + \frac{1}{2} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 v'}{\partial x_i \partial x_j} \leq C(1+|\nabla^2 v|').$$

By combining the above two inequalities, we arrive at

$$\eta^2 \sum_{i,j=1}^{2n-1} F'_{ij} D_{ij} w - 2 \sum_{i,j=1}^{2n-1} F'_{ij} D_i \eta^2 D_j w \leq \tag{52}$$

$$-2K\lambda(|\nabla^2 v|')^2 \eta^4 + 6(\sum_{i,j=1}^{2n-1} F'_{ij} D_i \eta D_j \eta) w - 2\eta(\sum_{i,j=1}^{2n-1} F'_{ij} D_{ij} \eta) w - C(1+K)\eta^4(1+|\nabla^2 v|')$$
$$\leq -A\lambda w^2 + C_A,$$

where the constant C_A depends on n, λ_1, μ_i $(i=0,1,2)$ and M_1. (Remark that C_A does not depend on $\lambda \in (0,1)$, for we have not yet used the boundary

condition (46)).
Next, by differentiating (46) in the direction of ξ_k, ξ_l,

$$\lambda D_{\xi_k} v + < \nabla(D_{\xi_k} v), \gamma > + < \nabla v, D_{\xi_k}\gamma > = D_{\xi_k}\overline{g}, \tag{53}$$

$$\begin{aligned} \lambda D_{\xi_k\xi_l} v + < \nabla(D_{\xi_k\xi_l} v), \gamma > + < \nabla(D_{\xi_k} v), D_{\xi_l}\gamma > + < \nabla(D_{\xi_l} v), D_{\xi_k}\gamma > \\ + < \nabla v, D_{\xi_k\xi_l}\gamma > = D_{\xi_k\xi_l}\overline{g}. \end{aligned} \tag{54}$$

Since

$$\frac{\partial w}{\partial x_i} = 2\frac{w}{\eta}\frac{\partial \eta}{\partial x_i} + \eta^2\left(\frac{\partial z}{\partial x_i} + A\frac{\partial v'}{\partial x_i}\right),$$

$$\begin{aligned} &\lambda w + < \nabla w, \gamma > -2\frac{w}{\eta} < \nabla \eta, \gamma > \\ &\qquad = \eta^2 < \nabla z, \gamma > + \eta^2 A < \nabla v', \gamma > + \lambda\eta^2(z + Av'), \end{aligned}$$

and from (54),

$$= \eta^2 A < \nabla v', \gamma > + \lambda\eta^2 A v' - \eta^2 < \nabla v, D_{\xi_k\xi_l}\gamma > -2\eta^2 < \nabla(D_{\xi_k} v), D_{\xi_k}\gamma > .$$

From (22) and (44),

$$|v'|, \quad |D_{\xi_k}\gamma|, \quad |D_{\xi_k\xi_l}\gamma| \le K \qquad \text{any} \quad 1 \le k, l \le n-1,$$

and by (53) $< \nabla v', \gamma >$ and $< \nabla(D_{\xi_k} v), D_{\xi_k}\gamma >$ are bounded. Therefore, we can fix A so that

$$\lambda w + < \nabla w, \gamma > -2\frac{w}{\eta} < \nabla \eta, \gamma > \le C_1\eta^2,$$

where $C_1 > 0$ depends on n, λ_1, μ_i $(i = 0, 1, 2)$, K and M_1. (In particular, C_1 is independent of $\lambda \in (0, 1)$.) Now, fix

$$\eta(x, \xi) = [1 - 4\{|x'|^2 + (x_n - \overline{\varepsilon} r)^2\}/r^2 - |\xi|^2\}^+,$$

where for

$$T = \{x \in B_r, \quad x_n = 0\}, \qquad N = \{(x, \xi) \in \mathbf{R}^{2n-1} | \quad \eta(x, \xi) > 0\},$$

$$\overline{\varepsilon} = \zeta/\sqrt{1 + \zeta^2}, \qquad \zeta = \sup_T \frac{|\gamma|}{\gamma_n} \le C.$$

Then, on $T \cap \partial N \cap \{w \geq 0\}$

$$< \nabla w, \gamma > + \lambda w \leq C_2,$$

where C_2 is independent of $\lambda \in (0,1)$. We take $\overline{w} = w + C_3 {\lambda_1}^{-1} x_n$ so that

$$< \nabla \overline{w}, \gamma > = < \nabla w, \gamma > + \gamma_n \frac{C_3}{\lambda_1} \gamma \leq C_2 - \lambda w + \gamma_n \frac{C_3}{\lambda_1} \leq 0.$$

From the definition of w, the above constant C_3 can be taken uniformly in $\lambda \in (0,1)$. By applying the maximum principle to $\overline{w}$, instead of w, we obtain

$$D_{\xi\xi} v(0) \leq C, \tag{55}$$

for any $\xi = (\xi_1, ..., \xi_{n-1}, 0)$ ($|\xi| = 1$), where $C > 0$ depends only on η, λ_1, μ_i ($i = 0, 1, 2$), M_1, Ω and K. (C is independent of $\lambda \in (0,1)$.) As for the remaining inequalities, the same argument in [31]n is available. That is, by regarding

$$G(x) = \lambda v + < \nabla v, \gamma > - g(x)$$

as a function in $B(0,r)$ ($0 \in \partial\Omega$, γ and g are extendable to some neighborhood of $\partial\Omega$ (22)),

$$\left| \sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 G}{\partial x_i \partial x_j} \right| \leq C(1 + M_2) \qquad (M_2 = \sup_{\Omega} |\nabla^2 v|) \quad \text{in} \quad B(0,r),$$

$$G = 0 \qquad \text{on} \quad \partial\Omega,$$

where C depends on n, M_1, μ_1, K, and does not depend on $\lambda \in (0,1)$. From this, the barrier argument leads

$$|DG(0)| \leq C\sqrt{1 + M_2}, \tag{56}$$

and we can extend the inequality (55) to

$$D_{\xi\xi} v(0) \leq C \qquad \text{any} \quad |\xi| = 1, \quad \xi \in \mathbf{R}^n. \tag{57}$$

Then, by the uniform ellipticity (36), the usual argument leads

$$\sup_{\partial\Omega} |\nabla^2 v| \leq C \qquad \text{any} \quad |\xi| = 1, \quad \xi \in \mathbf{R}^n, \tag{58}$$

where C is independent of $\lambda \in (0,1)$. From (36), by coupling (58) with the global Dirichlet bound for (45)-(46) leads (47), and Lemma 2.3 was proved.

We complete the proof of Theorem 2.1.

Theorem 2.4. *Assume that Ω is (7), and that (4), (6), (21) and (22) hold. Then there exists a number d and a function $u(x) \in C^{1,1}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ ($\alpha \in (0,1)$) which satisfy (1)-(2).*

Proof of Theorem 2.4. From (23)-(25) and the Evans-Krylov estimate, we can extract a subsequence $\lambda' \downarrow 0$ such that there exist a number d and $u(x) \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$, and

$$\lim_{\lambda' \downarrow 0} \lambda' u_{\lambda'}(x) = d, \quad \lim_{\lambda' \downarrow 0} (u_{\lambda'} - u_{\lambda'})(x_0) = u(x) \qquad \text{uniformly on} \quad \overline{\Omega}. \tag{59}$$

From the usual stability result ([15]n), it is clear that the pair (d,u) satisfies (1)-(2).

As for the uniqueness of the number d, we give the following theorem in which we consider (1)-(2) in the framework of viscosity solutions.

Theorem 2.5. *Assume that Ω is (7), and that (4), (5), (6) and (22) hold. Then, the number d such that (1)-(2) has a viscosity solution u is unique.*

Proof of Theorem 2.5. We argue by contradiction. Let (d_1, u_1) and (d_2, u_2) be two pairs satisfying (1)-(2) in the sense of viscosity solutions. We assume $d_1 > d_2$. First, we show the following Lemma.

Lemma 2.6. *Let $v = u_1 - u_2$. Then, v satisfies*

$$-M^+(\nabla^2 v) + \inf_{\alpha \in A} \{ -\sum_{i=1}^n b_i^\alpha \frac{\partial v}{\partial x_i} \} \leq 0 \qquad \textit{in} \quad \Omega, \tag{60}$$

$$< \nabla v, \gamma > \leq d_2 - d_1 < 0 \qquad \textit{on} \quad \partial\Omega, \tag{61}$$

where

$$M^+(X) = \sup_{\lambda_1 I \leq A \leq \Lambda_1 I} Tr(AX) \qquad X \in \mathbf{S}^n. \tag{62}$$

Proof of Lemma 2.6. Let $\phi \in C^2(\overline{\Omega})$ be such that $u - \phi$ takes its local strict maxixum at $\overline{x} \in \overline{\Omega}$. From the definition of viscosity solutions, we are to show the following.

(i) If $\overline{x} \in \Omega$,

$$-M^+(\nabla^2\phi(\overline{x})) + \inf_{\alpha \in A}\{< -b^\alpha(\overline{x}), \phi(\overline{x}) >\} \leq 0.$$

(ii) If $\overline{x} \in \partial\Omega$,

$$-M^+(\nabla^2\phi(\overline{x})) + \inf_{\alpha \in A}\{< -b^\alpha(\overline{x}), \phi(\overline{x}) >\} \leq 0,$$

or

$$< \phi(\overline{x}), \gamma(\overline{x}) > \leq d_2 - d_1.$$

Step 1. We shall show (i) by the contradiction argument. Thus, assume

$$-M^+(\nabla^2\phi(\overline{x})) + \inf_{\alpha \in A}\{< -b^\alpha(\overline{x}), \phi(\overline{x}) >\} > 0, \tag{63}$$

and we shall look for a contradiction. Define, for $\beta > 0$

$$\Psi_\beta(x, y) = u_1(x) - u_2(y) - \phi(\frac{x+y}{2}) - \beta|x-y|^2 \qquad \text{in} \quad \Omega \times \Omega,$$

and let (x_β, y_β) be the maximum point of Ψ_β. It is well known (see [15]n) that

$$(x_\beta, y_\beta) \to (\overline{x}, \overline{x}), \quad \beta|x_\beta - y_\beta|^2 \to 0 \qquad \text{as} \quad \beta \to \infty,$$

and that for any $\varepsilon > 0$, there exist X, $Y \in \mathbf{S}^n$ such that

$$(\frac{1}{2}\nabla\phi(\frac{x_\beta + y_\beta}{2}) + 2\beta(x_\beta - y_\beta), X) \in J^{2,+}_{\Omega} u_1(x_\beta),$$

$$(-\frac{1}{2}\nabla\phi(\frac{x_\beta + y_\beta}{2}) + 2\beta(x_\beta - y_\beta), Y) \in J^{2,-}_{\Omega} u_2(y_\beta),$$

and

$$-(\frac{1}{\varepsilon} + \|A\|)I \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \varepsilon A^2, \tag{64}$$

where by denoting $\psi(x,y)=\phi(\frac{x+y}{2})+\beta|x-y|^2$,

$$A=D^2\psi(x_\beta,y_\beta)\in\mathbf{S}^{2n},\quad \|A\|=\sup\{|<A\xi,\xi>|:|\xi|\le 1\}.$$

