FOURIER COEFFICIENTS OF NONCONGRUENCE CUSPFORMS

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Abstract. Given a finite index subgroup of $SL_2(\mathbb{Z})$ with modular curve defined over $\mathbb{Q}$, under the assumption that the space of weight k (≥ 2) cusp forms is 1-dimensional, we show that a form in this space with Fourier coefficients in $\mathbb{Q}$ has bounded denominators if and only if it is a congruence modular form.

1. Introduction

Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. The space of cusp forms of weight k for Γ, denoted by $S_k(\Gamma)$, consists of holomorphic functions f on the Poincaré upper half plane $\mathfrak{H}$ satisfying

(1) $f|\gamma = f$ for all $\gamma \in \Gamma$, where $|\gamma$ stands for the standard stroke operator (cf. [Shi71]);
(2) f is holomorphic at all cusps of Γ;
(3) f vanishes at all cusps of Γ.

Suppose the cusp ∞ of Γ has width μ. A form $f \in S_k(\Gamma)$ has Fourier expansions of the form

$$f(z) = \sum_{n \geq 1} a(n) q^{n/\mu}, \text{ where } q = e^{2\pi i z}.$$

When all Fourier coefficients $a(n)$ are algebraic, we say that f has bounded denominators if there exists a nonzero algebraic integer c such that $c \cdot a(n)$ is algebraically integral for all n.

For a congruence subgroup Γ, the space $S_k(\Gamma)$ has a basis consisting of forms with integral Fourier coefficients (cf. [Shi71]), resulting from the fact that Hecke operators have integral eigenvalues. Consequently, any form in $S_k(\Gamma)$ with algebraic coefficients has bounded denominators. When Γ is a noncongruence subgroup, which is majority, the situation is unknown. However, a folklore conjecture states that the bounded denominator property should characterize the congruence forms (cf. [Bir94]). More precisely,

Conjecture 1.1 (Unbounded denominators conjecture)**.** A meromorphic weight k modular form for a finite index subgroup Γ of $SL_2(\mathbb{Z})$ that is holomorphic on $\mathfrak{H}$ (such as the modular j function) with algebraic Fourier coefficients has bounded denominators if and only if it is a cusp form for a congruence subgroup.

This conjecture has impacts beyond the fundamental developments of modular forms. For instance, it implies a conjecture in the theory of vertex operator algebras known and believed by physicists, which says that the graded dimension of any C_2-cofinite, rational vertex operator algebra over $\mathbb{C}$ is a congruence modular function (cf. [Zhu96, CG99, DLM00]).

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Given a noncongruence subgroup Γ, denote by Γ^c its congruence closure, i.e., the smallest congruence subgroup containing Γ. Suppose $\Gamma^c = \cup_\gamma \Gamma\gamma$. Then $f \mapsto \sum_\gamma f|\gamma$ defines the trace map

$$\mathrm{Tr}_\Gamma^{\Gamma^c} : S_k(\Gamma) \to S_k(\Gamma^c).$$

As shown in Berger [Ber94] (see also [Sch97]), $S_k(\Gamma)$ decomposes into the direct sum of $S_k(\Gamma^c)$ and the kernel $S_k^{\mathrm{prim}}(\Gamma)$ of $\mathrm{Tr}_\Gamma^{\Gamma^c}$. The forms in $S_k^{\mathrm{prim}}(\Gamma)$ are thus genuinely noncongruence, called primitive noncongruence in [Sch97]. The conjecture above then says that any nonzero form in $S_k^{\mathrm{prim}}(\Gamma)$ and hence in $S_k(\Gamma) \setminus S_k(\Gamma^c)$ with algebraic Fourier coefficients must have unbounded denominators.

The purpose of this note is to give a partial supportive answer to the above conjecture.

Theorem 1. *Suppose that the modular curve X_Γ of Γ has a model defined over $\mathbb{Q}$ so that the cusp at ∞ is $\mathbb{Q}$-rational, $k \geq 2$ and $S_k(\Gamma)$ is 1-dimensional. Then a form in $S_k(\Gamma)$ with Fourier coefficients in $\mathbb{Q}$ has bounded denominators if and only if it is a congruence modular form.*

We exhibit two special occasions to which the above theorem can be applied. The first is when the modular curve X_Γ is an elliptic curve defined over $\mathbb{Q}$, the space $S_2(\Gamma)$ is 1-dimensional generated by a form from a holomorphic 1-form on X_Γ. The second is when an elliptic modular surface fibred over X_Γ is a K3 surface defined over $\mathbb{Q}$, $S_3(\Gamma)$ is generated by a form arising from a holomorphic 2-form on the elliptic modular surface.

