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ON SUMS INVOLVING PRODUCTS OF THREE BINOMIAL COEFFICIENTS

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Abstract. In this paper we study congruences for sums of terms related to cubes of central binomial coefficients. Let $p > 3$ be a prime. We show that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0\pmod{p^2}$$

for all $d\in\{0,\ldots,p-1\}$ with $d\equiv (p+1)/2 \pmod 2$. We also solve the remaining open cases of Rodriguez-Villegas' conjectured congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2.

1. Introduction

Let p be an odd prime. It is known that (see, e.g., S. Ahlgren [A], L. van Hammer [H], T. Ishikawa [I] and K. Ono [O])

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{-1/2}{k}^3$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (4\mid x-1\ \&\ 2\mid y),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

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Clearly,

$$\binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \quad \text{for all } k \in \mathbb{N} = \{0,1,2,3,\ldots\},$$

and

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \quad \text{for any } k = \frac{p+1}{2},\ldots,p-1.$$

After the work in [Su1], the author [Su2] raised many conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ mod p^2 where $m \in \{1,-8,16,-64,256,-512,4096\}$; for example, the author conjectured that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \& \ p=x^2+7y^2 \ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1, \text{ i.e., } p\equiv 3,5,6 \pmod{7}, \end{cases} \tag{1.1}$$

where $(-)$ denotes the Legendre symbol. (It is known that if $(\frac{p}{7})=1$ then $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$, see, e.g., [C].) Quite recently the author's twin brother Zhi-Hong Sun [S2] made important progress on those conjectures; in particular, he proved (1.1) in the case $(\frac{p}{7})=-1$ and confirm the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/(-8)^k$ mod p^2.

Let $p=2n+1$ be an odd prime. It is easy to see that for any $k=0,\ldots,n$ we have

$$\binom{n+k}{2k} = \frac{\prod_{j=1}^k(-(2j-1)^2)}{4^k(2k)!}\prod_{j=1}^k\left(1-\frac{p^2}{(2j-1)^2}\right) \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}. \tag{1.2}$$

Based on this observation Z. H. Sun [S2] studied the polynomial

$$f_n(x) = \sum_{k=0}^n \binom{n+k}{2k}\binom{2k}{k}^2 x^k$$

and found the key identity

$$f_n(x(x+1)) = D_n(x)^2 \tag{1.3}$$

in his approach to (1.1), where

$$D_n(x) := \sum_{k=0}^n \binom{n+k}{2k}\binom{2k}{k}x^k = \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k}x^k.$$

Note that those numbers $D_k = D_k(1)$ $(k = 0, 1, 2, \dots)$ are the so-called central Delannoy numbers and $P_n(x) := D_n((x-1)/2)$ is the Legendre polynomial of degree n.

Recall that Catalan numbers are those integers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}\ (n \in \mathbb{N})$$

while Schröder numbers are given by

$$S_n := \sum_{k=0}^{n}\binom{n+k}{2k}C_k = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}.$$

We define the Schröder polynomial of degree n by

$$S_n(x) := \sum_{k=0}^{n}\binom{n+k}{2k}C_k x^k. \tag{1.4}$$

For basic information on D_n and S_n, the reader may consult [CHV], [Sl], and p. 178 and p. 185 of [St].

Via Schröder polynomials and the Zeilberger algorithm (cf. [PWZ]), we obtain the following results.

Theorem 1.1. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k} \equiv 0 \pmod{p^2} \tag{1.5}$$

for all $d \in \{0, 1, \dots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$. If $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k} \equiv (2p + 2 - 2^{p-1})\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2} \tag{1.6}$$

Theorem 1.2. *Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with x odd and y even. Provided that*

$$S_{(p-1)/2} \equiv (-1)^{(p-1)/4}2\left(2x - \frac{p}{x}\right) \pmod{p^2}, \tag{1.7}$$

we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k} \equiv 2p - 2x^2 \pmod{p^2} \tag{1.8}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -2p \pmod{p^2}. \tag{1.9}$$