Now, by using the definition of viscosity solution for u_i $(i=1,2)$,

$$F(x_\beta,\frac{1}{2}\nabla\phi(\frac{x_\beta+y_\beta}{2})+2\beta(x_\beta-y_\beta),X)\le 0,$$

$$F(y_\beta,-\frac{1}{2}\nabla\phi(\frac{x_\beta+y_\beta}{2})+2\beta(x_\beta-y_\beta),Y)\ge 0,$$

and by taking the differences of two inequalities, using the form of (3), for any small $\delta>0$ there exists a control $\alpha'\in A$ such that

$$\{-Tr(A^{\alpha'}(x_\beta)X)-<\frac{1}{2}\nabla\phi(\frac{x_\beta+y_\beta}{2}),b^{\alpha'}(x_\beta)>\}$$
$$-\{-Tr(A^{\alpha'}(y_\beta)Y)-<\frac{1}{2}\nabla\phi(\frac{x_\beta+y_\beta}{2}),b^{\alpha'}(y_\beta)>\}\le\delta. \tag{65}$$

By taking $\varepsilon=\frac{1}{\beta}$ in (64), and multiplying the rightmost inequality in (64) by the symmetric matrix

$$\begin{pmatrix}\sigma^{\alpha'}(x_\beta)^t\sigma^{\alpha'}(x_\beta) & \sigma^{\alpha'}(y_\beta)^t\sigma^{\alpha'}(x_\beta)\\ \sigma^{\alpha'}(x_\beta)^t\sigma^{\alpha'}(y_\beta) & \sigma^{\alpha'}(y_\beta)^t\sigma^{\alpha'}(y_\beta)\end{pmatrix},$$

and taking traces, we have

$$Tr(A^{\alpha'}(x_\beta)X)-Tr(A^{\alpha'}(y_\beta)Y)-Tr(\nabla^2\phi(\overline{x})A^{\alpha'}(\overline{x}))\le L^2\beta|x_\beta-y_\beta|^2+o(\beta^{-1})$$

as $\beta\to\infty$, where $L>0$ is the Lipschitz constant in (5) (or K in (21)). (See [15], H. Ishii and P.-L. Lions [24]n for this techniques.) Therefore from (65), for any $\varepsilon>0$ there exists $\alpha'\in A$ such that

$$-Tr(\nabla^2\phi(\overline{x})A^{\alpha}(\overline{x}))-<\nabla\phi(\overline{x}),b^{\alpha'}(\overline{x})>\le\delta+o(\beta^{-1}),$$

which contradicts to (63), since $\delta>0$ is arbitrary. Thus, we showed (i).

Step 2. We shall prove (ii). First of all, from the usual technique to treat the Neumann boundary condition in the theory of viscosity solutions, we may replace the conditions to

$$d_1+<\nabla u_1,\gamma>-g(x)\le-\delta \qquad \text{on} \quad \partial\Omega, \tag{66}$$

$$d_2 + < \nabla u_2, \gamma > - g(x) \geq \delta \qquad \text{on} \quad \partial\Omega, \tag{67}$$

where $\delta > 0$ is a small number. (See [15]n.) Then, we assume that (ii) does not hold, and shall look for a contradiction. So, let

$$-M^+(\nabla^2 \phi(\overline{x})) + \inf_{\alpha \in A} \{ < -b^\alpha(\overline{x}), \nabla \phi(\overline{x}) > \} > 0, \tag{68}$$

$$< \nabla \phi(\overline{x}), \gamma(\overline{x}) >> d_2 - d_1. \tag{69}$$

It is well known ([27]n) that since $\partial\Omega$ is $C^{3,1}$, by putting

$$L(x,y) = \inf\{ \int_0^1 c_{ij}(\xi(t)) \dot{\xi}_i \dot{\xi}_j dt \quad | \quad \xi \in C^1([0,1]; \mathbf{R}^n), \quad \xi(0) = y, \quad \xi(1) = x \},$$

where $c_{ij}(x)$ is a smooth function, say in $C^3(\overline{\Omega})$ such that for $\mathbf{n} = (n_i)_i$

$$\sum_j c_{ij}(x) \gamma_j(x) = n_i(x) \qquad \text{any} \quad 1 \underline{<} i \underline{<} n, \quad x \in \partial\Omega,$$

we have:

$$< \gamma(x), \nabla_x L(x,y) >< \frac{1}{C} |y - x|^2 \qquad \text{any} \quad x \in \partial\Omega, \quad y \in \Omega, \tag{70}$$

where $C > 0$ is a constant. Define, for $\beta > 0$

$$\Psi_\beta(x,y) = u_1(x) - u_2(y) - \phi(\frac{x+y}{2}) - \beta L(x,y)$$

$$+ (d_1 - g) < \gamma(\overline{x}), x - y > + |x - \overline{x}|^4 + \frac{1}{2} < \nabla \phi(\overline{x}), x - y > \qquad \text{in} \quad \Omega \times \Omega.$$

Set

$$\psi(x,y) = \phi(\frac{x+y}{2}) + \beta L(x,y) - (d_1 - g) < \gamma(\overline{x}), x - y > - |x - \overline{x}|^4$$
$$- \frac{1}{2} < \nabla \phi(\overline{x}), x - y > .$$

Let (x_β, y_β) be the maximum point of Ψ_β. As in Step1, it is known (see [15]) that

$$(x_\beta, y_\beta) \to (\overline{x}, \overline{x}), \quad \beta |x_\beta - y_\beta|^2 \to 0 \qquad \text{as} \quad \beta \to \infty,$$

and that for any $\varepsilon > 0$, there exist X, $Y \in \mathbf{S}^n$ such that

$$(\nabla_x \psi(x_\beta, y_\beta), X) \in J_\Omega^{2,+} u_1(x_\beta), \quad (-\nabla_y \psi(x_\beta, y_\beta), Y) \in J_\Omega^{2,-} u_2(y_\beta),$$

which satisfy (64) with $A = D^2\psi \in \mathbf{S}^{2n}$.
If $(x_\beta, y_\beta) \in \partial\Omega$, by using (70) we calculate

$$
\begin{aligned}
&< \nabla\psi(x_\beta, y_\beta), \gamma(x_\beta) > +d_1 - g(x_\beta) =< \frac{1}{2}\nabla\phi(\frac{x_\beta + y_\beta}{2}), \gamma(x_\beta) > \\
&+2\beta < \gamma(x_\beta), \nabla_x L(x_\beta, y_\beta) > -(d_1 - g) < \gamma(x_\beta), \gamma(\overline{x}) > \\
&-4|x_\beta - \overline{x}|^2 < \gamma(x_\beta), x_\beta - \overline{x} > - < \gamma(x_\beta), \frac{1}{2}\nabla\phi(\overline{x}) > +d_1 - g \\
&\geq -\frac{\beta}{C}|x_\beta - y_\beta|^2 + O(|x_\beta - z|^3) \geq o(1) \qquad \text{as} \quad \beta \to \infty.
\end{aligned}
$$

$$
\begin{aligned}
&< -\nabla\psi(x_\beta, y_\beta), \gamma(y_\beta) > +d_2 - g(y_\beta) =< -\frac{1}{2}\nabla\phi(\frac{x_\beta + y_\beta}{2}), \gamma(y_\beta) > \\
&-2\beta < \gamma(y_\beta), \nabla_y L(x_\beta, y_\beta) > -(d_1 - g) < \gamma(y_\beta), \gamma(\overline{x}) > \\
&+ < \gamma(y_\beta), \frac{1}{2}\nabla\phi(\overline{x}) > +d_2 - g \\
&\leq \frac{\beta}{C}|x_\beta - y_\beta|^2 + d_2 - d_1 + o(1) \leq o(1) \qquad \text{as} \quad \beta \to \infty.
\end{aligned}
$$

(In the last inequality, we used the assumption $d_1 > d_2$.)
Therefore, by taking account of (66) and (67), regardless the fact that x_β, y_β $\in \Omega$ or $\in \partial\Omega$, we have the following.

$$F(x_\beta, \nabla\psi(x_\beta, y_\beta), X) \leq o(1) \qquad \text{as} \quad \beta \to \infty,$$

$$F(y_\beta, -\nabla\psi(x_\beta, y_\beta), Y) \geq o(1) \qquad \text{as} \quad \beta \to \infty.$$

The rest of the argument to obtain a contradiction from the above two inequalities is similar to that of Step 1, and we omit it here.

Now, we go back to the proof of Theorem 2.5, which is immediate from Lemma 2.6. From the strong maximum principle (Lemma A), v, which is not constant, attains its maximum at some point $x_1 \in \partial\Omega$

$$v(x_1) > v(x) \qquad \text{any} \quad x \in \Omega.$$

However, as we have seen in the proof of Theorem 2.1 in Step 1, this is not compatible with $< \nabla v, \gamma > \leq d_2 - d_1$ on $\partial\Omega$, in the sense of viscosity solutions. Thus, we have proved $d_1 = d_2$ must be hold.

If we consider the uniqueness of d in the framework of the $C^{1,1}(\overline{\Omega})$ solutions, the proof is much simpler. We add this as follows.

Proposition 2.7. *Assume that Ω is (7), and that (4), (5) and (6) hold. Moreover, assume that F satisfies the following comparison: for a subsolution u and a supersolution v of (1) such that $u \le v$ on $\partial\Omega$, $u \le v$ in $\overline{\Omega}$. Then, the number d such that (1)-(2) has a solution $u \in C^{1,1}(\overline{\Omega})$ is unique.*

Proof of Proposition 2.7. We assume that there are two pairs (d_1, u_1) and (d_2, u_2) which satisfy (1)-(2) such that $d_1 > d_2$ and $u_i \in C^{1,1}(\overline{\Omega})$ $(i = 1, 2)$. By adding a constant if necessary, we may assume that there is a point $x_0 \in \partial\Omega$ such that $u_1(x_0) = u_2(x_0)$ and

$$u_1(x) \le u_2(x) \qquad \text{on} \quad \partial\Omega.$$

Put $v = u_2 - u_1$, which satisfies

$$< \nabla v(x), \gamma(x) >= d_1 - d_2 > 0, \quad v(x) \ge 0 \qquad \text{on} \quad \partial\Omega.$$

From the comparison for (1),

$$v(x) \ge 0 \qquad \text{any} \quad x \in \overline{\Omega}.$$

However, at $x_0 \in \partial\Omega$, $v(x_0) = 0$ and $< \nabla v(x_0), \gamma(x_0) >> 0$ in the classical sense. Thus, we get a contradiction and $d_1 = d_2$.

3 Long time averaged reflection force in half spaces.

In this section, the existence and uniqueness of the number d in (1)-(2) is shown in the case that Ω satisfies (8), with a supplement boundary condition at $x_n = \infty$. We denote

$$\Omega = \{(x', x_n) | \quad x_n \ge f(x'), \quad x' \in (\mathbf{R} \backslash \mathbf{Z})^{n-1}\},$$

$$\Gamma_0 = \partial\Omega = \{(x', x_n) | \quad x_n = f(x'), \quad x' \in (\mathbf{R} \backslash \mathbf{Z})^{n-1}\},$$

where $f(x')$ is periodic in $x' \in (\mathbf{R}\backslash\mathbf{Z})^{n-1}$ and is $C^{3,1}$. Our goal is to find a unique number d which admits a viscosity solution u of (1)-(2) such that

$$u \quad \text{is bounded and periodic in } x'. \tag{71}$$

We begin with the uniqueness of d.

Theorem 3.1. *Assume that Ω is (8), and that (4), (5), (6) and (22) hold. Moreover, assume that*

$$b_n^\alpha(x) \leq 0 \qquad \text{any} \quad x \in \Omega, \quad \alpha \in A. \tag{72}$$

Then, the number d such that (1)-(2) and (71) has a viscosity solution u is unique.

