

# On the asymptotic distribution of singular values of products of large rectangular random matrices

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## Abstract

We consider products of independent large random rectangular matrices with independent entries. The limit distribution of the expected empirical distribution of singular values of such products is computed. The distribution function is described by its Stieltjes transform, which satisfies some algebraic equation. In the particular case of square matrices we get a well-known distribution which moments are Fuss-Catalan numbers.

## 1 Introduction

Let  $m \geq 1$  be a fixed integer. For every  $n \geq 1$  consider a nondecreasing set of  $m + 1$  integers  $p_0 = n, p_1, \dots, p_m$  where  $p_\nu = p_\nu(n)$  for  $\nu = 1, \dots, m$ , depending on  $n$  and  $p_\nu \geq n$ . For every  $n \geq 1$  we consider an array of independent complex random variables  $X_{jk}^{(\nu)}$ ,  $1 \leq j \leq p_{\nu-1}, 1 \leq k \leq p_\nu$ ,  $\nu = 1, \dots, m$  defined on a common probability space  $\{\Omega_n, \mathbb{F}_n, \Pr\}$  with  $\mathbf{E} X_{jk}^{(\nu)} = 0$  and let  $\mathbf{E} |X_{jk}^{(\nu)}|^2 = 1$ . Let  $\mathbf{X}^{(\nu)}$  denote the  $p_{\nu-1} \times p_\nu$  matrix with entries  $[\mathbf{X}^{(\nu)}]_{jk} = \frac{1}{\sqrt{p_\nu}} X_{jk}^{(\nu)}$ , for  $1 \leq j \leq p_{\nu-1}, 1 \leq k \leq p_\nu$ . The random variables

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$X_{jk}^{(\nu)}$  may depend on  $n$  but for simplicity we shall not make this explicit in our notations. Denote by  $s_1 \geq \dots \geq s_n$  the singular values of the random matrix  $\mathbf{W} := \prod_{\nu=1}^m \mathbf{X}^{(\nu)}$  and define the empirical distribution of its squared singular values by

$$\mathcal{F}_n(x) = \frac{1}{n} \sum_{k=1}^n I_{\{s_k^2 \leq x\}},$$

where  $I_{\{B\}}$  denotes the indicator of an event  $B$ . We shall investigate the approximation of the expected spectral distribution  $F_n(x) = \mathbf{E} \mathcal{F}_n(x)$  by the distribution function  $G_{\mathbf{y}}(x)$  which defined by its Stieltjes transform  $s_{\mathbf{y}}(z)$  in the equation (1.2) below.

We consider the Kolmogorov distance between the distributions  $F_n(x)$  and  $G_{\mathbf{y}}(x)$

$$\Delta_n := \sup_x |F_n(x) - G_{\mathbf{y}}(x)|.$$

The main result of this paper is the following

**Theorem 1.1.** *Let  $\mathbf{E} X_{jk}^{(\nu)} = 0$ ,  $\mathbf{E} |X_{jk}^{(\nu)}|^2 = 1$ . Assume the Lindeberg condition holds, i.e. for any  $\tau > 0$*

$$L_n(\tau) := \max_{\nu=1, \dots, m} \frac{1}{n^2} \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} |X_{jk}^{(\nu)}|^2 I_{\{|X_{jk}^{(\nu)}| \geq \tau \sqrt{n}\}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Assume that  $\lim_{n \rightarrow \infty} \frac{n}{p_l} = y_l \in (0, 1]$ . Then,

$$\lim_{n \rightarrow \infty} \sup_x |F_n(x) - G_{\mathbf{y}}(x)| = 0.$$

*Remark 1.2.* For  $m = 1$  we get the well-known result of Marchenko-Pastur for sample covariance matrices [11].

*Remark 1.3.* In the case  $y_1 = y_2 = \dots = y_m = 1$  the distribution  $G_{\mathbf{y}}$  has moments  $M_k$  defined by

$$M_k = \int_0^{\infty} x^k dG_{\mathbf{y}}(x) = \frac{1}{mk+1} \binom{k}{mk+k},$$

the so called Fuss–Catalan numbers.

The Fuss-Catalan numbers satisfy the following simple recurrence relation

$$M_k = \sum_{k_0 + \dots + k_m = k-1} \prod_{\nu=0}^m M_{k_{\nu}}. \quad (1.1)$$

Denote by  $s(z)$  Stieltjes transform of the distribution  $G$  determined by its moments  $M_k$ ,

$$s(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dG(x).$$

Using equality (1.1), we may show that this Stieltjes transform  $s(z)$  satisfies the equation

$$1 + zs(z) + (-1)^{m+1} z^m s(z)^{m+1} = 0.$$

In the general case ( $y_l \neq 1$ ) Stieltjes transform satisfies the following equation

$$1 + zs_{\mathbf{y}}(z) - s(z) \prod_{l=1}^m (1 - y_l - zy_l s_{\mathbf{y}}(z)) = 0, \quad (1.2)$$

where  $0 \leq y_l \leq 1$ . For more details about the moments of such distributions see [3].

The result of Theorem 1.1 is the first attempt in the Random Matrix Theory to describe the asymptotic of distribution of the singular spectrum of a product of rectangular random matrices. For rectangular random matrices there is no easily available analog in free probability to describe the limit law. The Theorem 1.1 was formulated in [2]. In the case of squared matrices ( $y_1 = y_2 = \dots = y_m = 1$ ) there is an analog in the form of product of so-called free  $\mathcal{R}$ -diagonal elements. It was studied for instance in Oravecz, [12]. It is well-known that the moments of distribution of a product of free  $\mathcal{R}$ -diagonal elements are Fuss-Catalans numbers (compare Remark 1.3). In [1] it has been shown by the method of moments that the limit distribution of singular values of powers of random matrices is the distribution  $G_{\mathbf{y}}$  with  $y_1 = \dots = y_m = 1$ . In Banica and others [4] the result of Theorem 1.1 was obtained for square Gaussian matrices (see Theorem 6.1 in [4]), using tools of Free Probability theory. For a description of the distribution of  $G_{\mathbf{y}}$  for the special case  $y_1 = \dots = y_m = 1$ , see Speicher and Mingo [13] as well.

In the the following we shall give the proof of Theorem 1.1. We shall investigate the Stieltjes transform  $s_n(z)$  of distribution function  $F_n(x)$ . We show that  $s_n(z)$  satisfies an approximate equation

$$1 + zs_n(z) - s_n(z) \prod_{l=1}^m (1 - y_l - zy_l s_n(z)) = \delta_n(z)$$

where  $\delta_n(z) \rightarrow 0$  as  $n \rightarrow \infty$ . This relation together with relation (1.2) implies that  $s_n(z)$  converges to  $s(z)$  uniformly on any compact set in the upper half-plane  $\mathcal{K} \subset \mathcal{C}^+$ . The last claim is equivalent to weak convergence of the distribution functions  $F_n(x)$  to the distribution function  $F(x)$ .

By  $C$  (with an index or without it) we shall denote generic absolute constants, whereas  $C(\cdot, \cdot)$  will denote positive constants depending on arguments.

## 2 Auxiliary results

In this Section we describe a symmetrization of a one-sided distribution and give a special representation for symmetrized distribution of the squared singular values of random matrices. Furthermore, we prove some lemmas about truncation of entries of random matrices.

## 2.1 Symmetrization

We shall use the following ‘‘symmetrization’’ of one-sided distributions. Let  $\xi^2$  be a positive random variable with distribution function  $F(x)$ . Define  $\tilde{\xi} := \varepsilon\xi$  where  $\varepsilon$  denotes a Rademacher random variable with  $\Pr\{\varepsilon = \pm 1\} = 1/2$  which is independent of  $\xi$ . Let  $\tilde{F}(x)$  denote the distribution function of  $\tilde{\xi}$ . It satisfies the equation

$$\tilde{F}(x) = 1/2(1 + \operatorname{sgn}\{x\}F(x^2)), \quad (2.1)$$

We apply this symmetrization to the distribution of the squared singular values of the matrix  $\mathbf{W}$ . Introduce the following matrices

$$\mathbf{V} = \begin{pmatrix} \mathbf{W} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^* \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I}_{p_m} \\ \mathbf{I}_{p_0} & \mathbf{O} \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{V}} = \mathbf{V}\mathbf{J}$$

Here and in the what follows  $\mathbf{A}^*$  denotes the adjoined (transposed and complex conjugate) matrix  $\mathbf{A}$  and  $\mathbf{I}_k$  denotes unit matrix of order  $k$ . Note that  $\hat{\mathbf{V}}$  is Hermitian matrix. The eigenvalues of the matrix  $\hat{\mathbf{V}}$  are  $-s_1, \dots, -s_n, s_n, \dots, s_1$  and  $p_m - n$  zeros. Note that the symmetrization of the distribution function  $\mathcal{F}_n(x)$  is a function  $\tilde{\mathcal{F}}_n(x)$  which is the empirical distribution function of the non-zero eigenvalues of matrix  $\hat{\mathbf{V}}$ . By (2.1), we have

$$\Delta_n = \sup_x |\tilde{F}_n(x) - \tilde{G}_y(x)|,$$

where  $\tilde{F}_n(x) = \mathbf{E} \tilde{\mathcal{F}}_n(x)$  and  $\tilde{G}(x)$  denotes the symmetrization of the distribution function  $G(x)$ .

## 2.2 Truncation

We shall now modify the random matrix  $\mathbf{X}$  by truncation of its entries. Since the function  $G_y(x)$  is continuous with respect to  $y_l$  we may assume that  $y_l = \frac{n}{p_l}$ ,  $l = 1, \dots, m$ . Furthermore, there exists a constants  $c > 0$  and  $C > 0$  such that  $Cn \geq p_l \geq cn$  for any  $l = 1, \dots, m$ . We note that there exists a sequence  $\tau_n \rightarrow 0$  as  $n \rightarrow \infty$  such that  $\frac{1}{\tau_n^2} L_n(\tau_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Introduce the random variables  $X_{jk}^{(\nu, c)} = X_{jk}^{(\nu)} I_{\{|X_{jk}^{(\nu)}| \leq c\tau_n \sqrt{n}\}}$  and the matrix  $\mathbf{X}^{(\nu, c)} = \frac{1}{\sqrt{p_\nu}} (X_{jk}^{(\nu, c)})$ . Denote by  $s_1^{(c)} \geq \dots \geq s_n^{(c)}$  the singular values of the random matrix  $\mathbf{W}^{(c)} := \prod_{\nu=1}^m \mathbf{X}^{(\nu, c)}$ . Introduce the matrix  $\mathbf{V}^{(c)} := \begin{pmatrix} \mathbf{W}^{(c)} & \mathbf{O} \\ \mathbf{O} & \mathbf{W}^{(c)*} \end{pmatrix}$ .

We define its empirical distribution by  $\tilde{\mathcal{F}}_n^{(c)}(x) = \frac{1}{2n} \sum_{k=1}^n I_{\{s_k^{(c)} \leq x\}} + \frac{1}{2n} \sum_{k=1}^n I_{\{-s_k^{(c)} \leq x\}}$ . Let  $s_n(z)$  and  $s_n^{(c)}(z)$  denote Stieltjes transforms of the distribution functions  $\tilde{F}_n(x)$  and  $\tilde{F}_n^{(c)}(x) = \mathbf{E} \tilde{\mathcal{F}}_n^{(c)}(x)$  respectively. Define the resolvent matrices  $\mathbf{R} = (\hat{\mathbf{V}} - z\mathbf{I})^{-1}$  and  $\mathbf{R}^{(c)} = (\hat{\mathbf{V}}^{(c)} - z\mathbf{I})^{-1}$ , where  $\mathbf{I}$  denotes the unit matrix of corresponding dimension. Note that

$$s_n(z) = \frac{1}{2n} \mathbf{E} \operatorname{Tr} \mathbf{R} + \frac{1 - y_m}{2y_m z}, \quad \text{and} \quad s_n^{(c)}(z) = \frac{1}{2n} \mathbf{E} \operatorname{Tr} \mathbf{R}^{(c)} + \frac{1 - y_m}{2y_m z}.$$

Applying the resolvent equality

$$(\mathbf{A} + \mathbf{B} - z\mathbf{I})^{-1} = (\mathbf{A} - z\mathbf{I})^{-1} - (\mathbf{A} - z\mathbf{I})^{-1}\mathbf{B}(\mathbf{A} + \mathbf{B} - z\mathbf{I})^{-1},$$

we get

$$|s_n(z) - s_n^{(c)}(z)| \leq \frac{1}{2n} \mathbf{E} |\mathrm{Tr} \mathbf{R}^{(c)}(\mathbf{V} - \mathbf{V}^{(c)}) \mathbf{J} \mathbf{R}|. \quad (2.2)$$

Let

$$\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{X}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1)*} \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{(\nu,c)} = \begin{pmatrix} \mathbf{X}^{(\nu,c)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-\nu+1,c)*} \end{pmatrix}$$

Introduce the matrices

$$\mathbf{V}_{\alpha,\beta} = \prod_{q=a}^b \mathbf{H}^{(q)}, \quad \mathbf{V}_{\alpha,\beta}^{(c)} = \prod_{q=a}^b \mathbf{H}^{(q,c)}.$$