Remark 1.1. We conjecture that (1.7) holds for any prime $p\equiv 1 \pmod 4$. By (1.2),

$$S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k} \pmod{p^2}. \tag{1.10}$$

Via the Gosper algorithm (cf. [PWZ]), we find that

$$8\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{(-16)^k}+\sum_{k=0}^{n}\frac{\binom{2k}{k}C_k}{(-16)^k}=\frac{(2n+1)^2}{(n+1)(-16)^n}\binom{2n}{n}^2\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{n}(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}=\frac{(2n+1)^2}{(-16)^n}\binom{2n}{n}^2\equiv 0 \pmod{p^2},$$

where $n=(p-1)/2$.

Motivated by his study related to K3 surfaces and Calabi-Yau manifolds, in 2003 Rodriguez-Villegas [RV] raised some conjectures on congruences. In particular, he conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p) \pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p) \pmod{p^2}, \tag{1.11}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}-a(p) \pmod{p^2} & \text{if } p\equiv 5 \pmod{12},\\ a(p) \pmod{p^2} & \text{otherwise},\end{cases} \tag{1.12}$$

where

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6=\eta(4z)^6,$$

$$\sum_{n=1}^{\infty}b(n)q^n=q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3=\eta^3(6z)\eta^3(2z),$$

$$\sum_{n=1}^{\infty}c(n)q^n=q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2=\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),$$

and the Dedekind η-function is given by

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\quad(\mathrm{Im}(z)>0\text{ and }q=e^{2\pi iz}).$$

In 1892 F. Klein and R. Fricke proved that (see also [SB])

$$a(p)=\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 4\text{ and }p=x^2+y^2\ (2\nmid x),\\ 0 & \text{if } p\equiv 3\pmod 4.\end{cases}$$

By [SB] we also have

$$b(p)=\begin{cases}4x^2-2p & \text{if } p\equiv 1\pmod 3\text{ and }p=x^2+3y^2\text{ with }x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2\pmod 3;\end{cases}$$

and

$$c(p)=\begin{cases}4x^2-2p & \text{if } (\frac{-2}{p})=1\text{ and }p=x^2+2y^2\text{ with }x,y\in\mathbb{Z},\\ 0 & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8.\end{cases}$$

Via an advanced approach involving the p-adic Gamma function and Gauss and Jacobi sums, E. Mortenson [M] managed to provid a partial solution of (1.11) and (1.12), with the following things open:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)=0\pmod{p^2}\quad\text{if } p\equiv 5\pmod 6,\tag{1.13}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2}\quad\text{if } p\equiv 3\pmod 4,\tag{1.14}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv -a(p)\pmod{p^2}\quad\text{if } p\equiv 5\pmod 6.\tag{1.15}$$

Concerning (1.13)-(1.15), Mortenson only showed that for each of them the squares of both sides of the congruence are congruent modulo p^2.

Our following theorem confirms (1.13)-(1.15) and hence completes the proof of (1.11) and (1.12). Now, all conjectures of Rodriguez-Villegas [RV] involving at most three products of binomial coefficients have been proved!

Theorem 1.3. *Let $p>3$ be a prime. For each $d=0,\ldots,(p-1)/2$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k}\equiv 0\pmod{p^2}\quad\textit{if } p\equiv 5\pmod 6,\tag{1.16}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{4k}{2k}}{256^k}\equiv 0\pmod{p^2}\quad\textit{if } p\equiv 5,7\pmod 8,\tag{1.17}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 0\pmod{p^2}\quad\textit{if } p\equiv 3\pmod 4.\tag{1.18}$$

Also, when $p\equiv 3\pmod 8$ *and* $p=x^2+2y^2$ *with* $x,y\in\mathbb{Z}$, *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 4x^2-2p\pmod{p^2};\tag{1.19}$$

when $p\equiv 5\pmod{12}$ *and* $p=x^2+y^2$ *with* $2\nmid x$ *and* $2\mid y$, *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 2p-4x^2\pmod{p^2}.\tag{1.20}$$

We will prove Theorems 1.1-1.2 in the next section, and show Theorem 1.3 in Section 3.

2. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *For any positive integer* n *we have*

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}x^{k-1}(x+1)^{k+1}=n(n+1)S_n(x)^2\tag{2.1}$$

and

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1}=\frac{S_n(x)}{2}(D_{n-1}(x)+D_{n+1}(x)).\tag{2.2}$$

Proof. (i) Observe that

$$S_n(x)^2=\sum_{k=0}^{n}\binom{n+k}{2k}C_kx^k\sum_{l=0}^{n}\binom{n+l}{2k}C_lx^l=\sum_{m=0}^{2n}a_m(n)x^m,$$

where

$$a_m(n):=\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}C_{m-k}.$$

Also, the coefficient of x^m in the left-hand side of (2.1) coincides with

$$\begin{aligned}b_m(n):=&\sum_{k=1}^{m+1}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\binom{k+1}{m+1-k}\\=&\sum_{k=0}^{m}\binom{n+k+1}{2k+2}\binom{2k+2}{k+1}\binom{2k+2}{k}\binom{k+2}{m-k}.\end{aligned}$$

Thus, for the validity of (2.1) it suffices to show that $b_m(n) = n(n+1)a_m(n)$ for all $m = 0, 1, \ldots$. Obviously, $a_0(n) = 1$ and $b_0(n) = n(n+1)$. Also, $a_1(n) = n(n+1)$ and $b_1(n) = n^2(n+1)^2$. By the Zeilberger algorithm via `Mathematica 7` (version 7) we find that both $u_m = a_m(n)$ and $u_m = b_m(n)$ satisfy the following recursion:

$$\begin{aligned}&(m+2)(m+3)(m+4)u_{m+2}\\ =&2(2mn^2+5n^2+2mn+5n-m^3-6m^2-11m-6)u_{m+1}\\ &-(m+1)(m-2n)(m+2n+2)u_m.\end{aligned}$$

Therefore $b_m(n) = n(n+1)a_m(n)$ by induction. This proves (2.1).

(ii) Note that

$$S_n(x)(D_{n-1}(x)+D_{n+1}(x)) = \sum_{m=0}^{2n+1} c_m(n)x^m$$

where

$$\begin{aligned}c_m(n) =& \sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{2m-2k}{m-k}\left(\binom{n-1+m-k}{2m-2k}+\binom{n+1+m-k}{2m-2k}\right)\\ =&2\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}\binom{2m-2k}{m-k}\frac{(m+n-k)^2-n(2m-2k-1)}{(m+n-k)(n-m+k+1)}.\end{aligned}$$

By the Zeilberger algorithm we find that $u_m = c_m(n)/2$ satisfies the recursion

$$\begin{aligned}&(m+2)(m+3)^2(m^2+5m+6+4n(n+1))u_{m+2}+2P(m,n)u_{m+1}\\ =&(m+2)((2n+1)^2-m^2)(m^2+7m+12+4n(n+1))u_m\end{aligned} \tag{2.3}$$

where $P(m,n)$ denotes the polynomial

$$\begin{aligned}&m^5+11m^4+45m^3+83m^2+64m+12+20n^4-40n^3-58n^2-38n\\ &-25mn+m^2n+2m^3n-33mn^2+m^2n^2+2m^3n^2-16mn^3-8mn^4.\end{aligned}$$

Clearly the coefficient of x^m on the left-hand side of (2.2) coincides with

$$d_m(n) = \sum_{k=0}^{m}\binom{n+k}{2k}\binom{2k}{k}^2\binom{k+1}{m-k}\frac{2k+1}{(k+1)^2}.$$

By the Zeilberger algorithm $u_m = d_m(n)$ also satisfies the recursion (2.3). Thus we have $d_m(n) = c_m(n)$ by induction on m. So (2.2) also holds.

