

Time scales in large systems of Brownian particles with stochastic synchronization

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Abstract

We consider a system $x(t) = (x_1(t), \dots, x_N(t))$ consisting of N Brownian particles with synchronizing interaction between them occurring at random time moments $\{\tau_n\}_{n=1}^{\infty}$. Under assumption that the free Brownian motions and the sequence $\{\tau_n\}_{n=1}^{\infty}$ are independent we study asymptotic properties of the system when both the dimension N and the time t go to infinity. We find three time scales $t = t(N)$ of qualitatively different behavior of the system.

1 Introduction

Mathematical models with stochastic synchronization between components take their origin from paper [1] where some two-dimensional system related with parallel computations was studied. A very good explanation of the role of synchronizations in asynchronous parallel and distributed algorithms can be found in [2]. It is rather natural that further mathematical interest to such models was moved to considerations of high dimensions and to studies of a long time behavior. It was discovered soon [3, 4, 5] that it is very convenient to interpret synchronization models as particle systems with very special interaction. It is worth to note that in the “traditional” mathematical theory of interacting particle systems such interactions were never considered before that time. In [11] one can find a short overview of the subject.

The present paper is a small contribution to the following general problem: how to describe a qualitative behavior of a multidimensional Markov (or semi-Markov) process $x(t) = (x_1(t), x_2(t), \dots, x_N(t))$ for large N and $t = t(N)$. We chose as an object of our study the system of N Brownian particles perturbed by synchronizing jumps at some random time moments. Reasons of such choice are the following. Synchronization models driven by Brownian motion were not studied yet, all papers mentioned above considered random walks on lattices or deterministic motions as non-perturbed dynamics. The second reason is that, as it will be shown here, a Markovian synchronization model based on the Brownian motions admits an explicit solution. This feature give us possibility to write very short and clear proofs of our main result on the existence of three different time stages of qualitative behavior of the particle system. We believe that such results hold also for very general multidimensional synchronization models. There are already many particular examples justifying this belief. Thus the existence of the three time stages in the long time behavior was already proved for

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system with two types of deterministic particles and pairwise stochastic synchronizations [6], for discrete time random walks with a 3-particle anisotropic interaction [5], for continuous time random walks with symmetric k -particle synchronizations [7].

The explanatory goals of this paper force us to choose the following organization of sections. In Section 2 we define and study a sequence of Markov models with pairwise synchronization between particles and constant coefficients in the front of the free dynamics and the interaction. This lets us avoid cumbersome notation in proofs (Section 3). Section 4 is devoted to generalizations of the model of Section 2. The first generalization to the case of coefficients varying with N is quite straightforward and is based on a careful analysis of the proofs in Section 3. The next extension of the main results is done for general symmetric k -particle synchronizing interaction. In Subsection 4.3 we discuss generalizations to the case when epochs of synchronization form a general renewal process and hence the particle system is no more a Markov process. Corresponding results are obtained by using the Laplace transform and are presented in Theorem 6.

2 Model with pairwise interaction

2.1 Definition and assumptions

We study a multi-dimensional stochastic process

$$x(t) = (x_1(t), x_2(t), \dots, x_N(t)) \in \mathbb{R}^N, \quad t \in \mathbb{R}_+,$$

which can be regarded mathematically as a special class of interacting particle systems. But from the view point of possible applications it would be better to consider this process as a multi-component stochastic system.

Here N is the number of particles and $x_i(t) \in \mathbb{R}^1$ is a coordinate of the i -th particle at time t . Denote $\mathcal{N}_N = \{1, \dots, N\}$. To give a precise construction of the process $(x(t), t \geq 0)$ we fix on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$

- (a) $B(t) = (B_1(t), \dots, B_N(t))$ — the N -dimensional standard Brownian motion,
- (b) a random sequence $\{\tau_n\}_{n=1}^\infty$ of time moments

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots$$

- (c) a random initial configuration of particles $x(0) = (x_1(0), x_2(0), \dots, x_N(0))$.

Main assumption is that $(B(t), t \geq 0)$, $\{\tau_n\}_{n=1}^\infty$ and $x(0)$ are *independent*.

We consider also another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ corresponding to the independent sequence

$$(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n), \dots \tag{1}$$

of *equiprobable* ordered pairs (i, j) such that $i, j \in \mathcal{N}_N$, $i \neq j$. In the next we will put simply $\omega' = ((i_1, j_1), (i_2, j_2), \dots, (i_n, j_n), \dots)$ and will use coordinate functions $i_n(\omega') = i_n$ and $j_n(\omega') = j_n$.

