SUPPORTS OF REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA OF TYPE A

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Abstract. We first consider the rational Cherednik algebra corresponding to the action of a finite group on a complex variety, as defined by Etingof. We define a category of representations of this algebra which is analogous to "category $\mathcal{O}$" for the rational Cherednik algebra of a vector space. We generalise to this setting Bezrukavnikov and Etingof's results about the possible support sets of such representations. Then we focus on the case of S_n acting on $\mathbb{C}^n$, determining which irreducible modules in this category have which support sets. We also show that the category of representations with a given support, modulo those with smaller support, is equivalent to the category of finite dimensional representations of a certain Hecke algebra.

1. Introduction

1.1. **Linear actions.** Let W be a finite group acting faithfully on a finite dimensional $\mathbb{C}$-vector space $\mathfrak{h}$. The Weyl algebra $D(\mathfrak{h})$ of $\mathfrak{h}$ admits an action of W, so $\mathbb{C}[W] \otimes_{\mathbb{C}} D(\mathfrak{h})$ becomes an algebra in a natural way. We denote this algebra by $\mathbb{C}[W] \ltimes D(\mathfrak{h})$. The *rational Cherednik algebra*, defined by Etingof and Ginzburg [10], is a universal flat deformation of this algebra. It is named thus because it is a degeneration of the *double affine Hecke algebra* defined by Cherednik [6]. We recall the definition of the rational Cherednik algebra below:

Definition 1.1. *We define the set of* reflections *in W to be*

$$S = \{s \in W \mid \mathrm{rk}(s-1) = 1\}.$$

For $s \in S$, let $\alpha_s^\vee \in \mathfrak{h}$ and $\alpha_s \in \mathfrak{h}^*$ be the nontrivial eigenvectors of s, with eigenvalues λ_s^{-1} and λ_s, normalised so that $\langle \alpha_s^\vee, \alpha_s \rangle = 2$. Given a W-invariant function $c : S \to \mathbb{C}$, the rational Cherednik algebra $H_c(W, \mathfrak{h})$ *is the unital associative $\mathbb{C}$-algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and W, with relations*

$$\begin{aligned} wx &= {}^w x w, \\ wy &= {}^w y w, \\ [x, x'] &= 0, \\ [y, y'] &= 0, \\ [y, x] &= \langle y, x \rangle - \sum_{s \in S} c(s) \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle s, \end{aligned}$$

for x, $x' \in \mathfrak{h}^*$, y, $y' \in \mathfrak{h}$ and $w \in W$.

If there is no risk of confusion, we denote this algebra simply by H_c.

Much progress has been made in the representation theory of H_c by restricting attention to finitely generated modules on which $\mathfrak{h}$ acts locally nilpotently. The category of such modules, introduced by Opdam and Rouquier [14], is denoted by $\mathcal{O}(H_c)$ and displays many similarities with "category $\mathcal{O}$" for semisimple complex Lie algebras; this point of view is explained in [18]. The natural homomorphism $\mathbb{C}[\mathfrak{h}] \to H_c$ allows us to think of such modules as coherent sheaves on the complex variety $\mathfrak{h}$. By completing at

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various points of $\mathfrak{h}$, Bezrukavnikov and Etingof [4] characterised the possible support sets of such a module, showing in particular that any irreducible component of this set is the set of fixed points of some subgroup of W. Moreover they constructed the following flat connections from these modules (see Proposition 3.20 of [4]).

Proposition 1.2. *Suppose $M \in \mathcal{O}(H_c)$ and W' is a subgroup of W. Let Y be the set of points in $\mathfrak{h}$ whose stabiliser is W', and let $i_Y : Y \hookrightarrow \mathfrak{h}$ be the inclusion. Denoting by $Sh(M)$ the coherent sheaf on $\mathfrak{h}$ corresponding to the $\mathbb{C}[\mathfrak{h}]$-module M, there is a flat connection on the coherent sheaf pullback $i_Y^* Sh(M)$ determined by*

$$\nabla_y m = ym - \sum_{s \in S \setminus W'} c(s)\langle y, \alpha_s\rangle \frac{2}{1-\lambda_s}\frac{1}{\alpha_s}(s-1)m$$

for $m \in M$ and $y \in \mathfrak{h}^{W'}$.

This flat connection is a special case of Theorem 1.4(2) below. In fact this statement holds for any module M in the category $H_c-\mathrm{mod}_{coh}$ of modules finitely generated over $\mathbb{C}[\mathfrak{h}] \subseteq H_c$. This allows us to give the following alternative characterisation of the category $\mathcal{O}(H_c)$.

Proposition 1.3. *The category $\mathcal{O}(H_c)$ is a Serre subcategory of $H_c-\mathrm{mod}_{coh}$. Moreover given an irreducible $M \in H_c-\mathrm{mod}_{coh}$, let $W' \subseteq W$ be a subgroup whose fixed point set is a component of* $\operatorname{Supp} M$*. Then M lies in $\mathcal{O}(H_c)$ if and only if the flat connection of Proposition 1.2 has regular singularities.*

1.2. **Actions on Varieties.** Now suppose W acts on a smooth complex algebraic variety X, and ω is a W-invariant closed 2-form on X. We recall briefly the notion of twisted differential operators [1]. Let $\mathcal{D}_\omega(X)$ denote the sheaf of algebras generated over $\mathcal{O}_X$ by the tangent bundle $\mathcal{T}X$, with relations

$$xy - yx = [x, y] + \omega(x, y), \qquad xf - fx = x(f)$$

for vector fields x and y and regular functions f, where $[\cdot,\cdot]$ denotes the usual Lie bracket of vector fields (note that throughout this paper, scripted letters will generally denote sheaves of modules or algebras). To give an action of $\mathcal{D}_\omega(X)$ on a quasi-coherent sheaf $\mathcal{M}$ is equivalent to giving a connection on $\mathcal{M}$ with curvature ω. Given an immersion of a smooth curve $i : C \hookrightarrow X$, we obtain a connection on the pullback $i^*\mathcal{M}$ which is trivially flat, so $i^*\mathcal{M}$ may be thought of as an untwisted $\mathcal{D}$-module. We say $\mathcal{M}$ has regular singularities if $i^*\mathcal{M}$ has regular singularities in the usual sense for every such immersion. This definition was given by Finkelberg and Ginzburg [12] for 2-forms which are étale-locally exact. In fact we will only be interested in coherent sheaves $\mathcal{M}$ over $\mathcal{D}_\omega(X)$, and the existence of such a sheaf ensures that ω is Zariski-locally exact.

Any 1-form α gives rise to an isomorphism $\mathcal{D}_\omega(X) \cong \mathcal{D}_{\omega+d\alpha}(X)$. Therefore by patching sheaves of algebras of the form $\mathcal{D}_\omega(X)$, we obtain a sheaf of algebras $\mathcal{D}_\psi(X)$ corresponding to any class $\psi \in H^2(X, \Omega_X^{\geq 1})$, where $\Omega_X^{\geq 1}$ is the two step complex $\Omega_X^1 \to \Omega_X^{2,cl}$ lying in degrees 1 and 2, Ω_X^1 is the sheaf of 1-forms and $\Omega_X^{2,cl}$ the sheaf of closed 2-forms. When X is affine, any such class is represented by a global 2-form. Note that our definition of regular singularities depends on a global 2-form chosen to represent the class.

Etingof [9] has defined a sheaf of algebras $\mathcal{H}_{c,\psi}(W, X)$ on X/W, generalizing Definition 1.1. We will recall this definition below (see Definition 2.2) after developing some preliminaries. There is a natural copy of the structure sheaf $\mathcal{O}_X$ in $\mathcal{H}_{c,\psi}(W, X)$, and we will consider the full subcategory $\mathcal{H}_{c,\psi}-\mathrm{mod}_{coh}$ of $\mathcal{H}_{c,\psi}(W, X)$-mod, consisting of sheaves of modules which are coherent as $\mathcal{O}_X$-modules. Our first goal is to classify possible support sets of such modules, in analogy with the results of [4]. Explicitly, given a subgroup $W' \subseteq W$, let

$$\begin{aligned} X^{W'} &= \{x \in X \mid {}^w x = x \text{ for } w \in W'\}, \\ X^{W'}_{\text{reg}} &= \{x \in X \mid \operatorname{Stab}_W(x) = W'\}. \end{aligned}$$

Also define

$$P = \{Y \mid Y \text{ is a component of } X^{W'}_{\text{reg}} \text{ for some } W' \subseteq W\}.$$

These subsets are locally closed, and may be viewed as (non-affine) varieties. Let P' denote the set of all $Y \in P$ such that $H_c(W', T_x X / T_x X^{W'})$ admits a nonzero finite dimensional module, where x is any point of Y and $W' = \operatorname{Stab}_W(x)$. We will prove:

Theorem 1.4. *Suppose $\mathcal{M} \in \mathcal{H}_{c,\psi}\text{–}\mathrm{mod}_{coh}$.*

(1) *Suppose $Z \subseteq X$ is a closed W-invariant subset of X, and consider the subsheaf of "Z-torsion" elements in $\mathcal{M}$,*

$$\Gamma_Z(\mathcal{M})(U) = \{m \in \mathcal{M}(U) \mid \operatorname{Supp} m \subseteq Z\}.$$

That is, $\Gamma_Z(\mathcal{M})$ is the sum of all coherent subsheaves of $\mathcal{M}$ which are set-theoretically supported on Z. Then $\Gamma_Z(\mathcal{M})$ is an $\mathcal{H}_{c,\psi}$-submodule of $\mathcal{M}$.

(2) *Let $Y \in P$ and let $i_Y : Y \hookrightarrow X$ be the inclusion. The coherent sheaf pullback $i_Y^*(\mathcal{M})$ on Y admits a natural action of $\mathcal{D}_{i_Y^*\psi}(Y)$. In particular, if $\psi = 0$, then $i_Y^*(\mathcal{M})$ admits a natural flat connection.*

(3) *The set-theoretical support of $\mathcal{M}$ has the form*

$$\operatorname{Supp} \mathcal{M} = \bigcup_{Y \in P_{\mathcal{M}}} \overline{Y}$$

for some W-invariant subset $P_{\mathcal{M}} \subseteq P'$.

(4) *There is an integer $K > 0$, depending only on c, W and X, such that any such $\mathcal{M}$ is scheme-theoretically supported on the K^{th} neighbourhood of its set-theoretical support.*

(5) *Every object of $\mathcal{H}_{c,\psi}\text{–}\mathrm{mod}_{coh}$ has finite length.*

(6) *If $\mathcal{M}$ is irreducible then we may take $P_{\mathcal{M}}$ in part (3) to be a single W-orbit in P'.*

We would like a sensible subcategory of $\mathcal{H}_{c,\psi}\text{–}\mathrm{mod}_{coh}$ in which to study the representation theory of $\mathcal{H}_{c,\psi}$, analogous to the category $\mathcal{O}(H_c)$ in the linear case. Motivated by Proposition 1.3, we make the following definition. Again we need to choose a global 2-form ω representing the class ψ for this definition.

Definition 1.5. *Let $\mathcal{H}_{c,\omega}\text{–}\mathrm{mod}_{RS}$ denote the Serre subcategory of $\mathcal{H}_{c,\omega}\text{–}\mathrm{mod}_{coh}$, such that an irreducible $\mathcal{M} \in \mathcal{H}_{c,\omega}\text{–}\mathrm{mod}_{coh}$ lies in $\mathcal{H}_{c,\omega}\text{–}\mathrm{mod}_{RS}$ exactly when the connection on $i_Y^*(M)$ given in Theorem 1.4(2) has regular singularities, where $Y \in P_{\mathcal{M}}$ is as in Theorem 1.4(6).*

For a linear action, Proposition 1.3 shows that this category coincides with $\mathcal{O}(H_c)$. Nevertheless we will use the notation $\mathcal{H}_{c,\omega}\text{–}\mathrm{mod}_{RS}$ even in the linear case to avoid confusion with the structure sheaf of a variety.

1.3. **The Type A Case.** Taking X to be an open subset of a vector space, the above will be of use in the sequel, in which we study representations of $H_c = H_c(S_n, \mathbb{C}^n)$, where S_n is the symmetric group acting on $\mathbb{C}^n$ by permuting coordinates. The category $H_c\text{–}\mathrm{mod}_{RS}$ is semisimple unless c is rational with denominator between 2 and n (see [2]), so we take $c = \frac{r}{m}$ where $m \geq 2$ is coprime with r. It is shown in [4] (and follows from Theorem 1.4) that the support of any module in $H_c\text{–}\mathrm{mod}_{RS}$ is of the form

$$X_q = \{b \in \mathfrak{h} \mid \operatorname{Stab}_{S_n}(b) \cong S_m^q\}$$

for some integer q with $0 \leq q \leq \frac{n}{m}$. It is known that the irreducible modules in $H_c{-}\mathrm{mod}_{RS}$ are parameterised by the irreducible representations of $\mathbb{C}[S_n]$, which are in turn parameterised by partitions of n. Given a partition $\lambda \vdash n$, let τ_λ and $L(\tau_\lambda)$ denote the corresponding representation of S_n and H_c respectively. The support of the latter is determined by the following.

Theorem 1.6. *If $c > 0$, then the support of the H_c-module $L(\tau_\lambda)$ is $X_{q_m(\lambda)}$, where*

$$q_m(\lambda) = \sum_{i \geq 1} i \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{m} \right\rfloor .$$

If $c < 0$, the support of $L(\tau_\lambda)$ is $X_{q_m(\lambda')}$, where λ' is the transpose of λ.

In particular, this proves the following conjecture of Bezrukavnikov and Okounkov. While this paper was in preparation, this result was generalised to the cyclotomic case by Shan and Vasserot [20].

Corollary 1.7. *Consider the universal enveloping algebra A of the Heisenberg algebra, with generators $\{\alpha_i \mid i \in \mathbb{Z}, i \neq 0\}$ and relation $[\alpha_i, \alpha_j] = i\delta_{i,-j}$. Consider the grading on A defined by* $\deg(\alpha_i) = i$. *Let F denote* Fock space*, that is, the left A-module*

$$F = A/\mathrm{span}\{A\alpha_i \mid i > 0\}.$$

The number of irreducibles in $H_c{-}\mathrm{mod}_{RS}$ whose support is X_q is the dimension of the qm-eigenspace of the operator

$$\sum_{i>0} \alpha_{-im}\alpha_{im}$$

acting on the degree n part of F.

Moreover, denoting by $H_c{-}\mathrm{mod}^q_{RS}$ the Serre subcategory of $H_c{-}\mathrm{mod}_{RS}$ consisting of all modules supported on X_q, we will determine the structure of the quotient category $H_c{-}\mathrm{mod}^q_{RS}/H_c{-}\mathrm{mod}^{q+1}_{RS}$ (where $H_c{-}\mathrm{mod}^{\lfloor n/m \rfloor+1}_{RS}$ is the subcategory containing only the zero module). Explicitly, let $p = n - qm$ and $\mathbf{q} = e^{2\pi i c}$, and consider the *Hecke algebra* $H_{\mathbf{q}}(S_p)$ with generators $T_1, \ldots, T_{p-1}$ and relations

$$\begin{aligned} T_iT_j &= T_jT_i \text{ if } |i-j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \\ (T_i - 1)(T_i + \mathbf{q}) &= 0. \end{aligned}$$

We will show:

Theorem 1.8. *With $c = \frac{r}{m}$ and $\mathbf{q} = e^{2\pi i c}$, the category $H_c{-}\mathrm{mod}^q_{RS}/H_c{-}\mathrm{mod}^{q+1}_{RS}$ is equivalent to the category of finite dimensional modules over $\mathbb{C}[S_q] \otimes_{\mathbb{C}} H_{\mathbf{q}}(S_p)$.*

1.4. **Outline of the Paper.** The paper is organised as follows. In Section 2, after some algebraic geometry preliminaries we recall the definition of the rational Cherednik algebra of a variety, and prove Theorem 1.4. In Section 3 we state some known results about the representation theory for linear actions, and in particular for $H_{\frac{r}{m}}(S_n, \mathbb{C}^n)$. From this we prove Proposition 1.3 and deduce one direction of Theorem 1.6. We restrict to the case $c = \frac{1}{m}$ in Section 4, and construct an explicit equivalence from the category of minimally supported representations. This enables us, in Section 5, to prove Theorem 1.8 for $c = \frac{1}{m}$. From this we deduce Theorems 1.8 and 1.6 in general.

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2. Coherent Representations

Suppose X is a smooth algebraic variety over $\mathbb{C}$ and W a finite group acting on X, such that the set of points with trivial stabiliser is dense in X. Let ω be a W-invariant closed 2-form on X. Suppose for the moment that X is affine. In order to define the rational Cherednik algebra $\mathcal{H}_{c,\omega}(W, X)$, we require the following lemma, which is shown in Section 2.4 of [9].

Lemma 2.1. *Suppose $Z \subseteq X$ is a smooth closed subscheme of codimension 1. Let $\mathcal{O}(X)$ denote the ring of regular functions on X, and $\mathcal{O}(X)\langle Z\rangle$ the space of rational functions on X whose only pole is along Z, with order at most 1. There is a natural $\mathcal{O}(X)$-module homomorphism $\xi_Z : TX \to \mathcal{O}(X)\langle Z\rangle/\mathcal{O}(X)$ whose kernel consists of all vector fields preserving the ideal sheaf of Z.*

Since TX is a projective $\mathcal{O}(X)$-module, we may lift ξ_Z along the surjection

$$\mathcal{O}(X)\langle Z\rangle \twoheadrightarrow \mathcal{O}(X)\langle Z\rangle/\mathcal{O}(X)$$

to an $\mathcal{O}(X)$-module homomorphism $\zeta_Z : TX \to \mathcal{O}(X)\langle Z\rangle$. It is known (and follows from Proposition 2.5 below) that $X^{W'}$ is a smooth closed subscheme for any subset $W' \subseteq W$.

Definition 2.2 (Definitions 2.7 and 2.8 of [9])**.** *Let S denote the set of pairs (Z, s), where $s \in W$ and Z is an irreducible component of X^s of codimension 1 in X. Let $c : S \to \mathbb{C}$ be a W-invariant function. Let X_{reg} denote the set of points in X with trivial stabiliser in W, and*

$$D_\omega(X_{\mathrm{reg}}) = \Gamma(X_{\mathrm{reg}}, \mathcal{D}_\omega(X))$$

the algebra of global algebraic twisted differential operators on the smooth scheme X_{reg}. For each vector field v on X, we define the Dunkl-Opdam operator $D_v \in \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}})$ by

$$D_v = v + \sum_{(Z,s)\in S} \frac{2c(Z, s)}{1 - \lambda_{Z,s}} \zeta_Z(v)(s - 1),$$

where $\lambda_{Z,s}$ is the determinant of s on T_xX^* for any $x \in Z$. The rational Cherednik algebra $H_{c,\omega}(W, X)$ is the unital $\mathbb{C}$-subalgebra of $\mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}})$ generated by $\mathbb{C}[W] \ltimes \mathcal{O}(X)$ and the D_v.

Remarks:

(1) Although D_v depends on the choice of lift ζ_Z, the algebra $H_{c,\omega}(W, X)$ does not.
(2) Proposition 2.3 below shows that this algebra behaves well with respect to étale morphisms. Moreover if α is any W-invariant 1-form on X, the isomorphism

$$\mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}}) \cong \mathbb{C}[W] \ltimes D_{\omega+d\alpha}(X_{\mathrm{reg}})$$

identifies $H_{c,\omega}(W, X)$ with $H_{c,\omega+d\alpha}(W, X)$. Therefore if we do not assume X is affine, and we take a W-invariant class $\psi \in H^2(X, \Omega_X^{\geq 1})^W$ rather than a global 2-form, we may patch algebras of the above form to construct a sheaf of algebras $\mathcal{H}_{c,\psi}(W, X)$ on X/W. Nevertheless, for the moment we will continue to assume X is affine and ω is a specified 2-form.

Proposition 2.3. *Suppose $p : U \to X$ is a W-equivariant étale morphism, with U affine. For each component Z' of U^s of codimension 1, the image of Z' is a component Z of X^s of codimension 1, and we set $c'(Z', s) = c(Z, s)$. There is a natural $\mathcal{O}(U)$-module isomorphism $\mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \xrightarrow{\sim} H_{c',p^*\omega}(W, U)$, whose composition with $H_{c,\omega}(W, X) \to \mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X)$ is an algebra homomorphism. Moreover given any $H_{c,\omega}(W, X)$-module M, there is a natural action of $H_{c',p^*\omega}(W, U)$ on $\mathcal{O}(U) \otimes_{\mathcal{O}(X)} M$.*

Proof. Since $\mathcal{O}(U)$ is flat over $\mathcal{O}(X)$, the inclusion $H_{c,\omega}(W,X) \subseteq \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}})$ induces an $\mathcal{O}(U)$-module monomorphism

$$\begin{aligned} i : \mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X) &\hookrightarrow \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}}) \\ &= \mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\mathrm{reg}}). \end{aligned}$$

Moreover, for an appropriate choice of the lifts ζ_Z, for any $v \in TX$ the image of $D_v \in H_{c,\omega}(W,X)$ under i is D_{p^*v}. Now $H_{c',p^*\omega}(W,U)$ is the subalgebra of $\mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\mathrm{reg}})$ generated by $\mathbb{C}[W] \ltimes \mathcal{O}(U)$ and $\{D_v \mid v \in TU\}$. But

$$TU = \mathcal{O}(U) \otimes_{\mathcal{O}(X)} TX,$$

and

$$[D_v, f] = v(f) + \sum_{(Z,s)\in S} \frac{2c(Z,s)}{1-\lambda_{Z,s}} \zeta_Z(v)({}^s f - f)s \in \mathbb{C}[W] \ltimes \mathcal{O}(U)$$

for any $v \in TU$ and $f \in \mathcal{O}(U)$. It follows that $H_{c',p^*\omega}(W,U)$ is spanned by elements of the form

$$f w D_{p^*v_1} D_{p^*v_2} \dots D_{p^*v_k}$$

for $f \in \mathcal{O}(U)$, $w \in W$ and $v_i \in TX$. Applying the same argument with p equal to the identity on X, we see that $H_{c,\omega}(W,X)$ is spanned by

$$f w D_{v_1} D_{v_2} \dots D_{v_k}$$

for $f \in \mathcal{O}(X)$, $w \in W$ and $v_i \in TX$. Thus the image of i is exactly $H_{c',p^*\omega}(W,U)$, giving the required isomorphism $j : \mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X) \xrightarrow{\sim} H_{c',p^*\omega}(W,U)$. Note that the composition of i with $H_{c,\omega}(W,X) \to \mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$ equals the composite

$$H_{c,\omega}(W,X) \subseteq \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}}) \to \mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\mathrm{reg}}),$$

which is an algebra homomorphism. In particular, j also preserves the right $\mathcal{O}(X)$-module structure.

Now suppose M is an $H_{c,\omega}(W,X)$-module. The multiplication map

$$H_{c',p^*\omega}(W,U) \otimes_{\mathbb{C}} \mathcal{O}(U) \to H_{c',p^*\omega}(W,U)$$

and the action map $H_{c,\omega}(W,X) \otimes_{\mathcal{O}(X)} M \to M$ give rise to a map

$$\begin{aligned} &H_{c',p^*\omega}(W,U) \otimes_{\mathbb{C}} \mathcal{O}(U) \otimes_{\mathcal{O}(X)} M \\ &\qquad \to H_{c',p^*\omega}(W,U) \otimes_{\mathcal{O}(X)} M \\ &\qquad \cong \mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X) \otimes_{\mathcal{O}(X)} M \\ &\qquad \to \mathcal{O}(U) \otimes_{\mathcal{O}(X)} M. \end{aligned}$$

It is straightforward to check that this defines an action. □

As in [4], we will study representations of $\mathcal{H}_{c,\omega}(W,X)$ by restricting to the formal neighbourhood of a point, or more generally a closed subset.

Proposition 2.4. *Suppose Z is a W-invariant closed subset of X, and let $I \subseteq \mathcal{O}(X)$ denote the ideal vanishing on Z. Consider the coordinate ring of the "formal neighbourhood" of Z,*

$$\hat{\mathcal{O}}_{X,Z} = \varprojlim_k \mathcal{O}(X)/I^k.$$

There is a natural algebra structure on $\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$, and this algebra acts naturally on $\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} M$ for any M in $H_{c,\omega}(W,X)-\mathrm{mod}_{coh}$.