Proof of Theorem 3.1. We argue by contradiction. Assume that there exist two pairs (d_1, u_1) and (d_2, u_2) which satisfy (1)-(2) and (71), and that $d_1 > d_2$. By using a similar argument to the proof of Lemma 2.6, $v = u_1 - u_2$ is a subsolution of

$$-M^+(\nabla^2 v) + \inf_\alpha\{< -b^\alpha(x), \nabla v >\} \leq 0 \qquad \text{in} \quad \Omega, \tag{73}$$

$$< \nabla v, \gamma(x) >= d_2 - d_1 < 0 \qquad \text{on} \quad \partial\Omega, \tag{74}$$

where M^+ is the Pucci operator defined in (62) (See [14]n). For $R > 0$ large enough, let

$$\Omega_R = \{(x', x_n) | \quad f(x') \leq x_n \leq R\},$$

and define

$$M_R = \sup_{\overline{\Omega_R}} |v|.$$

(Remark that v is periodic in $x' \in (\mathbf{R}\backslash\mathbf{Z})^{n-1}$ and the above supremum is well-definded.) Let $x_0 \in \Gamma_0$ be a point such that $v(x_0) = \sup_{x\in\Gamma_0} v(x) \equiv M_0$. Let $(x'_c, c) \in \Gamma_0$ be a point such that

$$c \leq x_n \qquad \text{any} \quad (x', x_n) \in \Gamma_0.$$

We take

$$w_R(x', x_n) \equiv \frac{M_R - M_0}{R - c}(x_n - c) + M_0 \qquad (x', x_n) \in \Omega. \tag{75}$$

Since $\frac{M_R - M_0}{R-c} \geq 0$, from (72)

$$-M^+(\nabla^2 w_R) + \inf_{\alpha}\{< -b^\alpha(x), \nabla w_R >\} \geq 0 \qquad \text{in} \quad \Omega_R,$$

$$w_{R|\Gamma_0} = \frac{M_R - M_0}{R-c}(x_n - c) + M_0 \geq M_0,$$

$$w_{R|\Gamma_R} = M_R.$$

Thus, by using the comparison argument, we get

$$v \leq w_R \qquad \text{in} \quad \overline{\Omega_R}, \quad \text{any} \quad R > 0 \quad \text{large enough.}$$

By (71), tending $R \to \infty$, this yields

$$v \leq M_0 \qquad \text{in} \quad \Omega.$$

Therefore, v takes its maximum on Γ_0. Finally, by using the strong maximum principle (Lemma A), (73) and (74) yields a contradiction as we argued in the proof of Theorem 2.1, Step1. Thus, $d_1 = d_2$ must hold.

Remark 3.1. (Counter example.) If we do not assume the boundary condition at infinity (71), d is not unique in general. For example, consider

$$-\Delta u = 0 \qquad \text{in} \quad \{x_n \geq 0\} \subset \mathbf{R}^n, \tag{76}$$

$$d + < \nabla u, \mathbf{n}(x) >= 0 \qquad \text{on} \quad \{x_n = 0\} \subset \mathbf{R}^n, \tag{77}$$

where $\mathbf{n}$ is the outward unit normal, and the solution u is periodic in $x' = (x_1, ..., x_{n-1})$. Then, for any c, $d \in R$, $u = -dx_n + c$ is the solution of (76)-(77). Thus, the number d in (77) is not unique. (Green's first identity does not hold in the half space.)

Next, for the existence of d we approximate (1)-(2) and (71) by

$$F(x, \nabla u_\lambda^R, \nabla^2 u_\lambda^R) = 0 \qquad \text{in} \quad \Omega_R = \{(x', x_n)| \quad f(x') \leq x_n \leq R\},$$

$$< \nabla u_\lambda^R, \mathbf{n}(x) >= 0 \qquad \text{on} \quad \Gamma_R = \{(x', x_n)| \quad x_n = R\}, \tag{78}$$

$$\lambda u_\lambda^R + < \nabla u_\lambda^R, \gamma(x) > -g(x) = 0 \qquad \text{on} \quad \partial\Omega = \Gamma_0 = \{x_n = f(x')\},$$

where $R > 0$ is large enough so that Γ_R and Γ_0 do not intersect, say $R \geq R_0$. We examine the regularity of u_λ^R uniformly in $\lambda \in (0, 1)$ and $R > R_0$.

Proposition 3.2. *Assume that Ω is (8), and that (4), (6), (21) and (22) hold. Let $R > R_0$ be fixed, and let u_λ^R be the solution of (78). Then, there exists a number d_R and a function u_R such that*

$$\begin{aligned} \lim_{\lambda \downarrow 0} \lambda u_\lambda^R(x) &= d_R, \\ \lim_{\lambda' \downarrow 0} (u_{\lambda'}^R(x) - u_{\lambda'}^R(x_0)) &= u_R(x) \qquad \textit{uniformly in} \quad \overline{\Omega_R}, \end{aligned} \tag{79}$$

where $\lambda' \to 0$ is a subsequence of $\lambda \to 0$, and x_0 is an arbitrarily fixed point in Ω_{R_0}. The pair (d_R, u_R) satisfies

$$\begin{gathered} F(x, \nabla u_R, \nabla^2 u_R) = 0 \qquad \textit{in} \quad \Omega_R, \\ < \nabla u_R, \mathbf{n}(x) >= 0 \qquad \textit{on} \quad \Gamma_R, \\ d_R + < \nabla u_R, \gamma(x) > - g(x) = 0 \qquad \textit{on} \quad \partial\Omega = \Gamma_0. \end{gathered} \tag{80}$$

The number d_R is the unique number such that (80) has a viscosity solution. Moreover, there exists a constant $M > 0$ such that

$$|u_R - u_R(x_0)|_{L^\infty(\overline{\Omega_R})} < M \qquad \textit{any} \quad R > R_0, \tag{81}$$

$$|\nabla u_R|_{L^\infty(\overline{\Omega_R})} < M \qquad \textit{any} \quad R > R_0. \tag{82}$$

Proof of Proposition 3.2. We devide the proof into three steps.
Step 1. First, we shall see

$$|u_\lambda^R(x) - u_\lambda^R(x_0)| \leq M \qquad \text{any} \quad \lambda \in (0,1), \quad R > R_0. \tag{83}$$

So, put $v_R = u_R - u_R(x_0)$. Assume that

$$(\varepsilon_\lambda^R)^{-1} \equiv |v_\lambda^R|_{L^\infty(\overline{\Omega_R})} \to \infty \qquad \text{as} \quad \lambda \to 0, \quad R \to \infty,$$

and we seek a contradiction. Put $w_\lambda^R \equiv \varepsilon_\lambda^R v_\lambda^R$ which satisfies

$$\begin{gathered} F(x, \nabla w_\lambda^R, \nabla^2 w_\lambda^R) = 0 \qquad \text{in} \quad \Omega_R, \\ < \nabla w_\lambda^R, \mathbf{n}(x) >= 0 \qquad \text{on} \quad \Gamma_R, \\ < \nabla w_\lambda^R, \gamma(x) >= \varepsilon_\lambda^R (g - \lambda u_\lambda^R) \qquad \text{on} \quad \Gamma_0. \end{gathered}$$

Since $|w_\lambda^R|_{L^\infty(\overline{\Omega_R})} = 1$ ($w_\lambda^R(x_0) = 0$),

$$w^*(x) = \limsup_{R\to\infty,\lambda\downarrow 0, y\to x} w_\lambda^R(y), \quad w_*(x) = \liminf_{R\to\infty,\lambda\downarrow 0, y\to x} w_\lambda^R(y),$$

are well-definded. From the uniform ellipticity (4) and the Krylov-Safonov interior estimate, for any $V \subset\subset \Omega$ there exists a constant $M_V > 0$ such that

$$|\nabla w_\lambda^R|_{L^\infty(V)} \le M_V \qquad \text{any} \quad \lambda \in (0,1), \quad R > R_0.$$

Thus, since $w_\lambda(x_0) = 0$ ($\forall \lambda \in (0,1)$),

$$w^*(x_0) = w_*(x_0) = 0. \tag{84}$$

Moreover from the strong maximum principle (Lemma A), for any $R > R_0$ and $\lambda \in (0,1)$, w_λ^R must take its maximum and minimum on Γ_0. (If it takes a maximun or a minimum on Γ_R, we have a contradiction to $< \nabla w_\lambda^R, \mathbf{n}(x) >= 0$ ($x \in \Gamma_R$) in the sense of viscosity solutions as we have seen in the proof of Theorem 2.1, Step 1.) Hence,

$$|w^*|_{L^\infty(\overline{\Omega_R})} = 1 \quad \text{or} \quad |w_*|_{L^\infty(\overline{\Omega_R})} = 1 \qquad \text{any} \quad R > R_0. \tag{85}$$

Hereafter, we assume that $|w^*|_{L^\infty(\overline{\Omega_R})} = 1$. (The case of $|w_*|_{L^\infty(\overline{\Omega_R})} = 1$ can be treated similarly.) The upper semicontinuous function w^* is a viscosity solution of

$$F(x, \nabla w^*, \nabla^2 w^*) \le 0 \qquad \text{in} \quad \Omega, \tag{86}$$

$$< \nabla w^*, \gamma(x) > \le 0 \qquad \text{on} \quad \Gamma_0. \tag{87}$$

We remark that w^* takes its maximum on Γ_0, as w_λ^R ($R > R_0$, $\lambda \in (0,1)$) does so. (w^* is periodic in $x' \in (\mathbf{R}\backslash\mathbf{Z})^{n-1}$.) Then, by the strong maximum principle (Lemma A) and the fact that w^* is not constant ((84), (85)), (86)-(87) lead a contradiction. (See the proof of Theorem 2.1, Step 1.) Therefore, there exists a constant $M > 0$ such that

$$|u_\lambda^R(x) - u_\lambda^R(x_0)| \le M \qquad \text{any} \quad \lambda \in (0,1), \quad R > R_0.$$

Step 2. Next, we shall show (79) and (82). For this purpose, we are to have the a priori estimates of $|\nabla u_\lambda^R|$ and $|\nabla^2 u_\lambda^R|$. Put

$$w_\lambda^R = \frac{u_\lambda^R - u_\lambda^R(x_0)}{|\nabla(u_\lambda^R - u_\lambda^R(x_0))|_{L^\infty(\overline{\Omega_R})}}. \tag{88}$$

Remark that w_λ^R is a solution of

$$F(x, \nabla w_\lambda^R, \nabla^2 w_\lambda^R) = 0 \qquad \text{in} \quad \Omega_R,$$

$$< \nabla w_\lambda^R, \mathbf{n}(x) >= 0 \qquad \text{on} \quad \Gamma_R, \tag{89}$$

$$\lambda w_\lambda^R + < \nabla w_\lambda^R, \gamma(x) > -\overline{g} = 0 \qquad \text{on} \quad \Gamma_0, \tag{90}$$

where

$$\overline{g} = \frac{g}{|\nabla(u_\lambda^R - u_\lambda^R(x_0))|_{L^\infty(\overline{\Omega_R})}}.$$

Taking account of the periodicity in x_i $(i = 1, ..., n-1)$, the above problem is reduced to the case of bounded domains treated in § 2. Despite the existence of the different boundary condition (89) on Γ_R, the argument in § 2 (and [31]n) works with a minor modification. (We do not rewrite it here.) Thus, the a priori estimate:

$$|\nabla^2 w_\lambda^R|_{L^\infty(\overline{\Omega_R})} \leq M \qquad \text{any} \quad \lambda \in (0, 1), \quad R > R_0,$$

where $M > 0$ is a constant, which leads

$$|\nabla^2 u_\lambda^R|_{L^\infty(\Omega_R)} \leq M(|\nabla u_\lambda^R|_{L^\infty(\Omega_R)} + 1) \qquad \text{any} \quad \lambda \in (0, 1), \quad R > R_0. \tag{91}$$

As in § 2, we use the interpolation inequality in Lemma B, with the function $u_\lambda^R - u_\lambda^R(x_0)$, $D = \Omega_R$, $j = 1$, $k = 2$ and $\alpha = \beta = 0$. That is, the interpolation inequality becomes:

$$|\nabla u_\lambda^R|_{L^\infty(\overline{\Omega_R})} \leq C_\varepsilon |u_\lambda^R - u_\lambda^R(x_0)|_{L^\infty(\overline{\Omega_R})} + \varepsilon |\nabla^2 u_\lambda^R|_{L^\infty(\overline{\Omega_R})}. \tag{92}$$

By combining (81), (91) and (92),

$$|\nabla^2 u_\lambda^R|_{L^\infty(\Omega_R)} \leq M \qquad \text{any} \quad \lambda \in (0, 1), \quad R > R_0,$$

$$|\nabla u_\lambda^R|_{L^\infty(\Omega_R)} \leq M \qquad \text{any} \quad \lambda \in (0, 1), \quad R > R_0.$$

Thus, by extracting a subsequence $\lambda' \downarrow 0$, there exists a number d_R and a function u_R such that

$$\lambda' u_{\lambda'}^R \to d_R, \qquad u_{\lambda'}^R - u_{\lambda'}^R(x_0) \to u_R,$$

and

$$|\nabla u_R|_{L^\infty(\overline{\Omega_R})} \leq M \qquad \text{any} \quad R > R_0.$$

Thus, we proved (79) and (82).