Let us introduce the new probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\tilde{\Omega} \times \Omega', \tilde{\mathcal{F}} \times \mathcal{F}', \tilde{\mathbb{P}} \times \mathbb{P}')$. *By formal definition* the process $(x(t), t \geq 0)$ has right-continuous trajectories $(x(t, \omega), t \geq 0)$, $\omega =$

$(\tilde{\omega}, \omega')$, satisfying to the following conditions:

$$\begin{aligned} x_k(s, \omega) - x_k(\tau_n(\tilde{\omega}), \omega) &= \sigma \cdot (B_k(s, \tilde{\omega}) - B_k(\tau_n(\tilde{\omega}), \tilde{\omega})), \\ &\quad \forall s \in [\tau_n(\tilde{\omega}), \tau_{n+1}(\tilde{\omega})], \quad \forall k \in \mathcal{N}_N, \\ x_{j_n(\omega')}(\tau_n(\tilde{\omega}), \omega) &= x_{i_n(\omega')}(\tau_n(\tilde{\omega}) - 0, \omega), \\ x_m(\tau_n(\tilde{\omega}), \omega) &= x_m(\tau_n(\tilde{\omega}) - 0, \omega) \quad \forall m \in \mathcal{N}_N \setminus \{j_n(\omega')\}. \end{aligned}$$

The scalar parameter $\sigma > 0$ is a diffusion coefficient.

Informally speaking the dynamics of the process $x(t)$ consists of two parts: free motion and pairwise interaction between particles. Namely, the *interaction* is possible only at random time moments

$$0 < \tau_1 < \tau_2 < \dots$$

and has the form of *synchronizing jumps*: at time τ_n with probability $\frac{1}{N(N-1)}$ a pair of particles (i, j) is chosen and the particle j jumps to the particle i :

$$(x_i, x_j) \rightarrow (x_i, x_i). \quad (2)$$

Inside the intervals (τ_k, τ_{k+1}) particles of the process $x(t)$ move as independent Brownian motions with diffusion coefficient σ (*free dynamics*).

In some sense the dynamics of the interacting particle system $x(t)$ can be considered as a perturbation of the stochastic dynamics $B(t)$. We are interested in the question how the synchronizing interaction will imply on a long time behavior of $x(t)$. We consider the following limiting situations:

- (i) N is fixed, $t \rightarrow \infty$;
- (ii) $N \rightarrow \infty$ is fixed, $t = t(N) \rightarrow \infty$ with different choices of the time scales $t(N)$.

We shall mainly be concerned here with the situation (ii) which is more important and more interesting.

To make our considerations more transparent in all subsequent sections we have the next assumption.

Assumption M. The moments $\{\tau_n\}_{n=1}^\infty$ are epochs of a Poisson flow of intensity δ , i.e., the sequence $\{\tau_n - \tau_{n-1}\}_{n=1}^\infty$ consists of independent random variables, having exponential distributions: $\mathbf{P}(\tau_n - \tau_{n-1} > s) = \exp(-\delta s)$.

Assumption M implies immediately that $(x(t), t \geq 0)$ is a Markov process on \mathbb{R}^N with symbolic generator

$$\sigma L_0^B + \delta L_S, \quad \sigma > 0, \quad \delta > 0,$$

where L_0^B is a generator of the standard N -dimensional Brownian motion and L_S corresponds to synchronizing jumps.

This assumption is not crucial for the validity of our asymptotic results. In Subsection 4.3 we shall discuss the case of a general *renewal process*.

2.2 Long time behavior for fixed N

We use notation $\mathcal{L}(\xi)$ for a distribution law of a random element ξ . Then $(\mathcal{L}(x(t)), t \geq 0)$ is a family of probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$.

Theorem 1 $\mathcal{L}(x(t))$ has no limit as $t \rightarrow \infty$.

We recall the well known fact that the Brownian motion $B(t)$ also has no limit on distribution as $t \rightarrow \infty$. But a long time behavior of the interacting particle system $x(t)$ strongly differs from the behavior of $B(t)$. Indeed, let us consider an “improved” process $x^\circ(t)$,

$$x_i^\circ(t) = x_i(t) - M(x(t)),$$

where $M(x) := \frac{1}{N} \sum_{m=1}^N x_m$ is the center of mass of the particle configuration $x = (x_1, \dots, x_N)$. In other words, $x^\circ(t)$ is the particle system $x(t)$ viewed by an observer placed in the center of mass $M(x(t))$.