In view of the above we have completed the proof of Lemma 2.1. □

Proof of Theorem 1.1. (i) We first determine $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{2k}{k+1}/64^k$ mod p^2 via Lemma 2.1, which actually led the author to the study of (1.5).

Recall the following combinatorial identity (cf. [Su2]):

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{C_k}{(-2)^k}=\begin{cases}(-1)^{(n-1)/2}C_{(n-1)/2}/2^n & \text{if } 2\nmid n,\\ 0 & \text{if } 2\mid n.\end{cases}$$

If we denote by $S(n)$ the sum of the left-hand side or the right-hand side of the identity, then we have the recursion $S(n+2)=-nS(n)/(n+3)$ ($n=1,2,3,\ldots$) by the Zeilberger algorithm.

Set $n=(p-1)/2$. Applying (2.1) with $x=-1/2$ we get

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-2)^{k-1}2^{k+1}}=n(n+1)S_n\left(-\frac{1}{2}\right)^2.$$

Thus, with the help of (1.2), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k}\equiv&\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-4)^k}\\ \equiv&-n(n+1)S_n\left(-\frac{1}{2}\right)^2\equiv\frac{1}{4}S_n\left(-\frac{1}{2}\right)^2\\ \equiv&\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4\\ C_{(n-1)/2}^2/2^{2n+2} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

In the case $p\equiv 3 \pmod 4$, clearly

$$\begin{aligned}\frac{C_{(n-1)/2}^2}{2^{2n+2}}=&\frac{\left(\binom{(p-1)/2}{(p+1)/4}\frac{2}{p-1}\right)^2}{4\times 2^{p-1}}\\ \equiv&\frac{1}{(1-2p)(1+p\,q_p(2))}\binom{(p-1)/2}{(p+1)/4}^2\\ \equiv&(1+2p-p\,q_p(2))\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2}\end{aligned}$$

where $q_p(2)=(2^{p-1}-1)/p$. Therefore (1.5) with $d=1$ holds if $p\equiv 1 \pmod 4$, and (1.6) is valid when $p\equiv 3 \pmod 4$.

(ii) For $d=0,1,\ldots,p-1$ set

$$u_d=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}=\sum_{k=d}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2u_d-(2d+3)^2u_{d+2}=\frac{(2p-1)^2(d+1)}{64^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1}=pC_{p-1}\equiv 0 \pmod p.$$

If $0\leqslant d<p-2$, then

$$\binom{2p}{p+d+1}=\frac{2p}{p+d+1}\binom{2p-1}{p+d}\equiv 0 \pmod p$$

and hence

$$(2d+1)^2u_d\equiv(2d+3)^2u_{d+2} \pmod{p^2}.$$

For $d\in\{0,\ldots,p-3\}$ with $d\equiv(p+1)/2 \pmod 2$, clearly $p\neq 2d+1<2p$ and hence

$$u_{d+2}\equiv 0 \pmod{p^2} \implies u_d\equiv 0 \pmod{p^2}.$$

If $p\equiv 3 \pmod 4$ then $p-1\equiv(p+1)/2 \pmod 2$; if $p\equiv 1 \pmod 4$ then $p-2\equiv(p+1)/2 \pmod 2$ and $p-2\geqslant(p+1)/2$. Thus, if $d\in\{p-1,p-2\}$ and $d\equiv(p+1)/2 \pmod 2$, then $d\geqslant(p+1)/2$ and hence $u_d\equiv 0 \pmod{p^2}$. It follows that $u_d\equiv 0 \pmod{p^2}$ (i.e., (1.5) holds) for all $d\in\{0,\ldots,p-1\}$ with $d\equiv(p+1)/2 \pmod 2$.