Step 3. We shall complete the proof by showing that the above limit d_R is the unique number such that (80) has a viscosity solution (and is independent of the choice of $\lambda' \to 0$). We argue by contradiction, and assume that there exist two pairs (d_R, u_R) and (d'_R, u'_R) $(d_R > d'_R)$ satisfying (80). Denote $v = u_R - u'_R$. A similar argument used in the proof of Lemma 2.6 leads

$$-M^+(\nabla^2 v) + \inf_{\alpha \in A}\{< -b^\alpha(x), \nabla v >\} \leq 0 \qquad \text{in} \quad \Omega_R,$$

$$< \nabla v, \mathbf{n}(x) > \leq 0 \qquad \text{on} \quad \Gamma_R,$$

$$< \nabla v, \gamma(x) > \leq d'_R - d_R \qquad \text{on} \quad \Gamma_0.$$

Since v is not constant, from the strong maximum principle (Lemma A), v attains its maximum at $x_0 \in \Gamma_0$:

$$v(x_0) > v(x) \qquad \text{any} \quad x \in \Omega_R.$$

However, as we have seen in the proof of Theorem 2.1 Step1, since $d'_R - d_R < 0$, it is not compatible with the preceding boundary conditions on Γ_0 and Γ_R. Therefore, we get a contradiction and $d_R = d'_R$ must hold.

Theorem 3.3. *Assume that Ω is (8), and that (4), (6), (21) and (22) hold. Then, there exists a unique number d such that (1)-(2) and (71) has a viscosity solution u.*

Proof of Theorem 3.3. By comparison, there exists a constant $C > 0$ such that

$$|\lambda u^R_\lambda|_{L^\infty(\overline{\Omega_R})} \leq C \qquad \text{any} \quad \lambda \in (0, 1), \quad R > R_0,$$

and thus $|d_R| < C$ for any $R > R_0$. Therefore, by using (81) and (82), we can extract a subsequence $R' \to \infty$ such that there exist a number d and a function u such that

$$d_{R'} \to d \qquad \text{as} \quad R' \to \infty,$$

$$u_{R'} \to u \qquad \text{as} \quad R' \to \infty, \quad \text{locally uniformly in} \quad \overline{\Omega}.$$

From the stability results,

$$F(x, \nabla u, \nabla^2 u) = 0 \qquad \text{in} \quad \Omega,$$

$$d+ < \nabla u, \gamma(x) > -g(x) = 0 \qquad \text{on} \quad \Gamma_0,$$
$$|u|_{L^\infty(\overline{\Omega})} < M.$$

The uniqueness of d was proved in Theorem 3.1, and we can end the proof.

Remark 3.2. From the view point of the stochastic process (13), the approximating system (80) gives a kind of boundary condition at infinity. It forces the admissible trajectories of (13) (corresponding to (1)-(2) and (71)) to be pushed back inward at some finite $x_n = R$. Therefore, the condition (72) is quite reasonable. (In [10]n, the ergodic problem in unbounded domain (not on the boundary like (2)) is solved with the condition $\lim_{x\to\infty} b_n^\alpha(x) = -\infty$, which is stronger than (72).)

4 Remarks on some degenerate cases.

The number d in (1)-(2) exists even for degenerate operators. In this section, we give a sufficient condition for the existence (in a weeker sense) and two classes of operators satisfying the sufficient condition. The following two examples illustrate the existence and non-uniqueness of d. In the case of degenerate operators, the uniqueness does not hold in general.

Example 4.1. *Consider*

$$|\nabla u| = 0 \qquad in \quad \Omega,$$
$$d+ < \nabla u, \mathbf{n}(x) > -g(x) = 0 \qquad on \quad \partial\Omega, \tag{93}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open domain with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal to Ω, and g is Lipschitz continuous on $\partial\Omega$. Then, any d such that

$$d \leq \min_{x\in\partial\Omega} g(x)$$

and $u \equiv C$ (constant) satisfies (93) in the sense of viscosity solutions. In fact, it is clear that u satisfies the equation in Ω. To see the boundary condition in the viscosity sense,

$$\max\{|\nabla u|, d+ < \nabla u, \mathbf{n}(x) > -g(x)\} \geq 0 \qquad on \quad \partial\Omega,$$

shows that u is a supersolution on $\partial\Omega$. For any $\phi \in C^1$ such that $u-\phi$ takes its strict maximum at $x_0 \in \partial\Omega$, if $d \leq \min_{\partial\Omega} g$ then

$$< \nabla\phi, \mathbf{n}(x) > \quad \leq 0 \quad \leq g(x) - d \qquad on \quad \partial\Omega.$$

Thus,

$$\min\{|\nabla u|, d+ < \nabla u, \mathbf{n}(x) > -g(x)\} \leq 0 \qquad on \quad \partial\Omega,$$

in the sense of viscosity solutions, and u is a subsolution on $\partial\Omega$.

Example 4.2. *Let $\Omega = (\mathbf{R}/\mathbf{Z}) \times (0,1) \subset \mathbf{R}^2$ (periodic in x_1). Consider*

$$-\frac{\partial^2 u}{\partial x_1^2} + |\frac{\partial u}{\partial x_2}| = 0 \qquad in \quad \Omega,$$

$$d+ < \nabla u, \mathbf{n}(x) > -g(x) = 0 \qquad on \quad \partial\Omega, \tag{94}$$

where $\mathbf{n}$ is the outward unit normal to Ω, g is Lipschitz continuous on $\partial\Omega$. Then, any d such that

$$d \leq \min_{x\in\partial\Omega} g(x)$$

and $u \equiv C$ (constant) satisfies (94) in the sense of viscosity solutions. In fact clearly, u is a viscosity solution in Ω. To see that u is a supersolution on $\partial\Omega$, suppose for $\phi \in C^1$, $u - \phi$ takes its strict minimum at $x_0 \in \partial\Omega$. Since $u = C$ on $x_1 = 0, 1$, we remark that such $\phi \in C^2$ must not satisfy $-\frac{\partial^2\phi}{\partial x_1^2}(x_0) \leq 0$. Thus,

$$-\frac{\partial^2 \phi}{\partial x_1^2}(x_0) + |\frac{\partial \phi}{\partial x_2}(x_0)| \geq 0,$$

and u is a viscosity super solution on $\partial\Omega$. The fact that u is a subsolution on $\partial\Omega$ is same to Example 4.1.

Remark 4.1. In the above examples the numbers d are not unique.

The operators F studied here are given in (3) with degenerate coefficients. For such operators, we approximate (1)-(2) by

$$-\varepsilon\Delta u_\varepsilon + F(x, \nabla u_\varepsilon, \nabla^2 u_\varepsilon) = 0 \qquad \text{in} \quad \Omega, \tag{95}$$

$$d_\varepsilon + < \nabla u_\varepsilon, \gamma(x) > -g(x) = 0 \qquad \text{on} \quad \partial\Omega, \tag{96}$$

where $\varepsilon \in (0,1)$. The domain Ω is either (7) or (8), and in the case of (8) the condition at infinity (71) is added. For any $\varepsilon > 0$, the existence and the uniqueness of d_ε and the existence of u_ε come from Theorems 2.4, 2.5, and 3.3, for (95) is uniformly elliptic.

Proposition 4.1. *Let Ω be a domain either (7) or (8). In the case of (7), assume all conditions but (4) in Theorems 2.4 and 2.5. In the case of (8), assume all conditions but (4) in Theorem 3.3. (Thus, F is possibly degenerate.) Let d_ε ($\varepsilon > 0$) be the number such that (95)-(96) (and (71) in the case of (8)) has a viscosity solution u_ε. Assume that there is a number $M > 0$ such that*

$$|u_\varepsilon - u_\varepsilon(x_0)|_{L^\infty(\Omega)} < M \qquad \textit{any} \quad \varepsilon \in (0,1). \tag{97}$$

Then, there exists a number d (not necessarily unique) such that (1)-(2) (and (71) in the case of (8)) has a viscosity subsolution $\underline{u}$ and a supersolution $\overline{u}$.

Proof of Proposition 4.1. Put $v_\varepsilon = u_\varepsilon - u_\varepsilon(x_0)$. Since d_ε is bounded in $\varepsilon \in (0,1)$, we can take a subsequence $\varepsilon' \to 0$ such that $\lim_{\varepsilon' \to 0} d_\varepsilon = d$ holds for a constant d. From (97),

$$v^*(x) = \limsup_{\varepsilon' \downarrow 0, y \to x} v_\varepsilon(y), \quad v_*(x) = \liminf_{\varepsilon' \downarrow 0, y \to x} v_\varepsilon(y)$$

are well-definded. Then, from the usual stability result (see [15]n), (d, v^*) and (d, v_*) are respectively viscosity sub and super solutions of (1)-(2) (and (71) in case of (8)).

Remark 4.2. In the above proposition $v^* \neq v_*$ in general, and thus the result is weaker than uniformly elliptic cases.