Theorem 2 For any $\sigma > 0$, $\delta > 0$ the Markov process $x^\circ(t)$ is ergodic. Hence there exists a probability distribution μ^N on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ such that $\mathcal{L}(x^\circ(t)) \rightarrow \mu^N$ as $t \rightarrow \infty$.

The idea of the proof is to show that $x^\circ(t)$ satisfies the Doeblin property. Similar arguments were used in [4, 8]. So we omit here the proofs of Theorems 1 and 2.

The result of Theorem 2 is close to the *shift-compactness* property of measure-valued stochastic processes [9].

It would be interesting to answer the following main questions. What is a typical “size” of the configuration (x_1, \dots, x_N) under the distribution μ^N ? How large (with respect to N is a domain where μ^N is supported with probability close to 1? To do this we let the dimension N and the time t grow to infinity in order to find on which time scale $t = t(N)$ the process $x(t)$ will approach μ^N .

2.3 Time scales

In collective behavior of a particle system with synchronization we observe a superposition of *two opposite tendencies*: with the course of time the free dynamics *increases* the *spread* of the particle system while the synchronizing interaction tries to *decrease* it.

To formalize the notion of a “size” or a “spread” we consider the following function on the state space

$$V : \mathbb{R}^N \rightarrow \mathbb{R}_+, \quad V(x) := \frac{1}{N-1} \sum_{m=1}^N (x_m - M(x))^2,$$

where $M(x)$ is the center of mass as defined above. In statistics the function V is known as the empirical variance. We introduce also the function $R_N : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending on the time $t \geq 0$ as

$$R_N(t) := \mathbb{E} V(x(t)). \tag{3}$$

It appears that the function $R_N(t(N))$ has completely different asymptotic behavior for different choices of the time scale $t = t(N)$. Before proving this result we start from the following explicit formula.

Theorem 3 There exist a number $\varkappa > 0$ such that

$$R_N(t) = \sigma^2 \delta^{-1} l_N (1 - \exp(-\delta t/l_N)) + \exp(-\delta t/l_N) R_N(0),$$

where $l_N = N(N-1)/\varkappa$.

This statement shows that the function $R_N(t)$ satisfies to a very simple differential equation

$$\frac{d}{dt}R_N(t) = \sigma^2 - \frac{\delta}{l_N} R_N(t).$$

So the choice of R_N in (3) was really good from the point of view of subsequent asymptotic analysis.

In the next theorem we assume that $N \rightarrow \infty$, $t = t(N) \rightarrow \infty$.

Theorem 4 (On three time scales) *Let $\sup_N R_N(0) < \infty$. Then*

I. *If $\frac{t(N)}{N^2} \rightarrow 0$, then $R_N(t(N)) \sim \sigma^2 t(N)$.*

II. *If $t(N) = cN^2 / (\varkappa\delta)$, $c > 0$, then $R_N(t(N)) \sim \frac{1 - e^{-c}}{c} \sigma^2 t(N)$.*

III. *If $\frac{t(N)}{N^2} \rightarrow \infty$, then $R_N(t(N)) \sim \left(\frac{\sigma^2}{\varkappa\delta}\right) N^2$.*

Remark 1. In case $\delta = 0$ when there is no synchronization and $x(t)$ behaves as the Brownian motion $\sigma B(t)$ the function R_N can be calculated explicitly: $R_N(t) = \sigma^2 t$.

Remark 2. The function f in the item II is strictly decreasing:

$$f(c) = \frac{1 - e^{-c}}{c}, \quad f(0) = 1, \quad f'(c) < 0, \quad f(+\infty) = 0.$$

Remark 3. For the pairwise synchronization (1) and (2) considered in the present section $\varkappa = 2$. Details will be given at the end of Subsection 4.2.

2.4 Discussion of collective behavior

We can easily observe from Theorem 4 that for the slowest time scale (case I) asymptotic behavior of $R_N(t(N))$ is the same as for non-perturbed dynamics. This means that a cumulative effect of synchronization jumps on time intervals of the form $(0, o(N^2))$ is negligible with respect to the influence of the free dynamics. Next observation is that on the fastest time scale (case III) asymptotics of $R_N(t(N))$ does not depend on the rate of grow of $t = t(N)$. We interpret this phenomenon as follows: synchronization dominates heavily on the free motion and the asymptotics $(\sigma^2/2\delta)N^2$ corresponds to the averaging of the function $f(x)$ with respect to the limiting distribution μ^N . The asymptotics on the middle time scale (case II, time intervals of the form (c_1N^2, c_2N^2)) “continuously joins” the asymptotics of the slowest and the fastest time stages.