We have

$$\mathbf{V} - \mathbf{V}^{(c)} = \sum_{q=1}^{m-1} \mathbf{V}_{1,q-1}^{(c)} (\mathbf{H}^{(q)} - \mathbf{H}^{(q,c)}) \mathbf{V}_{q+1,m}. \quad (2.3)$$

Applying  $\max\{\|\mathbf{R}\|, \|\mathbf{R}^{(c)}\|\} \leq v^{-1}$ , inequality (2.2), and the representations (2.3) together, we get

$$|s_n(z) - s_n^{(c)}(z)| \leq \frac{C}{\sqrt{n}} \sum_{q=1}^m \mathbf{E}^{\frac{1}{2}} \|\mathbf{X}^{(q+1)} - \mathbf{X}^{(q+1,c)}\|_2^2 \frac{1}{\sqrt{n}} \mathbf{E}^{\frac{1}{2}} \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R} \mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2. \quad (2.4)$$

Applying well-known inequalities for matrix norms, we get

$$\mathbf{E} \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R} \mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2 \leq \frac{C}{v^4} \mathbf{E} \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{V}_{q+1,m}\|_2^2$$

In view of Lemma 5.2, we obtain

$$\mathbf{E} \|\mathbf{V}_{1,q-1}^{(c)} \mathbf{R} \mathbf{R}^{(c)} \mathbf{V}_{q+1,m}\|_2^2 \leq \frac{Cn}{v^4}. \quad (2.5)$$

Direct calculations show that

$$\frac{1}{n} \mathbf{E} \|\mathbf{X}^{(q)} - \mathbf{X}^{(q,c)}\|_2^2 \leq \frac{C}{n^2} \sum_{j,k=1}^n \mathbf{E} |X_{jk}^{(q)}|^2 I_{\{|X_{jk}^{(q)}| \geq c\tau_n \sqrt{n}\}} \leq CL_n(\tau_n). \quad (2.6)$$

Inequalities (2.4), (2.5) and 2.6) together imply

$$|s_n(z) - s_n^{(c)}(z)| \leq \frac{C\sqrt{L_n(\tau_n)}}{v^2}. \quad (2.7)$$

Furthermore, by definition of  $X_{jk}^{(c)}$ , we have

$$|\mathbf{E} X_{jk}^{(q,c)}| \leq \frac{1}{c\tau_n\sqrt{n}} \mathbf{E} |X_{jk}^{(q)}|^2 I_{\{|X_{jk}^{(q)}| \geq c\tau_n\sqrt{n}\}}.$$

This implies that

$$\|\mathbf{E} \mathbf{X}^{(q,c)}\|_2^2 \leq \frac{C}{n} \sum_{j=1}^{p_{q-1}} \sum_{k=1}^{p_q} |\mathbf{E} X_{jk}^{(q,c)}|^2 \leq \frac{CL_n(\tau_n)}{c\tau_n^2}. \quad (2.8)$$

We denote  $\tilde{\mathbf{H}}^{(\nu,c)} := \begin{pmatrix} \mathbf{X}^{(\nu,c)} - \mathbf{E} \mathbf{X}^{(\nu,c)} & \mathbf{O} \\ \mathbf{O} & (\mathbf{X}^{(\nu,c)} - \mathbf{E} \mathbf{X}^{(\nu,c)})^* \end{pmatrix}$  and define the respectively matrices  $\tilde{\mathbf{W}}^{(c)}$ ,  $\tilde{\mathbf{V}}^{(c)}$ ,  $\tilde{\mathbf{V}}_{a,b}^{(c)}$ . Denote by  $\tilde{\mathcal{F}}_n^{(c)}(x)$  the empirical distribution of the squared singular values of the matrix  $\tilde{\mathbf{V}}^{(c)} \mathbf{J}$ . Let  $\tilde{s}_n^{(c)}(z)$  denote the Stieltjes transform of the distribution function  $\tilde{F}_n^{(c)} = \mathbf{E} \tilde{\mathcal{F}}_n^{(c)}$ ,

$$\tilde{s}_n^{(c)}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\tilde{F}_n^{(c)}(x).$$

Similar to inequality (2.4) we get

$$|s_n^{(c)} - \tilde{s}_n^{(c)}(z)| \leq \sum_{q=0}^{m-1} \frac{1}{\sqrt{n}} \|\mathbf{E} \mathbf{X}^{(q,c)}\|_2 \frac{1}{\sqrt{n}} \mathbf{E}^{\frac{1}{2}} \|\tilde{\mathbf{V}}_{0,q}^{(c)} \mathbf{R}^{(c)} \tilde{\mathbf{R}}^{(c)} \tilde{\mathbf{V}}_{q+1,m}^{(c)}\|_2^2.$$

Analogously to inequality (2.5), we get

$$\frac{1}{n} \mathbf{E} \|\tilde{\mathbf{V}}_{0,q}^{(c)} \mathbf{R}^{(c)} \tilde{\mathbf{R}}^{(c)} \tilde{\mathbf{V}}_{q+1,m}^{(c)}\|_2^2 \leq \frac{C}{v^4}.$$

By inequality (2.8),

$$\|\mathbf{E} X^{(q,c)}\|_2 \leq \frac{C\sqrt{L_n(\tau_n)}}{c\tau_n}.$$

The last two inequalities together imply that

$$|s_n^{(c)} - \tilde{s}_n^{(c)}(z)| \leq \frac{C\sqrt{L_n(\tau_n)}}{\sqrt{n}\tau_n v^2} \quad (2.9)$$

Inequalities (2.7) and (2.9) together imply that matrices  $\mathbf{W}$  and  $\tilde{\mathbf{W}}^{(c)}$  have the same limit distribution. In the what follows we may assume without loss of generality that for any  $n \geq 1$  and  $\nu = 1, \dots, m$  and any  $l = 1, \dots, m$  and  $j = 1, \dots, p_{l-1}$ ,  $k = 1, \dots, p_l$ ,

$$\mathbf{E} X_{jk}^{(\nu)} = 0, \quad \mathbf{E} X_{jk}^{(\nu)2} = 1, \quad \text{and} \quad |X_{jk}^{(\nu)}| \leq c\tau_n\sqrt{n} \quad (2.10)$$

with

$$\tau_n \rightarrow 0 \quad \text{and} \quad \frac{L_n(\tau_n)}{\tau_n^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

### 3 The proof of the main result for $m = 2$

Introduce the matrices  $\mathbf{H}^{(q)}$ ,  $q = 1, \dots, m$ , and  $\mathbf{J}$  by equalities

$$\mathbf{H}^{(q)} = \begin{pmatrix} \mathbf{X}^{(q)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-q+1)*} \end{pmatrix}, \quad \mathbf{J} := \begin{pmatrix} \mathbf{O} & \mathbf{I}_{p_m} \\ \mathbf{I}_{p_0} & \mathbf{O} \end{pmatrix},$$

Recall that  $\mathbf{A}^*$  denotes the adjoint matrix  $\mathbf{A}$  and  $\mathbf{I}_k$  denotes the identity matrix of order  $k$  (sometimes we shall omit the sub-index in the notation of the unit matrix). Let  $\widehat{\mathbf{V}} := \prod_{q=1}^m \mathbf{H}^{(q)} \mathbf{J}$ , and let  $\mathbf{R}(z)$  denote the resolvent matrix of the matrix  $\widehat{\mathbf{V}}$ ,

$$\mathbf{R}(z) := (\widehat{\mathbf{V}} - z\mathbf{I}_{p_m+p_0})^{-1}.$$

We note that the symmetrization of the distribution function  $G_{\mathbf{y}}(x)$  has the Stieltjes transform  $s_{\mathbf{y}}(z)$  (in the what follows we shall omit index  $\mathbf{y}$  in the notation for this Stieltjes transform) which satisfies the following equation

$$1 + zs(z) - \frac{s(z)}{z} \prod_{l=1}^m (1 - y_l - zy_l s(z)) = 0. \quad (3.1)$$

First, we prove Theorem 1.1 for  $m = 2$ . We start from the simple equality

$$1 + zs_n(z) = \frac{1}{2n} \mathbf{E} \operatorname{Tr} \mathbf{V} \mathbf{R}(z). \quad (3.2)$$

Using the definition of the matrices  $\mathbf{V}$ ,  $\mathbf{H}^{(q)}$  and  $\mathbf{J}$ , we get

$$1 + zs_n(z) = \frac{1}{2n\sqrt{p_1}} \sum_{j=1}^n \sum_{k=1}^{p_1} \mathbf{E} X_{jk}^{(1)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{kj} + \frac{1}{2n\sqrt{p_2}} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} \mathbf{E} X_{jk}^{(2)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j+p_1, k+n}. \quad (3.3)$$

In the what follows we shall use the notation  $\varepsilon_n(z)$  as generic error function such that  $|\varepsilon_n(z)| \leq \frac{C\tau_n}{v^4}$ . By Lemma 5.7 in the Appendix, we get

$$\begin{aligned} 1 + zs_n(z) &= \frac{1}{2np_1} \sum_{j=1}^n \sum_{k=1}^{p_1} \mathbf{E} \left[ \frac{\partial \mathbf{H}^{(2)} \mathbf{J} \mathbf{R}}{\partial X_{jk}^{(1)}} \right]_{kj} \\ &\quad + \frac{1}{2np_2} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} \mathbf{E} \left[ \frac{\partial \mathbf{H}^{(2)} \mathbf{J} \mathbf{R}}{\partial X_{jk}^{(2)}} \right]_{j+p_1, k+n} + \varepsilon_n(z), \end{aligned} \quad (3.4)$$

where  $|\varepsilon_n(z)| \leq \frac{C\tau_n}{v^4}$ .

Put  $n_1 = \max\{2p_1, n + p_2\}$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_{n_1}$  be an orthonormal basis of  $\mathbb{R}^{n_1}$ . First we note that, for  $j = 1, \dots, n$  and for  $k = 1, \dots, p_1$

$$\frac{\partial \mathbf{H}^{(1)}}{\partial X_{jk}^{(1)}} = \frac{1}{\sqrt{p_1}} \mathbf{e}_j \mathbf{e}_k^T, \quad \frac{\partial \mathbf{H}^{(2)}}{\partial X_{jk}^{(2)}} = \frac{1}{\sqrt{p_1}} \mathbf{e}_{k+p_1} \mathbf{e}_{j+p_2}^T, \quad (3.5)$$

and for  $j = 1, \dots, p_1$  and  $k = 1, \dots, p_2$ ,

$$\frac{\partial \mathbf{H}^{(1)}}{\partial X_{jk}^{(2)}} = \frac{1}{\sqrt{p_2}} \mathbf{e}_{k+n} \mathbf{e}_{j+p_1}^T, \quad \frac{\partial \mathbf{H}^{(2)}}{\partial X_{jk}^{(2)}} = \frac{1}{\sqrt{p_2}} \mathbf{e}_j \mathbf{e}_k^T. \quad (3.6)$$

We first compute the derivatives of the resolvent matrix as follows

$$\begin{aligned} \frac{\partial \mathbf{R}}{\partial X_{jk}^{(1)}} &= -\frac{1}{\sqrt{p_1}} \mathbf{R} \mathbf{e}_j \mathbf{e}_k^T \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_1}} \mathbf{R} \mathbf{H}^{(1)} \mathbf{e}_{k+p_1} \mathbf{e}_{j+p_2}^T \mathbf{J} \mathbf{R}, \\ \frac{\partial \mathbf{R}}{\partial X_{jk}^{(2)}} &= -\frac{1}{\sqrt{p_2}} \mathbf{R} \mathbf{e}_{k+n} \mathbf{e}_{j+p_1}^T \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_2}} \mathbf{R} \mathbf{H}^{(1)} \mathbf{e}_j \mathbf{e}_k^T \mathbf{J} \mathbf{R}, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \frac{\partial (\mathbf{H}^{(2)} \mathbf{J} \mathbf{R})}{\partial X_{jk}^{(1)}} &= \frac{1}{\sqrt{p_1}} \mathbf{e}_{k+p_1} \mathbf{e}_{j+p_2}^T \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_1}} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{e}_j \mathbf{e}_k^T \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \\ &\quad - \frac{1}{\sqrt{p_1}} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)} \mathbf{e}_{k+p_1} \mathbf{e}_{j+p_2}^T \mathbf{J} \mathbf{R}, \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \frac{\partial (\mathbf{H}^{(2)} \mathbf{J} \mathbf{R})}{\partial X_{jk}^{(2)}} &= \frac{1}{\sqrt{p_2}} \mathbf{e}_j \mathbf{e}_k^T \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_2}} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{e}_{k+n} \mathbf{e}_{j+p_1}^T \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \\ &\quad - \frac{1}{\sqrt{p_2}} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)} \mathbf{e}_j \mathbf{e}_k^T \mathbf{J} \mathbf{R}. \end{aligned} \quad (3.9)$$

The equalities (3.3) and (3.8) and (3.9) together imply that

$$1 + z s_n(z) = A_1 + A_2 + A_3 + \varepsilon_n(z), \quad (3.10)$$

where

$$\begin{aligned} A_1 &:= -\frac{1}{2np_1} \sum_{j=1}^n \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{jk}^2 - \frac{1}{2np_2} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j+p_1, k+n}^2, \\ A_2 &:= -\frac{1}{2np_1} \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k, k+p_1} \sum_{j=1}^n [\mathbf{J} \mathbf{R}]_{j+p_2, j}, \\ A_3 &:= -\frac{1}{2np_2} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k+p_1, k} \sum_{j=1}^{p_2} [\mathbf{J} \mathbf{R}]_{j, j+n}. \end{aligned}$$

We prove that the first summand is negligible and the main asymptotic terms are given by  $A_2$  and  $A_3$ . We now start the investigation of these summands.