Next, we give a class of operators satisfying (97). The first class admits the existence of the uniformly elliptic part:

there exists a point $x_0 \in \Omega$ such that in a small neighborhood $B(x_0, r) \subset \Omega$ ($r > 0$), there exist constants λ_2 and Λ_2 such that $0 < \lambda_2 \leq \Lambda_2$ and

$$\lambda_2 I \leq (a_{ij}^\alpha)_{1 \leq i,j \leq n} \leq \Lambda_2 \qquad \text{any} \quad \alpha \in A, \quad x \in B(x_0, r). \tag{98}$$

The second class admits the existence of the "controllability" part (see [2]):

there exists a point $x_0 \in \Omega$ such that for a small neighborhood $B(x_0, r) \subset \Omega$ $(r > 0)$,

$$\lim_{|p|\to\infty} F(x,p,X) \to \infty \qquad \text{uniformly in} \quad x \in \Omega, \quad X \in \mathbf{S}^n. \tag{99}$$

Thorem 4.2. *Let Ω be a domain either (7) or (8). In the case of (7), assume all conditions but (4) in Theorems 2.4 and 2.5. In the case of (8), assume all conditions but (4) in Theorem 3.3. (Thus, F is possibly degenerate.) Assume also that F satisfies (31), (32) and (33), and that either (98) or (99) holds. Then, the solutions u_ε $(\varepsilon > 0)$ of (95)-(96) (and (71) in the case of (8)) satisfy (97). Moreover, there exists a number d (not necessarily unique) such that (1)-(2) (and (71) in the case of (8)) has a viscosity subsolution $\underline{u}$ and a supersolution $\overline{u}$.*

Proof of Theorem 4.2. Assume that (97) does not hold, and we shall look for a contradiction. Let x_0 be a point satisfying (98) or (99), and assume that $|u_\varepsilon - u_\varepsilon(x_0)|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon > 0$ goes to 0. Put

$$v_\varepsilon = \frac{u_\varepsilon - u_\varepsilon(x_0)}{|u_\varepsilon - u_\varepsilon(x_0)|_{L^\infty(\Omega)}}.$$

The function v_ε satisfies

$$-\varepsilon \Delta v_\varepsilon + F(x, \nabla v_\varepsilon, \nabla^2 v_\varepsilon) = 0 \qquad \text{in} \quad \Omega,$$

$$< \nabla v_\varepsilon, \gamma >= \frac{g(x) - d_\varepsilon}{|u_\varepsilon - u_\varepsilon(x_0)|_{L^\infty(\Omega)}} \qquad \text{on} \quad \partial\Omega.$$

Since $|v_\varepsilon|_{L^\infty(\Omega)} = 1$,

$$v^*(x) = \limsup_{\varepsilon\downarrow 0, y\to x} v_\varepsilon(y), \quad v_*(x) = \liminf_{\varepsilon\downarrow 0, y\to x} v_\varepsilon(y),$$

are well definded. Now, in the case of (98), we use the Krylov-Safonov inequality as before to have

$$v^*(x_0) = v_*(x_0) = 0. \tag{100}$$

In the case of (99), by using the argument in [24], [27]n we have also the uniform continuity of u_ε ($\varepsilon \in (0,1)$) in $B(r,x_0)$, and (100) holds. In conclusion, (100) holds in both cases of (98) and (99).
We continue the proof, and see easily either $|v^*|_{L^\infty(\overline{\Omega})} = 1$ or $|v_*|_{L^\infty(\overline{\Omega})} = 1$ holds. If $|v^*|_{L^\infty(\overline{\Omega})} = 1$, since

$$F(x, \nabla v^*, \nabla^2 v^*)\underline{<}0 \qquad \text{in} \quad \Omega,$$

$$< \nabla v^*, \gamma > \underline{<}0 \qquad \text{on} \quad \partial\Omega,$$

the strong maximum principle (Lemma A) leads a contradiction, for v^* is not constant (100). (See the proof of Theorem 2.1, Step 1.) If $|v_*|_{L^\infty(\Omega)} = 1$, the same argument works, too. Therefore, u_ε satisfies (97), and Proposition 4.1 leads the remained claim.

As for the uniqueness of d, we do not have the general result, and shall give the following Example in which the uniqueness holds.

Example 4.3. *Let* $\Omega = \{(x_1, x_2)| \quad x_1 \in \mathbf{R}\backslash\mathbf{Z}, x_2 > 0\} \subset \mathbf{R}^2$ *(periodic in* x_1*). Assume that there exists a number* d *such that*

$$-\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial u}{\partial x_1} = 0 \qquad in \quad \Omega,$$

$$d + < \nabla u, \mathbf{n}(x) > - g(x) = 0 \qquad on \quad \partial\Omega,$$

where u *is bounded, and* $\mathbf{n}$ *is the outward unit normal to* Ω*. Then,* $d = \int_0^1 g(x_1, 0) dx_1$*.*

In fact, by integrating the above problem in $x_1 \in [0,1]$*,* $\overline{u}(x_2) = \int_0^1 u(x_1, x_2) dx_1$ *satisfies*

$$-\frac{\partial^2 \overline{u}(x_2)}{\partial x_2^2} = 0 \qquad in \quad (0, \infty),$$

$$d - \frac{\partial \overline{u}(0)}{\partial x_2} - \int_0^1 g(x_1, 0) dx_1 = 0 \qquad on \quad x_2 = 0,$$

and $\overline{u}$ *is bounded. From Theorem 3.3, we know that such a number* d *is unique. Since* $d = \int_0^1 g(x_1) dx_1$ *and* $\overline{u} \equiv C$ *(constant) satisfy the above, we proved the claim.*

5 Homogenization of oscillating Neumann type boundary conditions.

In this section, we study the following homogenization problem.

$$G(x,\nabla u_\varepsilon,\nabla^2 u_\varepsilon)=\sup_{\alpha\in A}\{-\sum_{ij=1}^{2}a_{ij}^{\alpha}(x)\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}-\sum_{i=1}^{2}b_i^{\alpha}(x)\frac{\partial u_\varepsilon}{\partial x_i}\}=0 \qquad (101)$$

$$\text{in}\quad \Omega_\varepsilon=\{(x_1,x_2)|\quad -a\le x_1\le a,\quad f_0(x_1)+\varepsilon f_1(x_1,\frac{x_1}{\varepsilon})\le x_2\le b\}\subset \mathbf{R}^2,$$

$$<\nabla u_\varepsilon,\mathbf{n}_\varepsilon>+c(x_1,\frac{x_1}{\varepsilon})u_\varepsilon=g(x_1,\frac{x_1}{\varepsilon}) \qquad (102)$$

$$\text{on}\quad \Gamma_\varepsilon=\{(x_1,x_2)|\quad -a\le x_1\le a,\quad x_2=f_0(x_1)+\varepsilon f_1(x_1,\frac{x_1}{\varepsilon})\},$$

$$u_\varepsilon=0\qquad \text{on}\quad \partial\Omega_\varepsilon\backslash\Gamma_\varepsilon, \qquad (103)$$

where $\varepsilon>0$, $a_{ij}^{\alpha}(x)$, $b_i^{\alpha}(x)$ are Lipschitz in x satisfying (5), $\mathbf{n}_\varepsilon(x)$ is the outward unit normal to Ω_ε,

$$c,\quad g,\quad f_1(x_1,\xi_1)\quad \text{are defined in}\quad \Omega_\varepsilon\times\mathbf{R},\quad \text{periodic in}\quad \xi_1\in\mathbf{R}\backslash\mathbf{Z}, \qquad (104)$$

$$0\le f_1(x_1,\xi_1),\quad 0<C<c(x,\xi_1)\qquad \text{in}\quad \Omega_\varepsilon\times\mathbf{R}\backslash\mathbf{Z}, \qquad (105)$$

where $C>0$ is a constant,

$$f_0'(\pm a)=0,\quad \frac{\partial f_1}{\partial \xi_1}(\pm a,\xi_1)=0, \qquad (106)$$

denoting $A_\alpha=(a_{ij}^{\alpha}(x))_{1\le i,j\le n}$,

$$\lambda_1\le A_\alpha\le\Lambda_1\qquad \text{any}\quad \alpha\in A. \qquad (107)$$

We are interested in the limit of u_ε of (101)-(103) as ε goes to 0. Remark that this problem is a straightforward generalization of Example 1.2, a similar case of which was treated in [22] by the variational method. For our nonlinear problem, we need further assumptions listed in the following. These assumptions come from the formal asymptotic expansion of u_ε which we describe in below. (See also Remark 5.1 and Lemma 5.1 in below.)

$$b_1^\alpha \equiv 0, \quad b_2^\alpha = a_{11}^\alpha f_0'' \qquad \text{any} \quad \alpha \in A, \quad x \in \Omega_\varepsilon, \tag{108}$$

$$\{a_{11}^\alpha(1+f_0'^2) - 2a_{12}^\alpha f_0' + a_{22}^\alpha\}^2 \geq 4(a_{11}^\alpha a_{22}^\alpha - a_{12}^{\alpha\,2}) \qquad \text{forall} \quad \alpha \in A, \quad x \in \Omega_\varepsilon, \tag{109}$$

and for

$$O(x_1) = \{(\xi_1, \xi_2) | \quad \xi_2 \geq f_1(x_1, \xi_1), \quad \text{periodic in} \quad \xi_1\},$$
$$\partial O(x_1) \quad is \quad C^{3,1}. \tag{110}$$

The existence and uniqueness of u_ε ($\varepsilon > 0$) is established in the general viscosity solutions theory. (See [15]n.) Our goal is to show the existence of $u(x)$ such that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x) \qquad \text{uniformly in} \quad \overline{\Omega}, \tag{111}$$

where $\Omega = \{(x_1, x_2) | \quad -a \underline{\leq} x_1 \underline{\leq} a, \quad f_0(x_1) \underline{\leq} x_2 \underline{\leq} b\}$, and to find the effective limit P.D.E. and B.C. for u. As for (111), we remark that our convergence is in L^∞, while in [22]n the convergence was in H^1. The limit (effective) P.D.E. and B.C. are given by using the long time averaged result in § 3. Let us begin by deriving the cell problem for (101)-(103) by the formal asymptotic expansions method:

$$u_\varepsilon = u(x) + \varepsilon v(\frac{x_1}{\varepsilon}, \frac{x_2 - f_0(x_1)}{\varepsilon}) + O(\varepsilon^2), \tag{112}$$

where we are assuming that "the corrector" v depends only on $\xi_1 = \frac{x_1}{\varepsilon}$ and $\xi_2 = \frac{x_2 - f_0(x_1)}{\varepsilon}$ (ξ_1, ξ_2 are rescaled variables.) From (112), we obtain

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x_1} &= \frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial \xi_1} - f_0'(x_1)\frac{\partial v}{\partial \xi_2} + O(\varepsilon), \\ \frac{\partial u_\varepsilon}{\partial x_2} &= \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial \xi_2} + O(\varepsilon), \end{aligned} \tag{113}$$

$$\begin{aligned} \frac{\partial^2 u_\varepsilon}{\partial x_1^2} &= \frac{\partial^2 u}{\partial x_1^2} - f_0''(x_1)\frac{\partial v}{\partial \xi_2} + \frac{1}{\varepsilon}\{\frac{\partial^2 v}{\partial \xi_1^2} - 2f_0'(x_1)\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} + (f_0')^2\frac{\partial^2 v}{\partial \xi_2^2}\} + O(\varepsilon), \\ \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} &= \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{\varepsilon}(\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} - f_0'(x_1)\frac{\partial^2 v}{\partial \xi_2^2}) + O(\varepsilon), \\ \frac{\partial^2 u_\varepsilon}{\partial x_2^2} &= \frac{\partial^2 u}{\partial x_2^2} + \frac{1}{\varepsilon}\frac{\partial^2 v}{\partial \xi_2^2} + O(\varepsilon), \end{aligned} \tag{114}$$

First, by introducing (113) and (114) into

$$
\begin{aligned}
&-\sum_{i,j=1}^{2} a_{ij}^{\alpha}\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} - \sum_{i=1}^{2} b_i^{\alpha}\frac{\partial u_\varepsilon}{\partial x_i} = \\
&= -\{a_{11}^{\alpha}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}^{\alpha}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}^{\alpha}\frac{\partial^2 u}{\partial x_2^2} - a_{11}^{\alpha} f_0''(x_1)\frac{\partial v}{\partial \xi_2} \\
&+b_1^{\alpha}(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial \xi_1} - f_0'\frac{\partial v}{\partial \xi_2}) + b_2(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial \xi_2})\} \\
&-\frac{1}{\varepsilon}[a_{11}^{\alpha}\{\frac{\partial^2 v}{\partial \xi_1^2} - 2f_0'(x_1)\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} + (f_0')^2\frac{\partial^2 v}{\partial \xi_2^2}\} + 2a_{12}^{\alpha}(\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} - f_0'(x_1)\frac{\partial^2 v}{\partial \xi_2^2}) \\
&+a_{22}^{\alpha}\frac{\partial^2 v}{\partial \xi_2^2}]
\end{aligned}
$$

and by using (108),

$$
\begin{aligned}
&= -(a_{11}^{\alpha}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}^{\alpha}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}^{\alpha}\frac{\partial^2 u}{\partial x_2^2}) \qquad (115)\\
&-\frac{1}{\varepsilon}[a_{11}^{\alpha}\frac{\partial^2 v}{\partial \xi_1^2} + 2(a_{12}^{\alpha} - a_{11}^{\alpha} f_0'(x_1))\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} + \{a_{11}^{\alpha}(f_0')^2 - 2a_{12}^{\alpha} f_0' + a_{22}^{\alpha}\}\frac{\partial^2 v}{\partial \xi_2^2}].
\end{aligned}
$$

Remark 5.1. The condition (108) was used to efface the dependence on ξ (microscopic variable) in the ordinary order ($O(1)$) part in (115).