As in [6] one can call these consecutive stages correspondingly:

I initial desynchronization

II critical slowdown of desynchronization

III final stabilization.

3 Proofs

3.1 Proof of Theorem 3

Let $\Pi_t = \max\{m : \tau_m \leq t\}$ and $\tau_t^* = \tau_{\Pi_t}$. Obviously, $\tau_t^* = \max\{\tau_i : \tau_i \leq t\}$. To get $R_N(t)$ we shall calculate the chain of conditional expectations as follows

$$\mathbb{E}(\cdot) = \mathbb{E}\left(\mathbb{E}\left(\mathbb{E}\left(\cdot \mid \{\tau_j\}_{j=1}^\infty\right) \mid \Pi_t\right)\right).$$

Lemma 1

$$\mathbb{E}\left(V(x(t) \mid \{\tau_j\}_{j=1}^\infty)\right) = \sigma^2 \sum_{i=0}^{\Pi_t-1} k_N^{\Pi_t-i} \cdot (\tau_{i+1} - \tau_i) + \sigma^2 \cdot (t - \tau_t^*) + k_N^{\Pi_t} R_N(0) \quad (4)$$

where $k_N := \left(1 - \frac{\lambda}{N(N-1)}\right)$.

To take expectation $\mathbb{E}(\cdot \mid \Pi_t)$ from the both sides of equation (4) we need to know the joint distribution of the following form

$$\mathbb{P}\{\tau_q - \tau_{q-1} \in (x, x + dx), \Pi_t = n\}, \quad q \leq n$$

and the expectation of the *spent waiting time* $(t - \tau_t^*)$ in terms of [10].

Lemma 2 *If Assumption M holds we have that $(\Pi_t, t \geq 0)$ is the Poisson process and*

$$\mathbb{E}(\tau_q - \tau_{q-1} \mid \Pi_t = n) = \mathbb{E}(t - \tau_t^* \mid \Pi_t = n) = \frac{t}{n+1}$$

Keeping in mind Lemmas 1 and 2 we can easily proceed with calculation of $R_N(t)$. Under Assumption M

$$\begin{aligned} \mathbb{E}(V(x(t) \mid \Pi_t = n)) &= \sigma^2 \sum_{i=0}^{n-1} k_N^{n-i} \frac{t}{n+1} + \sigma^2 \frac{t}{n+1} = \\ &= \sigma^2 \frac{t}{n+1} \sum_{j=0}^n k_N^j = \sigma^2 \frac{t}{n+1} \frac{1 - k_N^{n+1}}{1 - k_N} + k_N^n R_N(0) \end{aligned}$$

Moreover, for given $t > 0$ the random variable Π_t has the Poisson distribution with mean δt . Using identity

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{(n+1)!} = \alpha^{-1} (e^\alpha - 1)$$

we get

$$\begin{aligned} R_N(t) &= \sum_{n=0}^{\infty} \mathbb{E}(V(x(t) \mid \Pi_t = n)) \frac{(\delta t)^n}{n!} \exp(-\delta t) = \\ &= \frac{\sigma^2 t}{1 - k_N} \left(\frac{\exp(\delta t) - 1}{\delta t} - \frac{(\exp(k_N \delta t) - 1) k_N}{k_N \delta t} \right) \exp(-\delta t) + \exp(-(1 - k_N)\delta t) R_N(0) = \\ &= \frac{\sigma^2}{\delta} \frac{1 - \exp(-(1 - k_N)\delta t)}{1 - k_N} + \exp(-(1 - k_N)\delta t) R_N(0). \end{aligned}$$

Putting $l_N = (1 - k_N)^{-1}$ we obtain statement of Theorem 3.

3.2 Proof of Theorem 4

Our task is to analyze asymptotic behavior of

$$R_N(t(N)) = \sigma^2 \delta^{-1} l_N (1 - \exp(-\delta t(N)/l_N)) + \exp(-\delta t(N)/l_N) R_N(0), \quad l_N = N(N-1)/\varkappa,$$

for different choices of $t = t(N)$. Let $N \rightarrow \infty$.