**Lemma 3.1.** *Under conditions of Theorem 1.1 we have*

$$\begin{aligned} \left| A_2 + \left( \frac{1}{2} \frac{1}{p_1} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k, k+p_1} \right) \left( \frac{1}{n} \sum_{j=1}^n \mathbf{E} [\mathbf{J} \mathbf{R}]_{j+p_2, j} \right) \right| &\leq \frac{C}{nv^4}, \\ \left| A_3 + \left( \frac{1}{2} \frac{1}{p_2} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k+p_1, k} \right) \left( \frac{1}{n} \sum_{j=1}^{p_2} \mathbf{E} [\mathbf{J} \mathbf{R}]_{j, j+n} \right) \right| &\leq \frac{C}{nv^4}. \end{aligned} \quad (3.11)$$

*Proof.* Applying Lemma 5.5 with  $m = 2$  and  $a = 1$  and Lemma 5.4 (see Appendix), we obtain

$$\begin{aligned} &\left| A_2 + \left( \frac{1}{2} \frac{1}{p_1} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k, k+p_1} \right) \left( \frac{1}{n} \sum_{j=1}^n \mathbf{E} [\mathbf{J} \mathbf{R}]_{j+p_2, j} \right) \right| \\ &\leq \mathbf{E}^{\frac{1}{2}} \left| \frac{1}{2} \frac{1}{p_1} \left( \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k, k+p_1} - \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k, k+p_1} \right) \right|^2 \\ &\quad \times \mathbf{E}^{\frac{1}{2}} \left| \frac{1}{n} \sum_{j=1}^n ([\mathbf{J} \mathbf{R}]_{j+p_2, j} - \mathbf{E} \mathbf{J} \mathbf{R})_{j+p_2, j} \right|^2 \leq \frac{C}{nv^4} \end{aligned}$$

Similar we prove the second inequality in (3.11). Thus the Lemma is proved.  $\square$

Note that

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbf{E} [\mathbf{J} \mathbf{R}]_{j+p_2, j} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \mathbf{R}_{jj} = s_n(z), \\ \frac{1}{n} \sum_{k=1}^{p_2} \mathbf{E} [\mathbf{J} \mathbf{R}]_{k, k+n} &= \frac{1}{n} \sum_{j=1}^{p_2} \mathbf{E} [\mathbf{R}]_{j+n, j+n} = s_n(z) - \frac{1-y_2}{y_2 z}. \end{aligned} \quad (3.12)$$

Lemma 3.1, equalities (3.12) and the definition of matrix  $\mathbf{H}$  together imply

$$A_2 = -\frac{1}{2} s_n(z) \frac{1}{2p_1 \sqrt{p_2}} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} \mathbf{E} X_{jk}^{(2)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j, k+n} + \varepsilon_n(z), \quad (3.13)$$

and similar

$$A_3 = -\frac{1}{2} \left( s_n(z) - \frac{1-y_2}{y_2 z} \right) \frac{1}{2p_2 \sqrt{p_1}} \sum_{j=1}^n \sum_{k=1}^{p_1} \mathbf{E} X_{jk}^{(1)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{k+p_1, j} + \varepsilon_n(z).$$

Applying Lemma 5.7 and equalities (3.5)–(3.9), we get

$$\begin{aligned}
A_2 &= -s_n(z) \frac{1}{2p_1 p_2} \left( p_1 - \sum_{j=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{j,j} \right) \sum_{k=1}^{p_2} \mathbf{E} [\mathbf{J} \mathbf{R}]_{k,k+n} + A_4 + \varepsilon_n(z) \\
A_3 &= -\left( s_n(z) - \frac{1-y_2}{y_2 z} \right) \frac{1}{2p_2 p_1} \left( p_1 - \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k+p_1, k+p_1} \right) \sum_{j=1}^n \mathbf{E} [\mathbf{J} \mathbf{R}]_{j+p_2, j} \\
&\quad + A_5 + \varepsilon_n(z), \tag{3.14}
\end{aligned}$$

where

$$\begin{aligned}
A_4 &= s_n(z) \frac{1}{2p_1 p_2} \sum_{j=1}^{p_1} \sum_{k=1}^{p_2} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j, k+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j+p_1, k+n}, \\
A_5 &= \left( s_n(z) - \frac{1-y_2}{y_2 z} \right) \frac{1}{2p_1 p_2} \sum_{j=1}^n \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k+p_1, j} [\mathbf{J} \mathbf{R}]_{k, j}
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{j,j} + \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)}]_{k+p_1, k+p_1} &= \mathbf{E} \operatorname{Tr} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} \mathbf{H}^{(1)} \\
&= \mathbf{E} \operatorname{Tr} \mathbf{H}^{(1)} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R} = \mathbf{E} \operatorname{Tr} \mathbf{V} \mathbf{J} \mathbf{R}.
\end{aligned}$$

By resolvent equality  $\mathbf{I} + z\mathbf{R} = \mathbf{V} \mathbf{J} \mathbf{R}$ , we have

$$\frac{1}{2n} \left( \sum_{j=1}^n \mathbf{E} [\mathbf{H}^{(1)} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j,j} + \sum_{j=1}^{p_2} \mathbf{E} [\mathbf{H}^{(1)} \mathbf{H}^{(2)} \mathbf{J} \mathbf{R}]_{j+n, j+n} \right) = 1 + z s_n(z). \tag{3.16}$$

Equalities (3.2), (3.14) and (3.16) together imply

$$A_2 + A_3 = \frac{s_n(z)}{z} (1 - y_1 - z y_1 s_n(z)) (1 - y_2 - z y_2 s_n(z)) + A_4 + A_5 + \varepsilon_n(z). \tag{3.17}$$

**Lemma 3.2.** *Under condition of Theorem 1.1 we have*

$$\max\{|A_1|, |A_4|, |A_5|\} \leq \frac{C}{nv^2}. \tag{3.18}$$

*Proof.* We shall consider the bound for the quantity  $A_5$  only. The others are similar. By Hölder's inequality, we have

$$|A_4| \leq \frac{1}{n^2 v} \mathbf{E} \|\mathbf{H}^{(2)} \mathbf{J} \mathbf{R}\|_2^2,$$

where  $\|\cdot\|_2$  denotes Hilbert-Schmidt norm of matrix. Continuing the last inequality, we may write

$$|A_4| \leq \frac{C}{n^2 v^3} \mathbf{E} \|\mathbf{H}^{(2)}\|_2^2.$$

A simple calculation shows that

$$\mathbf{E} \|\mathbf{H}^{(2)}\|_2^2 \leq Cn \quad (3.19)$$

The last two inequalities together imply

$$|A_5| \leq \frac{C}{nv^3}.$$

Thus the Lemma is proved.  $\square$

Relation (3.17) and Lemma 3.2 together imply

$$1 + zs_n(z) = \frac{s_n(z)}{z}(1 - y_1 - zy_1s_n(z))(1 - y_2 - zy_2s_n(z)) + \delta_n(z) \quad (3.20)$$

where  $|\delta_n(z)| \leq \frac{C}{nv^4} + \frac{CT_n}{v^2}$ .

**Lemma 3.3.** *Under conditions of Theorem 1.1 for  $v \geq 3$  we have for sufficiently large  $n$ ,*

$$|s(z) - s_n(z)| \leq \frac{C|\delta_n(z)|}{v}. \quad (3.21)$$

*Proof.* We rewrite the equation (3.20) as follows

$$1 + zs_n(z) = \frac{1}{z}s_n(z)(1 - y_1)(1 - y_2) - zs_n^3(z) + s^2(z)(y_1(1 - y_2) + y_2(1 - y_1)) + \delta_n(z). \quad (3.22)$$

Introduce the notations

$$\begin{aligned} d &= \frac{(1 - y_1)(1 - y_2)}{z} \\ d_n &= z(s_n(z)^2 + s_n(z)s(z) + s^2(z)) \\ h_n &= (s(z) + s_n(z))(y_1(1 - y_2) + y_2(1 - y_1)). \end{aligned}$$

Then we may rewrite equality (3.22) as follows

$$s_n(z) - s(z) = \frac{\delta_n(z)}{-z + d + d_n + h_n}$$

First we note that

$$\operatorname{Im}\{d\} \leq 0. \quad (3.23)$$

Furthermore, note that

$$|zs_n(z)| \leq 1 + \frac{\mathbf{E}^{\frac{1}{2}}\|V\|_2^2}{nv} \leq 1 + \frac{1}{v}. \quad (3.24)$$

Using that  $\max\{|s(z)|, |s_n(z)|\} \leq \frac{1}{v}$  and (3.24), we get

$$\max\{|h_n|, |d_n(z)|\} \leq \left(1 + \frac{1}{v}\right)\frac{1}{v}. \quad (3.25)$$

We take  $v \geq 3$ . Equalities (3.23), (3.25) together complete the proof of lemma.  $\square$

The last Lemma implies that in  $\mathcal{C}^+$  there exists an open set with non-empty interior such that on this set  $s_n(z)$  converges to  $s(z)$ . The Stieltjes transform of our random variables is an analytic function on  $\mathcal{C}^+$  and locally bounded (that is  $|s_n(z)| \leq v^{-1}$  for any  $v > 0$ ). By Montel's Theorem (see, for instance, [16], p. 153, Theorem 2.9)  $s_n(z)$  converges to  $s(z)$  uniformly on any compact set  $\mathcal{K} \subset \mathcal{C}^+$  in the upper half-plane. This implies that  $\Delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the proof of Theorem 1.1 in the case  $m = 2$  is complete.

## 4 The proof of the main result in the general case

Recall that  $\mathbf{H}^{(q)}$  and  $\mathbf{J}$  are defined by following equalities, with  $q = 1, \dots, m$ ,

$$\mathbf{H}^{(q)} = \begin{pmatrix} \mathbf{X}^{(q)} & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{(m-q+1)*} \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I}_{p_m} \\ \mathbf{I}_{p_0} & \mathbf{O} \end{pmatrix}, \quad (4.1)$$

where  $\mathbf{I}_k$  denotes the identity matrix of dimension  $k$ . Note that  $\mathbf{H}^{(q)}$  is a  $(p_{q-1} + p_{m-q+1}) \times (p_q + p_{m-q})$ -matrix. Let  $\mathbf{V} := \prod_{q=1}^m \mathbf{H}^{(q)} \mathbf{J}$ , and denote by  $\mathbf{R}$  its resolvent matrix,

$$\mathbf{R} := (\mathbf{V} - z\mathbf{I})^{-1}.$$

We shall use the following ‘‘symmetrization’’ of one-sided distributions. Let  $\xi^2$  be a positive random variable. Define  $\tilde{\xi} := \varepsilon\xi$  where  $\varepsilon$  denotes a Rademacher random variable with  $\Pr\{\varepsilon = \pm 1\} = 1/2$  which independent of  $\xi$ . We apply this symmetrization to the distribution of the singular values of the matrix  $\mathbf{X}^2$ . Note that the symmetrized distribution function  $\tilde{F}_n(x)$  satisfies the equation

$$\tilde{F}_n(x) = 1/2(1 + \operatorname{sgn}\{x\} F_n(x^2)),$$

and this function is the empirical spectral distribution function of the random matrix  $\mathbf{W}$ . Furthermore, we note that the symmetrization of the distribution function  $G(x)$  has the Stieltjes transform  $s(z)$  which satisfies the following equation

$$1 + zs(z) - \frac{s(z)}{z} \prod_{\nu=1}^m (1 - y_\nu - zy_\nu s(z)) = 0. \quad (4.2)$$

In the rest of paper we shall prove that Stieltjes transform of expected spectral distribution function  $s_n(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\mathbf{E} \tilde{F}_n(x)$  satisfies the equation

$$1 + zs_n(z) - \frac{s_n(z)}{z} \prod_{\nu=1}^m (1 - y_\nu - zy_\nu s_n(z)) = \delta_n(z), \quad (4.3)$$

where  $\delta_n(z)$  denotes some function such that  $\delta_n(z) \rightarrow 0$  as  $n \rightarrow \infty$ .