Let $(x, r, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^2$ ($p = (p_1, p_2)$) be arbitrarily fixed, and define the following operators.

$$
\begin{aligned}
&P_{x,r,p}^{\alpha}(D_\xi^2 v(\xi_1, \xi_2)) \equiv \qquad (116)\\
&\equiv -[a_{11}^{\alpha}\frac{\partial^2 v}{\partial \xi_1^2} + 2(a_{12}^{\alpha} - a_{11}^{\alpha} f_0')\frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} + \{a_{11}^{\alpha}(f_0')^2 - 2a_{12}^{\alpha} f_0' + a_{22}^{\alpha}\}\frac{\partial^2 v}{\partial \xi_2^2}]
\end{aligned}
$$

in $O(x_1)$, and

$$
P_{x,r,p}(D_\xi^2 v(\xi_1, \xi_2)) \equiv \sup_{\alpha \in \mathrm{A}}\{P_{x,r,p}^{\alpha}(D_\xi^2 v(\xi_1, \xi_2))\} \qquad \text{in} \quad O(x_1). \qquad (117)
$$

Next, by introducing (113) into (102), we have

$$
\frac{1}{\sqrt{1 + (f_0' + \frac{\partial f}{\partial \xi_1})^2}}\{(f_0' + \frac{\partial f_1}{\partial \xi_1})\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2}\}
$$

$$= g(x,\xi_1) - c(x,\xi_1)u - \frac{1}{\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}}\{(f_0'+\frac{\partial f_1}{\partial \xi_1})(\frac{\partial v}{\partial \xi_1} - f_0'\frac{\partial v}{\partial \xi_2}) - \frac{\partial v}{\partial \xi_2}\}.$$

By denoting the outward unit normal to the boundary of

$$\Omega = \{(x_1,x_2)| \quad -a \underline{\leq} x_1 \underline{\leq} a, \quad x_2 \geq f_0(x_1)\}$$

as

$$\nu = \frac{1}{\sqrt{1+(f_0')^2}}(f_0', -1),$$

the above equation on the boundary becomes

$$\begin{aligned} <\nabla u, \nu> \quad &= \quad \frac{1}{\sqrt{1+(f_0')^2}}[-\frac{\partial u}{\partial x_1}\frac{\partial f_1}{\partial \xi_1} - \sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}(cu-g) \\ &\quad -(f_0'+\frac{\partial f_1}{\partial \xi_1})\frac{\partial v}{\partial \xi_1} + \{f_0'(f_0'+\frac{\partial f_1}{\partial \xi_1})+1\}\frac{\partial v}{\partial \xi_2}]. \end{aligned} \tag{118}$$

Let

$$\gamma(\xi_1,\xi_2) = \frac{(f_0'+\frac{\partial f_1}{\partial \xi_1}, -\{f_0'(f_0'+\frac{\partial f_1}{\partial \xi_1})+1\})}{\sqrt{1+(f_0')^2}} \qquad \text{on} \quad \partial O(x_1), \tag{119}$$

and for $(x,r,p) \in \Omega \times \mathbf{R} \times \mathbf{R}^2$

$$H(x,r,p,\xi) = \frac{1}{\sqrt{1+(f_0')^2}}\{-\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}(c(x,\xi_1)r-g) - p_1\frac{\partial f_1}{\partial \xi_1}\}. \tag{120}$$

Then, (118) becomes

$$<\nabla u, \nu> = -\{<\gamma, \nabla_\xi v> - H(x,r,p,\xi)\}. \tag{121}$$

From (115), (116), (117) and (121), the cell problem for (101)-(103) should be the following: for any fixed $(x,r,p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, find a unique number $d(x,p,r)$ such that the following problem has a viscosity solution (corrector) $v(\xi_1,\xi_2)$.

$$\begin{aligned} &P_{x,r,p}(D_\xi^2 v(\xi_1,\xi_2)) = 0 \qquad \text{in} \quad O(x_1), \\ &d(x,r,p) + <\nabla_\xi v, \gamma> - H(x,r,p,\xi) = 0 \qquad \text{on} \quad \partial O(x_1), \\ &v \quad \text{is bounded in} \quad \overline{O(x_1)}. \end{aligned} \tag{122}$$

Lemma 5.1. *Let (109) hold. Then, the operators $P^{\alpha}_{x,r,p}(\xi_1,\xi_2)$ are uniformly elliptic operators uniformly in $\alpha\in A$: there exist constants $0<\lambda_1'<\Lambda_1'$ such that*

$$\lambda_1' I \leq \begin{pmatrix} a^{\alpha}_{11} & a^{\alpha}_{12}-a^{\alpha}_{11}f_0' \\ a^{\alpha}_{12}-a^{\alpha}_{11}f_0' & a^{\alpha}_{22}-2a^{\alpha}_{12}f_0'+a^{\alpha}_{11}f_0' \end{pmatrix} \leq \Lambda_1' I \qquad \text{any} \quad \alpha\in A.$$

Proof of Lemma 5.1. The claim can easily confirmed by an elementary calculation. And we leave it to the readers.

Lemma 5.2. *Let $\alpha\in \mathrm{A}$ and (x,r,p) be fixed, and let $O(x_1)$, $P^{\alpha}_{x,r,p}(D^2_{\xi})$, $\gamma(\xi)$ and $H(x,r,p,\xi)$ be defined in (110), (116), (119) and (120). Assume that (104)-(110) hold. Then, there exists a unique number $d^{\alpha}(x,r,p)$ such that the following problem has a viscosity solution $v(\xi_1,\xi_2)$.*

$$\begin{aligned} &P^{\alpha}_{x,r,p}(D^2_{\xi}v(\xi_1,\xi_2)=0 \qquad \text{in} \quad O(x_1),\\ &d^{\alpha}(x,r,p)+<\nabla_{\xi}v,\gamma> -H(x,r,p,\xi)=0 \qquad \text{on} \quad \partial O(x_1),\\ &v \quad \text{is bounded in} \quad \overline{O(x_1)}. \end{aligned} \tag{123}$$

Proof of Lemma 5.2. From (119), we confirm easily that there exists a positive constant $\gamma_1>0$ such that

$$<\gamma,\zeta> \quad >\gamma_1>0 \qquad \text{on} \quad \partial O(x_1),$$

where $\zeta=\frac{(\frac{\partial f_1}{\partial \xi_1},-1)}{\sqrt{(\frac{\partial f_1}{\partial \xi_1})^2+1}}$ the outward unit normal to $\partial O(x_1)$. Then from Theorem 3.3, there exists a unique number $d^{\alpha}(x,r,p)$ such that (123) has a viscosity solution v.

Lemma 5.3. *We assume the same assumptions as in Lemma 5.2. For any fixed (x,r,p), there exists a unique number d such that (122) has a viscosity solution $v(\xi_1,\xi_2)$. Moreover,*

$$d(x,r,p)\leq d^{\alpha}(x,r,p) \qquad \text{any} \quad \alpha\in \mathrm{A}. \tag{124}$$

Proof of Lemma 5.3. From Theorem 3.3, there exists a unique number $d(x, r, p)$ such that (122) has a viscosity solution v. The inequality (124) comes from the construction of the number d and d^α in the proofs of Proposition 3.2. and Theorem 3.3. That is,

$$d = \lim_{R\to\infty} d_R, \quad d^\alpha = \lim_{R\to\infty} d^\alpha_R,$$

where d and d_R ($R \in \mathbf{N}$) are characterized by the following: for $O_R(x_1) = O(x_1) \cap \{\xi_2 \leq R\}$

$$\begin{aligned}
&P_{x,r,p}(D^2_\xi v_R(\xi_1, \xi_2) = 0 \qquad \text{in} \quad O_R(x_1),\\
&d_R(x, r, p) + < \nabla_\xi v_R, \gamma > -H(x, r, p, \xi) = 0 \qquad \text{on} \quad \partial O(x_1),\\
&< \nabla_\xi v_R, \mathbf{n} >= 0 \qquad \text{on} \quad \{\xi_2 = R\},
\end{aligned}$$

and

$$\begin{aligned}
&P^\alpha_{x,r,p}(D^2_\xi v^\alpha_R(\xi_1, \xi_2) = 0 \qquad \text{in} \quad O_R(x_1),\\
&d^\alpha_R(x, r, p) + < \nabla_\xi v^\alpha_R, \gamma > -H(x, r, p, \xi) = 0 \qquad \text{on} \quad \partial O(x_1),\\
&< \nabla_\xi v^\alpha_R, \mathbf{n} >= 0 \qquad \text{on} \quad \{\xi_2 = R\},
\end{aligned}$$

where $\mathbf{n}$ is the outward unit normal to $\partial O_R(x_1)$ on $\{\xi_2 = R\}$. From the stochastic representations (15) of d_R and d^α_R in the approximating problems (78), we see that

$$d_R \leq d^\alpha_R \qquad \text{any} \quad R \in \mathbf{N}.$$

Therefore, (124) was proved.

Since the oscillating Neumann boundary condition prevent us from obtaining the uniform gradient bounds of u_ε ($\varepsilon > 0$), we need to treat the upper and lower envelopes.

Lemma 5.4. *Assume that (5), (104)-(110) hold. Let u_ε be the solution of (101)-(103). Then, there exists a constant $M > 0$ such that*

$$|u_\varepsilon| < M \qquad \textit{any} \quad \varepsilon \in (0, 1). \tag{125}$$

Proof of Lemma 5.4. Let $x_0 = (0, b + r) \in \mathbf{R}^2$, where $r > 0$. Define

$$v(x) = A(r^{-p} - |x - x_0|^{-p}) \qquad x \in \Omega_\varepsilon.$$

Then, for $A > 0$ large enough, v is a super solution of (101)-(103) for any $\varepsilon \in (0,1)$. From the comparison result for (101)-(103), we get (125).

From (125),

$$u^*(x) = \limsup_{\varepsilon \downarrow 0, y \to x} u_\varepsilon(y), \quad u_*(x) = \liminf_{\varepsilon \downarrow 0, y \to x} u_\varepsilon(y) \qquad x \in \overline{\Omega},$$

are well-definded. Moreover, from (107) and the Krylov-Safonov inequality we can extract a subsequence $\varepsilon' \to 0$ such that

$$\lim_{\varepsilon' \downarrow 0} u_{\varepsilon'} = u \qquad \text{locally uniformly in} \quad \Omega, \quad u^* \geq u \geq u_*. \tag{126}$$

We claim the following.