By the above we have completed the proof of Theorem 1.1. □

Lemma 2.2. *Let $p\equiv 1 \pmod 4$ be a prime. Write $p=x^2+y^2$ with x odd and y even. Then*

$$D_{(p-1)/2}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right) \pmod{p^2}. \tag{2.4}$$

Proof. In view of (1.2), (2.4) has the following equivalent form:

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right) \pmod{p^2},$$

which was conjectured by the author [Su2] and confirmed by Z. H. Sun [S1]. This proves (2.5). □

Remark 2.1. If p is a prime with $p\equiv 3 \pmod 4$, then $n=(p-1)/2$ is odd and hence

$$\begin{aligned}D_n&\equiv\sum_{k=0}^{n}(-1)^k\frac{\binom{2k}{k}^2}{16^k}=\sum_{k=0}^{n}(-1)^k\binom{-1/2}{k}^2\\&\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}^2=0 \pmod p.\end{aligned}$$

The following result was conjectured by the author [Su2] and confirmed by Z. H. Sun [S2].

Lemma 2.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 4\mid p-1\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{2.5}$$

Remark 2.2. Since [S2] is not yet publicly available, we mention that (1.2) and (1.3) yield

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 2^k=D_n^2\pmod{p^2}$$

where $n=(p-1)/2$. Hence (2.5) follows from Lemma 2.2 and Remark 2.1.

Proof of Theorem 1.2. Write $p=2n+1$. By (2.1)

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k=\frac{n(n+1)}{2}S_n^2.$$

Thus, if (1.7) holds, then by (1.2) we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}&\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k\\&\equiv\frac{p^2-1}{8}4(4x^2-4p)\pmod{p^2}\end{aligned}$$

and hence (1.8) holds.

Now we consider (1.9). Observe that

$$\binom{2k}{k+1}^2=\left(1-\frac{2k+1}{(k+1)^2}\right)\binom{2k}{k}^2\quad\text{for } k=0,1,2,\ldots,$$

and

$$\binom{2(p-1)}{p-1}\binom{2(p-1)}{(p-1)+1}^2=\frac{p}{2p-1}\binom{2p-1}{p-1}\binom{2p-2}{p-2}^2\equiv -p\pmod{p^2}.$$

Thus

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -p+\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(-8)^k}-\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\pmod{p^2}.\tag{2.6}$$

By (1.2) and (2.2) with $x = 1$,

$$\begin{aligned}\sum_{k=0}^{n} \frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k} \equiv& \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}^2 \frac{(2k+1)2^k}{(k+1)^2} \\ =& \frac{S_n}{4}(D_{n-1} + D_{n+1}) \pmod{p^2}.\end{aligned}$$

It is known (cf. [Sl] and [St]) that

$$(n+1)D_{n+1} = 3(2n+1)D_n - nD_{n-1} \quad \text{and} \quad D_{n+1} - 3D_n = 2nS_n.$$

Thus

$$\begin{aligned} n(D_{n-1} + D_{n+1}) =& 3(2n+1)D_n - D_{n+1} \\ =& 3(2n+1)D_n - (3D_n + 2nS_n) = 2n(3D_n - S_n)\end{aligned}$$

and hence

$$\sum_{k=0}^{n} \frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k} \equiv \frac{S_n}{2}(3D_n - S_n) \pmod{p^2}.$$

Now assume that (1.7) holds. Then, with the help of (2.4), we have

$$\frac{S_n}{2}(3D_n - S_n) \equiv \left(2x - \frac{p}{x}\right)\left(3\left(2x - \frac{p}{2x}\right) - \left(4x - \frac{2p}{x}\right)\right) \pmod{p^2}$$

and hence

$$\sum_{k=0}^{n} \frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k} \equiv 4x^2 - p \pmod{p^2}.$$

Combining this with (2.5) and (2.6), we immediately obtain (1.9).