We start from the simple equality

$$1 + zs_n(z) = \frac{1}{2n} \operatorname{Tr} \mathbf{V} \mathbf{R}. \quad (4.4)$$

By definition of the matrices  $\mathbf{V}$ ,  $\mathbf{H}^{(q)}$  and  $\mathbf{J}$ , we get

$$1 + z s_n(z) = \frac{1}{2n\sqrt{p_1}} \sum_{j=1}^{p_0} \sum_{k=1}^{p_1} \mathbf{E} X_{jk}^{(1)} [\mathbf{V}_{2,m} \mathbf{J} \mathbf{R}]_{kj} + \frac{1}{2n\sqrt{p_m}} \sum_{j=1}^{p_{m-1}} \sum_{k=1}^{p_m} \mathbf{E} X_{jk}^{(m)} [\mathbf{V}_{2,m} \mathbf{J} \mathbf{R}]_{j+p_1, k+p_0}, \quad (4.5)$$

where  $\mathbf{V}_{\alpha, \beta} = \prod_{q=\alpha}^{\beta} \mathbf{H}^{(q)}$ . To simplify the calculations we assume that  $X_{jk}^{(\nu)}$  are i.i.d. Gaussian random variables, and we shall use the following well-known equality for a Gaussian r.v.  $\xi$

$$\mathbf{E} \xi f(\xi) = \mathbf{E} f'(\xi), \quad (4.6)$$

for every differentiable function  $f(x)$  such that both sides exist. By Lemma 5.7, we obtain that the error involved in this Gaussian assumption is of order  $O(\tau_n)$ . Recall the notation  $\varepsilon_n(z)$  for generic error functions such that  $|\varepsilon_n(z)| \leq C\tau_n v^{-q}$ , for some  $q \geq 0$ . Let  $p_0 = n$  and  $n_1 = \max_{0 \leq \nu \leq m-1} \{p_\nu + p_{m-\nu}\}$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_{n_1}$  be an orthonormal basis of  $\mathbb{R}^{n_1}$ . First we note that, for  $j = 1, \dots, p_{q-1}$  and  $k = 1, \dots, p_q$ ,

$$\frac{\partial \mathbf{H}^{(q)}}{\partial X_{jk}^{(q)}} = \frac{1}{\sqrt{p_q}} \mathbf{e}_j \mathbf{e}_k^T, \quad \frac{\partial \mathbf{H}^{(m-q+1)}}{\partial X_{jk}^{(q)}} = \frac{1}{\sqrt{p_q}} \mathbf{e}_{k+p_{m-q}} \mathbf{e}_{j+p_{m-q+1}}^T, \quad (4.7)$$

and, for  $j = 1, \dots, p_{m-q}$  and  $k = 1, \dots, p_{m-q+1}$

$$\frac{\partial \mathbf{H}^{(m-q+1)}}{\partial X_{jk}^{(m-q+1)}} = \frac{1}{\sqrt{p_{m-q+1}}} \mathbf{e}_j \mathbf{e}_k^T, \quad \frac{\partial \mathbf{H}^{(q)}}{\partial X_{jk}^{(m-q+1)}} = \frac{1}{\sqrt{p_{m-q+1}}} \mathbf{e}_{k+p_{q-1}} \mathbf{e}_{j+p_q}^T. \quad (4.8)$$

Now we may compute the derivatives of the matrix  $\mathbf{V}_{2,m} \mathbf{J} \mathbf{R}$  as follows

$$\begin{aligned} \frac{\partial (\mathbf{V}_{2,m} \mathbf{J} \mathbf{R})}{\partial X_{jk}^{(1)}} &= \frac{1}{\sqrt{p_1}} \mathbf{V}_{2,m-1} \mathbf{e}_{k+p_{m-1}} \mathbf{e}_{j+p_m}^T \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_1}} \mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \\ &\quad - \frac{1}{\sqrt{p_1}} \mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1} \mathbf{e}_{k+p_{m-1}} \mathbf{e}_{j+p_m}^T \mathbf{J} \mathbf{R}. \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \frac{\partial (\mathbf{V}_{2,m} \mathbf{J} \mathbf{R})}{\partial X_{jk}^{(m)}} &= \frac{1}{\sqrt{p_m}} \mathbf{V}_{2,m-1} \mathbf{e}_j \mathbf{e}_k^T \mathbf{J} \mathbf{R} - \frac{1}{\sqrt{p_m}} \mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{e}_{k+n} \mathbf{e}_{j+p_1}^T \mathbf{V}_{2,m-1} \mathbf{J} \mathbf{R} \\ &\quad - \frac{1}{\sqrt{p_m}} \mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1} \mathbf{e}_j \mathbf{e}_k^T \mathbf{J} \mathbf{R}. \end{aligned} \quad (4.10)$$

The equalities (4.5) and (4.9) together imply

$$1 + z s_n(z) = A_1 + A_2 + A_3 + B_1 + B_2 + B_3 + \varepsilon_n(z), \quad (4.11)$$

where

$$\begin{aligned}
A_1 &:= \frac{1}{2np_1} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{V}_{2,m-1}]_{k,k+p_{m-1}} \sum_{j=1}^n [\mathbf{JR}]_{j+p_m,j}, \\
A_2 &= -\frac{1}{2np_1} \mathbf{E} \sum_{j=1}^n \sum_{k=1}^{p_1} [\mathbf{V}_{2,m} \mathbf{JR}]_{k,j}^2, \\
A_3 &:= -\frac{1}{2np_1} \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{V}_{2,m} \mathbf{JR} \mathbf{V}_{1,m-1}]_{k,k+p_{m-1}} \sum_{j=1}^n [\mathbf{JR}]_{j+p_m,j}
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \frac{1}{2np_m} \mathbf{E} \sum_{j=1}^{p_{m-1}} [\mathbf{V}_{1,m-1}]_{j+p_1,j} \sum_{k=1}^{p_m} [\mathbf{JR}]_{k,k+n}, \\
B_2 &= -\frac{1}{2np_m} \mathbf{E} \sum_{j=1}^{p_{m-1}} \sum_{k=1}^{p_m} [\mathbf{V}_{2,m} \mathbf{JR}]_{j+p_1,k+n}^2, \\
B_3 &:= -\frac{1}{2np_m} \mathbf{E} \sum_{j=1}^{p_{m-1}} [\mathbf{V}_{2,m} \mathbf{JR} \mathbf{V}_{1,m-1}]_{j+p_1,j} \sum_{k=1}^{p_m} [\mathbf{JR}]_{k,k+n}.
\end{aligned}$$

**Lemma 4.1.** *Under the conditions of Theorem 1.1 there exists a constant  $C > 0$  such that the following inequality holds*

$$\max\{|A_2|, |B_2|\} \leq \frac{C}{nv^2}. \quad (4.12)$$

*Proof.* Note that

$$|A_2| \leq \frac{1}{n^2} \mathbf{E} \|\mathbf{V}_{2,m} \mathbf{JR}\|_2^2 \leq \frac{C}{n^2 v^2} \mathbf{E} \|\mathbf{V}_{2,m}\|_2^2 \quad (4.13)$$

By Lemma 5.2,

$$\mathbf{E} \|\mathbf{V}_{2,m}\|_2^2 \leq Cn \quad (4.14)$$

The last two inequalities conclude the proof. The bound for  $|B_2|$  is similar. Thus the Lemma is proved.  $\square$

**Lemma 4.2.** *Under conditions of Theorem 1.1 there exists a constant  $C > 0$  such that the following inequality holds*

$$\max\{|A_1|, |B_1|\} \leq \frac{C}{nv^2}.$$

*Proof.* We consider the quantity  $A_1$  only. The bound for  $B_1$  is similar. By Lemma 5.5, we have

$$|A_1| - \frac{1}{2p_1} \sum_{k=1}^{p_1} \mathbf{E} [\mathbf{V}_{1,m-1}]_{k,k+n} \frac{1}{n} \sum_{j=1}^n \mathbf{E} [\mathbf{JR}]_{j+n,j} \leq \frac{C}{nv^2}.$$

Direct calculation shows that

$$\mathbf{E}[\mathbf{V}_{1,m-1}]_{k,k+n} = 0.$$

Thus the Lemma is proved.  $\square$

**Lemma 4.3.** *Under conditions of Theorem 1.1 there exists a constant  $C > 0$  such that the following inequality holds*

$$\begin{aligned} |A_3 + \frac{1}{2p_1} \sum_{k=1}^{p_1} \mathbf{E}[\mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1}]_{k,k+p_{m-1}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}[\mathbf{J} \mathbf{R}]_{j+p_m,j}| &\leq \frac{C}{nv^2}, \\ |B_3 + \frac{1}{2n} \sum_{k=1}^{p_{m-1}} \mathbf{E}[\mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1}]_{k+p_1,k} \frac{1}{p_m} \sum_{j=1}^{p_m} \mathbf{E}[\mathbf{J} \mathbf{R}]_{j,j+n}| &\leq \frac{C}{nv^2}. \end{aligned}$$

*Proof.* Applying Hölder's inequality and Lemmas 5.5 and 5.4 together, we get

$$\begin{aligned} &|A_3 + \frac{1}{2p_1} \sum_{k=1}^{p_1} \mathbf{E}[\mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1}]_{k,k+p_{m-1}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}[\mathbf{J} \mathbf{R}]_{j+p_m,j}| \\ &\leq \mathbf{E}^{\frac{1}{2}} \left| \frac{1}{2n} \left( \sum_{k=1}^{p_1} [\mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1}]_{k,k+p_{m-1}} - \mathbf{E} \sum_{k=1}^{p_1} [\mathbf{V}_{2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-1}]_{k,k+p_{m-1}} \right) \right|^2 \\ &\quad \times \mathbf{E}^{\frac{1}{2}} \left| \frac{1}{n} \left( \sum_{j=1}^n [\mathbf{J} \mathbf{R}]_{j+p_m,j} - \mathbf{E} \sum_{j=1}^n [\mathbf{J} \mathbf{R}]_{j+p_m,j} \right) \right|^2 \leq \frac{C}{nv^2}. \end{aligned}$$

Thus the Lemma is proved.  $\square$

Introduce the following notations, for  $\alpha, \beta = 1, \dots, m$ ,

$$f_{\alpha,\beta} = \frac{1}{p_{\alpha-1}} \sum_{k=1}^{p_{\alpha-1}} \mathbf{E}[\mathbf{V}_{\alpha,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,\beta}]_{k,k+p_\beta}, \quad g_{\alpha,\beta} = \frac{1}{p_{\beta+1}} \sum_{k=1}^{p_\beta} \mathbf{E}[\mathbf{V}_{\alpha,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,\beta}]_{k+p_{\alpha-1},k},$$

and

$$f_{m+1,0} = \frac{1}{p_m} \sum_{k=1}^{p_m} \mathbf{E}[\mathbf{J} \mathbf{R}]_{k,k+p_0}, \quad g_{m+1,0} = \frac{1}{p_0} \sum_{k=1}^n \mathbf{E}[\mathbf{J} \mathbf{R}]_{k+p_m,k},$$

It is straightforward to check that

$$\begin{aligned} f_{m+1,0} &= \frac{1}{p_m} \sum_{j=1}^{p_m} \mathbf{E}[\mathbf{R}]_{k+n,k+n} = \frac{1}{z} (1 - y_m - zy_m s_n(z)), \\ g_{m+1,0} &= \frac{1}{p_0} \sum_{j=1}^n \mathbf{E}[\mathbf{R}]_{j,j} = s_n(z). \end{aligned} \tag{4.15}$$

By Lemma 4.3 and equality (4.15), we may write

$$A_3 + B_3 = -\frac{1}{2}s_n(z)f_{2,m-1} - \frac{1}{2}(-y_m s_n(z) + \frac{1-y_m}{z})g_{2,m-1} + \varepsilon_n(z). \quad (4.16)$$

Now we investigate the behavior of the coefficients  $f_{\alpha,m-\alpha+1}$  and  $g_{\alpha,m-\alpha+1}$ , for  $\alpha = 2, \dots, m$ . Assume that  $\alpha \leq m - \alpha$ . We have

$$\begin{aligned} f_{\alpha,m-\alpha+1} &= \frac{1}{p_{\alpha-1}\sqrt{p_\alpha}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} X_{j,k}^{(\alpha)} [\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha+1}]_{k,j+p_{m-\alpha+1}} \\ g_{\alpha,m-\alpha} &= \frac{1}{p_{m-\alpha}\sqrt{p_{m-\alpha+1}}} \sum_{j=1}^{p_{m-\alpha}} \sum_{k=1}^{p_{m-\alpha+1}} \mathbf{E} X_{j,k}^{(m-\alpha+1)} [\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha+1}]_{j+p_{\alpha-1},k}. \end{aligned} \quad (4.17)$$

It is straightforward to check that

$$\begin{aligned} \frac{\partial(\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha+1})}{\partial X_{j,k}^{(\alpha)}} &= \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{\alpha+1,m-\alpha} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1} I\{\alpha \leq m - \alpha\} \\ &+ \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{[\alpha+1,m]} \mathbf{JRV}_{1,\alpha-1} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{\alpha+1,m-\alpha+1} I\{\alpha \leq m - \alpha\} + \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \\ &\quad - \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{[\alpha+1,m]} \mathbf{JRV}_{1,\alpha-1} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1} \\ &\quad - \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1}. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial(\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha})}{\partial X_{j,k}^{(m-\alpha+1)}} &= \frac{1}{\sqrt{p_{m-\alpha+1}}} \mathbf{V}_{\alpha+1,m-\alpha} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1} I\{\alpha \leq m - \alpha\} \\ &+ \frac{1}{\sqrt{p_{m-\alpha+1}}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,\alpha-1} \mathbf{e}_{k+p_{\alpha-1}} \mathbf{e}_{j+p_\alpha}^T \mathbf{V}_{\alpha+1,m-\alpha+1} I\{\alpha \leq m - \alpha\} + \frac{1}{\sqrt{p_\alpha}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha} \mathbf{e}_j \mathbf{e}_k^T \\ &\quad - \frac{1}{\sqrt{p_{m-\alpha+1}}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,\alpha-1} \mathbf{e}_{k+p_{\alpha-1}} \mathbf{e}_{j+p_\alpha}^T \mathbf{V}_{\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1} \\ &\quad - \frac{1}{\sqrt{p_{m-\alpha+1}}} \mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1}. \end{aligned}$$