There is another approach to the unbounded denominators conjecture using modular functions and p-adic arguments, see [KL08, KL09].

2. Galois representations attached to $S_k(\Gamma)$

Recall that for $k \geq 2$ to the 1-dimensional space $S_k(\Gamma)$ there are associated compatible family of 2-dimensional ℓ-adic representations $\rho_{\Gamma,k,\ell}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. When $k = 2$ this is the dual of the Tate modules of the Jacobian of X_Γ. For $k \geq 4$ even this is constructed by Scholl in [Sch85]. As remarked in [Sch85] the same should hold for $k \geq 3$ odd, which we shall assume.

At a prime $p \neq \ell$ where $\rho_{\Gamma,k,\ell}$ is unramified, the characteristic polynomial of $\rho_{\Gamma,k,\ell}$ of the Frobenius Frob_p at p has the form $T^2 - b(p)T + \chi(p)p^{k-1}$, where χ is a real Dirichlet character of $\mathbb{Z}$ and $b(p) \in \mathbb{Z}$ with complex absolute value $|b(p)| \leq 2p^{k-1}$ (cf. [Sch85]).

It follows from the former Serre's conjecture that when ℓ is a large prime, the dual of $\rho_{\Gamma,k,\ell}$ is modular (cf. [ALLL10]). In other words, there is a normalized newform g of weight k level N and character χ such that its eigenvalue with respect to the Hecke operator T_p at any prime $p \nmid N$ is $b(p)$. Write g in its Fourier expansion $g(z) = \sum_{n\geq 1} b(n)q^n$.

Let $f = \sum_{n\geq 1} a(n)q^{n/\mu}$ be a nonzero form in $S_k(\Gamma)$ with rational Fourier coefficients. By Scholl [Sch85, Theorem 5.4], f satisfies the congruences predicted by Atkin and Swinnerton-Dyer (ASD) [ASD71], namely, there is an integer N_1 such that for every prime $p > N_1$

$$a(np) - b(p)a(n) + \chi(p)p^{k-1}a(n/p) \equiv 0 \mod p^{(k-1)(1+\mathrm{ord}_p n)}, \quad \forall n \geq 1. \tag{1}$$

When this happens, f and g are said to satisfy the Atkin and Swinnerton-Dyer congruence relations in [LLY05].

We shall prove the theorem below, from which Theorem 1 will follow as $S_k(\Gamma)$ is 1-dimensional.

Theorem 2. *Suppose $k \geq 2$ and $f(z) = \sum_{n\geq 1} a(n)q^{n/\mu}$ is a form in $S_k(\Gamma)$ with Fourier coefficients in $\mathbb{Q}$ satisfying (1) with a normalized weight k newform $g(z) = \sum_{n\geq 1} b(n)q^n$ for all primes $p > N_1$. If f has bounded denominators, then $f \notin S_k^{\mathrm{prim}}(\Gamma)$.*

Since f in Theorem 2 has bounded denominators, we may assume that all Fourier coefficients $a(n)$ are in $\mathbb{Z}$.

3. Comparing Fourier coefficients of f and g

We begin by observing a bound on Fourier coefficients of f.

Theorem 3 (Selberg [Sel65]). *Let $f(z) = \sum a(n)q^{n/\mu}$ be a weight k cusp form for some finite index subgroup of the modular group. Then there exists a constant $C > 1$ depending on f such that the complex absolute value of $a(n)$ satisfies the bound*

$$|a(n)| < Cn^{k/2-1/5} \text{ for all } n \geq 1. \tag{2}$$

Since the coefficients of g satisfy the Ramanujan bound which implies, for any $\varepsilon > 0$, $|b(n)| = O(n^{(k-1)/2+\varepsilon})$ for all $n \geq 1$, we may further assume that

$$|b(n)| < Cn^{k/2-1/5} \text{ for all } \ n \geq 1. \tag{3}$$

Suppose $a(m) \neq 0$ for some positive integer m. Let $A(m) = m^{k/2-1/5}$. Fix an integer $P(m) > N_1$ such that

$$3C^2 A(m)^2 n^{k/2-1/5} < n^{k-1}$$

for every integer $n > P(m)$. We proceed to compare $a(mn)$ with $b(n)$. In what follows we use the convention that $a(x) = 0$ if x is not a positive integer.