The proof of Theorem 1.2 is now complete. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let p be an odd prime. Then, for any p-adic integer $x \not\equiv 0, -1 \pmod{p}$ we have*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \left(\frac{-x}{64}\right)^k \equiv \left(\frac{x+1}{p}\right) \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{x}{64(x+1)^2}\right)^k \pmod{p}. \tag{3.1}$$

Proof. Taking $n = (p-1)/2$ in the MacMahon identity (see, e.g., [G, (6.7)])

$$\sum_{k=0}^{n}\binom{n}{k}^3 x^k = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}x^k(1+x)^{n-2k}$$

and noting (1.2) and the basic facts

$$\binom{n}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod p$$

and

$$\binom{n-k}{k} \equiv \binom{-1/2-k}{k} = \frac{\binom{4k}{2k}}{(-4)^k} \pmod p,$$

we immediately get (3.1). □

Proof of Theorem 1.3. (i) For $d = 0, 1, \ldots, (p-1)/2$, we define

$$f(d) = \sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{3k}{k}}{108^k}, \quad g(d) = \sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{2k}{k}\binom{4k}{2k}}{256^k},$$

and

$$h(d) = \sum_{k=0}^{p-1}\frac{\binom{2k}{k+2d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$

By the Zeilberger algorithm, we find the recursive relations:

$$\begin{aligned}&(3d+1)(6d+1)f(d) - (3d+2)(6d+5)f(d+1)\\ =&\frac{(3p-1)(3p-2)(2d+1)}{2^{2p-1}27^{p-1}p}\binom{2p}{p+2d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1},\end{aligned}$$

$$\begin{aligned}&(8d+1)(8d+3)g(d) - (8d+5)(8d+7)g(d+1)\\ =&\frac{(4p-1)(4p-3)(2d+1)}{2^{8(p-1)}p}\binom{2p}{p+2d+1}\binom{2p-2}{p-1}\binom{4p-4}{2p-2},\end{aligned}$$

and

$$\begin{aligned}&(12d+1)(12d+5)h(d) - (12d+7)(12d+11)h(d+1)\\ =&\frac{(6p-1)(6p-5)(2d+1)}{2^{6(p-1)}27^{p-1}p}\binom{2p}{p+2d+1}\binom{3p-3}{p-1}\binom{6p-6}{3p-3}.\end{aligned}$$

Recall that $\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod p$. Also,

$$(3p-2)\binom{3p-3}{p-1} = p\binom{3p-2}{p} \equiv 0 \pmod p,$$
$$(4p-3)\binom{4p-4}{2p-2} = p\binom{4p-2}{2p} \equiv 0 \pmod p,$$
$$(6p-5)\binom{6p-6}{3p-3} = \frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)}\binom{6p-3}{3p} \equiv 0 \pmod p.$$

If $d < (p-1)/2$, then

$$\binom{2p}{p+2d+1} = \binom{2p}{p-1-2d} \equiv 0 \pmod p$$

and hence by the above we have

$$(3d+1)(6d+1)f(d) \equiv (3d+2)(6d+5)f(d+1) \pmod{p^2}, \tag{3.2}$$

$$(8d+1)(8d+3)g(d) \equiv (8d+5)(8d+7)g(d+1) \pmod{p^2}, \tag{3.3}$$

$$(12d+1)(12d+5)h(d) \equiv (12d+7)(12d+11)h(d+1) \pmod{p^2}. \tag{3.4}$$

Fix $0 \leqslant d < (p-1)/2$. If $p \equiv 5 \pmod 6$, then $3d+1, 6d+1 \not\equiv 0 \pmod p$ and hence by (3.2) we have

$$f(d+1) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.$$

If $p \equiv 5, 7 \pmod 8$, then $8d+1, 8d+3 \not\equiv 0 \pmod p$ (since $8d+3 < 4p$ and $8d+1, 8d+3 \notin \{p, 2p, 3p\}$) and hence by (3.3) we have

$$g(d+1) \equiv 0 \pmod{p^2} \implies g(d) \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$, then $12d+1, 12d+5 \not\equiv 0 \pmod p$ (since $12d+5 < 7p$ and $12d+1, 12d+3 \notin \{p, 3p, 5p\}$) and hence (3.4) yields