Applying the Lemmas 5.7 and 5.5, we obtain the following relation

$$f_{\alpha,m+1-\alpha} = D_1 + \dots + D_5 + \varepsilon_n(z),$$



where

$$\begin{aligned}
D_1 &= \frac{1}{p_{\alpha-1}p_\alpha} \mathbf{E} \sum_{j=1}^{p_\alpha} [\mathbf{V}_{\alpha+1, m-\alpha}]_{k, k+p_{m-\alpha}} \sum_{k=1}^{p_{\alpha-1}} [\mathbf{V}_{m-\alpha+2, m} \mathbf{JRV}_{1, m-\alpha+1}]_{j+p_{m-\alpha+1}, j+p_{m-\alpha+1}} I\{\alpha \leq m-\alpha\} \\
D_2 &= \frac{1}{p_{\alpha-1}p_\alpha} \mathbf{E} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, \alpha-1}]_{k, j} [\mathbf{V}_{\alpha+1, m-\alpha+1}]_{k, j+p_{m-\alpha+1}} I\{\alpha \leq m-\alpha\} \\
D_3 &= \frac{1}{p_\alpha} \sum_{k=1}^{p_\alpha} \mathbf{E} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, m-\alpha}]_{k, k+p_{m-\alpha}} \\
D_4 &= -\frac{1}{p_{\alpha-1}p_\alpha} \mathbf{E} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, \alpha-1}]_{k, j} [\mathbf{V}_{\alpha+2, m} \mathbf{JRV}_{1, m-\alpha}]_{k, j+p_{m-\alpha+1}} \\
D_5 &= -\frac{1}{p_{\alpha-1}p_\alpha} \mathbf{E} \sum_{k=1}^{p_\alpha} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, m-\alpha}]_{k, k+p_{m-\alpha}} \\
&\quad \times \sum_{j=1}^{p_{\alpha-1}} [\mathbf{V}_{m-\alpha+2, m} \mathbf{JRV}_{1, m-\alpha+1}]_{j+p_{m-\alpha+1}, j+p_{m-\alpha+1}}.
\end{aligned}$$

Similar we get the representation for  $g_{\alpha, m-\alpha}$ .

$$g_{\alpha, m+1-\alpha} = \bar{D}_1 + \dots + \bar{D}_5 + \varepsilon_n(z),$$

where

$$\begin{aligned}
\bar{D}_1 &= \frac{1}{p_{m-\alpha+1}p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1, m-\alpha}]_{j+p_\alpha, j} \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{m-\alpha+2, m} \mathbf{JRV}_{1, m-\alpha+1}]_{k, k} I\{\alpha \leq m-\alpha\} \\
\bar{D}_2 &= \frac{1}{p_{m-\alpha+1}p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, \alpha-1}]_{j+p_\alpha, k+p_{\alpha-1}} [\mathbf{V}_{\alpha+1, m-\alpha+1}]_{j+p_\alpha, k} I\{\alpha \leq m-\alpha\} \\
\bar{D}_3 &= \frac{1}{p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, m-\alpha}]_{j+p_\alpha, j} \\
\bar{D}_4 &= -\frac{1}{p_{m-\alpha+1}p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, \alpha-1}]_{j+p_\alpha, k+p_{\alpha-1}} \\
&\quad \times [\mathbf{V}_{\alpha+2, m} \mathbf{JRV}_{1, m-\alpha+1}]_{j+p_\alpha, k} \\
\bar{D}_5 &= -\frac{1}{p_{m-\alpha+1}p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1, m} \mathbf{JRV}_{1, m-\alpha}]_{j+p_\alpha, j} \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{m-\alpha+2, m} \mathbf{JRV}_{1, m-\alpha+1}]_{k, k}.
\end{aligned}$$

**Lemma 4.4.** *Under the conditions of Theorem 1.1 there exists a constant  $C > 0$  such that the following inequality holds*

$$\max\{|D_2|, |\bar{D}_2|\} \leq \frac{C}{nv}$$

and

$$\max\{|D_4|, |\overline{D}_4|\} \leq \frac{C}{nv^2}$$

*Proof.* We describe the bound for  $D_2$  first. Applying Hölder's inequality, we get

$$|D_2| \leq \frac{1}{n^2} \mathbf{E} \|\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,\alpha-1}\|_2 \|\mathbf{V}_{\alpha+1,m-\alpha}\|_2$$

Applying Hölder's inequality again, we get

$$|D_2| \leq \frac{1}{n^2 v} \mathbf{E}^{\frac{1}{2}} \left\| \prod_{\nu=1, \nu \neq \alpha}^m \mathbf{H}^{(\nu)} \right\|_2^2 \mathbf{E}^{\frac{1}{2}} \|\mathbf{V}_{\alpha+1,m-\alpha}\|_2^2.$$

Applying Lemma 5.2 now, we obtain

$$|D_2| \leq \frac{C}{nv}.$$

Recall that  $\|\cdot\|_2$  denotes the Frobenius norm of a matrix. The proof of the bound for  $\overline{D}_2$ ,  $D_4$  and  $\overline{D}_4$  are similar. Thus the Lemma is proved.  $\square$

**Lemma 4.5.** *Under the conditions of Theorem 1.1 there exists a constant  $C > 0$  such that the following inequality holds*

$$\max\{|D_1|, |\overline{D}_1|\} \leq \frac{C}{nv}.$$

*Proof.* Applying Hölder's inequality and Lemma 5.5, we get

$$\begin{aligned} |D_1| &= \frac{1}{p_{\alpha-1}} \mathbf{E} \sum_{j=1}^{p_{\alpha-1}} [\mathbf{V}_{\alpha+1,m-\alpha}]_{k,k+p_{m-\alpha}} \\ &\quad \times \frac{1}{p_{\alpha}} \mathbf{E} \sum_{k=1}^{p_{\alpha}} [\mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha}]_{j+p_{m-\alpha+1}, j+p_{m-\alpha+1}} \leq \frac{C}{nv}. \end{aligned}$$

Thus the Lemma is proved.  $\square$

**Lemma 4.6.** *Under the conditions of Theorem 1.1 there exists a constant  $C > 0$  that the following inequality holds*

$$\begin{aligned} |D_5| &= \frac{1}{p_{\alpha}} \mathbf{E} \sum_{k=1}^{p_{\alpha}} [\mathbf{V}_{\alpha+1,m} \mathbf{JRV}_{1,m-\alpha}]_{k,k+p_{m-\alpha}} \\ &\quad + \frac{1}{p_{\alpha-1}} \mathbf{E} \sum_{j=1}^{p_{\alpha-1}} [\mathbf{V}_{m-\alpha+2,m} \mathbf{JRV}_{1,m-\alpha+1}]_{j+p_{m-\alpha+1}, j+p_{m-\alpha+1}} \leq \frac{C}{nv^2}. \end{aligned}$$

and

$$\begin{aligned} & |\bar{D}_5 + \frac{1}{p_{m-\alpha}} \mathbf{E} \sum_{j=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha}]_{j+p_{m-\alpha},j} \\ & \frac{1}{p_{m-\alpha+1}} \mathbf{E} \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{m-\alpha+2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{k,k} \leq \frac{C}{nv^2}. \end{aligned}$$

*Proof.* Applying Hölder's inequality and Lemma 5.5, we conclude the result.  $\square$

Using the obvious equality  $\text{Tr} \mathbf{A} \mathbf{B} = \text{Tr} \mathbf{B} \mathbf{A}$ , it is straightforward to check that

$$\begin{aligned} & \frac{1}{p_{\alpha-1}} \sum_{j=1}^{p_{\alpha-1}} \mathbf{E} [\mathbf{V}_{m-\alpha+2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{j+p_{m-\alpha+1},j+p_{m-\alpha+1}} \\ & = \frac{1}{p_{\alpha-1}} \sum_{j=1}^{p_m} \mathbf{E} [\mathbf{V} \mathbf{J} \mathbf{R}]_{j+n,j+n} = y_{\alpha-1} (1 + z s_n(z)). \end{aligned}$$

This implies that

$$1 - \frac{1}{p_{\alpha-1}} \sum_{j=1}^{p_{\alpha-1}} \mathbf{E} [\mathbf{V}_{m-\alpha+2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{j+p_{m-\alpha+1},j+p_{m-\alpha+1}} = (1 - y_{\alpha-1} - z y_{\alpha-1} s_n(z))$$

Lemmas 4.4–4.6 and last equality together imply

$$f_{\alpha,m-\alpha+1} = -(1 - y_{\alpha-1} - z y_{\alpha-1} s(z)) f_{\alpha+1,m-\alpha} + \varepsilon_n(z). \quad (4.18)$$

Similar we show that

$$g_{\alpha,m-\alpha+1} = -(1 - y_{m-\alpha+1} - z y_{m-\alpha+1} s_n(z)) g_{\alpha+1,m-\alpha} + \varepsilon_n(z). \quad (4.19)$$

Note that

$$f_{m+1,0} = \frac{1}{p_m} \sum_{k=1}^{p_m} \mathbf{E} [\mathbf{J} \mathbf{R}]_{k,k+p_0} = \frac{n}{p_m} \frac{1}{n} \sum_{k=1}^{p_m} \mathbf{E} [\mathbf{R}]_{k+n,k+n} = -\frac{1}{z} (1 - y_m - z y_m s_n(z)) \quad (4.20)$$

and

$$g_{m+1,0} = \frac{1}{p_0} \sum_{k=1}^n \mathbf{E} [\mathbf{J} \mathbf{R}]_{k+p_m,k} = \frac{1}{n} \sum_{j=1}^n \mathbf{E} [\mathbf{R}]_{jj} = s_n(z) \quad (4.21)$$

Equalities (4.18)–(4.21) together imply

$$\begin{aligned} f_{2,m} &= (-1)^{m+1} \frac{1}{z} \prod_{q=1}^m (1 - y_q - z y_q s_n(z)) + \varepsilon_n(z) \\ g_{2,m} &= (-1)^{m+1} \frac{s_n(z)}{z} \prod_{q=1}^{m-1} (1 - y_q - z y_q s_n(z)) + \varepsilon_n(z). \end{aligned} \quad (4.22)$$

Equalities (4.16) and (4.22) together imply

$$1 + zs_n(z) = (-1)^{m+1} \frac{s_n(z)}{z} \prod_{q=1}^m (1 - y_q - zy_q s_n(z)) + \varepsilon_n(z).$$

We rewrite the last equation as follows

$$1 + zs_n(z) + (-1)^m \frac{s_n(z)}{z} \prod_{q=1}^m (1 - y_q - zy_q s_n(z)) = \varepsilon_n(z). \quad (4.23)$$

Let Stieltjes transform  $s(z)$  satisfies the equation

$$1 + zs(z) + (-1)^m \frac{s(z)}{z} \prod_{q=1}^m (1 - y_q - zy_q s(z)) = 0$$

Introduce the notations

$$Q_0 := \frac{1}{z} \prod_{q=1}^m (1 - y_q - zy_q s_n(z)),$$

$$Q_\nu := s(z) \prod_{q=1}^{\nu-1} (1 - y_q - zy_q s(z)) \prod_{q=\nu+1}^m (1 - y_q - zy_q s_n(z)).$$

Relations (4.23) and (4.24) together imply that, for

$$s_n(z) - s(z) = \frac{\varepsilon_n(z)}{z + (-1)^{m-1} \sum_{q=0}^m Q_q} \quad (4.24)$$

Note that

$$\max\{|zs(z)|, |zs_n(z)|\} \leq 1 + \frac{1}{v}$$

and

$$\max\{|s_n(z)|, |s(z)|\} \leq \frac{1}{v}$$

Applying these inequalities, we obtain

$$|Q_q| \leq \frac{1}{v} \left(1 + \frac{1}{v}\right)^m.$$

We may choose  $v \geq m + 1$ . Then  $\frac{1}{v} \left(1 + \frac{1}{v}\right)^m \leq \frac{\varepsilon}{v}$ . If we choose  $v$  such that  $\frac{\varepsilon}{v} \leq \frac{v}{2m}$ , we get

$$|z + (-1)^{m-1} \sum_{q=0}^m Q_q| \geq \frac{v}{2}.$$

This implies that, for  $v \geq V_1 := \sqrt{\frac{2m}{e}}$ ,

$$|s_n(z) - s(z)| \leq \frac{C|\varepsilon_n(z)|}{v} \leq C\tau_n \quad (4.25)$$

From inequality (4.25) we conclude that there exists an open set with non-empty interior where  $s_n(z)$  converges to  $s(z)$ . The Stieltjes transform of our random matrices is an analytic function on  $\mathcal{C}^+$  and locally bounded ( $|s_n(z)| \leq v^{-1}$  for any  $v > 0$ ). By Montel's Theorem (see, for instance, [16], p. 153, Theorem 2.9)  $s_n(z)$  converges to  $s(z)$  uniformly on any compact set  $\mathcal{K} \subset \mathcal{C}^+$  in the upper half-plane. This implies that  $\Delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the proof of Theorem 1.1 in the general case is complete.