Lemma 1. *For primes $p > P(m)$ and all integers $n \geq 1$, we have $a(mp^n) = b(p)a(mp^{n-1}) - \chi(p)p^{k-1}a(mp^{n-2})$. Further, if p is coprime to m, we have*

$$a(mp^n) = b(p^n)a(m). \tag{4}$$

Proof. Combining (2) and (3) we have for all $n \geq 0$

$$|a(mp^{n+1}) - b(p)a(mp^n) + \chi(p)p^{k-1}a(mp^{n-1})| < 3Cm^{(k/2-1/5)(n+1)}p^{(k/2-1/5)(n+1)} < p^{(k-1)(n+1)}.$$

The ASD congruence (1) gives

$$a(mp^{n+1}) - b(p)a(mp^n) + \chi(p)p^{k-1}a(mp^{n-1}) \equiv 0 \mod p^{(k-1)(n+1)}.$$

It follows from the choice of $P(m)$ that for a prime $p > P(m)$, the integer $a(mp^{n+1}) - b(p)a(mp^n) + \chi(p)p^{k-1}a(mp^{n-1})$ has to be zero for every $n \geq 0$. Next assume $(p, m) = 1$. When $n = 0$, this gives $a(mp) = b(p)a(m)$. We prove the statements by induction. Assume that $a(mp^n) = b(p^n)a(m)$ when $n < e$. Then

$$\begin{aligned} a(mp^e) &= b(p)a(mp^{e-1}) - \chi(p)p^{k-1}a(mp^{e-2}) \\ &= a(m)[b(p)b(p^{e-1}) - \chi(p)p^{k-1}b(p^{e-2})] = a(m)b(p^e). \end{aligned}$$

□

Next we prove a similar statement with multiple large prime factors.

Lemma 2. *Let $p_1, ..., p_r$ be r distinct primes greater than $P(m)$ and coprime to m. For all integers $e_i \geq 0$ we have*

$$a(mp_1^{e_1} \cdots p_r^{e_r}) = a(m)b(p_1^{e_1}) \cdots b(p_{r-1}^{e_{r-1}})b(p_r^{e^r}) = a(m)b(p_1^{e_1} \cdots p_r^{e_r}). \tag{5}$$

Proof. This is Lemma 1 when $r=1$, hence we assume $r\geq 2$. We will first use mathematical induction on the sum $e_1+e_2+\cdots+e_r$, and for fixed sum by induction on the number of nonzero e_i's, to show that

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) = a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}).$$

In view of Lemma 1, we may assume all $e_i\geq 1$ to begin with. Applying congruence (1) to prime p_1, we have

$$a(mp_1^{e_1}\cdots p_r^{e_r}) - b(p_1)a(mp_1^{e_1-1}p_2^{e_2}\cdots p_r^{e_r}) + \chi(p_1)p_1^{k-1}a(mp_1^{e_1-2}p_2^{e_2}\cdots p_r^{e_r}) \equiv 0 \mod p_1^{(k-1)e_1} \tag{6}$$

and

$$a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}}) - b(p_1)a(mp_1^{e_1-1}p_2^{e_2}\cdots p_{r-1}^{e_{r-1}}) + \chi(p_1)p_1^{k-1}a(mp_1^{e_1-2}p_2^{e_2}\cdots p_{r-1}^{e_{r-1}}) \equiv 0 \mod p_1^{(k-1)e_1}. \tag{7}$$

Now multiplying (6) by $a(m)$, we obtain

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(m)b(p_1)a(mp_1^{e_1-1}p_2^{e_2}\cdots p_r^{e_r}) + \chi(p_1)p_1^{k-1}a(m)a(mp_1^{e_1-2}p_2^{e_2}\cdots p_r^{e_r}) \equiv 0 \mod p_1^{(k-1)e_1}.$$

By induction assumption this can be rewritten as

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - b(p_1)a(mp_1^{e_1-1}p_2^{e_2}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) + \chi(p_1)p_1^{k-1}a(mp_1^{e_1-2}p_2^{e_2}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) \equiv 0 \mod p_1^{(k-1)e_1}. \tag{8}$$

So multiplying both sides of (7) by $a(mp_r^{e_r})$ and subtracting it from (8), we get

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) \equiv 0 \mod (p_1^{e_1})^{(k-1)}.$$

By symmetry, the same congruence holds when one replaces p_1 by p_i and e_1 by e_i for $1\leq i\leq r-1$. As $p_1,...,p_{r-1}$ are distinct primes, this gives