Case I: $t(N)/l_N \rightarrow 0$. Then

$$R_N(t(N)) \sim \sigma^2 \delta^{-1} l_N \delta t(N)/l_N = \sigma^2 t(N).$$

Case II: $t(N)/l_N \rightarrow c \delta^{-1}$ for some $c > 0$. We have

$$R_N(t(N)) \sim \sigma^2 \delta^{-1} t(N) c^{-1} \delta (1 - \exp(-c)) = \sigma^2 t(N) (1 - \exp(-c)) / c.$$

Case III: $t(N)/l_N \rightarrow +\infty$. Here we get

$$R_N(t(N)) \sim \sigma^2 \delta^{-1} l_N \sim \frac{\sigma^2}{\delta \varkappa} N^2.$$

Theorem 4 is proved.

3.3 Proofs of Lemmas

Let us introduce families of σ -algebras which are generated $\mathcal{F}_m = \sigma((x(s), s \leq \tau_m), \{\tau_i\}_{i=1}^\infty)$ as follows

$$\begin{aligned} \mathcal{F}_m &= \sigma((x(s), s \leq \tau_m), \{\tau_i\}_{i=1}^\infty), \quad m = 0, 1, \dots \\ \mathcal{F}_{m-} &= \sigma((x(s), s \leq \tau_m - 0), \{\tau_i\}_{i=1}^\infty), \quad m = 1, 2, \dots \end{aligned}$$

Denote also $\mathcal{F}_{0-} = \sigma(\{\tau_i\}_{i=1}^\infty)$. Evidently,

$$\mathcal{F}_{0-} \subset \mathcal{F}_0 \subset \dots \subset \mathcal{F}_{m-} \subset \mathcal{F}_m \subset \mathcal{F}_{(m+1)-} \subset \mathcal{F}_{m+1} \subset \dots$$

To prove Lemma 1 we shall use the following result related with synchronizing jumps.

Lemma 3 *There exists $\varkappa > 0$ such that for any $m \in \mathbb{N}$*

$$\mathbb{E}(V(x(\tau_m)) | \mathcal{F}_{m-}) = k_N V(x(\tau_m - 0)),$$

where $k_N = \left(1 - \frac{\varkappa}{N(N-1)}\right) \in (0, 1)$.

We postpone the proof of this lemma Subsection 4.2 where the same statement will be established for more general interactions. Here we just note that in the case of pairwise synchronizations $\varkappa = 2$.

Since the free dynamics of particles corresponds to Brownian motions independent of the sequence $\{\tau_i\}_{i=1}^\infty$ of synchronization moments, for any $m \in \mathbb{N}$ we have

$$\mathbb{E}(V(x(\tau_{m+1} - 0)) | \mathcal{F}_m) = V(x(\tau_m)) + \sigma^2 \cdot (\tau_{m+1} - \tau_m).$$

Using Lemma 3 we get

$$\mathbb{E}(V(x(\tau_{m+1})) | \mathcal{F}_m) = k_N V(x(\tau_m)) + k_N \sigma^2 \cdot (\tau_{m+1} - \tau_m) \quad (5)$$

Hence, iterating (5) we come to the equation

$$\begin{aligned} \mathbb{E}(V(x(\tau_{m+1})) | \mathcal{F}_{m-1}) &= \mathbb{E}(\mathbb{E}(V(x(\tau_{m+1})) | \mathcal{F}_m) | \mathcal{F}_{m-1}) = \\ &= k_N \mathbb{E}(V(x(\tau_m)) | \mathcal{F}_{m-1}) + k_N \sigma \cdot (\tau_{m+1} - \tau_m) \end{aligned}$$

By developing this recurrent equation we obtain

$$\mathbb{E}\left(V(x(\tau_n) | \{\tau_j\}_{j=1}^{\infty})\right) = \sigma^2 \sum_{i=0}^{n-1} k_N^{n-i} (\tau_{i+1} - \tau_i) + k_N^n R_N(0).$$

In a similar way we get for any nonrandom $t > 0$

$$\mathbb{E}\left(V(x(t) | \{\tau_j\}_{j=1}^{\infty})\right) = \sigma^2 \sum_{i=0}^{\Pi_t-1} k_N^{\Pi_t-i} \cdot (\tau_{i+1} - \tau_i) + \sigma^2 \cdot (t - \tau_t^*) + k_N^{\Pi_t} R_N(0).$$

Here we take into account that all τ_j have continuous distributions and, as usually, the sign “=” for conditional expectations is understood in the sense of “almost surely” [12].