Proof. We define an algebra filtration $H_{c,\omega}^{\leq d}$ of $H_{c,\omega} = H_{c,\omega}(W, X)$ as follows. Let $H_{c,\omega}^{\leq 0} = \mathbb{C}[W] \ltimes \mathcal{O}(X)$ and

$$H_{c,\omega}^{\leq 1} = H_{c,\omega}^{\leq 0} + \mathbb{C}[W]\{D_v \mid v \in TX\}.$$

This is independent of the choice of lift ζ_Z, since different choices of D_v differ by elements of $\mathbb{C}[W] \ltimes \mathcal{O}(X)$. Finally let

$$H_{c,\omega}^{\leq d} = \left(H_{c,\omega}^{\leq 1}\right)^d.$$

As in the previous proof, if $v_1, \ldots, v_m$ generate TX over $\mathcal{O}(X)$, then $H_{c,\omega}^{\leq d}$ is generated over $\mathcal{O}(X)$ by

$$wD_{v_{i_1}} \ldots D_{v_{i_k}}$$

for $w \in W$ and $k \leq d$. In particular, $H_{c,\omega}^{\leq d}$ is a finitely generated left $\mathcal{O}(X)$-module.

Now I is W-invariant, so inside $H_{c,\omega}$ we have

$$(\mathbb{C}[W] \ltimes \mathcal{O}(X))I = I(\mathbb{C}[W] \ltimes \mathcal{O}(X))$$

and

$$[D_v, I] \subseteq [D_v, \mathcal{O}(X)] \subseteq \mathbb{C}[W] \ltimes \mathcal{O}(X).$$

It follows by induction on k that $H_{c,\omega}^{\leq 1} I^{k+1} \subseteq I^k H_{c,\omega}^{\leq 1}$, so that $H_{c,\omega}^{\leq d} I^{k+d} \subseteq I^k H_{c,\omega}^{\leq d}$ for all d, $k \geq 0$. The multiplication map

$$H_{c,\omega}^{\leq d} \otimes H_{c,\omega}^{\leq e} \to H_{c,\omega}^{\leq d+e}$$

therefore naturally induces a map

$$H_{c,\omega}^{\leq d} \otimes \left(\mathcal{O}(X)/I^{d+k} \otimes_{\mathcal{O}(X)} H_{c,\omega}^{\leq e}\right) \to \mathcal{O}(X)/I^k \otimes_{\mathcal{O}(X)} H_{c,\omega}^{\leq d+e}.$$

Taking inverse limits we obtain $\hat{H}_{c,\omega}^{\leq d} \otimes \hat{H}_{c,\omega}^{\leq e} \to \hat{H}_{c,\omega}^{\leq d+e}$, where

$$\hat{H}_{c,\omega}^{\leq d} = \varprojlim_k \mathcal{O}(X)/I^k \otimes_{\mathcal{O}(X)} H_{c,\omega}^{\leq d} = \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} H_{c,\omega}^{\leq d}.$$

In this way the space

$$\hat{H}_{c,\omega} = \bigcup_{d \geq 0} \hat{H}_{c,\omega}^{\leq d} = \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} H_{c,\omega}$$

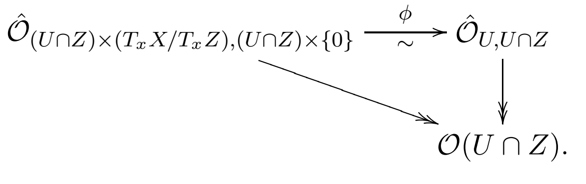
becomes an associative algebra. Moreover for any $M \in H_{c,\omega}-\mathrm{mod}_{coh}$, the action map induces

$$H_{c,\omega}^{\leq d} \otimes \left(\mathcal{O}(X)/I^{d+k} \otimes_{\mathcal{O}(X)} M\right) \to \mathcal{O}(X)/I^k \otimes_{\mathcal{O}(X)} M,$$

and taking inverse limit gives an action of $\hat{H}_{c,\omega}$ on $\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} M$. □

Next we show that the action of W on X looks, on the formal neighbourhood of the fixed point set, like a linear action.

Proposition 2.5. *Suppose $Z \subseteq X$ is a smooth closed subset which is fixed pointwise by the action of W. Then every $x \in Z$ admits an affine open W-invariant neighbourhood $U \subseteq X$ such that there is a W-equivariant ring isomorphism ϕ making the following diagram commute:*



Here W acts on the first ring according to its linear action on T_xX/T_xZ.

Proof. Both rings are inverse limits, so it suffices to construct compatible W-equivariant ring isomorphisms

$$\phi_k : \Gamma(X, \mathcal{O}_X/\mathcal{I}) \otimes \mathbb{C}[T_xX/T_xZ]/\mathfrak{m}^k \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X/\mathcal{I}^k),$$

such that ϕ_1 is the identity, where $\mathfrak{m} \subseteq \mathbb{C}[T_xX/T_xZ]$ is the ideal corresponding to the origin and $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf vanishing on Z.

Let $T_xZ^\perp$ denote the subspace of T_xX^* vanishing on T_xZ. This is the image of $\Gamma(X, \mathcal{I})$ under the gradient map $\Gamma(X, \mathcal{I}) \to T_xX^*$. Let $a_1, \dots, a_n$ be a basis for T_xX, and $b_1, \dots, b_r$ a basis for $T_xZ^\perp$, such that $\langle a_i, b_j \rangle = \delta_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq r$. Since W is finite and acts linearly on $\Gamma(X, \mathcal{I})$ and $T_xZ^\perp$, and we are working over characteristic zero, Maschke's theorem implies the existence of a W-equivariant $\mathbb{C}$-linear map $\beta : T_xZ^\perp \to \Gamma(X, \mathcal{I})$ which is right inverse to the surjection $\Gamma(X, \mathcal{I}) \twoheadrightarrow T_xZ^\perp$. Let $f_j \in \Gamma(X, \mathcal{I})$ be the image of b_j. Also choose $v_i \in TX$ mapping to $a_i \in T_xX$. Let U denote the open neighbourhood of x on which the matrix $(v_i(f_j))_{1 \leq i,j \leq r}$ is invertible. The functions $f_1, \dots, f_r$ have linearly independent gradients on U, so their zero set $Z' \subseteq U$ has codimension r. However $Z' \supseteq Z \cap U$ since $f_j \in \Gamma(X, \mathcal{I})$, and the dimension r of $T_xZ^\perp$ equals the codimension in X of the component of Z containing x. Therefore Z coincides with Z' on some neighbourhood of x. By shrinking U, we may suppose that U is an affine, W-invariant open neighbourhood of x such that $\Gamma(U, \mathcal{I})$ is generated by the f_j. We now inductively construct ring homomorphisms

$$\gamma_k : \Gamma(U, \mathcal{O}_X/\mathcal{I}) \to \Gamma(U, \mathcal{O}_X/\mathcal{I}^k)^W$$

for $k \geq 1$, compatible with the projections $\Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W \twoheadrightarrow \Gamma(U, \mathcal{O}_X/\mathcal{I}^k)^W$, and such that γ_1 is the identity (note that W acts trivially on $\Gamma(U, \mathcal{O}_X/\mathcal{I}) = \Gamma(U \cap Z, \mathcal{O}_Z)$, since Z is fixed pointwise by W). Suppose we have γ_k, where $k \geq 1$. Let $A = \mathbb{C}[x_1, \dots, x_m]$ be a polynomial ring mapping surjectively to $\Gamma(U, \mathcal{O}_X)$, and let $\mathfrak{p} \subseteq A$ be the inverse image of the ideal $\Gamma(U, \mathcal{I})$. Note that $\Gamma(U, \mathcal{O}_X/\mathcal{I}) = A/\mathfrak{p}$. Now choose $y_i \in \Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W$ mapping to $\gamma_k(x_i) \in \Gamma(U, \mathcal{O}_X/\mathcal{I}^k)^W$. We have a ring homomorphism $\gamma' : A \to \Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W$ sending x_i to y_i, and the composite with $\Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W \to \Gamma(U, \mathcal{O}_X/\mathcal{I}^k)^W$ factors through γ_k. In particular, the composite kills $\mathfrak{p}$, so $\gamma'(\mathfrak{p}) \subseteq \Gamma(U, \mathcal{I}^k/\mathcal{I}^{k+1})^W$. Also the composite of γ' with $\Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W \to \Gamma(U, \mathcal{O}_X/\mathcal{I})$ is the natural projection $A \to \Gamma(U, \mathcal{O}_X/\mathcal{I})$. It follows that the restriction

$$\delta = \gamma'|_{\mathfrak{p}} : \mathfrak{p} \to \Gamma(U, \mathcal{I}^k/\mathcal{I}^{k+1})^W$$

is an A-module homomorphism. Certainly then $\delta(\mathfrak{p}^2) = 0$. We have an exact sequence

$$0 \to \mathfrak{p}^2 \to \mathfrak{p} \to A/\mathfrak{p} \otimes_A T(\operatorname{Spec} A)^* \to T(\operatorname{Spec} A/\mathfrak{p})^*,$$

where the map $\mathfrak{p} \to A/\mathfrak{p} \otimes_A T(\operatorname{Spec} A)^*$ is the gradient map. Since

$$\operatorname{Spec} A/\mathfrak{p} = U \cap Z$$

is smooth, $T(\operatorname{Spec} A/\mathfrak{p})^*$ is a projective $A/\mathfrak{p}$-module. Therefore δ factors through the gradient map. Moreover $T(\operatorname{Spec} A)^*$ is freely generated over A by $dx_1, \dots, dx_m$. Therefore we may find $z_1, \dots, z_m \in \Gamma(U, \mathcal{I}^k/\mathcal{I}^{k+1})^W$ such that

$$\delta(f) = \sum_{i=1}^m \frac{\partial f}{\partial x_i} z_i.$$

Let $\gamma'' : A \to \Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W$ be the ring homomorphism sending x_i to $y_i - z_i$. Since $k \geq 1$, we have

$$\gamma''(f) = \gamma'(f) - \sum_{i=1}^m \frac{\partial f}{\partial x_i} z_i.$$

In particular, γ'' kills $\mathfrak{p}$, so it induces the required map $\gamma_{k+1} : A/\mathfrak{p} \to \Gamma(U, \mathcal{O}_X/\mathcal{I}^{k+1})^W$.

Now $\mathbb{C}[T_xX/T_xZ]$ is freely generated by $b_1,\ldots,b_r$, so we may extend γ_k to a homomorphism

$$\phi'_k : \Gamma(U,\mathcal{O}_X/\mathcal{I}) \otimes \mathbb{C}[T_xX/T_xZ] \to \Gamma(U,\mathcal{O}_X/\mathcal{I}^k)$$

by sending b_j to f_j. Note that the ϕ'_k are compatible as k varies, and are W-equivariant since β is. Since $f_j \in \Gamma(X,\mathcal{I})$ for each j, we have $\phi'_k(\Gamma(U,\mathcal{O}_X/\mathcal{I}) \otimes \mathfrak{m}^l) \subseteq \Gamma(U,\mathcal{I}^l/\mathcal{I}^k)$ for $0 \le l \le k$. Thus ϕ'_k induces a map

$$\phi_k : \Gamma(U,\mathcal{O}_X/\mathcal{I}) \otimes \mathbb{C}[T_xX/T_xZ]/\mathfrak{m}^k \to \Gamma(U,\mathcal{O}_X/\mathcal{I}^k),$$

and to prove ϕ_k is an isomorphism, it suffices to show that the induced maps

$$\Gamma(U,\mathcal{O}_X/\mathcal{I}) \otimes \mathfrak{m}^l/\mathfrak{m}^{l+1} \to \Gamma(U,\mathcal{I}^l/\mathcal{I}^{l+1})$$

are isomorphisms, for $0 \le l < k$. We took γ_1 to be the identity, so in fact this is a map of $\Gamma(U,\mathcal{O}_X/\mathcal{I})$-modules. That is, we are required to prove that $\Gamma(U,\mathcal{I}^l/\mathcal{I}^{l+1})$ is freely generated over $\Gamma(U,\mathcal{O}_X/\mathcal{I})$ by degree l monomials in the f_j. This is clear when $l = 0$. Moreover the monomials generate $\Gamma(U,\mathcal{I}^l)$ over $\Gamma(U,\mathcal{O}_X)$, since the f_j generate $\Gamma(U,\mathcal{I})$ over $\Gamma(U,\mathcal{O}_X)$. Finally suppose

$$\sum_\alpha g_\alpha f_1^{\alpha_1}\ldots f_r^{\alpha_r} \in \Gamma(U,\mathcal{I}^{l+1})$$

for some $g_\alpha \in \Gamma(U,\mathcal{O}_X)$, where each monomial has degree l. The matrix $(v_i(f_j))_{1\le i,j\le r}$ is invertible on U, so we may find vector fields $v'_1,\ldots,v'_r$ on U satisfying $v'_i(f_j) = \delta_{ij}$. Since $v'_i\Gamma(U,\mathcal{I}^{l+1}) \subseteq \Gamma(U,\mathcal{I}^l)$, we conclude that

$$\sum_\alpha \alpha_i g_\alpha f_1^{\alpha_1}\ldots f_i^{\alpha_i-1}\ldots f_r^{\alpha_r} \in \Gamma(U,\mathcal{I}^l)$$

for each $1 \le i \le r$. If $l > 0$ then for each α we have $\alpha_i \ne 0$ for some i, so by induction we conclude that $g_\alpha \in \Gamma(U,\mathcal{I})$, as required. □

The next proposition generalises Theorem 3.2 of [4], which applies to linear actions.

Proposition 2.6. *Suppose Z is a smooth connected closed subset of X, every point of which has the same stabiliser W' in W. Suppose the W-translates of Z are all equal to or disjoint with Z, and let WZ denote their union. Finally let W'' be the subgroup of W fixing Z setwise. Then*

$$\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X) \cong \mathrm{Mat}_{[W:W'']}(\mathbb{C}[W''] \otimes_{\mathbb{C}[W']} H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{X,Z})),$$

where $H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{X,Z})$ is an algebra depending only on the following data:

- *the ring $\hat{\mathcal{O}}_{X,Z}$,*
- *the action of W' on $\hat{\mathcal{O}}_{W,Z}$,*
- *the extension of ω to a map $\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \wedge \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \to \hat{\mathcal{O}}_{X,Z}$, and*
- *the parameters $c(Z',s)$ for $Z' \supseteq Z$.*

The isomorphism is natural up to a choice of coset representatives for W'' in W. There is a natural action of $\mathbb{C}[W''] \otimes_{\mathbb{C}[W']} H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{X,Z})$ on $\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} M$ for any $H_{c,\omega}(W,X)$-module M.

Remarks:

(1) The construction of $H_{c,\omega}(W,X)$ can be extended to allow X to be a formal scheme, and the algebra $H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{X,Z})$ is an example of this construction. Nevertheless we give a self-contained definition of this algebra below without making reference to formal schemes.

(2) When $\omega = 0$, the following proof can be simplified by embedding $H_{c,\omega}(W,X)$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(X))$ rather than $\mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}})$.

Proof. We have a natural isomorphism

$$\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,WZ}) \cong \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} TX,$$

so the closed 2-form

$$\omega : TX \otimes_{\mathcal{O}(X)} TX \to \mathcal{O}(X)$$

extends naturally to a closed 2-form $\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\hat{\mathcal{O}}_{X,WZ}} \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,WZ}) \to \hat{\mathcal{O}}_{X,WZ}$. We can now define an algebra $D_\omega(\hat{\mathcal{O}}_{X,WZ})$ in the same way that $D_\omega(X)$ was defined, and

$$D_\omega(\hat{\mathcal{O}}_{X,WZ}) \cong \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} D_\omega(X)$$

as an $\hat{\mathcal{O}}_{X,WZ}$-module. Let $K(\hat{\mathcal{O}}_{X,WZ})$ be the localisation of $\hat{\mathcal{O}}_{X,WZ}$ at all elements which are not zero divisors. There is a natural algebra structure on

$$K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\hat{\mathcal{O}}_{X,WZ}} D_\omega(\hat{\mathcal{O}}_{X,WZ}) = K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X).$$

Since X is smooth and X_{reg} is dense in X, the inclusion $\mathcal{O}(X) \hookrightarrow \hat{\mathcal{O}}_{X,Z}$ extends to a monomorphism $\Gamma(X_{\mathrm{reg}}, \mathcal{O}_X) \hookrightarrow K(\hat{\mathcal{O}}_{X,WZ})$. We therefore obtain an algebra monomorphism

$$D_\omega(X_{\mathrm{reg}}) = \Gamma(X_{\mathrm{reg}}, \mathcal{O}_X) \otimes_{\mathcal{O}(X)} D_\omega(X) \hookrightarrow K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X).$$

From this we construct a monomorphism

$$H_{c,\omega}(W,X) \hookrightarrow \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}}) \hookrightarrow \mathbb{C}[W] \ltimes (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X)).$$

Therefore $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$ may be naturally identified with a subalgebra of $\mathbb{C}[W] \ltimes (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X))$.

Let C be a set of left coset representatives for W'' in W. We choose $1 \in C$ to be the representative for W'' itself. We have

$$WZ = \coprod_{w \in C} wZ.$$

We have assumed that the closed sets on the right are pairwise disjoint, so

$$K(\hat{\mathcal{O}}_{X,WZ}) = \bigoplus_{w \in C} K(\hat{\mathcal{O}}_{X,wZ}),$$

where $K(\hat{\mathcal{O}}_{X,wZ})$ is the field of fractions of $\hat{\mathcal{O}}_{X,wZ}$. Let e denote the identity of $\hat{\mathcal{O}}_{X,Z}$ in the above direct sum. Note that W'' fixes e. Moreover for any $w \in C \setminus \{1\}$ we have $({}^w e)e = 0$, since ${}^w e$ is the identity of $\hat{\mathcal{O}}_{X,wZ}$ in this direct sum. It follows that there is an isomorphism

$$\phi : \mathbb{C}[W] \ltimes (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X)) \to \mathrm{Mat}_C(\mathbb{C}[W''] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))),$$

where Mat_C denotes the algbera of matrices with rows and columns indexed by C. Explicitly, for w_1, $w_2 \in C$ and $a \in \mathbb{C}[W''] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))$, ϕ sends $w_1 e a w_2^{-1}$ to the matrix with a in entry (w_1, w_2) and zeros elsewhere. Therefore to describe $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$, it suffices to determine its image under ϕ.

Now $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$ is generated as an algebra by $\mathbb{C}[W] \ltimes \hat{\mathcal{O}}_{X,WZ}$ and the Dunkl-Opdam operators. Under ϕ, the first subalgebra generates

$$\mathrm{Mat}_C(\mathbb{C}[W''] \ltimes \hat{\mathcal{O}}_{X,Z}) \subseteq \mathrm{Mat}_C(\mathbb{C}[W''] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))).$$

In particular, this contains $\mathrm{Mat}_C(\mathbb{C})$, so the image of $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W,X)$ is $\mathrm{Mat}_C(A)$, where A is the subalgebra of $\mathbb{C}[W''] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))$ generated by $\mathbb{C}[W''] \ltimes \hat{\mathcal{O}}_{X,Z}$ and the entries of the images of the Dunkl-Opdam operators under ϕ. Recall that these operators are given by

$$D_v = v + \sum_{(Z',s)\in S} \frac{2c(Z',s)}{1-\lambda_{Z',s}} \zeta_{Z'}(v)(s-1) \in \mathbb{C}[W] \ltimes D_\omega(X_{\mathrm{reg}}),$$

for $v \in TX$. Given $(Z', s) \in S$, if Z intersects Z', then there is a point in Z fixed by s, so $s \in W'$ and $Z \subseteq Z'$. On the other hand, if Z' is disjoint with Z, then $\zeta_{Z'}(v)$ defines a regular function in $\hat{\mathcal{O}}_{X,Z}$. Therefore A is generated by $\mathbb{C}[W''] \ltimes \hat{\mathcal{O}}_{X,Z}$ and the image of the map

$$D' : TX \to \mathbb{C}[W''] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X)),$$
$$D'(v) = v + \sum_{\substack{(Z',s)\in S\\ Z'\supseteq Z}} \frac{2c(Z',s)}{1-\lambda_{Z',s}} \zeta_{Z'}(v)(s-1).$$

Since $\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \cong \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} TX$, we may extend the $\mathcal{O}(X)$-linear maps $\zeta_{Z'} : TX \to \mathcal{O}(X)\langle Z'\rangle$ to $\hat{\mathcal{O}}_{X,Z}$-linear maps

$$\hat{\zeta}_{Z'} : \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \to \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} \mathcal{O}(X)\langle Z'\rangle \subseteq K(\hat{\mathcal{O}}_{X,Z}),$$

and the above formula then extends D' to an $\hat{\mathcal{O}}_{X,Z}$-linear map

$$\hat{D}' : \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \to \mathbb{C}[W'] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X)).$$

Since A contains $\hat{\mathcal{O}}_{X,Z}$ and the image of D', it also contains the image of $\hat{D}'$. Given $s \in W'$, let $I_s \subseteq \hat{\mathcal{O}}_{X,Z}$ denote the ideal generated by ${}^s\!f - f$ for $f \in \hat{\mathcal{O}}_{X,Z}$. Let Z' denote the component of X^s containing Z, and let $I_{Z'} \subseteq \mathcal{O}(X)$ be the ideal vanishing on Z'. Then $I_s = \hat{\mathcal{O}}_{X,Z} I_{Z'}$, so $(Z', s) \in S$ exactly when I_s is locally principal. Moreover if $f \in \Gamma(U, \mathcal{O}_X)$ generates $I_{Z'}$ on some open subset $U \subseteq X$, then it generates I_s on the corresponding open subset of $\mathrm{Spec}\, \hat{\mathcal{O}}_{X,Z}$, and

$$\hat{\zeta}_{Z'}(v) \in \frac{v(f)}{f} + \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} \Gamma(U, \mathcal{O}_X)$$

for any $v \in \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}, \hat{\mathcal{O}}_{X,Z})$. This formula determines $\hat{\zeta}_{Z'}(v)$ up to an element of $\hat{\mathcal{O}}_{X,Z}$. Thus $\hat{D}'$ is determined, up to a map $\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \to \mathbb{C}[W'] \ltimes \hat{\mathcal{O}}_{X,Z}$, by the action of W' on $\hat{\mathcal{O}}_{X,Z}$ and the parameters $c(Z', s)$ for $Z' \supseteq Z$. These data therefore determine the subalgebra

$$H_{c,\omega}(W', \mathrm{Spf}\, \hat{\mathcal{O}}_{X,Z}) \subseteq \mathbb{C}[W'] \ltimes (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))$$

generated by $\mathbb{C}[W'] \ltimes \hat{\mathcal{O}}_{X,Z}$ and the image of $\hat{D}'$. In particular, $H_{c,\omega}(W', \mathrm{Spf}\, \hat{\mathcal{O}}_{X,Z})$ is preserved by conjugation by W'', so $A = \mathbb{C}[W''] \otimes_{\mathbb{C}[W']} H_{c,\omega}(W', \mathrm{Spf}\, \hat{\mathcal{O}}_{X,Z})$, as required.

Finally consider any $H_{c,\omega}(W, X)$-module M. The previous proposition gives an action of $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X)$ on $\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} M$, so

$$\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} M = e\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} M$$

admits an action of $e\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X)e$. But $\phi(e)$ is the monomial matrix with a 1 in the $(1, 1)$ entry, so restricting ϕ gives an isomorphism

$$e\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X)e \cong \mathbb{C}[W''] \ltimes H_{c,\omega}(W', \mathrm{Spf}\, \hat{\mathcal{O}}_{X,Z}),$$

thus giving the required action. Note that this does not depend on the choice of coset representatives C, since we always take C to contain 1. □

We may now prove our first main result. In this proof we allow X to not be affine, and our class $\psi \in H^2(X, \Omega_X^{\geq 1})$ may not be represented by a global 2-form.