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) \equiv 0 \mod (p_1^{e_1}\cdots p_{r-1}^{e_{r-1}})^{k-1}. \tag{9}$$

On the other hand, similar to (6), we have

$$a(mp_1^{e_1}\cdots p_r^{e_r}) - b(p_r)a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}}p_r^{e_r-1}) + \chi(p_r)p_r^{k-1}a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}}p_r^{e_r-2}) \equiv 0 \mod p_r^{(k-1)e_r}.$$

Multiply $a(m)$ to the above and use induction assumption again to get

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - b(p_r)a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r-1}) + \chi(p_r)p_r^{k-1}a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r-2}) \equiv 0 \mod p_r^{(k-1)e_r}.$$

Apply $a(mp_r^{e_r}) = b(p_r)a(mp_r^{e_r-1}) - \chi(p_r)p_r^{k-1}a(mp_r^{e_r-2})$ from Lemma 1 to the above congruence to obtain

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) \equiv 0 \mod (p_r^{e_r})^{k-1}.$$

Combined with (9), this gives

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}) \equiv 0 \mod (p_1^{e_1}\cdots p_r^{e_r})^{k-1}.$$

By the choice of C,

$$|a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r})| < 2C^2A(m)^2(p_1^{e_1}\cdots p_r^{e_r})^{k/2-1/5} < (p_1^{e_1}\cdots p_r^{e_r})^{k-1}.$$

Since $a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) - a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r})$ is an integer, we conclude that

$$a(m)a(mp_1^{e_1}\cdots p_r^{e_r}) = a(mp_1^{e_1}\cdots p_{r-1}^{e_{r-1}})a(mp_r^{e_r}).$$

By induction hypothesis, this implies

$$a(m)^{r-1}a(mp_1^{e_1}\cdots p_r^{e_r}) = a(mp_1^{e_1})\cdots a(mp_{r-1}^{e_{r-1}})a(mp_r^{e^r}).$$

This is the same as (5) because of Lemma 1. □

Consequently, $a(mM) = a(m)b(M)$ as long as all prime factors of M are larger than $P(m)$ and coprime to m.

4. Proof of Theorem 2

Recall that a form $h \in S_k(\Gamma)$ is primitive noncongruence if $\mathrm{Tr}_\Gamma^{\Gamma^c} h = 0$. We study the primitive noncongruence condition. The lemma below says that the primitive noncongruence property is independent of the choice of the group which affords the given form.

Lemma 3. *Let h be a weight k modular form for a finite index subgroup Γ of $SL_2(\mathbb{Z})$ and $\Gamma_h = \{\gamma \in SL_2(\mathbb{Z}) : h|\gamma = h\}$ which is a supergroup of Γ. Then $h \in S_k^{prim}(\Gamma)$ if and only if $h \in S_k^{prim}(\Gamma_h)$.*

Proof. We first show that the set $\Gamma_h \cdot \Gamma^c = \{g_1g_2 : g_1 \in \Gamma_h, g_2 \in \Gamma^c\} = \Gamma_h^c$. It is easy to see $\Gamma_h \cdot \Gamma^c \subset \Gamma_h^c$. Let H be any principal congruence subgroup contained in Γ^c, then $\Gamma_h \cdot H = \{g_1g_2 : g_1 \in \Gamma_h, g_2 \in H\}$ is a group by the normality of H, and it is congruence and contained in Γ_h^c. Thus $\Gamma_h \cdot H = \Gamma_h^c$ which implies $\Gamma_h^c \subset \Gamma_h \cdot \Gamma^c$. It follows that $[\Gamma_h^c : \Gamma_h] = [\Gamma^c : \Gamma^c \cap \Gamma_h]$. Thus any transversal of $\Gamma^c \cap \Gamma_h$ in Γ^c is also a transversal of Γ_h in Γ_h^c. So $\mathrm{Tr}_{\Gamma_h}^{\Gamma_h^c} h = 0$ if and only if $\mathrm{Tr}_\Gamma^{\Gamma^c} h = \mathrm{Tr}_{\Gamma^c\cap\Gamma_h}^{\Gamma_h^c}\mathrm{Tr}_\Gamma^{\Gamma^c\cap\Gamma_h} h = [\Gamma^c \cap \Gamma_h : \Gamma]\mathrm{Tr}_{\Gamma_h\cap\Gamma^c}^{\Gamma^c} h = 0$ since h is a cusp form for $\Gamma^c \cap \Gamma_h$.