This completes the proof of the Lemma 1.

Lemma 2 follows from the well know facts of renewal processes theory [14, 13] or can be verified by a direct calculation in our concrete case.

4 Generalizations

4.1 Varying parameters

Since Theorems 3 and 4 deal with the sequence $\{x(t) = (x_1(t), \dots, x_N(t))\}_{N=1}^{\infty}$ of stochastic processes is it natural to ask whenever these statements remain true if we let the coefficients σ and δ depend on N . In other words under Assumption M we consider a family of Markov processes defined on the state spaces $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ with formal generators

$$\sigma_N L_0^B + \delta_N L_S, \quad \sigma_N > 0, \quad \delta_N > 0. \quad (6)$$

If we check carefully all calculations and arguments in the proofs of Theorems 3 and 4 we see that these proofs are valid *without any modification* for the case (6). The corresponding results are summarized in the next theorem.

Theorem 5

1. There exist a number $\varkappa > 0$ (not depending on N) such that

$$R_N(t) = \sigma_N^2 \delta_N^{-1} l_N (1 - \exp(-\delta_N t / l_N)) + \exp(-\delta_N t / l_N) R_N(0), \quad (7)$$

where $l_N = N(N-1)/\varkappa$.

2. Let $N \rightarrow \infty$, $t = t(N) \rightarrow \infty$. Assume that $\sup_N R_N(0) < \infty$. There are three different time stages in the collective behavior of the particle system:

	I	II	III
$t(N)$	$\alpha_N \rightarrow 0$	$\alpha_N \rightarrow c > 0$	$\alpha_N \rightarrow \infty$
$R_N(t(N)) \sim$	$\sigma_N^2 t(N)$	$(1 - e^{-c})c^{-1} \sigma_N^2 t(N)$	$\sigma_N^2 (\varkappa \delta_N)^{-1} N^2$

where $\alpha_N := \frac{\varkappa \delta_N t(N)}{N^2}$.

Remark 4. As it is seen from the representation (7) the assumption $\sup_N R_N(0) < \infty$ can be weakened. We can let some growth of $R_N(0)$ in the limit $N \rightarrow \infty$ and the statement 2 of Theorem 5 still remains true. But the conditions on the admissible growth will be different for each time stage.

4.2 k -particle synchronization

Recall our assumption (2) on *pairwise interaction*: we pick at random a pair of particles (x_i, x_j) and move this particles as follows $(x_i, x_j) \rightarrow (x_i, x_i)$. To study general problems of synchronization in stochastic systems with applications to wide classes of self-organizing systems we should face to so called *multi-particle interactions*. The most general rule of synchronizing jumps is

$$x = (x_1, \dots, x_N) \rightarrow x' = (x'_1, \dots, x'_N)$$

where $\{x'_1, \dots, x'_N\} \subset \{x_1, \dots, x_N\}$, $\{x'_1, \dots, x'_N\} \neq \{x_1, \dots, x_N\}$. Following the paper [7] we restrict ourself here to symmetric k -particle interactions based on synchronizing maps. Definition of synchronizing maps needs some preliminary notation. First we introduce a set $\mathcal{I} := \{(i_1, \dots, i_k) : i_j \in \mathcal{N}_N, i_p \neq i_q (p \neq q)\}$.

Fix integers $k \geq 2$ and $k_1 \geq 2, \dots, k_l \geq 2$: $k_1 + \dots + k_l = k$. The sequenced collection (k_1, \dots, k_l) will be called a *signature* of interaction. Given the signature (k_1, \dots, k_l) we introduce a map π_{k_1, \dots, k_l} defined on the set \mathcal{I} as follows: $\pi_{k_1, \dots, k_l} : (i_1, \dots, i_k) \mapsto (\Gamma_1, \dots, \Gamma_l)$, where $\Gamma_j = (g_j, \Gamma_j^\circ)$ with

$$\begin{aligned} g_1 &= i_1, & \Gamma_1^\circ &= (i_2, \dots, i_{k_1}), \\ & & \dots & \\ g_l &= i_{k_1 + \dots + k_{l-1} + 1}, & \Gamma_l^\circ &= (i_{k_1 + \dots + k_{l-1} + 1}, \dots, i_{k_1 + \dots + k_l}). \end{aligned}$$