Proof of Theorem 1.4. (1) Since being a submodule is a local property, we may suppose X is affine and that ψ is represented by $\omega \in (\Omega_X^{2,cl})^W$. Consider the module of global sections $M = \Gamma(X, \mathcal{M}) \in H_{c,\omega}(W, X)-$mod. Since Z is W-invariant, it is clear that $\Gamma_Z(M)$ is preserved by $\mathbb{C}[W] \ltimes \mathcal{O}(X)$. It suffices to show that D_v preserves $\Gamma_Z(M)$ for each $v \in TX$. Let $I \subseteq \mathcal{O}(X)$ denote the ideal vanishing on Z. Recall that $[D_v, \mathcal{O}(X)] \subseteq \mathbb{C}[W] \ltimes \mathcal{O}(X)$. Since $WI = IW$, it follows inductively that $I^{k+1}D_v \subseteq H_{c,\omega}I^k$ for each $k \geq 0$. Therefore if $m \in \Gamma_Z(M)$, then $I^k m = 0$ for some k, whence $I^{k+1}D_v m = 0$, so $D_v m \in \Gamma_Z(M)$.

(2) Consider $Y \in P$ and $x \in Y$, with stabiliser $W' \subseteq W$. Let $\mathfrak{h} = T_xX/T_xX^W$. Applying Proposition 2.5 to Y, there is a W'-invariant affine open neighbourhood U of x, with $U \cap Y$ closed in U, and a W'-equivariant isomorphism

$$\phi : \hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h}, U\cap Y} \xrightarrow{\sim} \hat{\mathcal{O}}_{U,U\cap Y},$$

where we identify $U\cap Y$ with $(U\cap Y)\times\{0\} \subseteq (U\cap Y)\times\mathfrak{h}$. Moreover ϕ induces the identity on $\mathcal{O}(U\cap Y)$. Let I be the kernel of the map $\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y} \twoheadrightarrow \mathcal{O}(U\cap Y)$, let C be a set of left coset representatives for W' in W, and let

$$\bar{U} = \coprod_{w\in C} wU.$$

We have a natural étale morphism $\bar{U} \to X$, so by Propositions 2.3 and 2.6, for any $\mathcal{M} \in \mathcal{H}_{c,\psi}-\mathrm{mod}_{coh}$ we have a natural action of $H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{U,U\cap Y})$ on

$$M = \hat{\mathcal{O}}_{U,U\cap Y} \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}),$$

where $\omega \in (\Omega_U^{2,cl})^W$ represents ψ on U. The isomorphism ϕ gives rise to an isomorphism

$$\phi_{c,\omega} : H_{c,\phi^*\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \xrightarrow{\sim} H_{c,\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{U,U\cap Y}).$$

Let ν denote the pullback of $i_{U\cap Y}^*\omega$ to $(U\cap Y)\times\mathfrak{h}$ under the projection map $(U\cap Y)\times\mathfrak{h} \to U\cap Y$. By abuse of notation, we will also use ν to denote the completed map

$$\mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \otimes_{\mathbb{C}} \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \to \hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}.$$

Then

$$\nu(v, v') - (\phi^*\omega)(v, v') \in I$$

for vector fields v, $v' \in \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y})$ which preserve I. It follows that $\nu - \phi^*\omega = d\alpha$ for some 1-form

$$\alpha : \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \to \hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}$$

which satisfies $\alpha(v) \in I$ whenever $v \in \mathrm{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y})$ preserves I. Since we are working over characteristic 0, we may suppose α is W'-invariant. This gives an isomorphism

$$\alpha_{c,\omega} : H_{c,\nu}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \xrightarrow{\sim} H_{c,\phi^*\omega}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}).$$

Proposition 2.6 also gives a natural isomorphism

$$H_{c,\nu}(W', \mathrm{Spf}\,\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}) \cong \hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y} \otimes_{\mathcal{O}(U\cap Y)\otimes\mathbb{C}[\mathfrak{h}]} H_{c,\nu}(W', (U\cap Y)\times\mathfrak{h}),$$

and $H_{c,\nu}(W', (U\cap Y)\times\mathfrak{h}) = D_{i_{U\cap Y}^*\omega}(U\cap Y) \otimes H_c(W', \mathfrak{h})$. Here for any $s \in W'$ acting by reflection on $\mathfrak{h}$, we take $c(s) = c(Z, s)$, where Z is the component of X^s containing Y. Composing these isomorphisms, we obtain an action of $D_{i_{U\cap Y}^*\omega}(U\cap Y)$ on M commuting with the action of $\phi(\mathbb{C}[\mathfrak{h}]) \subseteq \hat{\mathcal{O}}_{U,U\cap Y}$. Therefore $D_{i_{U\cap Y}^*\omega}(U\cap Y)$ acts on

$$M/\phi(\mathfrak{h}^*)M = M/\phi(I)M = \Gamma(U\cap Y, i_Y^*\mathcal{M}).$$

We will show that the latter action is independent of the choices of ϕ and α, thus proving that the action is natural and patches to give an action on all of Y.

Recall from the proof of Proposition 2.6 that $H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y})$ is a subalgebra of $\mathbb{C}[W'] \ltimes (K(\hat{\mathcal{O}}_{U,U\cap Y}) \otimes_{\mathcal{O}(U\cap Y)} D_\omega(U))$. The latter also contains a natural copy of $\operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y})$. We constructed an $\hat{\mathcal{O}}_{U,U\cap Y}$-linear map

$$\hat{D}' : \operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y}) \to H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y}).$$

If $v \in \operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y})$ is W'-invariant then

$$\hat{D}'(v) \in v + \mathbb{C}[W'] \ltimes \hat{\mathcal{O}}_{U,U\cap Y} \subseteq \mathbb{C}[W'] \ltimes (K(\hat{\mathcal{O}}_{U,U\cap Y}) \otimes_{\mathcal{O}(U\cap Y)} D_\omega(U)).$$

In particular $v \in H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y})$. Moreover by choosing $\hat{\zeta}_Z$ appropriately, we may ensure $\hat{D}'(v) \in v + \mathbb{C}[W'] \ltimes \phi(I)$ for any such v.

Consider a vector field v on $U \cap Y$. We may extend v naturally to a vector field on $(U \cap Y) \times \mathfrak{h}$, and therefore a derivation of $\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}$. Let $\bar{v}$ denote the pushforward of this derivation under ϕ. This has the following properties:

(a) $\bar{v} \in \operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y})$ is W' invariant.

(b) The following diagram commutes:

$$\begin{array}{ccc}
\hat{\mathcal{O}}_{U,U\cap Y} & \xrightarrow{\bar{v}} & \hat{\mathcal{O}}_{U,U\cap Y} \\
\downarrow & & \downarrow \\
\mathcal{O}(U\cap Y) & \xrightarrow{v} & \mathcal{O}(U\cap Y).
\end{array}$$

As noted above, the first property ensures $\bar{v} \in H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y})$. The action of v on M constructed above is exactly the action of

$$\bar{v} + \phi\alpha(v) \in H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y}) \subseteq \mathbb{C}[W'] \ltimes (K(\hat{\mathcal{O}}_{U,U\cap Y}) \otimes_{\mathcal{O}(U\cap Y)} D_\omega(U)).$$

However, $\phi\alpha(v) \in \phi(I)$, so v acts on $M/\phi(I)M$ as simply $\bar{v}$. Therefore it suffices to prove that the action of $\bar{v}$ on $M/\phi(I)M$ is determined by the above two properties. If $\bar{v}'$ also satisfies these properties, then

$$(\bar{v} - \bar{v}')(\hat{\mathcal{O}}_{U,U\cap Y}) \subseteq \ker(\hat{\mathcal{O}}_{U,U\cap Y} \to \mathcal{O}(U\cap Y)) = \phi(I).$$

That is, $\bar{v} - \bar{v}' \in \phi(I)\operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y})$. Since $\bar{v}$ and $\bar{v}'$ are fixed by W', this gives

$$\begin{aligned}
\bar{v} - \bar{v}' &\in \hat{D}'(\bar{v} - \bar{v}') + \mathbb{C}[W'] \ltimes \phi(I) \\
&\subseteq \phi(I)\hat{D}'(\operatorname{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{U,U\cap Y})) + \phi(I)\mathbb{C}[W'] \\
&\subseteq \phi(I)H_{c,\omega}(W', \operatorname{Spf} \hat{\mathcal{O}}_{U,U\cap Y}).
\end{aligned}$$

Thus $\bar{v} - \bar{v}'$ acts as zero on $M/\phi(I)M$, as required.

(3) It is well known that a coherent sheaf with a connection is locally free, so the previous part shows that each $Y \in P$ is either contained in or disjoint with $\operatorname{Supp} \mathcal{M}$. Since

$$X = \coprod_{Y\in P} Y,$$

we conclude that $\operatorname{Supp} \mathcal{M}$ is a disjoint union of sets in P. Moreover $\operatorname{Supp} \mathcal{M}$ is closed and the closure $\overline{Y}$ of any $Y \in P$ is irreducible, so the irreducible components of $\operatorname{Supp} \mathcal{M}$ have the form $\overline{Y}$ for some $Y \in P$. Let $P_{\mathcal{M}}$ denote the set of $Y \in P$ such that $\overline{Y}$ is an irreducible component of $\operatorname{Supp} \mathcal{M}$. The action of W on M ensures that $P_{\mathcal{M}}$ is W-invariant, so it suffices to prove that $P_{\mathcal{M}} \subseteq P'$.

Pick $Y \in P_{\mathcal{M}}$ and let x, W', U, M, $\mathfrak{h}$, ϕ and I be as above. Suppose $x \in \overline{Y}'$ for some $Y' \in P_{\mathcal{M}}$. Then $\overline{Y}'$ is a connected component of $X^{W''}$ for some $W'' \subseteq W$, and we must have $W'' \subseteq \operatorname{Stab}_W(x) = W'$. But then $\overline{Y} \subseteq X^{W''}$, so $\overline{Y} \subseteq \overline{Y}'$ since

$\overline{Y}$ is connected. Since $\overline{Y}$ is an irreducible component of $\operatorname{Supp}\mathcal{M}$, we conclude that $Y' = Y$. That is, $\operatorname{Supp}\mathcal{M}$ coincides with Y on some neighbourhood of x. By shrinking U, we suppose that this holds on U. Then some power of $\phi(I)$ kills M. It follows that

$$N = k_x \otimes_{\mathcal{O}(U\cap Y)} M$$

is finite dimensional, where k_x is the residue field of the point $x \in U \cap Y$, and the map $\mathcal{O}(U \cap Y) \to \hat{\mathcal{O}}_{U,U\cap Y}$ is given by the map ϕ. Moreover the action of $D_{i^*_{U\cap Y}\omega}(U \cap Y) \otimes H_c(W', \mathfrak{h})$ on M gives rise to an action of $H_{c,\omega}(W', \mathfrak{h})$ on N. Finally N is nonzero, since

$$\mathbb{C}[\mathfrak{h}]/\mathfrak{h}^*\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]} N$$

is the fibre of $\mathcal{M}$ at $x \in X$, which is nonzero by assumption.

(4) We keep the above notation. Since some power of I kills M, and the ring $\hat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h},U\cap Y}/I^l$ is finitely generated over $\mathcal{O}(U\cap Y)$ for any l, we conclude that M is a finitely generated module over $\mathcal{O}(U\cap Y)$. Since it admits a connection, it is locally free. We will show in Lemma 3.1 that there is an integer K_Y, depending only on $\mathfrak{h}$, W' and c, such that N is killed by $(\mathfrak{h}^*)^{K_Y}$. That is,

$$I_Y^{K_Y} M \subseteq \mathfrak{m}_x M,$$

where $I_Y \subseteq \mathcal{O}(U)$ is the ideal vanishing on $U \cap Y$, and $\mathfrak{m}_x \subseteq \mathcal{O}(U \cap Y)$ is the maximal ideal corresponding to the point x. Up to (non-canonical) isomorphism, the algebra $H_{c,\omega}(W', \mathfrak{h})$ is independent of the point $x \in Y$. Therefore this equation holds for any $x \in Y \cap U$. Together with local freeness, this ensures that $I_Y^{K_Y} M = 0$. Since $\operatorname{Supp}\mathcal{M}$ coincides with Y on U, we conclude that $\mathcal{I}^{K_Y}\mathcal{M}$ vanishes on U, where $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf vanishing on $\operatorname{Supp}\mathcal{M}$. Now U was chosen to contain an arbitrary point on Y, and $\mathcal{I}^{K_Y}$ is W-invariant, so $\mathcal{I}^{K_Y}\mathcal{M}$ vanishes on the union WY of all W-translates of Y. That is, $\mathcal{I}^{K_Y}\mathcal{M} \subseteq \Gamma_Z(M)$, where Z is the complement of WY in $\operatorname{Supp}\mathcal{M}$. Note that Z is closed and W-invariant. It now follows by induction on $\operatorname{Supp}\mathcal{M}$ that M is killed by $\mathcal{I}$ to the power of

$$\sum_{\substack{Y\in P' \\ Y\subseteq \operatorname{Supp}\mathcal{M}}} K_Y.$$

In particular, this proves the statement with $K = \sum_{Y\in P'} K_Y$.

(5) Let K be the integer constructed in the previous part. Again we prove the statement by induction on $\operatorname{Supp}\mathcal{M}$, and the case $\operatorname{Supp}\mathcal{M} = \emptyset$ is trivial. Suppose every module with smaller support has finite length. Choose $Y \in P_{\mathcal{M}}$, and let $\mathcal{I}_Y \subseteq \mathcal{O}_X$ denote the ideal sheaf vanishing on $\overline{Y}$. Let $U \subseteq X$ be some open affine subset intersecting Y, and let $I_Y = \Gamma(U, \mathcal{I}_Y)$. This is prime since Y is irreducible. Consider the ring

$$R = \mathcal{O}(U)_{(I_Y)}/I_Y^K\mathcal{O}(U)_{(I_Y)},$$

that is, the K^{th} formal neighbourhood of the (non-closed) generic point of Y. Then $I_Y^k R/I_Y^{k+1}R$ is finite dimensional over the field $R/I_Y R$ for each k, so R is Artinian. Since $\mathcal{M}$ is coherent, it follows that

$$R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M})$$

has finite length over R. We may therefore assume by induction that the statement also holds for modules with the same support, for which the above R-module has smaller length.

Again let Z be the complement of WY in $\operatorname{Supp}\mathcal{M}$. Let $\mathcal{I}_Z$ and $\mathcal{I}$ be the ideal sheaves vanishing on Z and $\operatorname{Supp}\mathcal{M}$ respectively, Let $\mathcal{M}' = \mathcal{M}/\Gamma_Z(\mathcal{M})$ and $\mathcal{M}'' = \mathcal{H}_{c,\psi}\mathcal{I}_Z^K\mathcal{M}'$. Since $\Gamma_Z(\mathcal{M})$ and $\mathcal{M}'/\mathcal{M}''$ are supported on Z, they have finite length by induction. If $\mathcal{M}''$ is zero or irreducible, we are done. Suppose otherwise, and let $\mathcal{N}'$ be a proper nonzero submodule of $\mathcal{M}''$, and let $\mathcal{N}$ be the inverse image of $\mathcal{N}'$ in $\mathcal{M}$. We have $\mathcal{I}_Y\mathcal{I}_Z \subseteq \mathcal{I}$, so

$$\mathcal{I}_Z^K\mathcal{I}_Y^K\mathcal{M} \subseteq \mathcal{I}^K\mathcal{M} = 0.$$

Therefore $\mathcal{I}_Y^K\mathcal{M} \subseteq \Gamma_Z(\mathcal{M})$, so $\mathcal{I}_Y^K\mathcal{M}' = 0$ and the map

$$\mathcal{O}(U)_{(I_Y)} \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}') \to R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}')$$

is an isomorphism. Since localisation is exact, we conclude that

$$R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{N}') \to R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}')$$

is injective. Therefore we have a short exact sequence

$$R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{N}') \hookrightarrow R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}') \twoheadrightarrow R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}'/\mathcal{N}').$$

We claim that the first and last modules are nonzero. This is equivalent to $\operatorname{Supp}\mathcal{N}'$ and $\operatorname{Supp}(\mathcal{M}'/\mathcal{N}')$ containing Y. Suppose the first fails. Then $\mathcal{N}'$ is supported on Z, so $\mathcal{I}_Z^K\mathcal{N}' = 0$. Thus

$$\mathcal{I}_Z^{2K}\mathcal{N} \subseteq \mathcal{I}_Z^K\Gamma_Z(\mathcal{M}) = 0.$$

Hence $\mathcal{N} \subseteq \Gamma_Z(\mathcal{M})$, so that $\mathcal{N}' = 0$, a contradiction. Now suppose $\mathcal{M}'/\mathcal{N}'$ is supported on Z. Then $\mathcal{I}_Z^K(\mathcal{M}'/\mathcal{N}') = 0$, so $\mathcal{N}' \supseteq \mathcal{I}_Z^K\mathcal{M}'$. Thus $\mathcal{N}' \supseteq \mathcal{H}_{c,\psi}\mathcal{I}_Z^K\mathcal{M}' = \mathcal{M}''$, contradicting the assumption that $\mathcal{N}'$ is a proper submodule of $\mathcal{M}''$. This proves the claim, and we conclude that

$$\operatorname{len}_R(R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{N}')), \operatorname{len}_R(R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M}'/\mathcal{N}')) < \operatorname{len}_R(R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M})),$$

so $\mathcal{N}'$ and $\mathcal{M}'/\mathcal{N}'$ have finite length by induction. Again, since $\Gamma_Z(\mathcal{M})$ has finite length, we are done.

(6) Again let Y be any element of $P_{\mathcal{M}}$ and let WY and Z be as above. Let $\mathcal{I}_{WY}$, $\mathcal{I}_Z$ and $\mathcal{I}$ be the ideal sheaves in $\mathcal{O}_X$ vanishing on $\overline{WY}$, Z and $\operatorname{Supp}\mathcal{M}$ respectively. Then

$$\mathcal{I}_Z^K\mathcal{I}_{WY}^K\mathcal{M} \subseteq \mathcal{I}^K\mathcal{M} = 0.$$

Thus $\mathcal{I}_{WY}^K\mathcal{M} \subseteq \Gamma_Z(\mathcal{M})$. Now $\Gamma_Z(\mathcal{M})$ is a submodule of $\mathcal{M}$ by (1), and it is proper since $Z \subsetneq \operatorname{Supp}\mathcal{M}$. Since $\mathcal{M}$ is irreducible, we conclude that $\Gamma_Z(\mathcal{M}) = 0$. Hence $\mathcal{I}_{WY}^K\mathcal{M} = 0$, so $\operatorname{Supp}\mathcal{M} = \overline{WY}$ as required.

□

3. Linear Actions

Now suppose $X = \mathfrak{h}$ is a finite dimensional vector space with a linear action. We briefly review some known results concerning representations of $H_c(W, \mathfrak{h})$; see [8]. We also prove Proposition 1.3 and one direction of Theorem 1.6.

3.1. **Verma modules.** Any W-module τ becomes a $\mathbb{C}[W] \ltimes \mathbb{C}[\mathfrak{h}^*]$-module by declaring that $\mathfrak{h}$ acts as 0. We may therefore construct an H_c-module

$$M(\tau) = H_c \otimes_{\mathbb{C}[W]\ltimes\mathbb{C}[\mathfrak{h}^*]} \tau.$$

This is the *Verma module* corresponding to τ. The multiplication map

$$\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \to H_c$$

is a vector space isomorphism, so $M(\tau) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \tau$ as a $\mathbb{C}[W] \ltimes \mathbb{C}[\mathfrak{h}]$-module. As a special case, when τ is the trivial representation, we obtain an action of H_c on $\mathbb{C}[\mathfrak{h}]$; this is called the *polynomial representation*. If τ is irreducible, then there is a unique maximal proper submodule $J(\tau)$ of $M(\tau)$, and the quotient $L(\tau)$ is irreducible. Let y_i be a basis of $\mathfrak{h}$, and x_i the dual basis of $\mathfrak{h}^*$. Define the *Euler element* by

$$\mathbf{eu} = \sum_i x_i y_i + \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s \in H_c.$$

This element has the useful property that $\operatorname{ad} \mathbf{eu}$ acts as 0 on W, as 1 on $\mathfrak{h}^* \subseteq H_c$, and as -1 on $\mathfrak{h} \subseteq H_c$. From this fact we deduce the following lemma, which was used in the proof of Theorem 1.4.

Lemma 3.1. *There exists a positive integer K, depending only on c and W, such that $\mathfrak{h}^K \subseteq H_c$ and $(\mathfrak{h}^*)^K \subseteq H_c$ annihilate any finite dimensional H_c-module.*

Proof. Let M be a finite dimensional H_c-module. Then M decomposes as

$$M = \bigoplus_{\lambda \in \Lambda} M_\lambda,$$

where M_λ is the generalised eigenspace of $\mathbf{eu}$ with eigenvalue λ, and $\Lambda \subseteq \mathbb{C}$ is the finite set of eigenvalues. Since $\operatorname{ad} \mathbf{eu}$ acts as -1 on $\mathfrak{h}$, it is clear that $\mathfrak{h}$ sends M_λ to $M_{\lambda-1}$. Thus $\mathfrak{h}^K$ kills M, where K is any integer larger than $d = \max(\mathrm{Re}(\Lambda)) - \min(\mathrm{Re}(\Lambda))$. The same is true of $\mathfrak{h}^*$, and it remains to bound d independently of M. Pick $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda)$ minimal. Then $\mathfrak{h}$ acts as 0 on M_λ. Since W commutes with $\mathbf{eu}$, it preserves the eigenspace M_λ. We may therefore find a subspace $\tau \subseteq M_\lambda$ which is irreducible under the action of W. Then

$$\mathbf{eu}|_\tau = \left(\sum_i x_i y_i + \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s \right) \Bigg|_\tau = \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s|_\tau.$$

This depends only on the action of W on τ. Since W has only finitely many irreducible modules, there are only finitely many possible values for λ, once c and W have been chosen. Similarly by writing

$$\mathbf{eu} = \sum_i y_i x_i - \dim \mathfrak{h} + \sum_{s \in S} c(s) \left(\frac{2}{\lambda_s - 1} + 2 \right) s,$$

we see that there are only finitely many possibilities for $\max(\mathrm{Re}(\Lambda))$. Thus d has only finitely many possible values, depending on W and c, and may therefore be bounded independently of M. □

For a module $M \in H_c{-}\mathrm{mod}_{coh}$, the following conditions are equivalent:

1. The action of $\mathbf{eu}$ on M is locally finite.
2. The action of $\mathfrak{h}$ on M is locally nilpotent.
3. Every composition factor of M is isomorphic to some $L(\tau)$.

Proposition 1.3 states that the category of modules satisfying these conditions is exactly the category $H_c{-}\mathrm{mod}_{RS}$ of Definition 1.5. We will require the following lemma to prove this.

Lemma 3.2. *Let $\mathfrak{h}$ be a finite dimensional vector space, and suppose $Z \subseteq \mathfrak{h}$ is the zero set of a homogeneous ideal in $\mathbb{C}[\mathfrak{h}]$ (that is, Z is a cone). Let $\xi \in D(\mathfrak{h} \setminus Z)$ denote the Euler vector field. Then ξ acts locally finitely on the global sections of any $\mathcal{O}$-coherent $\mathcal{D}$-module on $\mathfrak{h} \setminus Z$ with regular singularities.*

Proof. Let $\mathcal{M}$ be an $\mathcal{O}$-coherent $\mathcal{D}$-module on $\mathfrak{h} \setminus Z$ with regular singularities. Then $\mathcal{M}$ is locally free, so if U is an open subset of $\mathfrak{h} \setminus Z$, the restriction map

$$\Gamma(\mathfrak{h} \setminus Z, M) \to \Gamma(U, M)$$

is injective. We may therefore replace $\mathfrak{h} \setminus Z$ by any smaller $\mathbb{C}^*$-invariant open subset U. Denoting by $x_0, \ldots, x_n$ the coordinates on $\mathfrak{h}$, we suppose that U is affine and disjoint with the zero set of x_0. We have an isomorphism

$$\mathbb{C}[\mathfrak{h}][x_0^{-1}] \xrightarrow{\sim} \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[y_1, \ldots, y_n]$$

given by $x_0 \mapsto t$ and $x_i \mapsto ty_i$ for $i > 0$. Note the $\mathbb{C}^*$ action on the left, which scales each x_i, corresponds to the $\mathbb{C}^*$ action on the right scaling only t. Thus we have a $\mathbb{C}^*$-equivariant isomorphism

$$U \cong \mathbb{C}^* \times Y$$

where Y is some affine open subset of $\operatorname{Spec} \mathbb{C}[y_1, \ldots, y_n]$. In particular the vector field ξ on the left corresponds to $t\partial_t$ on $\mathbb{C}^*$.