## 5 Appendix

**Lemma 5.1.** *Under the conditions of Theorem 1.1 we have, for any  $j, k = 1, \dots, p_{\alpha-1} + p_\beta$ , and for any  $1 \leq \alpha \leq \beta \leq m$ ,*

$$\mathbf{E}[\mathbf{V}_{\alpha,\beta}]_{jk} = 0$$

*Proof.* For  $\alpha = \beta$  the claim is easy. Let  $\alpha < \beta$ . We consider the case  $j = 1, \dots, p_{\alpha-1}$  and  $k = 1, \dots, p_\beta$  only. The other cases are similar. Direct calculations show that

$$\mathbf{E}[\mathbf{V}_{\alpha,\beta}]_{jk} = \frac{1}{n^{\frac{\beta-\alpha}{2}}} \sum_{j_1=1}^{p_\alpha} \sum_{j_2=1}^{p_{\alpha+1}} \cdots \sum_{j_{\beta-\alpha}=1}^{p_{\beta-1}} \mathbf{E} X_{j,j_1}^{(\alpha)} X_{j_1,j_2}^{(\alpha+1)} \cdots X_{j_{\beta-\alpha},k}^{((\beta))} = 0$$

Thus the Lemma is proved.  $\square$

In all Lemmas below we shall assume that

$$\mathbf{E} X_{jk}^{(\nu)} = 0, \quad \mathbf{E} |X_{jk}^{(\nu)}|^2 = 1, \quad |X_{jk}^{(\nu)}| \leq c\tau_n \sqrt{n} \quad \text{a. s.} \quad (5.1)$$

**Lemma 5.2.** *Under the conditions of Theorem 1.1 assuming (5.1), we have, for any  $1 \leq \alpha \leq \beta \leq m$ ,*

$$\mathbf{E} \|\mathbf{V}_{\alpha,\beta}\|_2^2 \leq Cn \quad (5.2)$$

*Proof.* We shall consider the case  $\alpha < \beta$  only. The other cases are obvious. Direct calculation shows that

$$\mathbf{E} \|\mathbf{V}_{\alpha,\beta}\|_2^2 \leq \frac{C}{n^{\beta-\alpha+1}} \sum_{j=1}^n \sum_{j_1=1}^{p_{\alpha-1}} \sum_{j_2=1}^{p_\alpha} \cdots \sum_{j_{\beta-\alpha}=1}^{p_{\beta-1}} \sum_{k=1}^{p_\beta} \mathbf{E} [X_{j,j_1}^{(\alpha)} X_{j_1,j_2}^{(\alpha+1)} \cdots X_{j_{\beta-\alpha},k}^{(\beta)}]^2$$

By independents of random variables, we get

$$\mathbf{E} \|\mathbf{V}_{\alpha,\beta}\|_2^2 \leq Cn$$

Thus the Lemma is proved.  $\square$

**Lemma 5.3.** *Under the condition of Theorem 1.1 and assumption (5.1) we have, for any  $j = 1, \dots, p_{\alpha-1}$ ,  $k = 1 \dots p_{\beta}$  an  $r \geq 1$ ,*

$$\mathbf{E} \|\mathbf{V}_{\alpha,\beta} \mathbf{e}_k\|_2^{2r} \leq C_r, \quad \mathbf{E} \|\mathbf{V}_{\alpha,\beta} \mathbf{e}_{j+p_{\beta}}\|_2^{2r} \leq C_r \quad (5.3)$$

and

$$\mathbf{E} \|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} \leq C_r, \quad \mathbf{E} \|\mathbf{e}_{k+p_{\alpha-1}}^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} \leq C_r, \quad (5.4)$$

with some positive constant  $C_r$  depending on  $r$ .

*Proof.* By definition of the matrices  $\mathbf{V}_{\alpha,\beta}$ , we may write

$$\|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^2 = \frac{1}{p_{\alpha} \cdots p_{\beta}} \sum_{l=1}^{p_{\beta}} \left| \sum_{j_{\alpha}=1}^{p_{\alpha}} \cdots \sum_{j_{\beta-1}=1}^{p_{\beta-1}} X_{jj_{\alpha}}^{(\alpha)} \cdots X_{j_{\beta-1}l}^{(\beta)} \right|^2 \quad (5.5)$$

Using this representation, we get

$$\mathbf{E} \|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} = \frac{1}{p_{\alpha-1}^r \cdots p_{\beta-1}^r} \sum_{l_1=1}^{p_{\beta}} \cdots \sum_{l_r=1}^{p_{\beta}} \mathbf{E} \prod_{q=1}^r \left( \sum_{j_{\alpha}=1}^{p_{\alpha}} \cdots \sum_{j_{\beta-1}=1}^{p_{\beta-1}} \sum_{\hat{j}_{\alpha}=1}^{p_{\alpha}} \cdots \sum_{\hat{j}_{\beta-1}=1}^{p_{\beta-1}} A_{(j_{\alpha}, \dots, j_{\beta-1}, \hat{j}_{\alpha}, \dots, \hat{j}_{\beta-1})}^{(l_q)} \right) \quad (5.6)$$

where

$$A_{(j_{\alpha}, \dots, j_{\beta-1}, \hat{j}_{\alpha}, \dots, \hat{j}_{\beta-1})}^{(l_q)} = X_{jj_{\alpha}}^{(\alpha)} \overline{X}_{\hat{j}_{\alpha}}^{(\alpha)} X_{j_{\alpha}j_{\alpha+1}}^{(\alpha+1)} \overline{X}_{\hat{j}_{\alpha+1}}^{(\alpha+1)} \cdots X_{j_{\beta-2}j_{\beta-1}}^{(\beta-1)} \overline{X}_{\hat{j}_{\beta-2}}^{(\beta-1)} X_{j_{\beta-1}l_q}^{(\beta)} \overline{X}_{\hat{j}_{\beta-1}l_q}^{(\beta)}. \quad (5.7)$$

By  $\bar{x}$  we denote the complex conjugate of  $x$ . Rewriting the product on the r.h.s of (5.6), we get

$$\mathbf{E} \|\mathbf{e}_j^T \mathbf{V}_{\alpha,\beta}\|_2^{2r} = \frac{1}{p_{\alpha-1}^r \cdots p_{\beta-1}^r} \sum^{**} \mathbf{E} \prod_{q=1}^r A_{(j_{\alpha}^{(q)}, \dots, j_{\beta-1}^{(q)}, \hat{j}_{\alpha}^{(\nu)}, \dots, \hat{j}_{\beta-1}^{(q)})}^{(l_q)}, \quad (5.8)$$

where  $\sum^{**}$  is taken over all set of indices  $j_{\alpha}^{(q)}, \dots, j_{\beta-1}^{(q)}, l_q$  and  $\hat{j}_{\alpha}^{(\nu)}, \dots, \hat{j}_{\beta-1}^{(q)}$  where  $j_k^{(q)}, \hat{j}_k^{(q)} = 1, \dots, p_k$ ,  $k = \alpha, \dots, \beta - 1$ ,  $l_q = 1, \dots, p_{\beta}$  and  $q = 1, \dots, r$ . Note that the summands in the right hand side of (5.7) is equal 0 if there is at least one term in the product 5.7 which appears only one time. This implies that the summands in the right hand side of (5.7) is not equal zero only if the union of all sets of indices in r.h.s of (5.7) consist from at least  $r$  different terms and each term appears at least twice.

Introduce the random variables, for  $\nu = \alpha + 1, \dots, \beta - 1$ ,

$$\zeta_{j_{\nu-1}^{(1)}, \dots, j_{\nu-1}^{(r)}, j_{\nu}^{(1)}, \dots, j_{\nu}^{(r)}}^{(\nu)} = X_{j_{\nu-1}^{(1)}, j_{\nu}^{(1)}}^{(\nu)} \cdots X_{j_{\nu-1}^{(r)}, j_{\nu}^{(r)}}^{(\nu)} \overline{X}_{j_{\nu-1}^{(1)}, \hat{j}_{\nu}^{(1)}}^{(\nu)}, \dots, \overline{X}_{j_{\nu-1}^{(r)}, \hat{j}_{\nu}^{(r)}}^{(\nu)}, \quad (5.9)$$

and

$$\begin{aligned}\zeta_{j_1^{(1)}, \dots, j_1^{(r)}, \widehat{j}_1^{(1)}, \dots, \widehat{j}_1^{(r)}}^{(\alpha)} &= X_{j_1^{(1)}}^{(\alpha)} \cdots X_{j_a^{(r)}, j_{a+1}^{(r)}}^{(\alpha)} \overline{X}_{\widehat{j}_a^{(1)}}^{(\alpha)} \cdots \overline{X}_{\widehat{j}_a^{(r)}, \widehat{j}_{a+1}^{(r)}}^{(\alpha)} \\ \zeta_{j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}, l_q}^{(\beta)} &= X_{j_{\beta-1}^{(1)}}^{(\beta)} \cdots X_{j_{\beta-1}^{(r)}, l_q}^{(\beta)} \overline{X}_{\widehat{j}_{\beta-1}^{(1)}, l_q}^{(\beta)}, \dots, \overline{X}_{\widehat{j}_{\beta-1}^{(r)}, l_q}^{(\beta)}.\end{aligned}$$

Assume that the set of indices  $j_\alpha^{(1)}, \dots, j_\alpha^{(r)}, \widehat{j}_\alpha^{(1)}, \dots, \widehat{j}_\alpha^{(r)}$  contains  $t_\alpha$  different indexes, say  $i_1^{(\alpha)}, \dots, i_{t_\alpha}^{(\alpha)}$  with multiplicities  $k_1^{(\alpha)}, \dots, k_{t_\alpha}^{(\alpha)}$  respectively,  $k_1^{(\alpha)} + \dots + k_{t_\alpha}^{(\alpha)} = 2r$ . Note that  $\min\{k_1^{(\alpha)}, \dots, k_{t_\alpha}^{(\alpha)}\} \geq 2$ . Otherwise,

$|\mathbf{E} \zeta_{j_a^{(1)}, \dots, j_a^{(r)}, \widehat{j}_a^{(1)}, \dots, \widehat{j}_a^{(r)}}^{(\alpha)}| = 0$ . By assumption (5.1), we have

$$|\mathbf{E} \zeta_{j_\alpha^{(1)}, \dots, j_\alpha^{(r)}, \widehat{j}_\alpha^{(1)}, \dots, \widehat{j}_\alpha^{(r)}}^{(\alpha)}| \leq C(\tau_n \sqrt{n})^{2r-2t_\alpha} \quad (5.10)$$

Similar bounds we get for  $|\mathbf{E} \zeta_{j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}, l_q}^{(\beta)}|$ . Assume that the set of indexes

$\{j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}\}$  contains  $t_{\beta-1}$  different indices, say,  $i_1^{(\beta-1)}, \dots, i_{t_{\beta-1}}^{(\alpha)}$  with multiplicities  $k_1^{(\beta-1)}, \dots, k_{t_{\beta-1}}^{(\alpha)}$  respectively,  $k_1^{(\beta-1)} + \dots + k_{t_{\beta-1}}^{(\alpha)} = 2r$ . Then

$$|\mathbf{E} \zeta_{j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}, l_q}^{(\beta)}| \leq C(\tau_n \sqrt{n})^{2r-2t_{\beta-1}} \quad (5.11)$$

Furthermore, assume that for  $\alpha + 1 \leq \nu \leq \beta - 2$  there are  $t_\nu$  different pairs of indices, say,  $(i_\alpha, i'_\alpha), \dots, (i_{t_\beta}, i'_{t_\beta})$  in the set

$\{j_\alpha^{(1)}, \dots, j_\alpha^{(r)}, \widehat{j}_\alpha^{(1)}, \dots, \widehat{j}_\alpha^{(r)}, \dots, j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}, l_1, l_r\}$  with multiplicities  $k_1^{(\nu)}, \dots, k_{t_\nu}^{(\nu)}$ . Note that

$$k_1^{(\nu)} + \dots + k_{t_\nu}^{(\nu)} = 2r \quad (5.12)$$

and

$$|\mathbf{E} \zeta_{j_{\nu-1}^{(1)}, \dots, j_{\nu-1}^{(r)}, j_\nu^{(1)}, \dots, j_\nu^{(r)}, \widehat{j}_{\nu-1}^{(1)}, \dots, \widehat{j}_{\nu-1}^{(r)}, \widehat{j}_\nu^{(1)}, \dots, \widehat{j}_\nu^{(r)}}^{(\nu)}| \leq C(\tau_n \sqrt{n})^{2r-2t_\nu}. \quad (5.13)$$