□

An immediate consequence is

Corollary 1. *If $h \in S_k^{prim}(\Gamma)$, then $h \in S_k^{prim}(G)$ for any finite index subgroup G of Γ.*

Next, assuming a form is primitive noncongruence, we derive a few primitive noncongruence conclusions on forms related to this form.

Lemma 4. *Let $\gamma \in GL_2(\mathbb{Q})$ such that $\gamma^n \in \Gamma$ for some positive integer n. If h is primitive noncongruence, so is $h|\gamma$.*

Proof. Let $G = \cap_{j=1}^n \gamma^{-j}\Gamma\gamma^j$. It is normalized by γ as $\gamma^n \in \Gamma$. By Corollary 1, $h \in S_k^{\mathrm{prim}}(G)$. Further, $\gamma^{-1}G^c\gamma$ contains a congruence subgroup, hence $G^c \cap \gamma^{-1}G^c\gamma$ is a congruence subgroup containing G and contained in G^c, thus it is equal to G^c. This means that $\gamma^{-1}G^c\gamma \supseteq G^c$, or equivalently, $\gamma G^c\gamma^{-1} \subseteq G^c$. Thus $\gamma G^c\gamma^{-1}$ is a congruence subgroup containing $\gamma G\gamma^{-1} = G$, hence is equal to G^c. We have shown that G^c is also invariant under conjugation by γ. Since γ normalizes G, $h|\gamma \in S_k(G)$. We have $\mathrm{Tr}_G^{G^c} h|\gamma = (\mathrm{Tr}_G^{G^c} h)|\gamma = 0$. □

Corollary 2. *Let K be a positive integer. Suppose $h = \sum a(n)q^{n/\mu}$ is primitive noncongruence. Then the following two functions are also primitive noncongruence: (a) the subseries $\sum a(Kn)q^{Kn/\mu}$, and (b) $h \otimes \phi = \sum a(n)\phi(n)q^{n/\mu}$ for any Dirichlet character ϕ of conductor K.*

Proof. Let $\gamma = \begin{pmatrix} 1 & \mu/K \\ 0 & 1 \end{pmatrix}$. Note that $\gamma^K \in \Gamma$. The claims follow from the above lemmas since

$$\sum a(Kn)q^{Kn/\mu} = \frac{1}{K}\sum_{j=1}^{K} f|\gamma^j,$$

and as discussed in [AL78],

$$h \otimes \phi = \frac{1}{\mathbf{g}(\phi^{-1})} \sum_{j \mod K,\ (j,K)=1} \phi(j)^{-1} h|\gamma^j,$$

where $\mathbf{g}(\phi^{-1}) = \sum_{j \mod K,\ (j,K)=1} \phi(j)^{-1}e^{2\pi i j/K}$ is the Gauss sum attached to ϕ^{-1}. □

Lemma 5. *If $h(z)$ is a primitive noncongruence cusp form of weight k, then so is $h(z/K)$ for any positive integer K.*

Proof. Let $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$. Then $\gamma^{-1}\Gamma_0(K)\gamma = \Gamma^0(K)$, where $\Gamma_0(K) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod K\}$ and $\Gamma^0(K)$ is defined similarly using congruence with lower triangular matrices. Suppose that h is a weight k primitive noncongruence cusp form for Γ', then it is also primitive congruence for $G = \Gamma' \cap \Gamma_0(K)$ by Corollary 1. The form $h|\gamma(z) = K^{k/2}h(z/K)$ is a weight k cusp form for $\gamma^{-1}G\gamma$, which is a finite index subgroup of $\Gamma^0(K)$. The congruence closure G^c of G is contained in $\Gamma_0(K)$ so that its conjugate $\gamma^{-1}G^c\gamma$ is a congruence subgroup contained in $\Gamma^0(K)$. We claim that $\gamma^{-1}G^c\gamma$ is the congruence closure of $\gamma^{-1}G\gamma$. This is because

$$\gamma^{-1}G^c\gamma \supseteq (\gamma^{-1}G\gamma)^c \supseteq \gamma^{-1}G\gamma,$$

which is equivalent to

$$G^c \supseteq \gamma(\gamma^{-1}G\gamma)^c\gamma^{-1} \supseteq G.$$

Since $\gamma(\gamma^{-1}G\gamma)^c\gamma^{-1}$ is congruence, it is equal to G^c by the minimality of G^c. This proves the claim. Thus $\mathrm{Tr}_{\gamma^{-1}G\gamma}^{\gamma^{-1}G^c\gamma} h|\gamma = (\mathrm{Tr}_G^{G^c} h)|\gamma = 0$ if $\mathrm{Tr}_G^{G^c} h = 0$. □