It therefore suffices to consider a module M over $D(\mathbb{C}^* \times Y) = D(\mathbb{C}^*) \otimes_{\mathbb{C}} D(Y)$ which is $\mathcal{O}$-coherent with regular singularities. Moreover we may suppose that M is irreducible. The Riemann-Hilbert correspondence [7] implies that M is of the form $L \otimes_{\mathbb{C}} N$ for some irreducible modules $L \in D(\mathbb{C}^*)$-mod and $N \in D(Y)$-mod, and that $L = \mathbb{C}[t, t^{-1}]v$ with connection

$$\nabla_{\partial_t} f(t, t^{-1})v = (\partial_t f(t, t^{-1}))v + \lambda t^{-1} f(t, t^{-1})v$$

for some $\lambda \in \mathbb{C}$. Since $t\partial_t$ acts as $n + \lambda$ on $t^n v$, and $\{t^n v \mid n \in \mathbb{Z}\}$ is a basis for L, we are done. □

Proof of Proposition 1.3. First we show each $L(\tau)$ lies in $H_c\text{−mod}_{RS}$. Choose $Y \in P_{L(\tau)}$, and let $i_Y : Y \hookrightarrow X$ denote the inclusion. We are required to show that the connection on $i_Y^* Sh(L(\tau))$ has regular singularities. Certainly we have a surjection $i_Y^* Sh(M(\tau)) \twoheadrightarrow i_Y^* Sh(L(\tau))$ intertwining the connections, so it suffices to prove that the connection on $i_Y^* Sh(M(\tau))$ has regular singularities. However,

$$\Gamma(Y, i_Y^* Sh(M(\tau))) \cong \mathcal{O}(Y) \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \tau = \mathcal{O}(Y) \otimes_{\mathbb{C}} \tau.$$

Moreover τ is killed by $\mathfrak{h}$, so by Proposition 1.2, the connection on $i_Y^* Sh(M)$ is described by

$$\nabla_y m = -\sum_{s \in S \setminus W'} c(s) \langle y, \alpha_s \rangle \frac{2}{1 - \lambda_s} \frac{1}{\alpha_s} (s - 1)m$$

for $m \in \tau$ and $y \in \mathfrak{h}^{W'}$, where W' is the stabiliser of any point in Y. Since this expression only contains poles of first order, the connection has regular singularities. Since $H_c\text{−mod}_{RS}$ is a Serre subcategory of $H_c\text{−mod}_{coh}$ by definition, this proves any module satisfying condtion (3) above lies in $H_c\text{−mod}_{RS}$.

Conversely, we will show that any $M \in H_c\text{−mod}_{RS}$ satisfies condition (1) above. This condition is preserved by extensions, and Theorem 1.4(5) shows that M has finite length, so we may suppose M is irreducible. As usual let Y be in P_M with stabiliser W'. We may find a homogeneous polynomial $f \in \mathbb{C}[\mathfrak{h}]$ which vanishes on each $Y' \in P_M \setminus \{Y\}$, but not on Y itself. Note that the kernel of the map

$$M \to \bigoplus_{w \in W} M[({}^w f)^{-1}]$$

is $\Gamma_Z(M)$, where Z is the common zero set of all the ${}^w f$. This is zero by Theorem 1.4(1) and the irreducibility of M. Since $\operatorname{ad} \mathbf{eu}$ acts as $\deg f$ on each ${}^w f$, the action of $\mathbf{eu}$ on M extends naturally to one on $M[({}^w f)^{-1}]$. Therefore since $\mathbf{eu}$ is W-invariant, it suffices to show that the action of $\mathbf{eu}$ on $M[f^{-1}]$ is locally finite.

Let $U \subseteq \mathfrak{h}$ denote the affine open subset on which f is nonzero, and let $I_Y \subseteq \mathbb{C}[\mathfrak{h}][f^{-1}]$ denote the ideal vanishing on $U \cap Y$. Note that I_Y is generated by some subspace $V \subseteq \mathfrak{h}^*$, so $I_Y^k M[f^{-1}]$ is invariant under $\mathbf{eu}$ for each k. Moreover since $M[f^{-1}]$ is supported on $U \cap Y$, we have $I_Y^K M[f^{-1}] = 0$ for some $K > 0$. Therefore it suffices to show that

eu acts locally finitely on each $I_Y^k M[f^{-1}]/I_Y^{k+1}M[f^{-1}]$. Finally for each k we have a surjective map

$$V^{\otimes k} \otimes_{\mathbb{C}} M[f^{-1}]/I_Y M[f^{-1}] \twoheadrightarrow I_Y^k M[f^{-1}]/I_Y^{k+1}M[f^{-1}]$$

intertwining $1 \otimes \mathbf{eu}$ with $\mathbf{eu} - k \deg f$. We may therefore consider just $k = 0$. But

$$M[f^{-1}]/I_Y M[f^{-1}] = \mathcal{O}(U \cap Y) \otimes_{\mathbb{C}[\mathfrak{h}]} M$$

is an $\mathcal{O}$-coherent $\mathcal{D}$-module on $U \cap Y$ with connection given by Proposition 1.2. It has regular singularities by assumption. Note that $U \cap Y$ is the basic open subset of the vector space $\mathfrak{h}^{W'}$ on which $f|_{\mathfrak{h}^{W'}}$ is nonzero. Therefore the vector field ξ of Lemma 3.2 acts locally finitely. To describe the action of ξ explicitly, let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be dual bases for $\mathfrak{h}^*$ and $\mathfrak{h}$, such that $\{y_1, \ldots, y_r\}$ span $\mathfrak{h}^{W'}$. Then $x_{r+1}, \ldots, x_n$ are zero in $\mathcal{O}(U \cap Y)$, and a straighforward calculation shows that

$$\xi m = \mathbf{eu}\, m + \sum_{s \in S \setminus W'} \frac{2c(s)}{1 - \lambda_s} m + \sum_{s \in S \cap W'} \frac{2c(s)}{1 - \lambda_s} sm$$

for $m \in M$. Note that $\mathcal{O}(U \cap Y) \otimes_{\mathbb{C}[\mathfrak{h}]} M$ admits an action of W' commuting with $\mathcal{O}(U \cap Y)$. Since ξ and **eu** have the same commutator with any element of $\mathcal{O}(U \cap Y)$, this formula holds for any $m \in \mathcal{O}(U \cap Y) \otimes_{\mathbb{C}[\mathfrak{h}]} M$. Therefore since ξ and $\mathbb{C}[W']$ act locally finitely, so does **eu**, as required. ☐

3.2. **Characters of modules.** Let $\overline{\mathbb{Z}[[t]]}$ denote the space of formal $\mathbb{Z}$-linear combinations of powers of t, such that the exponents appearing belong to $A + \mathbb{Z}_{\geq 0}$ for some finite subset $A \subseteq \mathbb{C}$. Let $K(W\text{-mod}_{fd})$ denote the Grothendieck group of the category of finite dimensional representations of W. There is a homomorphism

$$Ch : K(H_c(W, \mathfrak{h})\text{-mod}_{RS}) \to K(W\text{-mod}_{fd}) \otimes_{\mathbb{Z}} \overline{\mathbb{Z}[[t]]}$$

sending $[M]$ to

$$\sum_{\lambda \in \mathbb{C}} [M_\lambda] t^\lambda,$$

where $M_\lambda \subseteq M$ is the generalised λ-eigenspace of **eu**, considered as a W-module. If τ is an irreducible W-module, then

$$Ch([M(\tau)]),\, Ch([L(\tau)]) \in [\tau] t^{h(\tau)} + K(W\text{-mod}_{fd}) \otimes \mathbb{Z}[[t]] t^{h(\tau)+1},$$

where $h(\tau)$ is the scalar by which

$$\sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s$$

acts on τ. It follows that Ch is injective, and both $\{[L(\tau)]\}$ and $\{[M(\tau)]\}$ form bases for $K(H_c(W, \mathfrak{h})\text{-mod}_{RS})$. Moreover the matrix relating the $[M(\tau)]$ to $[L(\tau)]$ is upper triangular, when the irreducibles are ordered by $\mathrm{Re}(h(\tau))$. It follows that the functor

$$\begin{aligned} Verma_W : W\text{-mod}_{fd} &\to H_c(W, \mathfrak{h})\text{-mod}_{RS}, \\ \tau &\mapsto M(\tau) \end{aligned}$$

induces an isomorphism on Grothendieck groups.

3.3. **Induction and restriction.** Bezrukavnikov and Etingof [4] also construct "parabolic induction and restriction" functors for rational Cherednik algebras. The following theorem summarises their Propositions 3.9, 3.10 and 3.14.

Theorem 3.3. *Consider a point $b \in \mathfrak{h}$, with stabiliser $W' \subseteq W$. There exist exact functors $Res_b : H_c(W, \mathfrak{h})\text{-mod}_{RS} \to H_c(W', \mathfrak{h})\text{-mod}_{RS}$ and $Ind_b : H_c(W', \mathfrak{h})\text{-mod}_{RS} \to H_c(W, \mathfrak{h})\text{-mod}_{RS}$ with the following properties:*

(1) *The functor Ind_b is right adjoint to Res_b.*

(2) *The support of $Res_b(M)$ is the union of the components of* $\operatorname{Supp} M$ *passing through b.*
(3) *The support of $Ind_b(N)$ is the union of W-translates of* $\operatorname{Supp} N$*.*
(4) *The induced maps $[Res_b]$ and $[Ind_b]$ on Grothendieck groups satisfy*

$$\begin{aligned} [Res_b][Verma_W] &= [Verma_{W'}][Res], \\ [Ind_b][Verma_{W'}] &= [Verma_W][Ind], \end{aligned}$$

where $Res : W-\mathrm{mod}_{fd} \to W'-\mathrm{mod}_{fd}$ and $Ind : W'-\mathrm{mod}_{fd} \to W-\mathrm{mod}_{fd}$ are the usual restriction and induction functors.

Moreover by its construction, Res_b respects monodromy in the following sense.

Proposition 3.4. *Suppose b, $v \in \mathfrak{h}$ have stabilisers W', $W'' \subseteq W$ respectively. Suppose that $W'' \subseteq W'$, and that $C \subseteq W'$ is a subgroup acting faithfully on $\mathbb{C}v$. Consider the map $\phi : \mathbb{C} \to \mathfrak{h}$ given by $\phi(z) = b + zv$. There is a Zariski open subset $U \subseteq \mathbb{C}^*$ such that ϕ maps U into $Y = \mathfrak{h}_{\mathrm{reg}}^{W''}$. Let $i_Y : Y \hookrightarrow \mathfrak{h}$ denote the inclusion. For any $M \in H_c(W, \mathfrak{h})-\mathrm{mod}_{RS}$, the $\mathcal{D}$-modules $i_Y^* Sh(M)$ and $i_Y^* Sh(Res_b M)$ satisfy*

$$\mathbb{C}((z)) \otimes_{\mathcal{O}_U} \phi|_U^* i_Y^* Sh(M) \cong \mathbb{C}((z)) \otimes_{\mathcal{O}_U} \phi|_U^* i_Y^* Sh(Res_b M)$$

as $\mathbb{C}[C] \ltimes \mathbb{C}((z))[\partial_z]$-modules. In particular, the monodromies about the origin of the $\mathcal{D}$-modules $\phi|_U^* i_Y^* Sh(M)$ and $\phi|_U^* i_Y^* Sh(Res_b M)$ are conjugate, and the same is true when these equivariant $\mathcal{D}$-modules are pushed down to U/C.

3.4. **Type A.** Now let $W = S_n$, the symmetric group S_n on n letters, acting on $\mathfrak{h} = \mathbb{C}^n$ by permuting coordinates. The reflections in this case are transpositions. As they are all conjugate, the function $c : S \to \mathbb{C}$ must be constant, and we identify it with its value in $\mathbb{C}$. Let $\mathfrak{h}/\mathbb{C}$ denote the quotient of $\mathfrak{h}$ by the line fixed by W. In this case we have the following simple criterion for when $H_c(W, \mathfrak{h}/\mathbb{C})$ admits a finite dimensional representation.

Theorem 3.5 ([3] Theorem 1.2)**.** *Suppose $n > 1$. The algebra $H_c(W, \mathfrak{h}/\mathbb{C})$ admits a nonzero finite dimensional representation if and only if $c = \frac{r}{n}$ for some integer r coprime with n. In this case the category of finite dimensional modules is semisimple with one irreducible. Moreover if $c = \frac{1}{n}$, this irreducible is one dimensional.*

Using this with Theorem 1.4(3) gives the following (see Example 3.25 of [4]).

Theorem 3.6. *Suppose $c = \frac{r}{m}$, where r and m are integers with m positive and coprime with r. For each nonnegative integer $q \le n/m$, let*

$$X'_q = \left\{ b \in \mathfrak{h} \;\middle|\; b_i = b_j \text{ whenever } \left\lceil \frac{i}{m} \right\rceil = \left\lceil \frac{j}{m} \right\rceil \le q \right\} \quad \text{and} \quad X_q = \bigcup_{w \in W} w X'_q.$$

Then any module in $H_c(W, \mathfrak{h})-\mathrm{mod}_{coh}$ is supported on one of the X_q. If c is irrational then every such module has full support.

We would like to determine more explicitly which irreducibles in $H_c-\mathrm{mod}_{RS}$ have which support sets. The irreducible representations of $W = S_n$ are well known to be parameterised by partitions of n. Given a partition $\lambda \vdash n$, the corresponding irreducible is denoted τ_λ and called the *Specht module* indexed by λ. We will represent a partition $\lambda \vdash n$ as a nonincreasing sequence of nonnegative integers, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, whose sum is n, where two sequences are identified if their nonzero entries agree. If $\lambda \vdash n$ and $\mu \vdash m$, we may obtain a partition $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$ of $n + m$. We first prove a lemma about the induction functor introduced in Theorem 3.3.

Lemma 3.7. *Suppose c is a positive real number, and suppose we have partitions $\lambda \vdash n$ and $\mu \vdash m$. Let b be a point in $\mathbb{C}^{n+m}$ whose stabiliser in S_{n+m} is $S_n \times S_m$. Then $Ind_b(L(\tau_\lambda) \otimes L(\tau_\mu))$ admits a nonzero map from $M(\tau_{\lambda+\mu})$.*

Proof. We first prove

Claim 1: the lowest order term in $Ch[Ind_b(M(\tau_\lambda) \otimes M(\tau_\mu))]$ is $[\tau_{\lambda+\mu}]t^{h(\tau_{\lambda+\mu})}$.

Indeed, the construction of Verma modules shows that $M(\tau_\lambda) \otimes M(\tau_\mu) \cong M(\tau_\lambda \otimes \tau_\mu)$, so Theorem 3.3(4) implies that

$$Ch[Ind_b(M(\tau_\lambda) \otimes M(\tau_\mu))] = Ch[M(Ind(\tau_\lambda \otimes \tau_\mu))].$$

The Littlewood-Richardson rule describes how $Ind(\tau_\lambda \otimes \tau_\mu)$ splits into Specht modules for S_{n+m}. In particular, given partitions α and β of $n+m$, say α *dominates* β, denoted $\alpha \geq \beta$, if

$$\sum_{i=1}^{p} \alpha_i \geq \sum_{i=1}^{p} \beta_i$$

for all $p \geq 1$. Then

$$[Ind(\tau_\lambda \otimes \tau_\mu)] = [\tau_{\lambda+\mu}] + \sum_{\nu \leq \lambda+\mu} c_{\lambda\mu}^{\nu}[\tau_\nu]$$

for some coefficients $c_{\lambda\mu}^{\nu} \in \mathbb{Z}$. Certainly $Ch[M(\tau_{\lambda+\mu})]$ has the required lowest order term, so it suffices to prove that $h(\tau_\alpha) < h(\tau_\beta)$ whenever $\alpha > \beta$. Recall that $h(\tau_\alpha)$ is the action of

$$-c \sum_{i \neq j} s_{ij}$$

on $S(\alpha)$. By the Frobenious character formula (see Exercise 4.17(c) of [13]), this is

$$h(\tau_\alpha) = -c \sum_{i \geq 1} \frac{1}{2}\alpha_i^2 - \left(i - \frac{1}{2}\right)\alpha_i.$$

Using the Abel summation formula,

$$\begin{aligned} h(\tau_\alpha) - h(\tau_\beta) &= -c \sum_{i \geq 1} (\alpha_i - \beta_i) \left(\frac{\alpha_i + \beta_i + 1}{2} - i \right) \\ &= -c \sum_{i \geq 1} \left(\sum_{j=1}^{i} (\alpha_j - \beta_j) \right) \left(\frac{\alpha_i - \alpha_{i+1} + \beta_i - \beta_{i+1}}{2} + 1 \right) \\ &< 0, \end{aligned}$$

if $\alpha > \beta$. Here we have used that $\sum_{j=1}^{i}(\alpha_j - \beta_j) = 0$ for i sufficiently large. This proves Claim 1. From this we will deduce

Claim 2: the lowest order term in $Ch[Ind_b(L(\tau_\lambda) \otimes M(\tau_\mu))]$ is $[\tau_{\lambda+\mu}]t^{h(\tau_{\lambda+\mu})}$.

We prove Claim 2 by descending induction on $h(\lambda)$; that is, suppose it holds for all pairs (ν, μ) with $h(\nu) > h(\lambda)$. Of course this assumption is vacuous when $h(\lambda)$ is maximal, so we need not prove the base case. We have

$$[M(\tau_\lambda)] = [L(\tau_\lambda)] + \sum_{h(\nu) > h(\lambda)} d_\lambda^\nu [L(\tau_\nu)]$$

for some nonnegative integers d_λ^ν. Tensoring by $M(\tau_\mu)$ and applying Ind_b and Ch, we obtain

$$\begin{aligned} &Ch[Ind_b(M(\tau_\lambda) \otimes M(\tau_\mu))] \\ &\qquad = Ch[Ind_b(L(\tau_\lambda) \otimes M(\tau_\mu))] + \sum_{h(\nu) > h(\lambda)} d_\lambda^\nu Ch[Ind_b(L(\tau_\nu) \otimes M(\tau_\mu))]. \end{aligned}$$

By Claim 1, the lowest order term in this expression is $[\tau_{\lambda+\mu}]t^{h(\tau_{\lambda+\mu})}$. However, each summand on the right has nonnegative integer coefficients. Therefore this must be the lowest order term of one of the summands. For $h(\nu) > h(\lambda)$, the lowest order term of $Ch[Ind_b(L(\tau_\nu) \otimes M(\tau_\mu))]$ is $[\tau_{\nu+\mu}]t^{h(\tau_{\nu+\mu})}$ by induction. Clearly if $\nu \neq \lambda$ then

$\nu + \mu \neq \lambda + \mu$, so the term $[\tau_{\lambda+\mu}]t^{h(\tau_{\lambda+\mu})}$ can only come from $Ch[Ind_b(L(\tau_\lambda) \otimes M(\tau_\mu))]$, as required. We now conclude

Claim 3: the lowest order term in $Ch[Ind_b(L(\tau_\lambda) \otimes L(\tau_\mu))]$ is $[\tau_{\lambda+\mu}]t^{h(\tau_{\lambda+\mu})}$.

Indeed this follows from Claim 2 using the same argument by which Claim 2 followed from Claim 1. This proves that the $h(\tau_{\lambda+\mu})$-eigenspace of **eu** in $Ind_b(L(\tau_\lambda) \otimes L(\tau_\mu))$ is isomorphic to $\tau_{\lambda+\mu}$ as an S_{n+m}-module, and is killed by $\mathfrak{h} \subseteq H_c(S_{n+m}, \mathfrak{h})$. We therefore have a nonzero $\mathbb{C}[S_{n+m}] \ltimes \mathbb{C}[\mathfrak{h}^*]$-module homomorphism

$$\tau_{\lambda+\mu} \to Ind_b(L(\tau_\lambda) \otimes L(\tau_\mu)),$$

where $\mathfrak{h}$ is defined to act as 0 on $\tau_{\lambda+\mu}$. By definition of the Verma module, we obtain a nonzero map

$$M(\tau_{\lambda+\mu}) \to Ind_b(L(\tau_\lambda) \otimes L(\tau_\mu)),$$

as required. □

Now suppose $c = \frac{r}{m}$ where r is coprime with m. We say λ is *m-regular* if no part of λ appears m or more times. Any partition $\lambda \vdash n$ can be written uniquely as $\lambda = m\mu + \nu$ such that ν' is m-regular. Let $q_m(\lambda) = |\mu|$. More explicitly

$$q_m(\lambda) = \sum_{i \geq 1} i \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{m} \right\rfloor .$$

We will eventually show, in Theorem 1.6, that the support of $L(\tau_\lambda)$ is $X_{q_m(\lambda)}$ for $c > 0$ and $X_{q_m(\lambda')}$ for $c < 0$. For the moment we prove one direction:

Theorem 3.8. *With $c = \frac{r}{m}$ and $\mathfrak{h} = \mathbb{C}^n$ as above, the support of the H_c-module $L(\tau_\lambda)$ is contained in $X_{q_m(\lambda)}$ if $c > 0$, and contained in $X_{q_m(\lambda')}$ if $c < 0$.*

Proof. We denote X_q by X_q^n throughout this proof, as we will be considering support sets for other values of n. Suppose $c > 0$.

Let $q = q_m(\lambda)$, and write $\lambda = m\mu + \nu$ for some partitions $\mu \vdash q$ and $\nu \vdash n - qm$, such that $q_m(\nu) = 0$. By Proposition 9.7 and Theorem 9.8 of [5], the support of $L(\tau_{m\mu})$ is contained in X_q^{qm}. Thus the support of $L(\tau_{m\mu}) \otimes L(\tau_\nu)$ is contained in $X_q^{qm} \times \mathbb{C}^{n-qm} \subseteq X_q^n$. By Theorem 3.3(3), the same is true of $Ind_b(L(\tau_{m\mu}) \otimes L(\tau_\nu))$, where $b \in \mathbb{C}^n$ is a point whose stabiliser in S_n is $S_{qm} \times S_{n-qm}$. By Lemma 3.7, we have a nonzero map

$$\phi : M(\tau_\lambda) \to Ind_b(L(\tau_{m\mu}) \otimes L(\tau_\nu)).$$

Now $L(\tau_\lambda)$ is the only irreducible quotient of $M(\tau_\lambda)$, so the image of ϕ admits $L(\tau_\lambda)$ as a quotient. Thus $L(\tau_\lambda)$ is a subquotient of $Ind_b(L(\tau_{m\mu}) \otimes L(\tau_\nu))$, so its support must be contained in X_q^n. This proves the $c > 0$ case.

There is an automorphism of $\mathbb{C}[W]$ sending $s \in S$ to $-s$. Twisting by this automorphism sends τ_λ to $\tau_{\lambda'}$. Moreover it extends to an isomorphism $H_c(W, \mathfrak{h}) \to H_{-c}(W, \mathfrak{h})$, which is the identity on $\mathfrak{h}$ and $\mathfrak{h}^*$. Therefore the statement for $c < 0$ follows from that for $c > 0$. □

Finally we will require the following classification of two-sided ideals in $H_c(W, \mathfrak{h})$, due to Losev.

Theorem 3.9 ([16] Theorem 4.3.1 and [11] Theorem 5.10)**.** *There are $\lfloor n/m \rfloor + 1$ proper two-sided ideals of $H_c(W, \mathfrak{h})$,*

$$0 = J_0 \subsetneq J_1 \subsetneq \ldots \subsetneq J_{\lfloor n/m \rfloor} \subsetneq H_c(W, \mathfrak{h}).$$

Moreover if $c > 0$ then the polynomial representation admits a filtration

$$0 = I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_{\lfloor n/m \rfloor} \subsetneq \mathbb{C}[\mathfrak{h}]$$

such that $Z(I_q) = X_q$ and $\mathrm{Ann}_{H_c(W,\mathfrak{h})}(\mathbb{C}[\mathfrak{h}]/I_q) = J_q$. Each I_q is radical if and only if $c = \frac{1}{m}$.