Lemma 5.5. *Assume that (104)-(110) hold. Then, u^* and u_* are respectively viscosity sub and super solutions of the following problem.*

$$\sup_{\alpha \in \mathrm{A}} \{ - \sum_{i,j=1}^{n} a_{ij}^{\alpha} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i^{\alpha} \frac{\partial u}{\partial x_i} \} = 0 \qquad in \quad \Omega, \tag{127}$$

$$< \nabla u, \nu > + \overline{L}(x, u, \nabla u) = 0 \qquad on \quad \Gamma_0, \tag{128}$$

where ν is the outward unit normal to Ω defined on

$$\Gamma_0 = \{(x_1, x_2) | \quad -a \underline{\leq} x_1 \underline{\leq} a, \quad x_2 = f_0(x_1)\},$$

and for $(x,r,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^2$,

$$\overline{L}(x,r,p) = -d(x,r,p), \tag{129}$$

where $d(x,r,p)$ is defined in (122).

Proof of Lemma 5.5. From (126) and by the usual stability results of the viscosity solutions, it is clear that (127) holds. In the following, we shall see (128).

<u>Step 1.</u> We shall show that u^* satisfies

$$< \nabla u^*, \nu > + \overline{L}(x, \nabla u^*, \nabla^2 u^*) \underline{\leq} 0 \qquad \text{on} \quad \Gamma_0,$$

in the sense of viscosity solutions. Remark that $\Omega_\varepsilon \subset \Omega$ for any $\varepsilon \in [0,1)$. Let $\phi \in C^2(\overline{\Omega})$ be such that $u^* - \phi$ takes its strict maximum at $x_0 = (x_{01}, x_{02}) \in \Gamma_0$ with $u^*(x_0) = \phi(x_0)$. From the definition of the Neumann type boundary condition in the sense of viscosity solutions, we are to show either

$$\sup_{\alpha \in \mathrm{A}} \{ -\sum_{ij} a_{ij}^\alpha \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0) - \sum_i b_i^\alpha \frac{\partial \phi}{\partial x_i}(x_0) \} \leq 0, \tag{130}$$

or

$$< \nabla \phi(x_0), \nu > + \overline{L}(x_0, \nabla \phi(x_0), \nabla^2 \phi(x_0)) \leq 0. \tag{131}$$

We shall assume that both (130) and (131) are not true, and shall seek a contradiction. Thus, assume there exist constants θ_1 and θ_2 such that

$$\sup_{\alpha \in \mathrm{A}} \{ -\sum_{ij} a_{ij}^\alpha \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0) - \sum_i b_i^\alpha \frac{\partial \phi}{\partial x_i}(x_0) \} \equiv \theta_1 > 0, \tag{132}$$

$$< \nabla \phi(x_0), \nu > + \overline{L}(x_0, \nabla \phi(x_0), \nabla^2 \phi(x_0)) \equiv \theta_2 > 0. \tag{133}$$

For $(x_0, r_0, p_0) = (x_0, \phi(x_0), \nabla \phi(x_0))$, from Lemma 5.2 there exists a number $d(x_0, r_0, p_0)$ and v of

$$\begin{aligned} &P_{x_0, r_0, p_0}(D_\xi^2 v(\xi_1, \xi_2)) = 0 \qquad \text{in} \quad O(x_{01}), \\ &d(x_0, r_0, p_0) + < \nabla_\xi v, \gamma > - H(x_0, r_0, p_0, \xi) = 0 \qquad \text{on} \quad \partial O(x_1). \end{aligned} \tag{134}$$

Since $\xi_2 \geq f_1(x_1, \xi_1)$ for any $(\xi_1, \xi_2) \in O(x_1)$, we may define

$$\phi_\varepsilon(x_1, x_2) = \phi(x_1, x_2) + \varepsilon v(\frac{x_1}{\varepsilon}, \frac{x_2 - f_0(x_1)}{\varepsilon}) \qquad \text{in} \quad \overline{\Omega_\varepsilon}.$$

We claim that ϕ_ε is the viscosity supersolution of

$$\sup_{\alpha \in \mathrm{A}} \{ -\sum_{ij} a_{ij}^\alpha \frac{\partial^2 \phi_\varepsilon}{\partial x_i \partial x_j} - \sum_i b_i^\alpha \frac{\partial \phi_\varepsilon}{\partial x_i} \} > \frac{1}{4}\theta_1 \qquad \text{in} \quad B(x_0, r) \cap \Omega_\varepsilon, \tag{135}$$

$$< \nabla \phi_\varepsilon, \mathbf{n}_\varepsilon > + c(x, \frac{x_1}{\varepsilon})\phi_\varepsilon - g(x, \frac{x_1}{\varepsilon}) > \frac{1}{4}\theta_2 \qquad \text{on} \quad B(x_0, r) \cap \Gamma_\varepsilon, \tag{136}$$

in the sense of viscosity solutions in some small neighborhood of x_0, $B(x_0, r)$ ($r > 0$ is uniform in $\varepsilon \in (0,1)$). To see this, assume for $\psi \in C^2(\overline{\Omega})$, $\phi_\varepsilon - \psi$ takes its minimum at $(\overline{x_1}, \overline{x_2})$ with $\phi_\varepsilon(\overline{x_1}, \overline{x_2}) = \psi_\varepsilon(\overline{x_1}, \overline{x_2})$.

First, let us assume that $(\overline{x_1}, \overline{x_2}) \in \Omega_\varepsilon$. We write

$$\eta(\xi_1, \xi_2) \equiv \frac{1}{\varepsilon}(\psi - \phi)(\varepsilon\xi_1, \varepsilon\xi_2 + f_0(\varepsilon\xi_1)) \qquad (\xi_1, \xi_2) \in O(x_1), \tag{137}$$

$$\overline{\xi_1} \equiv \frac{\overline{x_1}}{\varepsilon}, \quad \overline{\xi_2} \equiv \frac{\overline{x_2} - f_0(\overline{x_1})}{\varepsilon}.$$

Hence,

$$(v - \eta)(\overline{\xi_1}, \overline{\xi_2}) \leq (v - \eta)(\xi_1, \xi_2),$$

in a neighborhood of $(\frac{x_{01}}{\varepsilon}, \frac{x_{02} - f_0(x_{01})}{\varepsilon}) \equiv (\xi_{01}, \xi_{02})$. Now, from (137),

$$\begin{aligned} \frac{\partial \eta}{\partial \xi_1} &= \frac{\partial}{\partial x_1}(\psi - \phi) + \frac{\partial}{\partial x_2}(\psi - \phi) f_0'(\varepsilon\xi_1), \\ \frac{\partial \eta}{\partial \xi_2} &= \frac{\partial}{\partial x_2}(\psi - \phi), \end{aligned} \tag{138}$$

$$\begin{aligned} \frac{\partial^2 \eta}{\partial \xi_1^2} &= \varepsilon\{\frac{\partial^2}{\partial x_1^2}(\psi - \phi) + 2\frac{\partial^2}{\partial x_1 \partial x_2}(\psi - \phi) f_0' + \frac{\partial^2}{\partial x_2^2}(\psi - \phi)(f_0')^2 \\ &+ \frac{\partial}{\partial x_2}(\psi - \phi) f_0''\}, \\ \frac{\partial^2 \eta}{\partial \xi_1 \partial \xi_2} &= \varepsilon\{\frac{\partial^2}{\partial x_1 \partial x_2}(\psi - \phi) + \frac{\partial^2}{\partial x_2^2}(\psi - \phi)(f_0')\}, \\ \frac{\partial^2 \eta}{\partial \xi_2^2} &= \varepsilon \frac{\partial^2}{\partial x_2^2}(\psi - \phi). \end{aligned} \tag{139}$$

Since $v(\xi_1, \xi_2)$ is the viscosity solution of (134), by (137), (138) and (139), for any $\delta > 0$ there exists a control $\overline{\alpha} \in A$ such that

$$\begin{aligned} -[a_{11}^{\overline{\alpha}}\{\frac{\partial^2}{\partial x_1^2}(\psi - \phi) \quad &+ \quad 2\frac{\partial^2}{\partial x_1 \partial x_2}(\psi - \phi) f_0' + \frac{\partial^2}{\partial x_2^2}(\psi - \phi)(f_0')^2 + \frac{\partial}{\partial x_2}(\psi - \phi) f_0''\} \\ &+2(a_{12}^{\overline{\alpha}} - a_{11}^{\overline{\alpha}} f_0')\{\frac{\partial^2}{\partial x_1 \partial x_2}(\psi - \phi) + \frac{\partial^2}{\partial x_2^2}(\psi - \phi)(f_0')\} \\ &+(a_{22}^{\overline{\alpha}} - 2a_{12}^{\overline{\alpha}} f_0' + a_{11}^{\overline{\alpha}}(f_0')^2)\frac{\partial^2}{\partial x_2^2}(\psi - \phi)(\overline{x_1}, \overline{x_2})] \geq -\delta. \end{aligned}$$

We can simplify the above by using $a_{11}^{\overline{\alpha}} f_0'' = b_2^{\overline{\alpha}}$ ((108)) to

$$\begin{aligned} &(-\sum_{ij} a_{ij}^{\overline{\alpha}}(x_0)\frac{\partial^2 \psi}{\partial x_i \partial x_j} - \sum_i b_i^{\overline{\alpha}}(x_0)\frac{\partial \psi}{\partial x_i} + \sum_{ij} a_{ij}^{\overline{\alpha}}(x_0)\frac{\partial^2 \phi}{\partial x_i \partial x_j} \\ &+ \sum_i b_i^{\overline{\alpha}}(x_0)\frac{\partial \phi}{\partial x_i})(\overline{x_1}, \overline{x_2}) \geq -\delta. \end{aligned}$$

Thus, since $\delta > 0$ is arbitrary,

$$\sup_{\alpha\in\mathrm{A}}\{-\sum_{ij} a^{\alpha}_{ij}(x_0)\frac{\partial^2\psi}{\partial x_i\partial x_j} - \sum_i b^{\alpha}_i(x_0)\frac{\partial\psi}{\partial x_i}\}(\overline{x_1},\overline{x_2})$$
$$\geq (-\sum_{ij} a^{\overline{\alpha}}_{ij}(x_0)\frac{\partial^2\psi}{\partial x_i\partial x_j} - \sum_i b^{\overline{\alpha}}_i(x_0)\frac{\partial\psi}{\partial x_i})(\overline{x_1},\overline{x_2})$$
$$\geq -\delta + (-\sum_{ij} a^{\overline{\alpha}}_{ij}(x_0)\frac{\partial^2\phi}{\partial x_i\partial x_j} - \sum_i b^{\overline{\alpha}}_i(x_0)\frac{\partial\phi}{\partial x_i})(\overline{x_1},\overline{x_2}) \geq \frac{\theta_1}{2},$$

for $(\overline{x_1},\overline{x_2})$ is near to x_0, and for $r > 0$ small enough. Therefore, (135) was shown.