Now we proceed to prove Theorem 2. Assuming $f \in S_k^{\mathrm{prim}}(\Gamma)$, we shall derive a contradiction. Let m be an integer such that $a(m) \neq 0$. By Corollary 2 and Lemma 5, $h = \sum a(mn)q^{n/\mu}$ is a primitive noncongruence cusp form for a finite index subgroup $\tilde{G}$. Let K be the square of the product of all primes $\leq P(m)$ in §3, and let ϕ be a Dirichlet character of conductor K. Then $h \otimes \phi$ is a primitive noncongruence form for the group $G = \cap_{j=1}^{n} \gamma^{-j}\tilde{G}\gamma^j$, where $\gamma = \begin{pmatrix} 1 & \mu/K \\ 0 & 1 \end{pmatrix}$. It has Fourier expansion $(h \otimes \phi)(z) = \sum_{n \geq 1} a(nm)\phi(n)q^{n/\mu} \neq 0$. Moreover, by Lemmas 1 and 2, it agrees with $a(m)(g \otimes \phi)(z/\mu)$, a cusp form for a congruence subgroup Γ'. So it is a cusp form for the intersection $G \cap \Gamma'$, which is contained in the congruence subgroup $G^c \cap \Gamma'$. Note that $G \cap (G^c \cap \Gamma') = G \cap \Gamma'$ and there is a principal congruence subgroup H' of $G^c \cap \Gamma'$ such that $G \cdot H' = G^c$. Thus $[G^c : G] = [G^c \cap \Gamma' : G \cap \Gamma']$

and consequently, $0 = \mathrm{Tr}_G^{G^c}(h \otimes \phi) = \mathrm{Tr}_{G\cap\Gamma'}^{G^c\cap\Gamma'}(h \otimes \phi) = [G^c \cap \Gamma' : G \cap \Gamma'](h \otimes \phi)$ since $h \otimes \phi$ is a cusp form for Γ' and hence for $G^c \cap \Gamma'$. This then implies $h \otimes \phi = 0$, a contradiction.

References

[AL78] A. O. L. Atkin and W. C. Li, *Twists of newforms and pseudo-eigenvalues of W-operators*, Invent. Math. **48** (1978), no. 3, 221–243.

[ALLL10] A. O. L. Atkin, W.C. Li, T. Liu, and L. Long, *Galois representations with quaternion multiplications associated to noncongruence modular forms*, arXiv:1005.4105, (2010)

[ASD71] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Modular forms on noncongruence subgroups*, Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1971, pp. 1–25.

[Bir94] B. Birch, *Noncongruence subgroups, covers and drawings*, The Grothendieck theory of dessins d'enfants (Luminy, 1993), London Math. Soc. Lecture Note Ser., vol. 200, Cambridge Univ. Press, Cambridge, 1994, pp. 25–46.

[Ber94] G. Berger, *Hecke operators on noncongruence subgroups*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 9, 915–919.

[CG99] A. Coste and G. Gannon, *Congruence subgroups and rational conformal field theory*, `arXiv:math.NT/9909080`.

[DLM00] C. Dong, H. Li, and G. Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Comm. Math. Phys. **214** (2000), no. 1, 1–56.

[KL08] C. A. Kurth and L. Long, *On modular forms for some noncongruence subgroups of $SL_2(\mathbb{Z})$*, J. Number Theory **128** (2008), no. 7, 1989–2009.

[KL09] C. A. Kurth and L. Long, *On modular forms for some noncongruence subgroups of $SL_2(\mathbb{Z})$ II*, Bull. London Math. Soc **41** (2009), 589–598.

[LLY05] W.-C. W. Li, L. Long, and Z. Yang, *On Atkin and Swinnerton-Dyer congruence relations*, J. of Number Theory **113** (2005), no. 1, 117–148.

[Sch85] A. J. Scholl, *Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences*, Invent. Math. **79** (1985), no. 1, 49–77.

[Sch97] ———, *On the Hecke algebra of a noncongruence subgroup*, Bull. London Math. Soc. **29** (1997), no. 4, 395–399.

[Sel65] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math., Vol. VIII, Amer. Math. Soc., Providence, R.I., 1965, pp. 1–15.

[Shi71] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1.

[Zhu96] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), no. 1, 237–302.

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