4. Minimal Support for Type A

In this section we consider the algebra $H_c = H_{\frac{1}{m}}(S_n, \mathbb{C}^n)$, and study modules in $H_c-\text{mod}_{RS}$ whose support is the smallest possible set, namely X_q where $q = \lfloor n/m \rfloor$. In particular, we will show that the full subcategory of such modules is semisimple. We begin with a general lemma concerning the localisation functor for linear actions.

Lemma 4.1. *Consider a finite group W acting linearly on a finite dimensional vector space $\mathfrak{h}$. Suppose $\alpha \in \mathbb{C}[\mathfrak{h}]^W$ is a symmetric polynomial, and let $U \subseteq \mathfrak{h}$ denote the open set on which α is nonzero. Moreover suppose no nonzero module in $H_c(W, \mathfrak{h})-\text{mod}_{RS}$ is supported on the zero set of α. Then the localisation functor*

$$L : H_c(W, \mathfrak{h})-\text{mod}_{coh} \to H_c(W, U)-\text{mod}_{coh}$$

identifies $H_c(W, \mathfrak{h})-\text{mod}_{RS}$ with a full subcategory of $H_c(W, U)-\text{mod}_{coh}$ closed under subquotients.

Proof. Let $\mathcal{A}$ denote the full subcategory of $H_c(W, U)-\text{mod}_{coh}$ consisting of modules M such that every $m \in M$ is killed by $\mathfrak{h}^n \alpha^n$ for some n. This is clearly closed under subquotients.

Certainly L is exact, since $\mathbb{C}[\mathfrak{h}][\alpha^{-1}]$ is flat over $\mathbb{C}[\mathfrak{h}]$, and its image lies in $\mathcal{A}$. Conversely suppose $M \in \mathcal{A}$, and let $V \subseteq M$ be a finite dimensional subspace generating M over $\mathbb{C}[\mathfrak{h}][\alpha^{-1}]$. Multiplying V by some power of α, we may suppose that $\mathfrak{h}$ acts nilpotently on V. Since $H_c(W, \mathfrak{h}) = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]$, the $H_c(W, \mathfrak{h})$-submodule N of M generated by V is finitely generated over $\mathbb{C}[\mathfrak{h}]$, and $\mathfrak{h}$ acts locally nilpotently on N. Thus $N \in H_c(W, \mathfrak{h})-\text{mod}_{RS}$. Moreover M is generated by N over $\mathbb{C}[\mathfrak{h}][\alpha^{-1}]$, so $M \cong L(N)$ is in the image of the localisation functor. Therefore $\mathcal{A}$ is exactly the image of L.

Define the functor $E : H_c(W, U)-\text{mod} \to H_c(W, \mathfrak{h})-\text{mod}$ sending M to the subspace $E(M) \subseteq M$ on which $\mathfrak{h}$ acts nilpotently. Note that $E(M)$ is stable under the action of $H_c(W, \mathfrak{h})$. We will show that if $M \in H_c(W, \mathfrak{h})-\text{mod}_{RS}$, then $EL(M)$ is naturally isomorphic to M. The kernel of the natural map $M \to L(M)$ is exactly the submodule $\Gamma_{Z(\alpha)}(M)$ defined in Theorem 1.4(1). This is zero since we have assumed no modules are supported on $Z(\alpha)$. We may therefore identify M with a submodule of $EL(M)$. Consider any $v \in EL(M)$. As above, the $H_c(W, \mathfrak{h})$-submodule N of $L(M)$ generated by v is in $H_c(W, \mathfrak{h})-\text{mod}_{RS}$. Thus $M' = M + N \subseteq L(M)$ is also in $H_c(W, \mathfrak{h})-\text{mod}_{RS}$, and $L(M') \cong L(M)$. Using the exactness of localisation, we conclude that $L(M'/M) = 0$, so that M'/M is supported on $Z(\alpha)$. Again this implies that $M'/M = 0$, so $v \in M$. Thus M is exactly $EL(M)$. We have shown above that the localisation functor $L : H_c(W, \mathfrak{h})-\text{mod}_{RS} \to \mathcal{A}$ is essentially surjective, so E and L are mutually inverse functors. □

We will also need the following simple algebraic geometry lemma.

Lemma 4.2. *Suppose U is an open subset (not necessarily affine) of an affine integral Noetherian scheme* Spec A*, and $\mathcal{M}$ is a torsion free coherent sheaf on U. Then $\Gamma(U, \mathcal{M})$ is finitely generated over $\mathcal{O}(U)$.*

Proof. By Exercise II.5.15 of [15], there is a finitely generated A-module N such that $\mathcal{M} \cong Sh(N)|_U$. We may replace N by its image in $\Gamma(U, \mathcal{M})$. Since $A \to \Gamma(U, \mathcal{O}_U)$ is injective, N will then be torsion free. Therefore the natural map $N \to K \otimes_A N$ is injective, where K is the field of fractions of A. Since N is finitely generated, $K \otimes_A N$ is finite dimensional, with some basis $\{x_1, \ldots, x_n\}$. Dividing this basis by some nonzero element of A, we may suppose that the free A-module $F = Ax_1 \oplus \ldots \oplus Ax_n$ contains N. Hence

$$\Gamma(U, \mathcal{M}) = \Gamma(U, Sh(N)) \hookrightarrow \Gamma(U, Sh(F)) = \Gamma(U, \mathcal{O}_U)^{\oplus n}.$$

Since $\Gamma(U, \mathcal{O}_U)$ is Noetherian, the result follows. □

Finally we require the following well-known result.

Lemma 4.3. *Suppose a unital associative algebra H contains an idempotent e such that $HeH = H$. Then the functors*

$$H\text{−mod} \to eHe\text{−mod}, \qquad M \mapsto eM$$

and

$$eHe\text{−mod} \to H\text{−mod}, \qquad N \mapsto He \otimes_{eHe} N$$

are mutually inverse equivalences.

Now consider the algebra $H_c = H_{\frac{1}{m}}(S_n, \mathbb{C}^n)$. Let

$$\mathfrak{h} = \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n],$$

where S_n permutes the x_i. Let $q = \lfloor n/m \rfloor$ and write $n = qm + p$, so that $0 \le p < m$. Let $\mathfrak{h}' = \mathbb{C}^{q+p}$, with coordinate ring $\mathbb{C}[\mathfrak{h}'] = \mathbb{C}[z_1, \ldots, z_q,\, t_1, \ldots, t_p]$, on which $S_q \times S_p$ acts naturally. Define

$$\pi : \mathbb{C}[\mathfrak{h}] \twoheadrightarrow \mathbb{C}[\mathfrak{h}']$$

by

$$\pi(x_i) = \begin{cases} z_{\lceil i/m \rceil} & \text{if } i \le qm \\ t_{i-qm} & \text{if } i > qm. \end{cases}$$

This identifies $\mathfrak{h}'$ with one of the components of $X_q \subseteq \mathfrak{h}$. The map π restricts to a map $\pi : \mathbb{C}[\mathfrak{h}]^{S_n} \to \mathbb{C}[\mathfrak{h}']^{S_q \times S_p}$. Unfortunately this is not surjective if $p > 0$. We therefore localise as follows. Let $[n] = \{1, 2, \ldots, n\}$, and let

$$\alpha_d = \sum_{\substack{J \subseteq [n] \\ |J| = p}} \left(\sum_{j \in J} x_j^d \right) \prod_{j \in J,\, i \notin J} (x_i - x_j).$$

Clearly α_d is symmetric. Moreover using that $p < m$, it can be shown that

$$\pi(\alpha_d) = \left(\sum_{j=1}^{r} t_j^d \right) \prod_{\substack{1 \le i \le q \\ 1 \le j \le p}} (z_i - t_j)^m.$$

Let $U \subseteq \mathfrak{h}$ and $U' \subseteq \mathfrak{h}'$ denote the affine open subsets on which α_0 and $\pi(\alpha_0)$ are nonzero, respectively. Let $A = \mathcal{O}(U)^{S_n}$ and $B = \mathcal{O}(U')^{S_q \times S_p}$, and let $\phi : A \to B$ be the map induced by π. Now B is generated as a $\mathbb{C}$-algebra by $\pi(\alpha_0)^{-1}$, $\sum_{j=1}^p t_j^d$ and $\sum_{i=1}^q z_i^d$. We have

$$\sum_{j=1}^{p} t_j^d = \phi\left(p \frac{\alpha_d}{\alpha_0} \right) \qquad \text{and} \qquad \sum_{i=1}^{q} z_i^d = \phi\left(\frac{1}{m} \sum_{i=1}^{n} x_i^d - \frac{p}{m} \frac{\alpha_d}{\alpha_0} \right),$$

so ϕ is surjective. Moreover since X_q is the union of the S_n translates of $\mathfrak{h}'$, the kernel of ϕ is $(\mathcal{O}(U) I_q)^{S_n}$, where I_q is the ideal vanishing on X_q.

Now consider the idempotents

$$\begin{aligned} e &= \frac{1}{n!} \sum_{w \in S_n} w \in \mathbb{C}[S_n] \subseteq H_c(S_n, \mathfrak{h}) \subseteq H_c(S_n, U), \\ e' &= \frac{1}{q!p!} \sum_{w \in S_q \times S_p} w \in \mathbb{C}[S_q \times S_p] \subseteq H_{(m,c)}(S_q \times S_p, \mathfrak{h}') \subseteq H_{(m,c)}(S_q \times S_p, U), \end{aligned}$$

where (m, c) indicates the function which takes the value m on transpositions in S_q and c on those in S_p. For convenience, we omit the subscripts c and (c, m) for the rest of this section. It is known (see Corollary 4.2 of [4]) that $H(S_n, \mathfrak{h}) e H(S_n, \mathfrak{h}) = H(S_n, \mathfrak{h})$, so that $H(S_n, U) e H(S_n, U) = H(S_n, U)$, and $M \mapsto eM$ is an equivalence of categories from $H(S_n,U)$-mod to $eH(S_n,U)e$-mod by Lemma 4.3. Similar results hold for e'. In particular the faithful action of $H(S_q\times S_p,U')$ on $\mathcal{O}(U')$ gives a faithful action of $e'H(S_q\times S_p,U')e'$ on B. Also $\mathcal{O}(U)I_q$ is an $H(S_n,U)$-submodule of $\mathcal{O}(U)$ by Theorem 5.10 of [11], so $eH(S_n,U)e$ acts on $A/\ker\phi\cong B$.

Proposition 4.4. *The image of the homomorphism $\sigma : eH(S_n,U)e\to \mathrm{End}_{\mathbb{C}}(B)$ describing the above action is exactly $e'H(S_q\times S_p,U')e'\subseteq \mathrm{End}_{\mathbb{C}}(B)$, and the kernel is generated by eI_qe.*

Proof. Let $\{x_a^\vee\}$ and $\{z_i^\vee,t_j^\vee\}$ be the bases of $\mathfrak{h}$ and $\mathfrak{h}'$ dual to $\{x_a\}$ and $\{z_i,t_j\}$. For $d\geq 0$, let

$$p_x(d)=\sum_{a=1}^n x_a^d,$$

and similarly for p_t, p_z, $p_{x^\vee}$, $p_{z^\vee}$, $p_{t^\vee}$. It is known that $eH(S_n,\mathfrak{h})e$ is generated as an algebra by $e\mathbb{C}[\mathfrak{h}]e$ and $p_{x^\vee}(2)$ (see the proof of Proposition 4.9 of [10], and Corollary 4.9 of [2]). It follows that $eH(S_n,U)e$ is generated by Ae and $p_{x^\vee}(2)e$, and that

$$e'H(S_q\times S_p,U')e'$$

is generated by Be', $p_{z^\vee}(2)e'$ and $p_{t^\vee}(2)e'$. For f, $g\in A$, we have

$$\sigma(fe)\phi(g)=\phi(fg)=\phi(f)\phi(g),$$

so the image under σ of $Ae\subseteq eH(S_n,U)e$ is $Be'\subseteq e'H(S_q\times S_p,U')e'$. Next, recalling that $\phi:A\to B$ is the restriction of $\pi:\mathcal{O}(U)\to\mathcal{O}(U')$, we show that

$$\pi(\partial_{x_a^\vee}f)=\begin{cases}\frac{1}{m}\partial_{z_i^\vee}\phi(f) & \text{if } \pi(x_a)=z_i,\\ \partial_{t_j^\vee}\phi(f) & \text{if } \pi(x_a)=t_j\end{cases}\qquad \text{for } f\in A. \tag{1}$$

Indeed, this holds when $f=p_x(d)$, since

$$\frac{1}{m}\partial_{z_i^\vee}\phi(p_x(d))=\frac{1}{m}\partial_{z_i^\vee}(mp_z(d)+p_t(d))=dz_i^{d-1}$$

and

$$\partial_{t_j^\vee}\phi(p_x(d))=\partial_{t_j^\vee}(mp_z(d)+p_t(d))=dt_j^{d-1}.$$

Now as functions of f, both sides of (1) are $\mathbb{C}$-derivations from A to $\mathcal{O}(U')$. Therefore (1) holds on the subring generated by the $p_x(d)$, namely $\mathbb{C}[\mathfrak{h}]^{S_n}$, whence it holds on $\mathbb{C}[\mathfrak{h}]^{S_n}[\alpha_0^{-1}]=A$. Now for $f\in A$ we have

$$\begin{aligned}p_{x^\vee}(2)f &= \sum_{a=1}^n x_a^\vee\partial_{x_a^\vee}f\\ &= \sum_{a=1}^n\left(\partial^2_{x_a^\vee}f-c\sum_{b\neq a}\frac{\partial_{x_a^\vee}f-s_{ab}\partial_{x_a^\vee}f}{x_a-x_b}\right)\\ &= \triangle_x f-c\sum_{a\neq b}\frac{\partial_{x_a^\vee}f-\partial_{x_b^\vee}f}{x_a-x_b},\end{aligned}$$

where $\triangle_x$ denotes the Laplacian in the variables x_a. Similar formulae hold for the actions of $p_{z^\vee}(2)$ and $p_{t^\vee}(2)$ on B. Define $P\in e'H(S_q\times S_p,U')e'$ by

$$P=\frac{1}{m}p_{z^\vee}(2)e'+p_{t^\vee}(2)e'-2\sum_{\substack{1\leq i\leq q\\ 1\leq j\leq p}}\frac{1}{z_i-t_j}\left(\frac{1}{m}z_i^\vee-t_j^\vee\right)e'.$$

Note that

$$\sum_{\substack{1\leq i\leq q\\ 1\leq j\leq p}}\frac{1}{z_i-t_j}\left(\frac{1}{m}z_i^\vee-t_j^\vee\right)$$

is symmetric under $S_q \times S_p$, so final sum is an element of $e'H(S_q \times S_p, U')e'$, though the individual summands are not. We claim that

$$\phi(p_{x^\vee}(2)f) - P\phi(f) = 0$$

for $f \in A$. We first show that the left hand side is a derivation. Indeed

$$\begin{aligned}\phi(p_{x^\vee}(2)(fg)) &= \phi\left(fp_{x^\vee}(2)g + gp_{x^\vee}(2)f + \sum_a (\partial_{x_a^\vee} f)(\partial_{x_a^\vee} g)\right) \\ &= \phi(f)\phi(p_{x^\vee}(2)g) + \phi(g)\phi(p_{x^\vee}(2)f) \\ &\quad + \sum_i \frac{1}{m}(\partial_{z_i^\vee}\phi(f))(\partial_{z_i^\vee}\phi(g)) + \sum_j (\partial_{t_j^\vee}\phi(f))(\partial_{t_j^\vee}\phi(g)),\end{aligned}$$

using (1). Also

$$P(fg) = fP(g) + gP(f) + \sum_i \frac{1}{m}(\partial_{z_i^\vee} f)(\partial_{z_i^\vee} g) + \sum_j (\partial_{t_j^\vee} f)(\partial_{t_j^\vee} g). \tag{2}$$

Subtracting, we see that $\phi(p_{x^\vee}(2)f) - P\phi(f)$ is indeed a derivation in f. By the same argument as above, it suffices to check the equation when $f = p_x(d)$. We calculate

$$\phi(p_{x^\vee}(2)p_x(d)) = \phi\left(d(d-1)p_x(d-2) - cd\sum_{a\neq b}\frac{x_a^{d-1} - x_b^{d-1}}{x_a - x_b}\right).$$

If $\pi(x_a) = \pi(x_b) = z_i$, then

$$\pi\left(\frac{x_a^{d-1} - x_b^{d-1}}{x_a - x_b}\right) = \pi\left(\sum_{i=0}^{d-2} x_a^i x_b^{d-2-i}\right) = (d-1)z_i^{d-2}.$$

Therefore recalling that $c = \frac{1}{m}$,

$$\begin{aligned}&\phi(p_{x^\vee}(2)p_x(d)) \\ &= d(d-1)(mp_z(d-2) + p_t(d-2)) - cd(d-1)m(m-1)p_z(d-2) \\ &\quad - cdm^2\sum_{i\neq i'}\frac{z_i^{d-1} - z_{i'}^{d-1}}{z_i - z_{i'}} - 2cdm\sum_{i,j}\frac{z_i^{d-1} - t_j^{d-1}}{z_i - t_j} - cd\sum_{j\neq j'}\frac{t_j^{d-1} - t_{j'}^{d-1}}{t_j - t_{j'}} \\ &= \left[d(d-1)p_z(d-2) - md\sum_{i\neq i'}\frac{z_i^{d-1} - z_{i'}^{d-1}}{z_i - z_{i'}}\right] \\ &\quad + \left[d(d-1)p_t(d-2) - cd\sum_{j\neq j'}\frac{t_j^{d-1} - t_{j'}^{d-1}}{t_j - t_{j'}}\right] - 2d\sum_{i,j}\frac{z_i^{d-1} - t_j^{d-1}}{z_i - t_j} \\ &= p_{z^\vee}(2)p_z(d) + p_{t^\vee}(2)p_t(d) - 2\sum_{i,j}\frac{\partial_{z_i^\vee}p_z(d) - \partial_{t_j^\vee}p_t(d)}{z_i - t_j} \\ &= P(mp_z(d) + p_t(d)) \\ &= P\phi(p_x(d)),\end{aligned}$$

as required. Therefore $\operatorname{Im}\sigma$ is the subalgebra of $\operatorname{End}_{\mathbb{C}}(B)$ generated by Be' and P. Now (2) shows that for $fe' \in Be' \subseteq e'H(S_q \times S_p, U')e'$, we have

$$[P, fe'] = P(f)e' + \sum_i \frac{1}{m}(\partial_{z_i^\vee} f)z_i^\vee e' + \sum_j (\partial_{t_j^\vee} f)t_j^\vee e'.$$

In particular,

$$[P, \pi(\alpha_0)e'] = P(\pi(\alpha_0))e' + \sum_{\substack{1\le i\le q\\1\le j\le p}} \frac{m\pi(\alpha_0)}{z_i - t_j}\left(\frac{1}{m}z_i^\vee - t_j^\vee\right)e'.$$

Therefore $\operatorname{Im}\sigma$ contains

$$P + \frac{2}{m\pi(\alpha_0)}\left([P, \pi(\alpha_0)e'] - P(\pi(\alpha_0))e'\right) = \frac{1}{m}p_{z^\vee}(2)e' + p_{t^\vee}(2)e'.$$

We also have

$$[P, p_z(2)e'] = P(p_z(2))e' + \frac{2}{m}\sum_i z_i z_i^\vee e',$$

so $\operatorname{Im}\sigma$ contains

$$\left[\frac{1}{m}p_{z^\vee}(2)e' + p_{t^\vee}(2)e', \sum_i z_i z_i^\vee e'\right] = \frac{2}{m}p_{z^\vee}(2)e'.$$

Thus $\operatorname{Im}\sigma$ contains Be', $p_{z^\vee}(2)e'$ and $p_{t^\vee}(2)e'$, so it is exactly $e'H(S_q \times S_p, U')e'$.

Finally since $H(S_n, U)eH(S_n, U) = H(S_n, U)$, the two-sided ideals of the algebra $eH(S_n, U)e$ are in one to one correspondence with those in $H(S_n, U)$. The latter are determined by their inverse image in $H(S_n, \mathfrak{h})$. Clearly the kernel of σ is proper and contains eI_qe, so by Theorem 3.9 it is generated by eI_qe. □

We have identified $\mathfrak{h}'$ with a subspace of $\mathfrak{h}$, and the stabiliser of a generic point of $\mathfrak{h}'$ in S_n is a natural copy of $S_m^q \subseteq S_n$. Let $\mathfrak{h}'_{\mathrm{reg}}$ denote the elements of $\mathfrak{h}'$ with exactly this stabiliser. Note that $\mathfrak{h}'_{\mathrm{reg}}$ is contained in U, since the zero set of

$$\pi(\alpha_0) = p \prod_{\substack{1\le i\le q\\1\le j\le p}} (z_i - t_j)^m$$

consists of elements whose stabiliser in S_n is at least as large as $S_{m+1} \times S_m^{q-1}$. Also note that the normaliser subgroup $N_{S_n}(S_m^q) \subseteq S_n$ acts on $\mathfrak{h}'_{\mathrm{reg}}$. Each coset of S_m^q in $N_{S_n}(S_m^q)$ has a unique representative of minimal length, and these representatives form a subgroup isomorphic to $S_q \times S_p$. The induced action of $S_q \times S_p$ on $\mathfrak{h}' = \operatorname{Spec}\mathbb{C}[z_1, \dots, z_q, t_1, \dots, t_p]$ is the natural one. We now use the homomorphism σ to analyse the subcategory of $H(S_n, \mathfrak{h})-\mathrm{mod}_{RS}^q$ of minimally supported modules. Some of this information is generalised by Theorems 1.8 and 1.6.

Theorem 4.5. *Let $H(S_n, \mathfrak{h})-\mathrm{mod}_{RS}^q$ be the full subcategory of $H(S_n, \mathfrak{h})-\mathrm{mod}_{RS}$, consisting of modules supported on X_q. There is an equivalence of categories*

$$F : H(S_n, \mathfrak{h})-\mathrm{mod}_{RS}^q \to H(S_q \times S_p, \mathfrak{h}')-\mathrm{mod}_{RS},$$

such that $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} M \cong \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}']} F(M)$ as $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$-modules. In particular, $H(S_n, \mathfrak{h})-\mathrm{mod}_{RS}^q$ is semisimple, and its irreducibles are exactly the $L(\tau_\lambda)$ for which $q_m(\lambda) = q$.

Proof. Let $eH(S_n, U)e-\mathrm{mod}_{coh}$ denote the full subcategory of $eH(S_n, U)e-\mathrm{mod}$ consisting of modules finitely generated over $e\mathbb{C}[\mathfrak{h}]e$, and similarly for $H(S_q \times S_p, U')$. Recall the equivalences $H(S_n, U)-\mathrm{mod}_{coh} \xrightarrow{\sim} eH(S_n, U)e-\mathrm{mod}_{coh}$ and $H(S_q \times S_p, U')-\mathrm{mod}_{coh} \xrightarrow{\sim} e'H(S_q \times S_p, U')e'-\mathrm{mod}_{coh}$; we denote the functors by e and e'. By Lemma 4.1, we also have localisation functors $L : H(S_n, \mathfrak{h})-\mathrm{mod}_{RS} \to H(S_n, U)-\mathrm{mod}_{coh}$ and $L' : H(S_q \times S_p, \mathfrak{h}')-\mathrm{mod}_{RS} \to H(S_q \times S_p, U')-\mathrm{mod}_{coh}$ which identify their domains with full subcategories of their codomains closed under subquotients. Finally the previous proposition gives a functor σ^* identifying $e'H(S_q \times S_p, U')e'-\mathrm{mod}_{coh}$ with a full subcategory of $eH(S_n,U)e-\mathrm{mod}_{coh}$, again closed under subquotients.

$$\begin{array}{ccccc} H(S_n,\mathfrak{h})-\mathrm{mod}_{RS} & \xhookrightarrow{\quad L\quad} & H(S_n,U)-\mathrm{mod}_{coh} & \xrightarrow[\sim]{e} & e'H(S_q\times S_p,U')e'-\mathrm{mod}_{coh} \\ &&&& \uparrow{\scriptstyle\sigma^*} \\ H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS} & \xhookrightarrow{L'} & H(S_q\times S_p,U')-\mathrm{mod}_{coh} & \xrightarrow[\sim]{e'} & e'H(S_q\times S_p,U')e'-\mathrm{mod}_{coh}. \end{array}$$

Note that these functors are all exact. We make the following claims about them:

Claim 1: If $M\in H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$, then the $\mathcal{O}$-coherent $\mathcal{D}$-module $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}M$ on $\mathfrak{h}'_{\mathrm{reg}}$ has regular singularities and trivial monodromy around each irreducible component of $Z(\alpha_0)\cap\mathfrak{h}'$.