Inequalities (5.10)-(5.13) together yield

$$|\mathbf{E} \prod_{q=1}^r A_{(j_\alpha^{(q)}, \dots, j_{\beta-1}^{(q)}, \widehat{j}_1^{(q)}, \dots, \widehat{j}_{\beta-1}^{(q)})}^{(l_q)}| \leq C(\tau_n \sqrt{n})^{2r(\beta-\alpha)-2(t_1+\dots+t_{\beta-\alpha})}. \quad (5.14)$$

It is straightforward to check that the number  $\mathcal{N}(t_\alpha, \dots, t_\beta)$  of sequences of indices  $\{j_\alpha^{(1)}, \dots, j_\alpha^{(r)}, \widehat{j}_\alpha^{(1)}, \dots, \widehat{j}_\alpha^{(r)}, \dots, j_{\beta-1}^{(1)}, \dots, j_{\beta-1}^{(r)}, \widehat{j}_{\beta-1}^{(1)}, \dots, \widehat{j}_{\beta-1}^{(r)}, l_1, \dots, l_r\}$  with  $t_\alpha, \dots, t_\beta$  of different pairs satisfies the inequality

$$\mathcal{N}(t_\alpha, \dots, t_\beta) \leq Cn^{t_\alpha+\dots+t_\beta}, \quad (5.15)$$

with  $1 \leq t_i \leq r$ ,  $i = \alpha, \dots, \beta$ . By the assumption of Theorem 1.1, we have

$$cn \leq p_\nu \leq Cn \quad (5.16)$$

for any  $\nu = 1, \dots, m$ . Note that in the case  $t_\alpha = \dots = t_\beta = r$  the inequalities (5.10)–(5.13) imply

$$\mathbf{E} \zeta_{j_{\nu-1}^{(1)}, \dots, j_{\nu-1}^{(r)}, j_\nu^{(1)}, \dots, j_\nu^{(r)}, \widehat{j}_{\nu-1}^{(1)}, \dots, \widehat{j}_{\nu-1}^{(r)}, \widehat{j}_\nu^{(1)}, \dots, \widehat{j}_\nu^{(r)}}^{(\nu)} \leq C \quad (5.17)$$

Inequalities (5.15), (5.14), (5.17), and the representation (5.6) together conclude the proof.  $\square$

**Lemma 5.4.** *Under the conditions of Theorem 1.1 assuming (5.1), we have*

$$\mathbf{E} \left| \frac{1}{n} (\text{Tr} \mathbf{R} - \mathbf{E} \text{Tr} \mathbf{R}) \right| \leq \frac{C}{nv^2}.$$

*Proof.* Consider the matrix  $\mathbf{X}^{(\nu, j)}$  obtained from the matrix  $\mathbf{X}^{(\nu)}$  by replacing the  $j$ -th row by a row with zero-entries. We define the following matrices

$$\mathbf{H}^{(\nu, j)} = \mathbf{H}^{(\nu)} - \mathbf{e}_j \mathbf{e}_j^T \mathbf{H}^{(\nu)},$$

and

$$\widetilde{\mathbf{H}}^{(m-\nu+1, j)} = \mathbf{H}^{(m-\nu+1)} - \mathbf{H}^{(m-\nu+1)} \mathbf{e}_{j+p_{m-\nu+1}} \mathbf{e}_{j+p_{m-\nu+1}}^T.$$

For simplicity we shall assume that  $\nu \leq m - \nu + 1$ . Define

$$\mathbf{V}^{(\nu, j)} = \prod_{q=1}^{\nu-1} \mathbf{H}^{(q)} \mathbf{H}^{(\nu, j)} \prod_{q=\nu+1}^{m-\nu} \mathbf{H}^{(q)} \widetilde{\mathbf{H}}^{(m-\nu+1, j)} \prod_{q=m-\nu+2}^m \mathbf{H}^{(q)}.$$

We shall use the following inequality. For any Hermitian matrix  $\mathbf{A}$  and  $\mathbf{B}$  with spectral distribution function  $F_A(x)$  and  $F_B(x)$  respectively, we have

$$|\text{Tr}(\mathbf{A} - z\mathbf{I})^{-1} - \text{Tr}(\mathbf{B} - z\mathbf{I})^{-1}| \leq \frac{\text{rank}(\mathbf{A} - \mathbf{B})}{v}. \quad (5.18)$$

It is straightforward to show that

$$\text{rank}(\mathbf{V}\mathbf{J} - \mathbf{V}^{(\nu, j)}\mathbf{J}) \leq 4m. \quad (5.19)$$

Inequality (5.18) and (5.19) together imply

$$\left| \frac{1}{2n} (\text{Tr} \mathbf{R} - \text{Tr} \mathbf{R}^{(\nu, j)}) \right| \leq \frac{C}{nv}.$$

We may now apply a standard martingale expansion technique already used in Girko [7]. We may introduce  $\sigma$ -algebras  $\mathcal{F}_{\nu, j} = \sigma\{X_{lk}^{(\nu)}, j < l \leq p_{\nu-1}, k = 1, \dots, p_\nu; X_{pk}^{(q)}, q = \nu + 1, \dots, m, p = 1, \dots, p_{q-1}, k = 1, \dots, p_q\}$  and use the representation

$$\text{Tr} \mathbf{R} - \mathbf{E} \text{Tr} \mathbf{R} = \sum_{\nu=1}^m \sum_{j=1}^{p_{\nu-1}} (\mathbf{E}_{\nu, j-1} \text{Tr} \mathbf{R} - \mathbf{E}_{\nu, j} \text{Tr} \mathbf{R}),$$

where  $\mathbf{E}_{\nu, j}$  denotes the conditional expectation with respect to the  $\sigma$ -algebra  $\mathcal{F}_{\nu, j}$ . Note that  $\mathcal{F}_{\nu, p_{\nu-1}} = \mathcal{F}_{\nu+1, 0}$   $\square$



**Lemma 5.5.** *Under the conditions of Theorem 1.1 we have, for  $1 \leq a \leq m$ ,*

$$\mathbf{E} \left| \frac{1}{n} \left( \sum_{k=1}^{p_{m-a}} [\mathbf{V}_{a+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-a}]_{k,k+p_\alpha} - \mathbf{E} \sum_{k=1}^{p_{m-a}} [\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha}]_{kk+p_\alpha} \right) \right|^2 \leq \frac{C}{nw^4}.$$

and, for  $1 \leq \alpha \leq m-1$ ,

$$\mathbf{E} \left| \frac{1}{n} \left( \sum_{k=1}^{p_{m-\alpha+1}} [\mathbf{V}_{m-\alpha+2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{k,k} - \mathbf{E} \sum_{j=1}^{p_{m-\alpha+1}} [\mathbf{V}_{m-\alpha+2,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{kk} \right) \right|^2 \leq \frac{C}{nw^4}.$$

*Proof.* We prove the first inequality only. The proof of other one is similar. We introduce the following matrices, for  $\nu = 1, \dots, m$  and for  $j = 1, \dots, p_{\nu-1}$ ,  $\mathbf{X}^{(\nu,j)} = \mathbf{X}^{(\nu)} - \mathbf{e}_j \mathbf{e}_j^T \mathbf{X}^{(\nu)}$ , and  $\mathbf{H}^{(\nu,j)} = \mathbf{H}^{(\nu)} - \mathbf{e}_j \mathbf{e}_j^T \mathbf{H}^{(\nu)}$  and  $\tilde{\mathbf{H}}^{(m-\nu+1,j)} = \mathbf{H}^{(m-\nu+1,j)} - \mathbf{H}^{(m-\nu+1)} \mathbf{e}_{j+p_{m-\nu+1}} \mathbf{e}_{j+p_{m-\nu+1}}^T$ . Note that the matrix  $\mathbf{X}^{(\nu,j)}$  is obtained from matrix  $\mathbf{X}^{(\nu)}$  by replacing all entries of the  $j$ -th row by 0. Similar to the proof of the previous Lemma we introduce matrices  $\mathbf{V}_{c,d}^{(\nu,j)}$  by replacing in the definition of the matrix  $\mathbf{V}_{c,d}$  the matrix  $\mathbf{H}^{(\nu)}$  by  $\mathbf{H}^{(\nu,j)}$  and the matrix  $\mathbf{H}^{(m-\nu+1)}$  by  $\tilde{\mathbf{H}}^{(m-\nu+1,j)}$ . For instance, for  $c \leq \nu \leq m-\nu+1 \leq d$ , we get

$$\mathbf{V}_{c,d}^{(\nu,j)} = \prod_{q=a}^{\nu-1} \mathbf{H}^{(q)} \mathbf{H}^{(\nu,j)} \prod_{q=\nu+1}^{m-\nu} \mathbf{H}^{(q)} \tilde{\mathbf{H}}^{(m-\nu+1,j)} \prod_{q=m-\nu+1}^b \mathbf{H}^{(q)}.$$

Define as well  $\mathbf{V}^{(\nu,j)} := \mathbf{V}_{1,m}^{(\nu,j)}$  and  $\mathbf{R}^{(j)} := (\mathbf{V}^{(\nu,j)} - z\mathbf{I})^{-1}$ . Consider the following quantities, for  $\nu = 1, \dots, m$  and  $j = 1, \dots, p_{\nu-1}$ ,

$$\Xi_j := \sum_{k=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{kk+p_\alpha} - \sum_{k=1}^n [\mathbf{V}_{\alpha+1,m}^{(\nu,j)} \mathbf{J} \mathbf{R}^{(j)} \mathbf{V}_{1,m-\alpha+1}^{(\nu,j)}]_{kk+p_\alpha}$$

We represent it in the following form

$$\Xi_j := \Xi_j^{(1)} + \Xi_j^{(2)} + \Xi_j^{(3)},$$

where

$$\begin{aligned} \Xi_{\nu,j}^{(1)} &= \sum_{k=1}^{p_{m-\alpha}} [(\mathbf{V}_{\alpha+1,m} - \mathbf{V}_{\alpha+1,m}^{(\nu,j)}) \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1}]_{k,k+p_\alpha}, \\ \Xi_{\nu,j}^{(2)} &= \sum_{k=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1,m}^{(\nu,j)} \mathbf{J} (\mathbf{R} - \mathbf{R}^{(j)}) \mathbf{J} \mathbf{V}_{1,m-\alpha+1}]_{kk+p_\alpha}, \\ \Xi_{\nu,j}^{(3)} &= \sum_{k=1}^{p_{m-\alpha}} [\mathbf{V}_{\alpha+1,m}^{(j)} \mathbf{J} \mathbf{R}^{(j)} (\mathbf{V}_{1,m-\alpha+1} - \mathbf{V}_{1,m-\alpha+1}^{(\nu,j)})]_{kk+p_\alpha}. \end{aligned}$$

Note that

$$\begin{aligned}\mathbf{V}_{a+1,m} - \mathbf{V}_{a+1,m}^{(\nu,j)} &= \mathbf{V}_{a+1,\nu-1}(\mathbf{H}^{(\nu)} - \mathbf{H}^{(\nu,j)})\mathbf{V}_{\nu+1,m} \\ &\quad + \mathbf{V}_{a+1,\nu-1}\mathbf{H}^{(\nu,j)}\mathbf{V}_{\nu+1,m-\nu}(\tilde{\mathbf{H}}_{m-\nu+1} - \tilde{\mathbf{H}}_{m-\nu+1}^{(\nu,j)})\mathbf{V}_{m-\nu+2,m}.\end{aligned}$$

By definition of the matrices  $\mathbf{H}^{\nu,j}$  and  $\tilde{\mathbf{H}}^{m-\nu+1,j}$ , we have

$$\begin{aligned}\sum_{k=1}^{p_{m-a}} [(\mathbf{V}_{a+1,m} - \mathbf{V}_{a+1,m}^{(\nu,j)})\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-\nu+1}]_{k,k+p_a} &= [\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}]_{j,j} \\ &\quad + [\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,m-a+1}]_{j+p_{\nu-1},j+p_{\nu-1}},\end{aligned}$$

where

$$\tilde{\mathbf{J}} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

This equality implies that

$$\begin{aligned}|\Xi_j^{(1)}| &\leq |[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}]_{j,j+n}| \\ &\quad + |[\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,m-\nu+1}]_{j+p_{\nu-1},j+p_{\nu-1}}|.\end{aligned}$$

Using the obvious inequality  $\sum_{j=1}^n a_{jj}^2 \leq \|\mathbf{A}\|_2^2$  for any matrix  $\mathbf{A} = (\alpha_{jk})$ ,  $j, k = 1, \dots, n$ , we get

$$\begin{aligned}T_1 &:= \sum_{j=1}^n \mathbf{E} |\Xi_j^{(1)}|^2 \leq \mathbf{E} \|\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,\nu}\|_2^2 \\ &\quad + \mathbf{E} \|\mathbf{V}_{m-\nu+2,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,m-a+1}\tilde{\mathbf{J}}\mathbf{V}_{a+1,m-\nu+1}\|_2^2.\end{aligned}$$

By Lemma 5.2, we get

$$T_1 \leq \frac{C}{v^2} \mathbf{E} \|\mathbf{V}_{a+1,m}\mathbf{V}_{1,m-a+1}\|_2^2 \leq \frac{Cn}{v^2} \quad (5.20)$$