$$h(d+1) \equiv 0 \pmod{p^2} \implies h(d) \equiv 0 \pmod{p^2}.$$

Note that

$$f\left(\frac{p-1}{2}\right) = \frac{\binom{2p-2}{p-1}\binom{3p-3}{p-1}}{108^{p-1}} \equiv 0 \pmod{p^2},$$
$$g\left(\frac{p-1}{2}\right) = \frac{\binom{2p-2}{p-1}\binom{4p-4}{2p-2}}{256^{p-1}} \equiv 0 \pmod{p^2},$$
$$h\left(\frac{p-1}{2}\right) = \frac{\binom{3p-3}{p-1}\binom{6p-6}{3p-3}}{12^{3(p-1)}} \equiv 0 \pmod{p^2}.$$

So, by the above we have (1.16)-(1.18) for all $d\in\{0,1,\ldots,(p-1)/2\}$.

(ii) Assume that $p\equiv 3\pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$. Since $4x^2\not\equiv 0\pmod p$ and Mortenson [M] already proved that the squares of both sides of (1.19) are congruent mod p^2, (1.19) is reduced to its mod p form. Applying (3.1) with $x=1$ we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\pmod p.$$

By [A, Theorem 5(3)], we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}(-1)^k\equiv 4x^2-2p\pmod p,$$

where $n=(p-1)/2$. For $k=0,\ldots,n$ clearly

$$\begin{aligned}\binom{n}{k}^2\binom{n+k}{k}(-1)^k=&\binom{(p-1)/2}{k}^2\binom{-(p+1)/2}{k}\\ \equiv&\binom{-1/2}{k}^3=\frac{\binom{2k}{k}^3}{(-64)^k}\pmod p,\end{aligned}$$

therefore

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{-1}{p}\right)(4x^2-2p)\pmod p$$

and hence (1.19) follows.

(iii) Finally we suppose $p\equiv 5\pmod{12}$ and write $p=x^2+y^2$ with x odd and y even. Once again it suffices to show the mod p form of (1.20) in view of Mortenson's work [M]. As the author's twin brother Z. H. Sun observed,

$$\binom{(p-5)/6+k}{2k}\binom{2k}{k}\equiv\binom{k-5/6}{2k}\binom{2k}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{(-432)^k}\pmod p$$

for all $k=0,1,2,\ldots$. If $p/6<k<p/3$ then $p\mid\binom{6k}{3k}$; if $p/3<k<p/2$ then $p\mid\binom{3k}{k}$; if $p/2<k<p$ then $p\mid\binom{2k}{k}$. Thus

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv&\sum_{k=0}^{(p-5)/6}\binom{(p-5)/6+k}{2k}\binom{2k}{k}^2\left(-\frac{1}{4}\right)^k\\ =&D_{2n}\left(-\frac{1}{2}\right)^2\pmod p\quad\text{(by (1.3))},\end{aligned}$$

where $n = (p-5)/12$. Note that

$$D_{2n}\left(-\frac{1}{2}\right) = \frac{1}{(-4)^n}\binom{2n}{n}$$

by [G, (3.133) and (3.135)], and

$$\binom{(p-1)/2}{(p-1)/4} \equiv 12(-432)^n\binom{2n}{n} \pmod p$$

by P. Morton [Mo]. Therefore

$$D_{2n}\left(-\frac{1}{2}\right)^2 = \frac{1}{16^n}\binom{2n}{n}^2 \equiv \frac{\binom{(p-1)/2}{(p-1)/4}^2}{12^{6n+2}} \equiv \left(\frac{12}{p}\right)\binom{(p-1)/2}{(p-1)/4}^2 \pmod{p}.$$

Thus, by applying Gauss' congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$ (cf. [BEW, (9.0.1)] or [HW]) we immediately get the mod p form of (1.20) from the above.

The proof of Theorem 1.3 is now complete. □

Remark 3.1. We mention that the author [Su3] made a conjecture on $\sum_{k=0}^{p-1}\binom{6k}{3k}\binom{3k}{k}/864^k \bmod p^2$ for any prime $p > 3$, and its mod p version was recently confirmed by Z. H. Sun.

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