Claim 2: If $N\in H(S_q\times S_p,U')-\mathrm{mod}_{coh}$ is such that the $\mathcal{O}$-coherent $\mathcal{D}$-module $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U')}N$ on $\mathfrak{h}'_{\mathrm{reg}}$ has regular singularities and trivial monodromy around each component of $Z(\alpha_0)\cap\mathfrak{h}'$, then $N\cong L'(N')$ for some $N'\in H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS}$.

Claim 3: If $M\in H(S_n,U)-\mathrm{mod}_{coh}$ and $N\in H(S_q\times S_p,U')-\mathrm{mod}_{coh}$ are such that $eM\cong\sigma^*e'N$, then the $S_q\times S_p$-equivariant $\mathcal{D}$-modules $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U)}M$ and $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U')}N$ on $\mathfrak{h}'_{\mathrm{reg}}$ are isomorphic.

Let us first see how these results imply the statements of the theorem. The composites eL and $\sigma^*e'L'$ identify $H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$ and $H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS}$ with full subcategories of $eH(S_n,U)e-\mathrm{mod}_{coh}$ closed under subquotients, which we call the images of eL and $\sigma^*e'L'$. Now consider any $M\in H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$. Theorem 3.9 shows that I_q kills M, so $eL(M)$ is killed by eI_qe, and therefore by the kernel of σ. Thus $eL(M)\cong\sigma^*e'N$ for some $N\in H(S_q\times S_p,U')-\mathrm{mod}_{coh}$. By claim 3,

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U')}N\cong\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U)}L(M)\cong\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}M$$

as $D(\mathfrak{h}'_{\mathrm{reg}})$-modules. By claim 1, this $\mathcal{D}$-module has regular singularities and trivial monodromy around each component of $Z(\alpha_0)\cap\mathfrak{h}'$. Therefore by claim 2, $N\cong L'(N')$ for some $N'\in H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS}$, so that $eL(M)\cong\sigma^*e'L'(N')$. This proves that the image of eL is contained in that of $\sigma^*e'L'$, so there is an exact fully faithful embedding $F:H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}\to H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS}$ such that $eL=\sigma^*e'L'F$, and whose image is closed under subquotients. Now $H(S_q\times S_p,\mathfrak{h}')-\mathrm{mod}_{RS}$ is semisimple and has $\mathbf{p}_q\mathbf{p}_p$ (isomorphism classes of) irreducibles, where $\mathbf{p}_n$ is the number of partitions of n. On the other hand, by Theorem 3.8, the $\mathbf{p}_q\mathbf{p}_p$ distinct irreducibles

$$\{L(\tau_{m\lambda+\nu})\mid\lambda\vdash q,\,\nu\vdash p\}$$

lie in $H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$. It follows that F is an equivalence of categories, and that the above are all of the irreducibles in $H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$. Finally for $M\in H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$, we have $eL(M)\cong\sigma^*e'L'F(M)$, so claim 3 implies

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}M=\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U)}L(M)\cong\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathcal{O}(U')}L'F(M)=\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}']}F(M)$$

as $S_q\times S_p$-equivariant $\mathcal{D}$-modules on $\mathfrak{h}'_{\mathrm{reg}}$. It remains to prove the three claims.

Proof of claim 1: Fix $M\in H(S_n,\mathfrak{h})-\mathrm{mod}^q_{RS}$. By assumption, $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}M$ has regular singularities. Let $b\in\mathfrak{h}'$ be a "generic" point of $Z(\alpha_0)\cap\mathfrak{h}'$, that is, a point whose stabiliser is $W'\cong S_{m+1}\times S_m^{q-1}$. By Proposition 3.4, it suffices to show that the $H(W',\mathfrak{h})$-module $N=Res_bM$ is such that $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}N$ has trivial monodromy around b. This depends only on the action of $H(S_{m+1},\mathbb{C}^m)\subseteq H(W',\mathfrak{h})$, so we may suppose $n=m+1$. But N has minimal support by Theorem 3.3(2). The proof of Corollary 4.7 of [4] shows that the only irreducible in $H_c(S_{m+1},\mathbb{C}^m)-\mathrm{mod}_{RS}$ with minimal support is $L(\mathbb{C})$, where $\mathbb{C}$ denotes the trivial representation of S_{m+1}. Moreover Lemma 2.9 of [14] shows that $\mathrm{Ext}^1(L(\mathbb{C}),L(\mathbb{C}))=0$, so N is a direct sum of copies of $L(\mathbb{C})$. Thus $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\otimes_{\mathbb{C}[\mathfrak{h}]}N$ is a free module with trivial connection, and in particular has trivial monodromy.

Proof of claim 2: Suppose $N\in H(S_q\times S_p,U')-\mathrm{mod}_{coh}$ has the given properties. Let $U''\subseteq\mathfrak{h}'$ denote the subset of points with trivial stabiliser in $S_q\times S_p$, so that

$\mathfrak{h}'_{\mathrm{reg}} = U' \cap U''$. Then $Sh(N)|_{\mathfrak{h}'_{\mathrm{reg}}}$ is an $\mathcal{O}$-coherent $\mathcal{D}$-module on $\mathfrak{h}'_{\mathrm{reg}}$ with regular singularities and trivial monodromy around $Z(\pi(\alpha_0))$, so it extends to an $\mathcal{O}$-coherent $\mathcal{D}$-module $\mathcal{N}''$ on U'' with regular singularities. We may glue $Sh(N)$ and $\mathcal{N}''$ to obtain a coherent sheaf $\mathcal{N}_1$ on $U' \cup U''$. Note that the complement of $U' \cup U''$ in $\mathfrak{h}'$ has codimension 2, so $\Gamma(U' \cup U'', \mathcal{O}_{\mathfrak{h}'}) = \mathbb{C}[\mathfrak{h}']$. Also $\mathcal{N}''$ is torsion free since it is an $\mathcal{O}$-coherent $\mathcal{D}$-module. Therefore $\mathcal{N}_1$ is torsion free, so Lemma 4.2 shows that

$$N' = \Gamma(U' \cup U'', \mathcal{N}_1)$$

is finitely generated over $\mathbb{C}[\mathfrak{h}']$. Also $H(S_q \times S_p, U'') \cong D(U'')$ acts on $\mathcal{N}''$ consistently with the action of $H(S_q \times S_p, U')$ on N, so we obtain an action of $H(S_q \times S_p, \mathfrak{h}')$ on N'. Finally since $\mathcal{N}'|_{U''} = \mathcal{N}''$ has regular singularities, we have $N' \in H(S_q \times S_p, \mathfrak{h}')-\mathrm{mod}_{RS}$ and $L'(N') = \mathcal{O}(U') \otimes_{\mathbb{C}[\mathfrak{h}']} N' = N$.

Proof of claim 3: We have a natural $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$ action on $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} H(S_q \times S_p, U')$ given by

$$w(f \otimes a) = {}^w f \otimes wa$$

and

$$\nabla_y(f \otimes a) = y(f) \otimes a + f \otimes ya - \sum_{s \in S'} c_s \langle y, \alpha_s \rangle \frac{f}{\alpha_s} \otimes (s-1)a,$$

where $S' \subseteq S_q \times S_p$ is the set of reflections. For $N \in H(S_q \times S_p, U')-\mathrm{mod}_{coh}$, the natural isomorphism

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} H(S_q \times S_p, U') \otimes_{H(S_q \times S_p, U')} N \xrightarrow{\sim} \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} N$$

preserves the equivariant $D(\mathfrak{h}'_{\mathrm{reg}})$-module structures by Proposition 1.2. By Lemma 4.3, we may write the above as

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} N \cong L' \otimes_{e'H(S_q \times S_p, U')e'} e'N,$$

where L' is the $(\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}), e'H(S_q \times S_p, U')e')$-bimodule

$$L' = \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} H(S_q \times S_p, U')e'.$$

Since $H(S_q \times S_p, U') = \mathcal{O}(U')\mathbb{C}[(\mathfrak{h}')^*]\mathbb{C}[S_q \times S_p]$, we have

$$\begin{aligned} L' &= \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} H(S_q \times S_p, U')e'H(S_q \times S_p, U')e' \\ &= \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} \mathbb{C}[(\mathfrak{h}')^*]e'H(S_q \times S_p, U')e' \\ &= \bigcup_{d \geq 0} \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} \mathbb{C}[(\mathfrak{h}')^*]^{\leq d} e'H(S_q \times S_p, U')e'. \end{aligned}$$

But

$$f \otimes ya = \nabla_y(f \otimes a) - y(f) \otimes a + \sum_{s \in S'} c_s \langle y, \alpha_s \rangle \frac{f}{\alpha_s} \otimes (s-1)a,$$

so

$$\begin{aligned} &\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} \mathbb{C}[(\mathfrak{h}')^*]^{\leq d+1} e'H(S_q \times S_p, U')e' \\ &\qquad \subseteq D(\mathfrak{h}'_{\mathrm{reg}})\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} \mathbb{C}[(\mathfrak{h}')^*]^{\leq d} e'H(S_q \times S_p, U')e'. \end{aligned}$$

Since $\mathbb{C}[(\mathfrak{h}')^*]^{\leq 0} = \mathbb{C}$, it follows that L' is generated as a bimodule by $1 \otimes e'$. Now $H(S_q \times S_p, U')$ acts faithfully on $\mathcal{O}(U')$, so we have an injection

$$H(S_q \times S_p, U')e' \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(e'\mathcal{O}(U'), \mathcal{O}(U')).$$

Note that $e'\mathcal{O}(U') = \mathcal{O}(U')^{S_q \times S_p} = B$. Also $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})$ is flat over $\mathcal{O}(U')$, so we obtain an inclusion

$$i : L' \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(B, \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})).$$

Now $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$ and $e'H(S_q \times S_p, U')e'$ act on $\mathrm{Hom}_{\mathbb{C}}(B, \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}))$ from the left and right, respectively, via their inclusions in $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}))$ and $\mathrm{End}_{\mathbb{C}}(B)$, and i is a homomorphism of bimodules. Finally $i(1 \otimes e')$ is the natural map $\phi : B \to \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})$, so L' is the sub-bimodule of $\mathrm{Hom}_{\mathbb{C}}(B, \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}))$ generated by ϕ.

Similarly $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)$ admits an action of $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$ given by

$$w(f \otimes a) = {}^w f \otimes wa$$

and

$$\nabla_y(f \otimes a) = y(f) \otimes a + f \otimes ya - \frac{1}{m} \sum_{s \in S \setminus S_m^q} \langle y, \alpha_s \rangle \frac{f}{\alpha_s} \otimes (s-1)a,$$

where $S \subseteq S_n$ is the set of reflections. We are now identifying $S_q \times S_p$ with a subgroup of S_n as discussed before the theorem. Again for $M \in H(S_n, U)-\mathrm{mod}_{coh}$, we have a $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$-module isomorphism

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} M \cong \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)e \otimes_{eH(S_n,U)e} eM.$$

Therefore if $eM \cong \sigma^* e' N$, then

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} M \cong L \otimes_{e'H(S_q \times S_p, U')e'} e'N$$

where L is the $(\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}), e'H(S_q \times S_p, U')e')$-bimodule

$$L = \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)e \otimes_{eH(S_n,U)e} e'H(S_q \times S_p, U')e'.$$

The exact sequence

$$eH(S_n, U)I_q H(S_n, U)e \hookrightarrow eH(S_n, U)e \twoheadrightarrow e'H(S_q \times S_p, U')e'$$

gives rise to a right exact sequence of right $eH(S_n, U)e$-modules

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)I_q H(S_n, U)e \to \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)e \overset{\rho}{\twoheadrightarrow} L,$$

since $H(S_n, U)eH(S_n, U) = H(S_n, U)$. As above, we have

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)e = \bigcup_{d \geq 0} \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathbb{C}[\mathfrak{h}^*]^{\leq d} eH(S_n, U)e.$$

Now $\mathfrak{h}$ is spanned by $\mathfrak{h}'$ and $x_i^\vee - x_j^\vee$, for all $i \neq j$ such that $x_i - x_j$ vanishes on $\mathfrak{h}'$. Fix such i and j, and let $g \in \mathbb{C}[\mathfrak{h}]$ vanish on all components of X_q except $\mathfrak{h}'$, such that g is nonzero on $\mathfrak{h}'_{\mathrm{reg}}$. Then ${}^{s_{ij}}f - f$ is divisible by $x_i - x_j$ for any $f \in \mathbb{C}[\mathfrak{h}]$, so $g(s_{ij} - 1) \in H(S_n, \mathfrak{h})$ sends $\mathbb{C}[\mathfrak{h}]$ into I_q. But the annihilator of $\mathbb{C}[\mathfrak{h}]/I_q$ in $H(S_n, \mathfrak{h})$ is $H(S_n, \mathfrak{h})I_q H(S_n, \mathfrak{h})$, so

$$H(S_n, U)I_q H(S_n, U) \ni [x_i^\vee, g(s_{ij} - 1)] = [x_i^\vee, g](s_{ij} - 1) + g(x_i^\vee - x_j^\vee)s_{ij}.$$

Note that $[x_i^\vee, g](s_{ij} - 1) \in \mathbb{C}[\mathfrak{h}]\mathbb{C}[S_n]$. But g is invertible in $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})$, so

$$x_i^\vee - x_j^\vee \in \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} H(S_n, U)I_q H(S_n, U) + \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})\mathbb{C}[S_n],$$

whence

$$\begin{aligned} \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathbb{C}[\mathfrak{h}^*]^{\leq d+1} eH(S_n, U)e \quad \subseteq \quad & \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathfrak{h}' \mathbb{C}[\mathfrak{h}^*]^{\leq d} eH(S_n, U)e \\ & + \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathbb{C}[\mathfrak{h}^*]^{\leq d} eH(S_n, U)e \\ & + \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} H(S_n, U)I_q H(S_n, U)e. \end{aligned}$$

Applying ρ, and using the same argument as above, we obtain

$$\rho\left(\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathbb{C}[\mathfrak{h}^*]^{\leq d+1} eH(S_n, U)e\right) \subseteq D(\mathfrak{h}'_{\mathrm{reg}})\rho\left(\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} \mathbb{C}[\mathfrak{h}^*]^{\leq d} eH(S_n, U)e\right).$$

Thus L is generated as a bimodule by $\rho(1 \otimes e)$. Finally $H(S_n, U)$ acts on $\mathcal{O}(U)/I_q\mathcal{O}(U)$ with annihilator $H(S_n, U)I_q H(S_n, U)$, so we have an inclusion

$$H(S_n, U)e / H(S_n, U)I_q H(S_n, U)e \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(e(\mathcal{O}(U)/I_q\mathcal{O}(U)), \mathcal{O}(U)/I_q\mathcal{O}(U))$$

of $(\mathcal{O}(U)/I_q\mathcal{O}(U), eH(S_n, U)e)$-bimodules. Note that $e(\mathcal{O}(U)/I_q\mathcal{O}(U)) = B$. Since $\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})$ is flat over $\mathcal{O}(U)/I_q\mathcal{O}(U)$, we obtain an inclusion

$$j : L \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(B, \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}))$$

of $(\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}), e'H(S_q \times S_p, U')e')$-bimodules. Again j preserves the left action of

$$\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}),$$

and sends $\rho(1 \otimes e)$ to ϕ, so L is the sub-bimodule of $\mathrm{Hom}_{\mathbb{C}}(B, \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}))$ generated by ϕ. In particular, $L \cong L'$ as bimodules, so

$$\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U)} M \cong L \otimes_{e'H(S_q \times S_p, U')e'} e'N \cong L' \otimes_{e'H(S_q \times S_p, U')e'} e'N \cong \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathcal{O}(U')} N$$

as $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})$-modules, as required. □

5. The Monodromy Functors for Type A

We continue to study the case $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$ and $c = \frac{1}{m}$. Previously we studied the modules in $H_c{-}\mathrm{mod}_{RS}$ supported on X_q, where $q = \lfloor n/m \rfloor$. Now let q be any integer satisfying $0 \leq qm \leq n$, and let $H_c{-}\mathrm{mod}^q_{RS}$ denote the Serre subcategory of $H_c{-}\mathrm{mod}_{RS}$ consisting of all modules supported on X_q. Our goal in this section is to construct an equivalence of categories from $H_c{-}\mathrm{mod}^q_{RS}/H_c{-}\mathrm{mod}^{q+1}_{RS}$ to the category of finite dimensional representations of a certain Hecke algebra (where $H_c{-}\mathrm{mod}^{\lfloor n/m \rfloor+1}_{RS}$ is the subcategory containing only the zero module). From this we will deduce Theorem 1.6.

Let us fix an integer q as above, and put $p = n - qm$. Consider the natural copy $S^q_m \subseteq S_n$, and put $\mathfrak{h}' = \mathfrak{h}^{S^q_m}$ and $\mathfrak{h}'_{\mathrm{reg}} = \mathfrak{h}^{S^q_m}_{\mathrm{reg}}$. As in the previous section, we may identify $\mathfrak{h}'$ with $\mathrm{Spec}\, \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p]$ via the map

$$\pi : \mathbb{C}[\mathfrak{h}] \twoheadrightarrow \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p], \qquad \pi(x_i) = \begin{cases} z_{\lceil i/m \rceil} & \text{if } i \leq qm \\ t_{i-qm} & \text{if } i > qm. \end{cases}$$

Note that $\mathfrak{h}'$ is one of the components of $X_q \subseteq \mathfrak{h}$. By Proposition 1.2, we have a functor

$$\begin{aligned} Loc^q : H_c{-}\mathrm{mod}^q_{RS} &\to \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}){-}\mathrm{mod}_{RS}, \\ M &\mapsto \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} M, \end{aligned}$$

where $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}){-}\mathrm{mod}_{RS}$ denotes the category of coherent $S_q \times S_p$-equivariant $D(\mathfrak{h}'_{\mathrm{reg}})$-modules with regular singularities.

Lemma 5.1. *The functor Loc^q is exact, and induces a faithful functor*

$$H_c{-}\mathrm{mod}^q_{RS}/H_c{-}\mathrm{mod}^{q+1}_{RS} \to \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}){-}\mathrm{mod}_{RS},$$

which we also denote by Loc^q.

Proof. Any $M \in H_c{-}\mathrm{mod}^q_{RS}$ is annihilated by some power of I_q. By Theorem 3.9, $I_qM = 0$. Let b be any point in $\mathfrak{h}'_{\mathrm{reg}}$, and let $U \subseteq \mathfrak{h}$ denote the affine Zariski open set

$$U = \{x \in \mathfrak{h} \mid x_i \neq x_j \text{ whenever } b_i \neq b_j\}.$$

Then $U \cap \mathfrak{h}' = U \cap X_q = \mathfrak{h}'_{\mathrm{reg}}$, so $I_q\mathcal{O}(U) \to \mathcal{O}(U) \to \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})$ is right exact. Thus $\mathcal{O}(U) \otimes_{\mathbb{C}[\mathfrak{h}]} M \to \mathcal{O}(\mathfrak{h}'_{\mathrm{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} M$ is an isomorphism. Since $M \mapsto \mathcal{O}(U) \otimes_{\mathbb{C}[\mathfrak{h}]} M$ is exact, so is Loc^q. Clearly the objects killed by Loc^q are exactly those in $H_c{-}\mathrm{mod}^{q+1}_{RS}$. The result now follows from general categorical considerations. □

We now want to determine the "image" of Loc^q. The Riemann-Hilbert correspondence [7] gives an equivalence of categories

$$\begin{aligned} \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}}){-}\mathrm{mod}_{RS} &\xrightarrow{\sim} D(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p){-}\mathrm{mod}_{RS} \\ &\xrightarrow{\sim} \pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p){-}\mathrm{mod}_{\mathrm{fd}}, \end{aligned}$$

where the latter denotes the category of finite dimensional representations of the group $\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p)$ over $\mathbb{C}$. Let ϕ denote the composite functor. Under the identification $\mathfrak{h}' = \operatorname{Spec}\mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p] \cong \mathbb{C}^{q+p}$, the open subset $\mathfrak{h}'_{\mathrm{reg}}$ consists of points with all coordinates distinct. By the correspondence between covering spaces of $\mathfrak{h}'_{\mathrm{reg}}/S_{q+p}$ and subgroups of $\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_{q+p})$, we have a homomorphism

$$\mu : \pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_{q+p}) \twoheadrightarrow S_{q+p},$$

and $\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p) = \mu^{-1}(S_q \times S_p)$. The fundamental group $\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_{q+p})$ is known as the *braid group* B_{q+p}. We need to describe some explicit elements of B_{q+p}. Suppose $x_1, x_2, \ldots, x_{q+p} \in \mathbb{C}$ are the vertices of a convex $(q+p)$-gon in the complex plane, listed counterclockwise. We take the point

$$\mathbf{x} = (x_1, x_2, \ldots, x_{q+p}) \in \mathbb{C}^{q+p} \cong \mathfrak{h}'$$

as the basepoint of $\pi_1(\mathfrak{h}'_{\mathrm{reg}})$. Suppose $1 \le i, j \le q+p$ and $i \ne j$. Pick $\epsilon > 0$ such that $2\epsilon|x_i - x_j| < |x_i + x_j - 2x_k|$ for all k. Let $\gamma_{ij} : [0,1] \to \mathfrak{h}'_{\mathrm{reg}}$ be the path

$$\gamma_{ij}(t) = \begin{cases} ts_{ij}\mathbf{x} + (1-t)\mathbf{x} & \text{if } |t - \frac{1}{2}| > \epsilon \\ \frac{1}{2}(s_{ij}\mathbf{x} + \mathbf{x}) + i\epsilon e^{\frac{1}{2\epsilon}i\pi(t-\frac{1}{2})}(\mathbf{x} - s_{ij}\mathbf{x}) & \text{otherwise.} \end{cases}$$

from $\mathbf{x}$ to $s_{ij}\mathbf{x}$. Geometrically, as γ_{ij} is traversed, x_i and x_j switch positions linearly, traversing small semicircles counterclockwise around $\frac{1}{2}(x_i + x_j)$ to avoid intersecting. Pushing this path down to $\mathfrak{h}'_{\mathrm{reg}}/S_{q+p}$ gives an element $S_{ij} \in B_{q+p}$ such that $\mu(S_{ij}) = s_{ij}$. Note that there is a unique component Z of $\mathfrak{h}' \setminus \mathfrak{h}'_{\mathrm{reg}}$ passing through $\frac{1}{2}(s_{ij}\mathbf{x} + \mathbf{x})$, and for $M \in \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\mathrm{reg}})-\mathrm{mod}_{RS}$, the action of S_{ij} on $\phi(M)$ is conjugate to the monodromy around Z of the induced local system on $\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p$.