In other words the map π_{k_1, \dots, k_l} is a special regrouping of indices (i_1, \dots, i_k) :

$$(i_1, \dots, i_k) = (i_1, i_2, \dots, i_{k_1}, i_{k_1+1}, i_{k_1+2}, \dots, i_{k_1+k_2}, \dots, \dots, i_k)$$

$$\begin{array}{ccccccc} i_1, & \underbrace{i_2, \dots, i_{k_1}} & i_{k_1+1}, & \underbrace{i_{k_1+2}, \dots, i_{k_1+k_2}} & \dots & & \\ g_1 & \Gamma_1^\circ & g_2 & \Gamma_2^\circ & \dots & & \end{array}$$

The map π_{k_1, \dots, k_l} generates a family of *synchronizing maps* $\left\{ J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}, (i_1, \dots, i_k) \in \mathcal{I} \right\}$ defined on the set \mathbb{R}^N of particle configurations:

$$J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} : x = (x_1, \dots, x_N) \mapsto y = (y_1, \dots, y_N), \quad (8)$$

where

$$y_m = \begin{cases} x_m, & \text{if } m \notin (i_1, \dots, i_k), \\ x_{g_j}, & \text{if } m \in (i_1, \dots, i_k), m \in \Gamma_j. \end{cases}$$

We call the jump (8) a *synchronization* of the collection of particles x_{i_1}, \dots, x_{i_k} , corresponding to the signature (k_1, \dots, k_l) . The configuration $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} x$ has at least k_1 particles with coordinates that are equal to x_{g_1}, \dots , at least k_l particles at the point x_{g_l} .

We are ready now to define a particle system with *symmetric k -particle interaction* of the given signature (k_1, \dots, k_l) . To do this we repeat the strategy of Subsection 2.1 but with another definition of the probability space $(\Omega', \mathcal{F}', \mathbf{P}')$. Now the space $(\Omega', \mathcal{F}', \mathbf{P}')$ corresponds to the independent sequence

$$(i_1^1, \dots, i_k^1), \quad \dots, \quad (i_1^n, \dots, i_k^n), \quad \dots$$

of *equiprobable* elements of the set \mathcal{I} . As before the dynamics of $x(t)$ consists of two parts: free motion and interaction. Inside the intervals (τ_k, τ_{k+1}) particles of the process $x(t)$ move as independent Brownian motions with diffusion coefficient σ (*free dynamics*). *Interaction* is possible only at the epochs $0 < \tau_1 < \tau_2 < \dots$ and has the following form. At time τ_n with probability $\frac{1}{N(N-1)\dots(N-k+1)}$ a set of indices (i_1, \dots, i_k) is chosen and the particle configuration (x_1, \dots, x_N) instantly changes to (y_1, \dots, y_N) accordingly to the synchronizing map $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}$ (see (8)).

Note that the pairwise interaction defined in (2) is a particular case of the symmetric k -particle synchronizing interaction considered here. To see this put $k = 2, l = 1$, the signature $(k_1, \dots, k_l) = (2)$. Then $\pi_2 : (i_1, i_2) \mapsto \Gamma_1, \quad \Gamma_1 = (g_1, \Gamma_1^\circ) = (i_1, i_2), \quad g_1 = i_1, \Gamma_1^\circ = i_2$.

Let $L_{S, (k_1, \dots, k_l)}$ denote a formal generator corresponding to the symmetric k -particle interaction of the signature (k_1, \dots, k_l) . Main goal now is to generalize our results to the Markov process $x(t)$ with generator

$$\sigma_N L_0^B + \delta_N L_{S, (k_1, \dots, k_l)}, \quad \sigma_N > 0, \quad \delta_N > 0.$$

All arguments of the proof in Section 3 can be repeated as well for this case, we should only to take care about an analog of Lemma 3. Fortunately, the proof of Lemma 3 for the general symmetric k -particle interaction can be obtained by a slight modification of the proof of Lemma 2 in [7]. So there is no need to repeat that proof here. We mention only the explicit form of the constant \varkappa entering in definition of

$$k_N = 1 - \frac{\varkappa}{(N-1)N}.$$

It appears (see [7]) that $\varkappa = \sum_{j=1}^l k_j^2 - k$. It is easy to check that $\varkappa > 0$ for any $k_1 \geq 2, \dots, k_l \geq 2$ such that $k_1 + \dots + k_l = k$.