Next, we assume

$$(\overline{x_1},\overline{x_2}) \in \Gamma_\varepsilon. \tag{140}$$

Again, we use the same function η defined in (137) and denote $\xi_1 = \frac{x_1}{\varepsilon}$, $\xi_2 = \frac{x_2 - f_0(x_1)}{\varepsilon}$,

$$(\overline{\xi_1},\overline{\xi_2}) = (\frac{\overline{x_1}}{\varepsilon}, \frac{\overline{x_2} - f_0(\overline{x_1})}{\varepsilon}), \quad (\overline{\xi_{01}},\overline{\xi_{02}}) = (\frac{\overline{x_{01}}}{\varepsilon}, \frac{\overline{x_{02}} - f_0(\overline{x_{01}})}{\varepsilon}).$$

Thus,

$$(v-\eta)(\overline{\xi_1},\overline{\xi_2}) \underline{\leq} (v-\eta)(\xi_1,\xi_2), \tag{141}$$

in a small neighborhood of $(\overline{\xi_{01}},\overline{\xi_{02}})$. By (140) $\overline{x_2} = f_0(\overline{x_1}) + \varepsilon f_1(\overline{x}, \frac{\overline{x_1}}{\varepsilon})$, and

$$\overline{\xi_2} = f_1(\overline{x},\overline{\xi_1}), \quad (\overline{\xi_1},\overline{\xi_2}) \in \partial O(x_1).$$

Since v satisfies (134), from the definition of the viscosity solution

$$P_{x_0,\phi(x_0),\nabla\phi(x_0)}(D^2_\xi\eta)(\overline{\xi_1},\overline{\xi_2}) \geq 0, \tag{142}$$

or

$$d(x_0,\phi(x_0),\nabla\phi(x_0)) + <\nabla_\xi\eta,\gamma> (\overline{\xi_1},\overline{\xi_2}) - H(x_0,\phi(x_0),\nabla\phi(x_0),\overline{\xi_1},\overline{\xi_2}) \geq 0. \tag{143}$$

In the case of (142), as before we obtain

$$\sup_{\alpha\in A}\{-\sum_{ij} a^{\alpha}_{ij}(\overline{x})\frac{\partial^2\psi}{\partial x_i\partial x_j}(\overline{x}) - \sum_i b^{\alpha}_i(\overline{x})\frac{\partial\psi}{\partial x_i}(\overline{x})\} > \frac{1}{4}\theta_1. \tag{144}$$

In the case of (143), from (129), (120) and (143),

$$-\overline{L}(x_0,\phi(x_0),\nabla\phi(x_0))+\frac{1}{\sqrt{(f_0')^2+1}}<\nabla_\xi\eta,(f_0'+\frac{\partial f_1}{\partial \xi_1},-f_0'(f_0'+\frac{\partial f_1}{\partial \xi_1})-1)>$$

$$-\frac{1}{\sqrt{(f_0')^2+1}}(-\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}c(x,\xi_1)\phi-\frac{\partial \phi}{\partial x_1}\frac{\partial f_1}{\partial \xi_1}+\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}g)\geq 0. \tag{145}$$

Introducing (138) to (145)

$$-\overline{L}(x_0,\phi(x_0),\nabla\phi(x_0))+\frac{1}{\sqrt{(f_0')^2+1}}<\nabla(\psi-\phi)(x_0),(f_0'+\frac{\partial f_1}{\partial \xi_1},-1)>$$

$$-\frac{1}{\sqrt{(f_0')^2+1}}(-\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}c\phi-\frac{\partial \phi}{\partial x_1}\frac{\partial f_1}{\partial \xi_1}+\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}g)\geq o(\varepsilon),$$

and deviding the both hands sides of the above by $\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}$, by remarking that

$$\mathbf{n}_\varepsilon=(\frac{f_0'+\frac{\partial f_1}{\partial \xi_1}}{\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}},\frac{-1}{\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}})+o(\varepsilon),$$

we have

$$\frac{1}{\sqrt{(f_0')^2+1}}<\nabla\psi(x_0),\mathbf{n}_\varepsilon>-\frac{\overline{L}(x_0,\phi(x_0),\nabla\phi(x_0))}{\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}}$$

$$\geq\frac{1}{\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}}<\nabla\phi,\nu>-\frac{1}{\sqrt{(f_0')^2+1}}c\phi+\frac{1}{\sqrt{(f_0')^2+1}}g+o(\varepsilon).$$

By using (133) and multiplying the both hands sides of the above by $\sqrt{(f_0')^2+1}$, we get

$$<\nabla\psi(x_0),\mathbf{n}_\varepsilon>+c\phi(x_0)-g\geq\overline{L}(x_0,\phi(x_0),\nabla\phi(x_0))+<\nabla\phi(x_0),\nu>\equiv\theta_2>0,$$

and for $r>0$ and $\varepsilon>0$ small enough,

$$<\nabla\psi(x_1),\mathbf{n}_\varepsilon>+c\phi(x_1)-g\geq\frac{1}{2}\theta_2. \tag{146}$$

We have proved (136). Thus, in $B(x_0,r)\cap\overline{\Omega_\varepsilon}$, we have (135)-(136) and (1)-(2). Therefore,

$$\max_{B(x_0,r)\cap\Omega_\varepsilon}(u_\varepsilon-\phi_\varepsilon)=\max_{\partial(B(x_0,r)\cap\Omega_\varepsilon)}(u_\varepsilon-\phi_\varepsilon).$$

From (102) and (136), by using a similar argument in the proof of Lemma 2.6,

$$<\nabla(u_\varepsilon-\phi_\varepsilon),\mathbf{n}_\varepsilon>+c(u_\varepsilon-\phi_\varepsilon)<-\frac{1}{4}\theta_2<0 \qquad \text{on} \quad \Gamma_\varepsilon\cap B(x_0,r),$$

in the sense of viscosity solutions. By letting ε tends to zero, $\max(u_\varepsilon-\phi_\varepsilon)$ goes to zero and there exists $\varepsilon_0>0$ such that

$$<\nabla(u_\varepsilon-\phi_\varepsilon),\mathbf{n}_\varepsilon> \quad <-\frac{1}{8}\theta_2<0 \qquad \text{on} \quad \Gamma_\varepsilon\cap B(x_0,r) \qquad \text{any} \quad \varepsilon\in(0,\varepsilon_0).$$

From this, if $u_\varepsilon-\phi_\varepsilon$ ($\varepsilon\in(0,\varepsilon_0)$) takes its local maximum on $\Gamma_\varepsilon\cap B_r(x_0)$ the strong maximum principle (Lemma A) leads a contradiction. Thus, $u_\varepsilon-\phi_\varepsilon$ must take its maximum on $\partial\overline{B(x_0,r)\cap\Omega_\varepsilon}\backslash\Gamma_\varepsilon$, that is on $\partial B(x_0,r)$. However this contradicts to the fact that $u-\phi$ takes its strong maximum in $\overline{B(x_0,r)\cap\Omega}$ at x_0. Thus, we proved (130)-(131).

Step 2. The fact that u_* is a supersolution of

$$<\nabla u_*,\nu>+\overline{L}(x,\nabla u_*,\nabla^2u_*)\leq 0 \qquad \text{on} \quad \Gamma_0,$$

in the sense of viscosity solutions can be shown similarly to (and slightly easier than) Step 1. We omit the details, since the argument is parallel.

From the above, we complete the proof of Lemma 5.5.

Lemma 5.6. *Assume that (104)-(110) hold. Then,*

$$u^*=u_*=0 \qquad x\in\partial\Omega\backslash\Gamma_0.$$

Proof of Lemma 5.6. Let $x_0\in\partial\Omega_\varepsilon\backslash\Gamma_\varepsilon$ be arbitrarily fixed. We can take $\underline{v}$ and $\overline{v}$, sub and super solutions of

$$\sup_{\alpha\in A}\{-\sum_{ij}a^\alpha_{ij}\frac{\partial^2\underline{v}}{\partial x_i\partial x_j}-\sum_i b^\alpha_i\frac{\partial\underline{v}}{\partial x_i}\}\leq 0 \qquad \text{in} \quad \Omega_\varepsilon,$$

$$< \nabla \underline{v}, \mathbf{n}_\varepsilon > + c\underline{v} \leq g \qquad \text{on} \quad \Gamma_\varepsilon,$$

$$\underline{v}(x_0) = 0, \quad \underline{v}(x) \leq 0 \qquad \text{on} \quad \partial\Omega \backslash \Gamma_\varepsilon,$$

and

$$\sup_{\alpha \in \mathrm{A}} \{ -\sum_{ij} a_{ij}^\alpha \frac{\partial^2 \overline{v}}{\partial x_i \partial x_j} - \sum_i b_i^\alpha \frac{\partial \overline{v}}{\partial x_i} \} \geq 0 \qquad \text{in} \quad \Omega_\varepsilon,$$

$$< \nabla \overline{v}, \mathbf{n}_\varepsilon > + c\overline{v} \geq g \qquad \text{on} \quad \Gamma_\varepsilon,$$

$$\overline{v}(x_0) = 0, \quad \overline{v}(x) \geq 0 \qquad \text{on} \quad \partial\Omega \backslash \Gamma_\varepsilon.$$

From the comparison,

$$\underline{v} \leq u_\varepsilon \leq \overline{v} \qquad \text{any} \quad \varepsilon \in (0, 1),$$

and thus

$$\underline{v} \leq u_* \leq u^* \leq \overline{v} \qquad \text{any} \quad x \in \overline{\Omega}.$$

In particular, at x_0,

$$\underline{v}(x_0) = u_*(x_0) = u^*(x_0) = \overline{v}(x_0) = 0.$$

Lemma 5.7. *The function $\overline{L}(x, r, p)$ is increasing in r.*

Proof of Lemma 5.7. From the definition of $\overline{L}$, we are to show that $d(x, r, p)$ is decreasing in r. As we mentioned in the proof of (124) in Lemma 5.3, this fact is clear from the construction of d and its meaning in (15).

From Lemmas 5.5-5.7, we arrive at the following result.

Theorem 5.8. *Assume that (104)-(110) hold. Then, there exists a unique function $u(x)$ such that*

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon(x) = u(x) \qquad \textit{locally uniformly in} \quad \overline{\Omega},$$

which is the unique solution of (127), (128), and (103).

Proof of Theorem 5.8. From Lemmas 5.5, 5.6 and 5.7, the limit $u^* = u_* = u$ is unique and is a solution of the above problem. Moreover, since from Lemma 5.7 the uniqueness holds for (127)-(128) and (103), u is the unique solution. (We refer the readers to [15]) and G. Barles [8] for such uniqueness results. And, we proved the claim.

Remark 5.2. The effective boundary condition (128) is in general nonlinear. However, for the linear problem as in Example 1.2, (128) is lenear and matchs to the result in [22].

Example 5.1. *Let* $f_0' \equiv 0$*, and assume that* $a_{11} = a_{22} = 1$*,* $a_{12} = 0$*. Then,*

$$\overline{L}(x,r,p) = -d(x,r,p),$$

is obtained by the following long time averaged problem:

$$P_{x,r,p}(D_\xi^2 v(\xi_1,\xi_2)) = -\frac{\partial^2 v}{\partial \xi_1^2} - \frac{\partial^2 v}{\partial \xi_2^2} = 0 \qquad in \quad O(x_1),$$

$$d(x,r,p) - < \nabla_\xi v, (\frac{\partial f_1}{\partial \xi_1}, -1) > -\{-\sqrt{1+(\frac{\partial f_1}{\partial \xi_1})^2}(c(x,\xi_1)r - g) - p_1 \frac{\partial f_1}{\partial \xi_1}\} = 0 \qquad in \quad O(x_1),$$

where

$$O(x_1) = \{(\xi_1,\xi_2) | \quad periodic\ in \quad \xi_1 \in \mathbf{R}\backslash\mathbf{Z}, \quad \xi_2 \geq f_1(x,\xi_1)\}.$$

By integrating the above problem in $\xi_1 \in [0,1]$*, and by remarking that* f_1 *and* v *are periodic in* ξ_1*, we have*

$$d(x,r,p) = -r\int_0^1 \sqrt{1+(\frac{\partial f_1}{\partial \xi_1})^2} c(x,\xi_1) d\xi_1 + \int_0^1 \sqrt{1+(\frac{\partial f_1}{\partial \xi_1})^2} g d\xi_1.$$

Therefore, $\overline{L}(x,r,p)$ *is linear in* r*.*

Remark 5.3. Although in this paper we considered a particular exaple of the oscillating Neumann condition ((102)) in $\mathbf{R}^2$, we can apply the same method to more general homogenization of the oscillating boundary conditions in $\mathbf{R}^n$. We shall give more general formulation of this kind of problem in the future occassion.

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