Lemma 5.2. *We have a surjection $\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p) \twoheadrightarrow B_q \times B_p$ whose kernel is generated by $\{S_{ij}^2 \mid i \le q,\ j > q\}$.*

Proof. Let $U_q \subseteq \mathbb{C}^q$ and $U_p \subseteq \mathbb{C}^p$ denote the open subsets on which all coordinates are distinct. We have a natural continuous map $\mathfrak{h}'_{\mathrm{reg}} \to U_q \times U_p$, and the natural action of $S_q \times S_p$ on the latter space makes this map equivariant. We may therefore identify $\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p$ with an open subset of $(U_q/S_q) \times (U_p/S_p)$, and the complement Z is the image of

$$\{\mathbf{x} \in U_q \times U_p \subseteq \mathbb{C}^{q+p} \mid x_i = x_j \text{ for some } i \le q,\ j > q\}.$$

Note that $(U_q/S_q) \times (U_p/S_p)$ is a smooth complex variety, and Z is an irreducible divisor. Moreover Z is invariant under the action of $\mathbb{R}^*$, and is therefore contractible. Van Kampen's Theorem now gives a surjection

$$\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p) \twoheadrightarrow \pi_1((U_q/S_q) \times (U_p/S_p)) = B_q \times B_p,$$

whose kernel is generated by a loop around Z. That is, the kernel is generated by S_{ij}^2 for any $i \le q$ and $j > q$. □

The group B_q has a standard presentation; namely it is generated by $\{S_{i,i+1} \mid 1 \le i < q\}$ subject to the braid relations

$$\begin{aligned} S_{i,i+1}S_{i+1,i+2}S_{i,i+1} &= S_{i+1,i+2}S_{i,i+1}S_{i+1,i+2}, \\ S_{i,i+1}S_{j,j+1} &= S_{j,j+1}S_{i,i+1} \text{if } |i-j| > 1. \end{aligned}$$

The kernel of $B_q \to S_q$ is generated by $\{S_{ij}^2\}$. For any $\mathbf{q} \in \mathbb{C}^*$, the *Hecke algebra of type* A_{p-1} is the algebra

$$H_{\mathbf{q}}(S_p) = \mathbb{C}[B_p]/\langle (S_{ij} - 1)(S_{ij} + \mathbf{q}) \rangle.$$

We will take $\mathbf{q} = e^{2\pi i/m}$. Combining the above, we have a surjection

$$\psi : \mathbb{C}[\pi_1(\mathfrak{h}'_{\mathrm{reg}}/S_q \times S_p)] \twoheadrightarrow \mathbb{C}[S_q] \otimes_{\mathbb{C}} H_{\mathbf{q}}(S_p)$$

whose kernel is generated by

$$\{S_{ij}^2 - 1 \mid i \le q \text{ or } j \le q\} \cup \{(S_{ij} - 1)(S_{ij} + \mathbf{q}) \mid i, j > q\}. \tag{3}$$

This gives rise to a fully faithful embedding

$$\psi^* : \mathbb{C}[S_q] \otimes_{\mathbb{C}} H_{\mathbf{q}}(S_p)\text{−mod}_{fd} \to \pi_1(\mathfrak{h}'_{\text{reg}}/S_q \times S_p)\text{−mod}_{fd},$$

whose image is the full subcategory $\mathcal{A}$ of $\pi_1(\mathfrak{h}'_{\text{reg}}/S_q \times S_p)\text{−mod}_{fd}$ consisting of modules on which the relations (3) vanish.

Lemma 5.3. *The composite functor $\phi\, Loc^q$ sends each module into $\mathcal{A}$.*

Proof. Consider any $M \in H_c\text{−mod}_{RS}^q$. We must show that each of the relations (3) vanish on $\phi\, Loc^q(M)$. That is, for each component Z of $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$, we must show that the monodromy of the induced local system on $\mathfrak{h}'_{\text{reg}}/S_q \times S_p$ around Z satisfies the appropriate equation. Let $b \in Z$ be a "generic" point, that is, chosen from Z so that its stabiliser $W' \subseteq S_n$ is minimal. By Proposition 3.4, it suffices to prove the appropriate equation for the monodromy of the local system corresponding to $N = Res_b M$. There are three possibilities, depending on which two coordinates we have set equal.

Case 1: $W' \cong S_m^{q-2} \times S_{2m}$. We are required to show that the monodromy of the local system on $\mathfrak{h}'_{\text{reg}}$ (ignoring the equivariance structure) around Z is trivial. The monodromy depends only on the action of $H_c(S_{2m}, \mathbb{C}^{2m}) \subseteq H_c(W', \mathfrak{h})$, so we may suppose $n = 2m$. But N has minimal support by Theorem 3.3(2). By Theorem 4.5, it suffices to consider the module $F(N) \in H_m(S_2, \mathbb{C}^2)\text{−mod}_{RS}$. However, the latter category is semisimple with irreducibles $L(\tau_{(2)})$ and $L(\tau_{(1,1)})$, so it suffices to check that these modules give rise to local systems with trivial monodromy around the diagonal. This is clear from the description of the connection given in Proposition 1.2.

Case 2: $W' \cong S_m^{q-1} \times S_{m+1}$. Again we must show that the monodromy of the local system on $\mathfrak{h}'_{\text{reg}}$ around Z is trivial. Now the monodromy depends only on the action of $H_c(S_{m+1}, \mathbb{C}^{m+1}) \subseteq H_c(W', \mathfrak{h})$, so we may suppose $n = m + 1$. The result follows as in the proof of Claim 1 in Theorem 4.5.

Case 3: $W' \cong S_m^q \times S_2$. This case is well known (see [14] Theorem 5.13) but we give the calculation here for convenience. Now the monodromy depends on the action of $H_c(S_2, \mathbb{C}^2) \subseteq H_c(W', \mathfrak{h})$, so we may suppose $n = 2$. Since $\mathfrak{h}$ acts locally nilpotently on N, we can find a surjection

$$(H_c(S_2, \mathbb{C}^2)/H_c(S_2, \mathbb{C}^2)\mathfrak{h}^k)^{\oplus l} \twoheadrightarrow N$$

for some k and l. We may replace N by the former module. Then

$$V = (\mathbb{C}[S_2] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]/\mathfrak{h}^k)^{\oplus l} \subseteq N$$

freely generates N over $\mathbb{C}[\mathfrak{h}]$ and is invariant under the actions of $\mathfrak{h}$ and S_2. By Proposition 1.2, the corresponding D-module on $\mathfrak{h}_{\text{reg}}$ is freely generated over $\mathcal{O}(\mathfrak{h}_{\text{reg}})$ by V, and has connection

$$\nabla_y v = yv - \frac{\langle y, \alpha_s \rangle}{\alpha_s} c(s-1)v$$

for $v \in V$, where $S_2 = \{1, s\}$. In particular, the residue of this connection acts as 0 on the 1-eigenspace of s in V, and as $2c$ on the -1-eigenspace. Therefore after pushing down to $\mathfrak{h}/S_2$, the monodromy acts as 1 on the 1-eigenspace and $-e^{2\pi i c} = -\mathbf{q}$ on the -1-eigenspace. This proves the required relation. □

Proposition 5.4. *The functor Loc^q induces an equivalence*

$$H_c\text{−mod}_{RS}^q / H_c\text{−mod}_{RS}^{q+1} \cong \phi^{-1}\mathcal{A}.$$

Proof. By the above lemmas, Loc^q induces a faithful functor

$$H_c\text{-mod}^q_{RS}/H_c\text{-mod}^{q+1}_{RS} \to \phi^{-1}\mathcal{A},$$

and it remains to show that this functor is full and essentially surjective. Both statements will follow from the existence of a functor $G : \phi^{-1}\mathcal{A} \to H_c\text{-mod}^q_{RS}$ such that $Loc^q G$ is naturally equivalent to the identity functor. This will take some work, so we proceed in a sequence of lemmas.

Lemma 5.5. *Let $N \subseteq S_n$ denote the normaliser of the subgroup $S_m^q \subseteq S_n$, and choose a set C of left coset representatives for N in S_n. Let $\mathfrak{h}^\perp$ denote the orthogonal complement to $\mathfrak{h}'$ with respect to the natural S_n-invariant inner product on $\mathfrak{h} = \mathbb{C}^n$. There is a functor*

$$\bar{G} : \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\text{reg}})\text{-mod}_{RS} \to H_c(S_n, \mathfrak{h})\text{-mod}$$

such that

$$\bar{G}M \cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[N]} M = \bigoplus_{w \in C} wM$$

as an S_n-module, and the actions of $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ are given by

$$\begin{aligned} x(w \otimes v) &= w \otimes \pi(x^w)v, \\ y(w \otimes v) &= w \otimes \nabla_{\rho(y^w)} v - \frac{1}{m} \sum_{\substack{i,j \\ \pi(x_i) \neq \pi(x_j)}} \langle y^w, x_i - x_j \rangle (w + ws_{ij}) \otimes \frac{1}{\pi(x_i - x_j)} v, \end{aligned}$$

where $\pi : \mathbb{C}[\mathfrak{h}] \twoheadrightarrow \mathbb{C}[\mathfrak{h}']$ is as above, and $\rho : \mathfrak{h} \twoheadrightarrow \mathfrak{h}'$ is the projection with kernel $\mathfrak{h}^\perp$. Note that if $\pi(x_i) \neq \pi(x_j)$, then $\pi(x_i - x_j)$ is invertible in $\mathcal{O}(\mathfrak{h}'_{\text{reg}})$.

Proof. Under the natural identification $\mathfrak{h}' \times \mathfrak{h}^\perp \cong \mathfrak{h}$, the set $\mathfrak{h}'_{\text{reg}} \times \mathfrak{h}^\perp$ is identified with an open subset $U \subseteq \mathfrak{h}$. Let

$$\bar{U} = S_n \times_N U = \coprod_{w \in C} wU.$$

We have a natural étale morphism $\bar{U} \to \mathfrak{h}$, so by Propositions 2.3, 2.4 and 2.6 we have algebra homomophisms

$$H_c(S_n, \mathfrak{h}) \to H_c(S_n, \bar{U}) \to \text{Mat}_{[S_n:N]}(\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m^q]} H_c(S_m^q, \text{Spf}\, \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}})),$$

But

$$H_c(S_m^q, \text{Spf}\, \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}}) = \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}} \otimes_{\mathcal{O}(U)} H_c(S_m^q, U) = \left(\varprojlim \mathcal{O}(U)/I^k\right) \otimes_{\mathcal{O}(U)} H_c(S_m^q, U),$$

where $I \subseteq \mathcal{O}(U)$ is the kernel of $\mathcal{O}(U) \to \mathcal{O}(\mathfrak{h}'_{\text{reg}})$. Since $U \cong \mathfrak{h}'_{\text{reg}} \times \mathfrak{h}^\perp$, we have

$$H_c(S_m^q, U) = H_c(S_m^q, \mathfrak{h}^\perp) \otimes_{\mathbb{C}} D(\mathfrak{h}'_{\text{reg}}).$$

Moreover $\mathfrak{h}^\perp \cong (\mathbb{C}^m/\mathbb{C})^q$, so by Theorem 3.5 and Lemma 3.1 we have an algebra homormorphism $H_c(S_m^q, \mathfrak{h}^\perp) \to \mathbb{C}$ whose kernel contains $(\mathfrak{h}^\perp)^*$. This induces a homomorphism

$$H_c(S_m^q, \mathfrak{h}^\perp) \otimes_{\mathbb{C}} D(\mathfrak{h}'_{\text{reg}}) \to D(\mathfrak{h}'_{\text{reg}})$$

which kills $IH_c(S_m^q, \mathfrak{h}^\perp) \otimes_{\mathbb{C}} D(\mathfrak{h}'_{\text{reg}})$. This therefore factors through the completion, and we obtain

$$\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m^q]} H_c(S_m^q, \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}}) \to \mathbb{C}[N] \ltimes_{\mathbb{C}[S_m^q]} D(\mathfrak{h}'_{\text{reg}}) = \mathbb{C}[N/S_m^q] \ltimes D(\mathfrak{h}'_{\text{reg}}).$$

Since $N/S_m^q \cong S_q \times S_p$, we obtain functors

$$\begin{aligned} \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\text{reg}})\text{-mod}_{RS} &\to \mathbb{C}[N] \ltimes_{\mathbb{C}[S_m^q]} H_c(S_m^q, \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}})\text{-mod} \\ &\cong \text{Mat}_{[S_n:N]}(\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m^q]} H_c(S_m^q, \hat{\mathcal{O}}_{U, \mathfrak{h}'_{\text{reg}}}))\text{-mod} \\ &\to H_c(S_n, \mathfrak{h})\text{-mod}. \end{aligned}$$

Let $\bar{G}$ be the composite functor. Following through the construction in Proposition 2.6, we see that $\bar{G}M$ is exactly as descibed in the statement. $\square$

Now let

$$z_i^\vee = m\rho(x_{mi}^\vee) = \sum_{j=0}^{m-1} x_{mi-j}^\vee, \qquad t_a^\vee = \rho(x_{mq+a}^\vee) = x_{mq+a}^\vee.$$

These form the basis of $\mathfrak{h}'$ dual to the basis $\{z_i, t_a\}$ of $(\mathfrak{h}')^*$. As in the previous section, denote

$$p_{x^\vee}(2) = \sum_{i=1}^n (x_i^\vee)^2 \in H_c(S_n, \mathfrak{h}),$$

and similarly for $p_{z^\vee}(2)$ and $p_{t^\vee}(2)$.

Lemma 5.6. *The element* $\mathbf{eu} \in H_c(S_n, \mathfrak{h})$ *acts locally finitely on* $\bar{G}M$*. Also* $p_{x^\vee}(2)$ *acts on symmetric elements of* $\bar{G}M$ *via the formula*

$$p_{x^\vee}(2) \sum_{w \in S_n} w \otimes v = \sum_{w \in S_n} w \otimes \zeta v,$$

where $\zeta \in D(\mathfrak{h}'_{\text{reg}})$ *is the differential operator*

$$\zeta = \frac{1}{m} p_{z^\vee}(2) + p_{t^\vee}(2) - 2\sum_{i \neq j} \frac{z_i^\vee - z_j^\vee}{z_i - z_j} - \frac{2}{m}\sum_{i,a} \frac{z_i^\vee - m t_a^\vee}{z_i - t_a} - \frac{2}{m}\sum_{a \neq b} \frac{t_a^\vee - t_b^\vee}{t_a - t_b}.$$

Proof. Consider the Euler vector field

$$\xi = \sum_{i=1}^q z_i z_i^\vee + \sum_{i=1}^r t_i t_i^\vee \in D(\mathfrak{h}').$$

This acts locally finitely on M by Lemma 3.2. Calculating the action of the Euler element $\mathbf{eu} \in H_c$ on $v \in M \subseteq \bar{G}M$ gives

$$\mathbf{eu}\, v = \nabla_\xi v - \frac{n(n-1)}{2m} v.$$

Since $\mathbf{eu}$ centralises S_n, we conclude that $\mathbf{eu}$ acts locally finitely on $\bar{G}M$. The second statement also follows by a straightforward calculation. $\square$

Clearly the support of $\bar{G}M$ lies in X_q, and $\mathcal{O}(\mathfrak{h}'_{\text{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} \bar{G}M \cong M$. If $\bar{G}M$ were finitely generated over $C[\mathfrak{h}]$, this would be the required module in $H_c\text{-mod}^q_{RS}$. Unfortunately it is too large. We will construct the required module GM as a submodule of $\bar{G}M$. For each irreducible component Z of $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$, let $x_Z \in (\mathfrak{h}')^*$ be an element with kernel Z. Let

$$\alpha = \prod_Z x_Z^2 \in \mathbb{C}[\mathfrak{h}']^{S_q \times S_p}.$$

Note that the zero set of α is exactly $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$.

Lemma 5.7. *For each component* Z *of* $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$*, there is a subspace* $D_Z M \subseteq M$ *satisfying:*

1. $D_Z M$ *is functorial in* M*,*
2. $D_Z M$ *is preserved by the actions of* ζ *and* $\pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathbb{C}[\mathfrak{h}']$
3. *If* $U \subseteq \mathfrak{h}'$ *is a Zariski open subset such that* $V \subseteq \mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) \otimes_{\mathcal{O}(\mathfrak{h}'_{\text{reg}})} M$ *freely generates* $\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) \otimes_{\mathcal{O}(\mathfrak{h}'_{\text{reg}})} M$ *over* $\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}})$*, then setting*
$$U' = U \cap \operatorname{int}(\mathfrak{h}'_{\text{reg}} \cup Z),$$
we have $D_Z M \subseteq x_Z^{-K} \mathcal{O}(U') V$ *for some integer* K*.*
4. *If* $M \in \phi^{-1}\mathcal{A}$ *then* $M^{S_q \times S_p} = \mathbb{C}[\alpha^{-1}](D_Z M)^{S_q \times S_p}$*.*

Proof. Choose a generic point $b \in Z$ with stabiliser $W' \subseteq S_n$. Let $B \subseteq \mathfrak{h}'$ be an open ball around b which doesn't intersect the other components of $\mathfrak{h}' \setminus \mathfrak{h}'_{\mathrm{reg}}$. Since $Z \subseteq \mathfrak{h}'$ is a codimension 1 complex subspace, $\pi_1(B \cap \mathfrak{h}'_{\mathrm{reg}}) \cong \mathbb{Z}$. Let $\bar{B} \twoheadrightarrow B \cap \mathfrak{h}'_{\mathrm{reg}}$ denote the universal cover, and let $\mathcal{O}^{an}_B$ and $\mathcal{O}^{an}_{\bar{B}}$ denote the rings of analytic functions on B and $\bar{B}$ respectively. Now $\mathcal{O}^{an}_{\bar{B}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M$ is naturally an analytic $\mathcal{O}$-coherent $\mathcal{D}$-module on $\bar{B}$. Since $\bar{B}$ is simply connected, we have

$$\mathcal{O}^{an}_{\bar{B}} \otimes_{\mathbb{C}} M_{flat} \xrightarrow{\sim} \mathcal{O}^{an}_{\bar{B}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M,$$

where $M_{flat} \subseteq \mathcal{O}^{an}_{\bar{B}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M$ is the space of flat sections. Let $i : M \to \mathcal{O}^{an}_{\bar{B}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M$ denote the inclusion. Again we consider three cases, depending on which two coordinates are equal on Z.

Case 1: $W' \cong S_m^q \times S_2$, so Z is the kernel of $x_Z = t_a - t_b \in (\mathfrak{h}')^*$. Let $s \in S_q \times S_p$ be the transposition switching t_a with t_b, and let $\mathcal{O}^{an,s}_B$ denote the subspace of $\mathcal{O}^{an}_B$ fixed by s. We may think of $\mathcal{O}^{an,s}_B$ as consisting of functions involving only even powers of x_Z. Let $\lambda = \frac{2}{m} + 1$, and let

$$D_Z M = i^{-1}\left(\mathcal{O}^{an,s}_B M_{flat} + x_Z^\lambda \mathcal{O}^{an,s}_B M_{flat}\right),$$

noting that x_Z^λ is a well-defined function in $\mathcal{O}^{an}_{\bar{B}}$. We first check that this is independent of the choice of b and B. Suppose b' and B' are chosen to satisfy the same conditions. Since $\mathfrak{h}^{W'}_{\mathrm{reg}}$ is connected, we may find a path joining b to b', and some tubular neighbourhood T of this path in $\mathfrak{h}'$ will contain B and B', such that the inclusions $B \cap \mathfrak{h}'_{\mathrm{reg}} \hookrightarrow T \cap \mathfrak{h}'_{\mathrm{reg}}$ and $B' \cap \mathfrak{h}'_{\mathrm{reg}} \hookrightarrow T \cap \mathfrak{h}'_{\mathrm{reg}}$ are homotopy equivalences. This allows us to identify $\bar{B}$ and $\bar{B}'$ with open subsets of the universal cover $\bar{T}$ of $T \cap \mathfrak{h}'_{\mathrm{reg}}$. Thus i may be expressed as a composite

$$M \xrightarrow{j} \mathcal{O}^{an}_{\bar{T}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M \xrightarrow{k} \mathcal{O}^{an}_{\bar{B}} \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M.$$

Now $\mathcal{O}^{an,s}_T + x_Z^\lambda \mathcal{O}^{an,s}_T$ is the inverse image of $\mathcal{O}^{an,s}_B + x_Z^\lambda \mathcal{O}^{an,s}_B$ under $\mathcal{O}^{an}_{\bar{T}} \to \mathcal{O}^{an}_{\bar{B}}$, so

$$\mathcal{O}^{an,s}_T M_{flat} + x_Z^\lambda \mathcal{O}^{an,s}_T M_{flat} = k^{-1}(\mathcal{O}^{an,s}_B M_{flat} + x_Z^\lambda \mathcal{O}^{an,s}_B M_{flat}).$$

It follows that $D_Z M = j^{-1}(\mathcal{O}^{an,s}_T M_{flat} + x_Z^\lambda \mathcal{O}^{an,s}_T M_{flat})$, so using b' and B' instead of b and B produces the same subspace. Property (1) is clear.

Now $\pi : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}']$ is equivariant with respect to the action of $S_q \times S_p$, so

$$\pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathbb{C}[\mathfrak{h}']^{S_q \times S_p} \subseteq \mathcal{O}^{an,s}_B.$$

Therefore $D_Z M$ is preserved by the action of $\pi(\mathbb{C}[\mathfrak{h}]^{S_n})$. To show it is preserved by ζ, note that

$$\begin{aligned}
\zeta x_Z^\lambda &= 2\lambda(\lambda-1)x_Z^{\lambda-2} - \frac{4}{m}\lambda x_Z^{\lambda-2} - 2\sum_i \left(\frac{1}{t_a - z_i} - \frac{1}{t_b - z_i}\right)\lambda x_Z^{\lambda-1} \\
&\quad - \frac{2}{m}\sum_{c \neq a,b}\left(\frac{1}{t_a - t_c} - \frac{1}{t_b - t_c}\right)\lambda x_Z^{\lambda-1} \\
&= 2\sum_i \frac{1}{(t_a - z_i)(t_b - z_i)}\lambda x_Z^\lambda + \frac{2}{m}\sum_{c\neq a,b}\frac{1}{(t_a - t_c)(t_b - t_c)}\lambda x_Z^\lambda \\
&\in x_Z^\lambda \mathcal{O}^{an,s}_B.
\end{aligned}$$

Also setting $x_Z^\vee = t_a^\vee - t_b^\vee$, we may write

$$\zeta \in \frac{1}{2}(x_Z^\vee)^2 - \frac{2}{m}\frac{x_Z^\vee}{x_Z} + \sum_{y \in Z} \mathbb{C}y^2 + \mathcal{O}^{an}_B y,$$

so for $f \in \mathcal{O}_B^{an,s}$, we have

$$[\zeta, f] \in x_Z^\vee(f) x_Z^\vee - \frac{2}{m}\frac{x_Z^\vee(f)}{x_Z} + \sum_{y \in Z} \mathbb{C} y(f) y + \mathcal{O}_B^{an} y(f) + \mathcal{O}_B^{an,s}.$$

But $x_Z^\vee(f)/x_Z \in \mathcal{O}_B^{an,s}$ and s fixes ζ, so we conclude that

$$[\zeta, \mathcal{O}_B^{an,s}] \subseteq \mathcal{O}_B^{an,s} x_Z x_Z^\vee + \mathcal{O}_B^{an,s} Z + \mathcal{O}_B^{an,s}.$$

Hence

$$\begin{aligned} &\zeta(\mathcal{O}_B^{an,s} M_{flat} + x_Z^\lambda \mathcal{O}_B^{an,s} M_{flat}) \\ &\qquad \subseteq \; [\zeta, \mathcal{O}_B^{an,s}] M_{flat} + [\zeta, \mathcal{O}_B^{an,s}] x_Z^\lambda M_{flat} + \mathcal{O}_B^{an,s} \zeta(x_Z^\lambda) M_{flat} \\ &\qquad \subseteq \; \mathcal{O}_B^{an,s} M_{flat} + x_Z^\lambda \mathcal{O}_B^{an,s} M_{flat}, \end{aligned}$$

proving property (2).

Now suppose that U and V are as in property (3). If $U \cap Z = \emptyset$ then $\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) = \mathcal{O}(U')$, and the property is clear. Otherwise we may suppose that $B \subseteq U$. Then

$$\mathcal{O}_{\bar{B}}^{an} \otimes_{\mathbb{C}} M_{flat} = \mathcal{O}_{\bar{B}}^{an} \otimes_{\mathcal{O}(\mathfrak{h}'_{\text{reg}})} M = \mathcal{O}_{\bar{B}}^{an} \otimes_{\mathbb{C}} V,$$

so $M_{flat} = XV$, where $X \in GL_{\mathcal{O}_{\bar{B}}^{an}}(\mathcal{O}_{\bar{B}}^{an} \otimes_{\mathbb{C}} V)$ satisfies

$$\begin{aligned} X &\in \sum_{\substack{\mu \in \Lambda, \\ l \in L}} \operatorname{End}_{\mathbb{C}}(V) \otimes_{\mathbb{C}} \mathcal{O}_B^{an} x_Z^\mu (\log x_Z)^l, \\ X^{-1} &\in \sum_{\substack{\mu \in \Lambda, \\ l \in L}} \operatorname{End}_{\mathbb{C}}(V) \otimes_{\mathbb{C}} \mathcal{O}_B^{an} x_Z^{-\mu} (\log x_Z)^l \end{aligned}$$

for some finite subsets $\Lambda \subseteq \mathbb{C}$ and $L \subseteq \mathbb{Z}_{\geq 0}$. Thus

$$\begin{aligned} M_{flat} &\subseteq \sum_{\substack{\mu \in \Lambda, \\ l \in L}} \mathcal{O}_B^{an} x_Z^\mu (\log x_Z)^l V, \\ V &\subseteq \sum_{\substack{\mu \in \Lambda, \\ l \in L}} \mathcal{O}_B^{an} x_Z^{-\mu} (\log x_Z)^l M_{flat}. \end{aligned}$$

Thus

$$\begin{aligned} D_Z M &\subseteq (\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) \otimes_{\mathbb{C}} V) \cap \left(\sum_{\substack{\mu \in \Lambda, \\ l \in L}} (\mathcal{O}_B^{an} x_Z^\mu (\log x_Z)^l + \mathcal{O}_B^{an} x_Z^{\mu+\lambda} (\log x_Z)^l) V \right) \\ &\subseteq x_Z^{-K} \mathcal{O}(U') V \end{aligned}$$

for some integer K, proving (3).