So our final conclusion is the following one.

The both statements of Theorem 5 remains true for the particle system with symmetric k -particle interaction. Moreover, $\varkappa = \varkappa(k_1, \dots, k_l) = \sum_{j=1}^l k_j^2 - k$. The choice of the sequence $\{\alpha_N\}$ is the same: $\alpha_N = \frac{\varkappa \delta_N t(N)}{N^2}$.

Let us remark also that for the pairwise synchronization $\varkappa((2)) = 2$.

4.3 Nonmarkovian model: general renewal epochs for synchronization

The next step in generalization of the model is to consider more general sequences $\{\tau_n\}$. We replace Assumption M by the following one.

Assumption T_N . For each fixed $N \in \mathbb{N}$ the moments $\{\tau_n^{(N)}\}_{n=1}^{\infty}$ are epochs of some *renewal* process, i.e., the sequence $\{\tau_n^{(N)} - \tau_{n-1}^{(N)}\}_{n=1}^{\infty}$ consists of independent random variables, having common continuous distribution function $F_N(s) = \mathbb{P}\{\tau_n^{(N)} - \tau_{n-1}^{(N)} \leq s\}$ satisfying $F_N(s) = 0$ for $s \leq 0$. Intervals $\tau_n^{(N)} - \tau_{n-1}^{(N)}$ have finite mean $\mu_N > 0$ and variance d_N .

Expected result is the following one. Consider a stochastic process $x^{(N)}(t) = (x_1(t), \dots, x_N(t))$, $t \geq 0$, corresponding to N Brownian particles with diffusion coefficient $\sigma_N > 0$. Particles of $x^{(N)}(t)$ interact at epochs $\{\tau_n^{(N)}\}_{n=1}^{\infty}$ according to the *symmetric k -particle interaction* of the signature (k_1, \dots, k_l) . Let Assumption T_N holds.

Conjecture. Assume that $\sup_N R_N(0) < \infty$. Let $N \rightarrow \infty$, $t = t(N) \rightarrow \infty$. There are three different time stages in the collective behavior of the particle system:

	I	II	III
$t(N)$	$\alpha_N \rightarrow 0$	$\alpha_N \rightarrow c > 0$	$\alpha_N \rightarrow \infty$
$R_N(t(N)) \sim$	$\sigma_N^2 t(N)$	$(1 - e^{-c})c^{-1} \sigma_N^2 t(N)$	$\varkappa^{-1} \sigma_N^2 \mu_N N^2$

where $\alpha_N := \frac{\varkappa t(N)}{\mu_N N^2}$, $\varkappa = \sum_{j=1}^l k_j^2 - k$.

Evidently, $x^{(N)}(t)$ is not a Markov process. Of course, we can not expect to have here an explicit representation for $R_N(t)$ as in Theorem 3. Possible proofs of the Conjecture can be obtained by two different ways. The first one is close to Section 3 of the present paper. The idea is to represent $R_N(t)$ in term of generating function of the number of renewals Π_t : $g(t, \zeta) = \mathbb{E}\zeta^{\Pi_t}$ (we recall that under Assumption T_N (Π_t , $t \geq 0$) is not a Poisson process). We are interested in the long time behavior ($t \rightarrow \infty$), so we take the Laplace transform of the function $g(t, \zeta)$ in t (see [13, Section 3.2]),

$$g^*(s, \zeta) = \int_0^{+\infty} e^{-st} g(t, \zeta) dt,$$

to analyze its behavior for small s . Applying Tauberian theorems from [10, Ch. 13, Section 5] we come to the following statement.

Theorem 6 Assume that $\sup_N R_N(0) < \infty$, $N \rightarrow \infty$, $t(N) \rightarrow \infty$.

If $\alpha_N \rightarrow 0$, then $R_N(t(N)) \sim \sigma_N^2 t(N) L_1(t(N))$.

If $\alpha_N \rightarrow \infty$, then $R_N(t(N)) \sim \varkappa^{-1} \sigma_N^2 \mu_N N^2 L_2(t(N))$.

Here L_1 and L_2 are some slowly varying functions, notation α_N is the same as in Conjecture.

These results are slightly weaker than the corresponding items of Theorems 4 or 5 but this is the best we can do by this method. We omit details.

The second possible way of proving the above conjecture is an approach based on embedded Markov chains. It was very effective in [6] and [7]. We shall devote to it a separate paper.

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