Proceeding with the above notation, suppose now that $M \in \phi^{-1}\mathcal{A}$. Let S be the endomorphism of $\mathcal{O}_{\bar{B}}^{an} \otimes_{\mathcal{O}(\mathfrak{h}'_{\text{reg}})} M$ given by

$$S(f \otimes v) = (\bar{s}^* f) \otimes sv,$$

where $\bar{s}^* \in \operatorname{End}_{\mathbb{C}}(\mathcal{O}_{\bar{B}}^{an})$ is induced by the automorphism $\bar{s} : \bar{B} \to \bar{B}$ obtained by lifting $s : B \to B$. That is, $\bar{s}^*$ sends x_Z to $e^{-\pi i} x_Z$, and fixes any function killed by $x_Z^\vee$. The restriction of S to M_{flat} is exactly the monodromy of the corresponding local system on $\mathfrak{h}'_{\text{reg}}/S_q \times S_p$ around Z, which is assumed to satisfy

$$(S|_{M_{flat}} - 1)(S|_{M_{flat}} + \mathbf{q}) = 0.$$

Thus M_{flat} decomposes into eigenspaces M^{1}_{flat} and $M^{-\mathbf{q}}_{flat}$ for S. The above shows that

$$\begin{aligned} M &\subseteq \sum_{\substack{\mu\in\Lambda,\\ l\in L}} \mathcal{O}^{an}_{B} x_Z^{-\mu}(\log x_Z)^l M_{flat} \\ &= \sum_{\substack{\mu\in\Lambda,\\ l\in L}} \mathcal{O}^{an}_{B} x_Z^{-\mu}(\log x_Z)^l M^{1}_{flat} + \mathcal{O}^{an}_{B} x_Z^{-\mu}(\log x_Z)^l M^{-\mathbf{q}}_{flat}. \end{aligned}$$

But $-\mathbf{q} = e^{\lambda\pi i}$, so the elements fixed by s must be contained in

$$M^s \subseteq \mathcal{O}^{an,s}_{B}[x_Z^{-2}]M^{1}_{flat} + \mathcal{O}^{an,s}_{B}[x_Z^{-2}]x_Z^{\lambda} M^{-\mathbf{q}}_{flat}.$$

If $v \in M^{S_q\times S_p} \subseteq M^s$, then for some $K > 0$ we have

$$\alpha^K v \in M \cap \left(\mathcal{O}^{an,s}_{B} M^{1}_{flat} + \mathcal{O}^{an,s}_{B} x_Z^{\lambda} M^{-\mathbf{q}}_{flat}\right) \subseteq D_Z M.$$

Since α is also $S_q \times S_p$-invariant, we conclude that $M^{S_q\times S_p} \subseteq \mathbb{C}[\alpha^{-1}](D_Z M)^{S_q\times S_p}$, proving (4).

Case 2: $W' \cong S_m^{q-2} \times S_{2m}$, so Z is the kernel of $x_Z = z_i - z_j \in (\mathfrak{h}')^*$. Let $s \in S_q \times S_p$ be the transposition switching z_i with z_j, and again let $\mathcal{O}^{an,s}_{B}$ denote the subspace of $\mathcal{O}^{an}_{B}$ fixed by s. We now set $\lambda = 2m + 1$, and again define

$$D_Z M = i^{-1}\left(\mathcal{O}^{an,s}_{B} M_{flat} + x_Z^{\lambda}\mathcal{O}^{an,s}_{B} M_{flat}\right).$$

The arguments proceed as in the previous case, the only difference being the following two calculations. We have

$$\begin{aligned} \zeta x_Z^{\lambda} &= \frac{2\lambda(\lambda-1)}{m} x_Z^{\lambda-2} - 4\lambda x_Z^{\lambda-2} - 2\sum_{k\neq i,j}\left(\frac{1}{z_i - z_k} - \frac{1}{z_j - z_k}\right)\lambda x_Z^{\lambda-1} \\ &\quad - \frac{2}{m}\sum_a \left(\frac{1}{z_i - t_a} - \frac{1}{z_j - t_a}\right)\lambda x_Z^{\lambda-1} \\ &= 2\sum_{k\neq i,j}\frac{1}{(z_i - z_k)(z_j - z_k)}\lambda x_Z^{\lambda} + \frac{2}{m}\sum_a \frac{1}{(z_i - t_a)(z_j - t_a)}\lambda x_Z^{\lambda} \\ &\in x_Z^{\lambda}\mathcal{O}^{an,s}_{B}. \end{aligned}$$

Also, setting $x_Z^\vee = z_i^\vee - z_j^\vee$, we have

$$\zeta \in \frac{1}{2m}(x_Z^\vee)^2 - 2\frac{x_Z^\vee}{x_Z} + \sum_{y\in Z}\mathbb{C}y^2 + \mathcal{O}^{an}_{B} y,$$

so for $f \in \mathcal{O}^{an,s}_{B}$, we have

$$[\zeta, f] \in \frac{1}{m}x_Z^\vee(f)x_Z^\vee - 2\frac{x_Z^\vee(f)}{x_Z} + \sum_{y\in Z}\mathbb{C}y(f)y + \mathcal{O}^{an}_{B} y(f) + \mathcal{O}^{an,s}_{B}.$$

The properties follow as above.

Case 3: $W' \cong S_m^{q-1} \times S_{m+1}$, so Z is the kernel of $x_Z = z_i - t_a$. Let $x_Z^\vee = z_i^\vee - m t_a^\vee$, and define

$$\begin{aligned} \mathcal{O}^{an+}_{B} &= \{f \in \mathcal{O}^{an}_{B} \mid x_Z^\vee(f) \in x_Z\mathcal{O}^{an}_{B}\}, \\ D_Z M &= i^{-1}(\mathcal{O}^{an+}_{B} M_{flat}). \end{aligned}$$

Properties (1), (3) and (4) follow as in the previous two cases. Now suppose $f \in \mathbb{C}[\mathfrak{h}]^{S_n}$, and pick j and k such that $\pi(x_j) = z_i$ and $\pi(x_k) = t_a$. Then $(x_j^\vee - x_k^\vee)f$ is antisymmetric under s_{jk}, so $(x_j^\vee - x_k^\vee)f \in (x_j - x_k)\mathbb{C}[\mathfrak{h}]$. Using equation (1) from the proof of Proposition 4.4, applying π gives

$$\frac{1}{m} x_Z^\vee \pi(f) \in x_Z \mathbb{C}[\mathfrak{h}'].$$

Therefore $\pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathcal{O}_B^{an+}$, so $D_Z M$ is preserved by the action of $\pi(\mathbb{C}[\mathfrak{h}]^{S_n})$. Finally note that, for $j \neq i$ we have

$$\begin{aligned}
\frac{1}{z_j - z_i} - \frac{1}{z_j - t_a} &= \frac{x_Z}{(z_j - z_i)(z_j - t_a)} \in x_Z \mathcal{O}_B^{an} \text{ and} \\
x_Z^\vee \left(\frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} \right) &= \frac{m}{(z_j - z_i)^2} - \frac{m}{(z_j - t_a)^2} \\
&= \frac{m x_Z (2z_j - z_i - t_a)}{(z_j - z_i)^2 (z_j - t_a)^2} \in x_Z \mathcal{O}_B^{an}, \text{ whence} \\
\frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} &\in \mathcal{O}_B^{an+}.
\end{aligned}$$

Thus

$$\begin{aligned}
&-2\frac{z_i^\vee - z_j^\vee}{z_i - z_j} - \frac{2}{m}\frac{z_j^\vee - m t_a^\vee}{z_j - t_a} \\
&\quad = \frac{2}{m+1}\left(\frac{1}{z_j - z_i} - \frac{1}{z_j - t_a} \right)(z_i^\vee - m t_a^\vee) \\
&\quad\quad + \frac{2}{m+1}\left(\frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} \right)\left(z_i^\vee + t_a^\vee - (1 + 1/m) z_j^\vee\right) \\
&\quad \in x_Z \mathcal{O}_B^{an} x_Z^\vee + \mathcal{O}_B^{an+} Z.
\end{aligned}$$

Similarly for $b \neq a$, we have

$$\begin{aligned}
&-\frac{2}{m}\frac{z_i^\vee - m t_b^\vee}{z_i - t_b} - \frac{2}{m}\frac{t_a^\vee - t_b^\vee}{t_a - t_b} \\
&\quad = \frac{2}{m(m+1)}\left(\frac{1}{t_b - z_i} + \frac{1}{t_a - t_b} \right)(z_i^\vee - m t_a^\vee) \\
&\quad\quad + \frac{2}{m(m+1)}\left(\frac{m}{t_b - z_i} - \frac{1}{t_a - t_b} \right)\left(z_i^\vee + t_a^\vee - (m+1) t_b^\vee\right) \\
&\quad \in x_Z \mathcal{O}_B^{an} x_Z^\vee + \mathcal{O}_B^{an+} Z.
\end{aligned}$$

Finally

$$\frac{1}{m}(z_i^\vee)^2 + (t_a^\vee)^2 = \frac{1}{m(m+1)}(x_Z^\vee)^2 + \frac{1}{m+1}(z_i^\vee + t_a^\vee)^2,$$

so

$$\zeta \in \frac{1}{m(m+1)}(x_Z^\vee)^2 - \frac{2}{m}\frac{x_Z^\vee}{x_Z} + x_Z \mathcal{O}_B^{an} x_Z^\vee + \sum_{y \in Z} \mathbb{C} y^2 + \mathcal{O}_B^{an+} y.$$

Now suppose $f \in \mathcal{O}_B^{an+}$, so $x_Z^\vee(f)/x_Z \in \mathcal{O}_B^{an}$. We have $[x_Z^\vee, x_Z] = (m+1)$, so

$$\begin{aligned}
x_Z^\vee \zeta(f) &\in \frac{1}{m(m+1)}\left((x_Z^\vee)^2 x_Z - 2(m+1)x_Z^\vee\right)\frac{x_Z^\vee(f)}{x_Z} \\
&\quad + x_Z^\vee x_Z \mathcal{O}_B^{an} x_Z^\vee(f) + \sum_{y\in Z}\mathbb{C}x_Z^\vee y^2(f) + x_Z^\vee \mathcal{O}_B^{an+} y(f) \\
&\subseteq \frac{1}{m(m+1)} x_Z (x_Z^\vee)^2 \left(\frac{x_Z^\vee(f)}{x_Z}\right) \\
&\quad + x_Z^\vee\left(x_Z^2 \mathcal{O}_B^{an}\right) + \sum_{y\in Z}\mathbb{C}y^2 x_Z^\vee(f) + x_Z\mathcal{O}_B^{an} y(f) + \mathcal{O}_B^{an+} y x_Z^\vee(f) \\
&\subseteq x_Z\mathcal{O}_B^{an} + \sum_{y\in Z}\mathbb{C}y^2\left(x_Z\mathcal{O}_B^{an}\right) + \mathcal{O}_B^{an+} y\left(x_Z\mathcal{O}_B^{an}\right) \\
&\subseteq x_Z\mathcal{O}_B^{an}.
\end{aligned}$$

Thus $\zeta(f) \in \mathcal{O}_B^{an+}$, proving property (2). □

Lemma 5.8. *Consider the intersection*

$$DM = \bigcap_Z D_Z M$$

over all components of $\mathfrak{h}' \setminus \mathfrak{h}'_{\mathrm{reg}}$. This subspace has the following properties:

1. *DM is functorial in M.*
2. *DM is preserved by the actions of ζ and $\pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathbb{C}[\mathfrak{h}']$.*
3. *DM is finitely generated over $\pi(\mathbb{C}[\mathfrak{h}]^{S_n})$.*
4. *If $M \in \phi^{-1}\mathcal{A}$ then $M^{S_q\times S_p} = \mathbb{C}[\alpha^{-1}](DM)^{S_q\times S_p}$.*

Proof. Certainly (1), (2) and (4) follow immediately from the corresponding properties of $D_Z M$. To prove (3), let $\{U_i\}$ denote a Zariski open cover of $\mathfrak{h}'_{\mathrm{reg}}$ such that

$$\mathcal{O}(U_i) \otimes_{\mathcal{O}(\mathfrak{h}'_{\mathrm{reg}})} M \cong \mathcal{O}(U_i) \otimes_{\mathbb{C}} V_i.$$

We may suppose U_i consists of points in $\mathfrak{h}'_{\mathrm{reg}}$ where $g_i \neq 0$, for some $g_i \in \mathbb{C}[\mathfrak{h}']$. Moreover we may suppose g_i does not vanish on any component of $\mathfrak{h}'\setminus\mathfrak{h}'_{\mathrm{reg}}$. Now $\mathbb{C}[\mathfrak{h}']$ is a UFD, so we have the notion of the order of pole of any element of $\mathcal{O}(U_i) = \mathbb{C}[\mathfrak{h}'][\alpha^{-1}, g_i^{-1}]$ along some component Z. Property (3) of $D_Z M$ states that the coefficients of any $v \in D_Z M$ relative to V_i have poles along Z of order at most K, for some integer K. Thus

$$DM \subseteq \mathcal{O}(U_i')\alpha^{-K} V_i,$$

where $U_i' = \operatorname{Spec}\mathbb{C}[\mathfrak{h}'][g_i^{-1}] \subseteq \mathfrak{h}'$. In particular, $\mathcal{O}(U_i')DM$ is finitely generated over $\mathcal{O}(U_i')$, so $\mathcal{O}(U')DM$ is a coherent sheaf on U', where U' is the union of the U_i'. We have $\mathcal{O}(U')DM \subseteq M$, and the latter is locally free, so $\mathcal{O}(U')DM$ is torsion free. By Lemma 4.2, we conclude that $\mathcal{O}(U')DM$ is finitely generated over $\mathcal{O}(U')$. However, $\mathfrak{h}' \setminus U'$ is contained in $\mathfrak{h}' \setminus \mathfrak{h}'_{\mathrm{reg}}$, but doesn't contain any component of the latter, so it has codimension at least 2. Therefore $\mathcal{O}(U') = \mathbb{C}[\mathfrak{h}']$, so $\mathbb{C}[\mathfrak{h}']DM$ is finitely generated over $\mathbb{C}[\mathfrak{h}']$. Finally $\mathbb{C}[\mathfrak{h}']$ is finite over $\pi(\mathbb{C}[\mathfrak{h}]^{S_n})$, and the latter ring is Noetherian, so (3) follows. □

We may now complete the proof of Proposition 5.4. Consider the subspaces

$$\begin{aligned}
EM &= \left\{ \sum_{w\in S_n} w \otimes v \,\middle|\, v \in (DM)^{S_q\times S_p} \right\} \subseteq e\bar{G}M, \\
GM &= H_c EM,
\end{aligned}$$

where, as usual, $e = \frac{1}{n!}\sum_{w\in S_n} w$. Property (2) of DM implies that EM is preserved by the actions of $p_{x^\vee}(2)e \in H_c$ and $\mathbb{C}[\mathfrak{h}]^{S_n}e \subseteq H_c$. These generate the subalgebra

$eH_ce \subseteq H_c$, so EM is an eH_ce-submodule of $e\bar{G}M$. Also $EM \cong (DM)^{S_q \times S_p}$ as $\mathbb{C}[\mathfrak{h}]^{S_n}$-modules, so EM is finitely generated over $\mathbb{C}[\mathfrak{h}]^{S_n}$. Since $\mathbf{eu}$ acts locally finitely on $\bar{G}M$, we conclude that EM decomposes into finite dimensional generalised eigenspaces for $\mathbf{eu}$, with eigenvalues in $\Lambda + \mathbb{Z}_{\geq 0}$ for some finite subset $\Lambda \subset \mathbb{C}$. The same is true of GM, since H_c is finite over eH_ce and ad $\mathbf{eu}$ is locally finite on H_c. This ensures that $GM \in H_c\text{--mod}^q_{RS}$. Moreover the composite

$$\eta : Loc^q GM = \mathcal{O}(\mathfrak{h}'_{\text{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} GM \hookrightarrow \mathcal{O}(\mathfrak{h}'_{\text{reg}}) \otimes_{\mathbb{C}[\mathfrak{h}]} \bar{G}M \xrightarrow{\sim} M$$

is a homomorphism of $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\text{reg}})$-modules, by Proposition 1.2. But $GM \supseteq EM$, so if $M \in \phi^{-1}\mathcal{A}$, then property (4) of DM ensures that $\text{Im}\,(\eta)$ contains $M^{S_q \times S_p}$. Theorem 2.3 of [17] shows that

$$e' = \frac{1}{q!p!} \sum_{w \in S_q \times S_p} w$$

generates $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\text{reg}})$ as a two-sided ideal, so $M^{S_q \times S_p} = e'M$ generates M over $\mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}'_{\text{reg}})$. Therefore η is an isomorphism, proving the result. □

Although we have considered $c = \frac{1}{m}$ so far in this section, the following result will allow us to generalise to $c = \frac{r}{m}$. The first statement follows from Corollary 4.3 of [4] in the case $m = 2$, and from Theorem 5.12 and Proposition 5.14 of [19] when $m > 2$. The second is Theorem 5.10 of [19].

Theorem 5.9. *There is an equivalence of categories* $H_{\frac{r}{m}}\text{--mod}_{RS} \cong H_{\frac{1}{m}}\text{--mod}_{RS}$ *which identifies* $H_{\frac{r}{m}}\text{--mod}^q_{RS}$ *with* $H_{\frac{1}{m}}\text{--mod}^q_{RS}$. *There is a* $\mathbb{C}$*-algebra isomorphism* $H_{e^{2\pi i r/m}}(S_p) \cong H_{e^{2\pi i/m}}(S_p)$.

We may now combine the above to prove our remaining main results.

Proof of Theorem 1.8. Certainly when $c = \frac{1}{m}$, the above shows that $\phi\, Loc^q$ and ψ^* identify $H_c\text{--mod}^q_{RS}/H_c\text{--mod}^{q+1}_{RS}$ and $\mathbb{C}[S_q] \otimes_{\mathbb{C}} H_{\mathbf{q}}(S_p)\text{--mod}_{\text{fd}}$, respectively, with the full subcategory $\mathcal{A} \subseteq \pi_1(\mathfrak{h}'_{\text{reg}}/S_q \times S_p)\text{--mod}_{fd}$. The case $c = \frac{r}{m}$ follows immediately from the case $c = \frac{1}{m}$ by the previous theorem. □

Proof of Theorem 1.6. Consider the case $c > 0$; the case $c < 0$ follows from this as in the proof of Theorem 3.8. We will prove by induction on q that if $q_m(\lambda) = q$, then $\text{Supp}\,(L(\tau_\lambda)) = X_q$. Suppose it holds for $q' < q$. If the support of $L(\tau_\lambda)$ is X_q, then $q_m(\lambda) \leq q$ by Theorem 3.8. However we cannot have $q_m(\lambda) < q$ by the inductive hypothesis. Therefore the irreducibles in $H_c\text{--mod}^q_{RS}/H_c\text{--mod}^{q+1}_{RS}$ are a subset of

$$\Omega = \{L(\tau_\lambda) \mid q_m(\lambda) = q\}.$$

We have a bijection

$$\{\mu \vdash q\} \times \{\nu \vdash p \mid q_m(\nu) = 0\} \to \{\lambda \vdash n \mid q_m(\lambda) = q\}$$

given by $(\mu, \nu) \mapsto m\mu + \nu$. The irreducibles in $\mathbb{C}[S_q]\text{--mod}_{\text{fd}}$ are indexed by $\{\mu \vdash q\}$, and those in $H_{\mathbf{q}}(S_p)\text{--mod}_{\text{fd}}$ by $\{\nu \vdash p \mid q_m(\nu) = 0\}$. Therefore $|\Omega|$ is exactly the number of irreducibles in $\mathbb{C}[S_q] \otimes_{\mathbb{C}} H_{\mathbf{q}}(S_p)\text{--mod}_{\text{fd}}$. Since the latter category is equivalent to $H_c\text{--mod}^q_{RS}/H_c\text{--mod}^{q+1}_{RS}$, it follows that each module in Ω must be an irreducible in $H_c\text{--mod}^q_{RS}/H_c\text{--mod}^{q+1}_{RS}$; that is, they must all be supported on X_q. This completes the induction. □

Proof of Corollary 1.7. Let us again denote by $\mathbf{p}_n$ the number of partitions of n, and let $\mathbf{p}_{n,m} = |\{\lambda \vdash n \mid q_m(\lambda) = 0\}|$ denote the number of m-regular partitions of n. By the previous proof, the number of irreducibles in $H_{\frac{r}{m}}\text{-mod}_{RS}(S_n, \mathbb{C}^n)$ whose support is X_q is $\mathbf{p}_q\mathbf{p}_{n-qm,m}$. This is the coefficient of $s^n t^{qm}$ in the formal power series

$$N(s,t) = \sum_{p,q\geq 0} \mathbf{p}_q \mathbf{p}_{p,m} s^{qm+p} t^{qm}.$$

It is well known that

$$\sum_{n\geq 0} \mathbf{p}_n t^n = \prod_{n>0} \frac{1}{1-t^n}.$$

Every partition λ of n can be written uniquely as $\lambda = m\mu + \nu$ where ν is m-regular. Therefore

$$\prod_{n>0} \frac{1}{1-t^n} = \sum_{n\geq 0} \mathbf{p}_n t^n = \sum_{p,q\geq 0} \mathbf{p}_q \mathbf{p}_{p,m} t^{mq+p} = \left(\prod_{q>0} \frac{1}{1-t^{mq}}\right)\left(\sum_{p\geq 0} \mathbf{p}_{p,m} t^p\right),$$

giving

$$\sum_{p\geq 0} \mathbf{p}_{p,m} t^p = \prod_{\substack{n>0\\ m\nmid n}} \frac{1}{1-t^n}.$$

Thus

$$N(s,t) = \left(\sum_{p\geq 0} \mathbf{p}_{p,m} s^p\right)\left(\sum_{q\geq 0} \mathbf{p}_q (st)^{qm}\right) = \left(\prod_{\substack{p>0\\ m\nmid p}} \frac{1}{1-s^p}\right)\left(\prod_{q>0} \frac{1}{1-(st)^{qm}}\right).$$

Now consider the operator

$$A_m = \sum_{i>0} \alpha_{-im}\alpha_{im}$$

acting on Fock space F. The elements

$$\prod_{i>0} \alpha_{-i}^{\nu_i} + \operatorname{span}\{A\alpha_i \mid i > 0\} \in F$$

form a basis for F, where the ν_i are nonnegative integers with only finitely many nonzero. This element is an eigenvector for A_m with eigenvalue

$$\sum_{\substack{i>0\\ m\mid i}} i\nu_i.$$

Therefore

$$\begin{aligned}
\operatorname{tr}_F(s^{A_1} t^{A_m}) &= \sum_{\nu} \left(\prod_{i>0} s^{i\nu_i}\right)\left(\prod_{\substack{i>0\\ m\mid i}} t^{i\nu_i}\right)\\
&= \left(\prod_{\substack{i>0\\ m\nmid i}} \sum_{\nu_i\geq 0} s^{i\nu_i}\right)\left(\prod_{\substack{i>0\\ m\mid i}} \sum_{\nu_i\geq 0} (st)^{i\nu_i}\right)\\
&= \left(\prod_{\substack{i>0\\ m\nmid i}} \frac{1}{1-s^i}\right)\left(\prod_{i>0} \frac{1}{1-(st)^{mi}}\right)\\
&= N(s,t),
\end{aligned}$$

as required. □

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