Consider now

$$T_2 = \sum_{j=1}^n \mathbf{E} |\Xi_j^{(2)}|^2.$$

Using that  $\mathbf{R} - \mathbf{R}^{(j)} = -\mathbf{R}^{(j)}(\mathbf{V} - \mathbf{V}^{(\nu,j)})\mathbf{R}$ , we get

$$\begin{aligned}|\Xi_j^{(2)}| &\leq \left| \sum_{k=1}^{p_{a-1}} [\mathbf{V}_{a,m}^{(\nu,j)}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\nu-1}\mathbf{e}_j\mathbf{e}_j^T\mathbf{V}_{\nu,m}\mathbf{R}\mathbf{V}_{1,b}]_{k,k+p_{m-b}} \right| \\ &\leq [\mathbf{J}\mathbf{H}^{(\alpha+1)}\mathbf{V}_{\alpha+2,m-\alpha}\mathbf{H}^{(m-\alpha+1,j)}\mathbf{V}_{m-\alpha+2,m}\mathbf{R}\mathbf{V}_{1,m-\alpha}\mathbf{V}_{\alpha+1,m}^{(j)}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\alpha}]_{jj}.\end{aligned}$$

This implies that

$$T^{(2)} \leq C\mathbf{E} \|[\mathbf{V}_{\nu+1,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,b}\mathbf{V}_{a,m}\mathbf{J}\mathbf{R}\mathbf{V}_{1,\nu}]\|_2^2.$$

It is straightforward to check

$$T^{(2)} \leq \frac{C}{v^4} \mathbf{E} \|\mathbf{V}_{1,\alpha} \mathbf{J} \mathbf{H}^{(\alpha+1)} \mathbf{V}_{\alpha+2,m-\alpha} \mathbf{H}^{(m-\alpha+1,j)} \mathbf{V}_{m-\alpha+2,m}\|_2^2 = \mathbf{E} \|\mathbf{Q}\|_2^2 \quad (5.21)$$

The matrix on the right hand side of equation (5.21) may be represented in the form

$$Q = \prod_{\nu=1}^m \mathbf{H}^{(\nu)\varkappa_\nu},$$

where  $\varkappa_\nu = 0$  or  $\varkappa_\nu = 1$  or  $\varkappa_\nu = 2$ . Since  $X_{ss}^{(\nu)} = 0$ , for  $\varkappa = 1$  or  $\varkappa = 2$ , we have

$$\mathbf{E} |\mathbf{H}^{(\nu)\varkappa}_{kl}|^2 \leq \frac{C}{n}.$$

This implies that

$$T_2 \leq Cn. \quad (5.22)$$

Similar we prove that

$$T_3 := \sum_{j=1}^n \mathbf{E} |\Xi_j^{(3)}|^2 \leq Cn. \quad (5.23)$$

Inequality (5.20), (5.22) and (5.23) together imply

$$\sum_{j=1}^n \mathbf{E} |\Xi_j|^2 \leq Cn$$

Applying now a martingale expansion with respect to the  $\sigma$ -algebras  $\mathcal{F}_j$  generated the random variables  $X_{kl}^{(\alpha+1)}$  with  $1 \leq k \leq j$ ,  $1 \leq l \leq n$  and all other random variables  $X_{sl}^{(q)}$  except  $q = \alpha + 1$ , we get

$$\mathbf{E} \left| \frac{1}{n} \left( \sum_{k=1}^n [\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha}]_{kk+n} - \mathbf{E} \sum_{j=1}^n [\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha}]_{kk+n} \right) \right|^2 \leq \frac{C}{nv^4}.$$

Thus the Lemma is proved. □

**Lemma 5.6.** *Under the conditions of Theorem 1.1 we have, for  $\alpha = 1, \dots, m$ , that there exists a constant  $C$  such that*

$$\frac{1}{n^{\frac{3}{2}}} \mathbf{E} \left| \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} (X_{jk}^{(\alpha)} + X_{jk}^{(\alpha)3}) \left[ \frac{\partial^2 (\mathbf{V}_{\alpha+1,m} \mathbf{J} \mathbf{R} \mathbf{V}_{1,m-\alpha+1})}{\partial X_{jk}^{(\alpha)2}} (\theta_{jk}^{(\alpha)} X_{jk}^{(\alpha)}) \right]_{kj} \right| \leq C\tau_n v^{-4},$$

and

$$\frac{1}{n^{\frac{3}{2}}}\mathbf{E}\left|\sum_{j=1}^{p_{m-\alpha}}\sum_{k=1}^{p_{m-\alpha+1}}(X_{jk}^{(m-\alpha+1)}+X_{jk}^{(m-\alpha+1)^3})\times\left[\frac{\partial^2(\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,m-\alpha+1})}{\partial X_{jk}^{(m-\alpha+1)^2}}(\theta_{jk}^{(m-\alpha+1)}X_{jk}^{(m-\alpha+1)})\right]_{j+p_{\alpha-1},k}\right|\leq C\tau_n v^{-4},\quad (5.24)$$

where  $\theta_{jk}^{(\alpha)}$  and  $X_{jk}^{(\alpha)}$  are r.v. which are independent in aggregate for  $\alpha = 1, \dots, m$  and  $j = 1, \dots, p_{\alpha-1}$ ,  $k = 1, \dots, p_{\alpha}$ , and  $\theta_{jk}^{(\alpha)}$  are uniformly distributed on the unit interval. By  $\frac{\partial^2}{\partial X_{jk}^{(\alpha)^2}}\mathbf{A}(\theta_{jk}^{(\alpha)}X_{jk}^{(\alpha)})$  we denote the matrix obtained from  $\frac{\partial^2}{\partial X_{jk}^{(\alpha)^2}}\mathbf{A}$  by replacing its entries  $X_{jk}^{(\alpha)}$  by  $\theta_{jk}^{(\alpha)}X_{jk}^{(\alpha)}$ .

*Proof.* The proof of this lemma is rather technical. But for completeness we shall include it here. By the formula for the derivatives of a resolvent matrix, we have

$$\frac{\partial(\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,m-\alpha+1})}{\partial X_{jk}^{(\alpha)}}=\sum_{l=1}^5Q_l,\quad (5.25)$$

$$\begin{aligned} Q_1 &= \frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,\alpha-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{\alpha+1,m-\alpha+1}I_{\{\alpha\leq m-\alpha+1\}} \\ Q_2 &= \frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+p_{m-\alpha+1}} \\ Q_3 &= -\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,\alpha-1}\mathbf{e}_j\mathbf{e}_k^T\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,m-\alpha+1} \\ Q_4 &= -\frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m}\mathbf{JRV}_{1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+p_{m-\alpha+1}}^T\mathbf{V}_{m-\alpha+2,m}\mathbf{JRV}_{1,m-\alpha+1} \\ Q_5 &= \frac{1}{\sqrt{n}}\mathbf{V}_{\alpha+1,m-\alpha}\mathbf{e}_{k+p_{m-\alpha}}\mathbf{e}_{j+p_{m-\alpha+1}}^T\mathbf{V}_{m-\alpha+2,m}\mathbf{JRV}_{1,m-\alpha+1}I_{\{\alpha\leq m-\alpha+1\}}. \end{aligned}$$

Introduce the notations

$$\mathbf{U}_{\alpha}:=\mathbf{V}_{\alpha+1,m},\quad \mathbf{V}_{\alpha}=\mathbf{V}_{1,m-\alpha+1}.$$

From formula (5.25) it follows that

$$\frac{\partial^2(\mathbf{U}_{\alpha}\mathbf{JRV}_{\alpha})}{\partial X_{jk}^{(\nu)^2}}=\sum_{l=1}^5\frac{\partial Q_l}{\partial X_{jk}^{(\alpha)}}.$$

Since all other calculations will be similar we consider the case  $l = 3$  only. Simple calculations show that

$$\frac{\partial \mathbf{Q}_3}{\partial X_{jk}^{(\alpha)}} = \sum_{m=1}^7 \mathbf{P}^{(m)}, \quad (5.26)$$

where

$$\begin{aligned} \mathbf{P}^{(1)} &= -\frac{1}{n} \mathbf{V}_{\alpha+1, m-\alpha} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{U}_{m-\alpha+1} \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_\alpha \\ \mathbf{P}^{(2)} &= -\frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_{\alpha+1} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \\ \mathbf{P}^{(3)} &= -\frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{V}_{\alpha+1, m-\alpha} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{U}_{m-\alpha+1} \mathbf{JRV}_\alpha \\ \mathbf{P}^{(4)} &= -\frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_\alpha \\ \mathbf{P}^{(5)} &= \frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{\alpha+1} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{U}_{m-\alpha+1} \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_\alpha \\ \mathbf{P}^{(6)} &= \frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_{\alpha+1} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{U}_{m-\alpha+1} \mathbf{JRV}_\alpha \\ \mathbf{P}^{(7)} &= \frac{1}{n} \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2} \mathbf{e}_j \mathbf{e}_k^T \mathbf{U}_\alpha \mathbf{JRV}_\alpha. \end{aligned}$$

Consider now the quantity, for  $\mu = 1, \dots, 5$ ,

$$L_\mu = \frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} X_{j,k}^{(\alpha)3} \left[ \frac{\partial \mathbf{Q}_\mu}{\partial X_{jk}^{(\alpha)}} \right]_{kj}. \quad (5.27)$$

We bound  $L_3$  only. The others bounds are similar. First we note that

$$\sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} X_{j,k}^{(\alpha)3} [\mathbf{P}^{(\nu)}]_{kj} = 0, \quad \text{for } \nu = 1, 2, 3. \quad (5.28)$$

Furthermore,

$$\mathbf{E} |X_{j,k}^{(\alpha)}|^3 ||[\mathbf{P}^{(4)}]_{kj}| \leq \mathbf{E} |X_{j,k}^{(\alpha)}|^3 |[U_\alpha \mathbf{JRV}_{m-\alpha+2}]_{kj}|^2 |[U_\alpha \mathbf{JRV}_\alpha]_{kj}|. \quad (5.29)$$

Let  $\mathbf{U}_\alpha^{(jk)}$  ( $\mathbf{V}_\alpha^{(j,k)}$ ) denote matrix obtained from  $\mathbf{U}_\alpha$  ( $\mathbf{V}_\alpha$ ) by replacing  $X_{jk}^{(\alpha)}$  by zero. We may write

$$\mathbf{U}_\alpha = \mathbf{U}_\alpha^{(jk)} + \frac{1}{\sqrt{n}} X_{jk}^{(\alpha)} \mathbf{V}_{\alpha+1, m-\alpha+1} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T \mathbf{V}_{m-\alpha+2, m}. \quad (5.30)$$

and

$$\mathbf{V}_\alpha = \mathbf{V}_\alpha^{(j,k)} + \frac{1}{\sqrt{n}} X_{jk} \mathbf{V}_{1, m-\alpha+1} \mathbf{e}_{k+p_{m-\alpha}} \mathbf{e}_{j+p_{m-\alpha+1}}^T.$$

Using these representations and taking in account that

$$[\mathbf{V}_{\alpha+1, m-\alpha}]_{k, k+p_{m-\alpha}} = [\mathbf{V}_{1, m-\alpha}]_{k, k+p_{m-\alpha}} = 0, \quad (5.31)$$

we get

$$\mathbf{E} |X_{j,k}^{(\alpha)}|^3 |[\mathbf{P}^{(4)}]_{kj}| \leq \frac{1}{n} \mathbf{E} |X_{j,k}^{(\alpha)}|^3 |[\mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2}]_{kj}|^2 |[\mathbf{U}_\alpha^{(j,k)} \mathbf{JRV}_\alpha^{(j,k)}]_{kj}|. \quad (5.32)$$

Furthermore,

$$\begin{aligned} |[\mathbf{U}_\alpha \mathbf{JRV}_{m-\alpha+2}]_{kj}| &\leq \frac{1}{v} \|\mathbf{V}_{m-\alpha+2} \mathbf{e}_j\|_2 \|\mathbf{e}_k^T \mathbf{U}_\alpha\|_2 \\ |[\mathbf{U}_\alpha^{(j,k)} \mathbf{JRV}_\alpha^{(j,k)}]_{kj}| &\leq \frac{1}{v} \|\mathbf{V}_\alpha^{(j,k)} \mathbf{e}_k\|_2 \|\mathbf{e}_j^T \mathbf{U}_\alpha^{(j,k)}\|_2. \end{aligned} \quad (5.33)$$

Applying inequalities (5.32) and (5.33) and taking in account the independence of entries, we get

$$\mathbf{E} |X_{j,k}^{(\alpha)}|^3 |[\mathbf{P}^{(4)}]_{kj}| \leq \frac{1}{nv^2} \mathbf{E} |X_{j,k}^{(\alpha)}|^3 \mathbf{E} \|\mathbf{V}_{m-\alpha+2} \mathbf{e}_k\|_2^2 \|\mathbf{e}_j^T \mathbf{U}_\alpha\|_2^2 \|\mathbf{V}_\alpha^{(j,k)} \mathbf{e}_k\|_2 \|\mathbf{e}_j^T \mathbf{U}_\alpha^{(j,k)}\|_2 \quad (5.34)$$

Applying Lemma 5.3, we get

$$\frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} |X_{jk}^{(\alpha)}|^3 |[\mathbf{P}^{(4)}]_{kj}| \leq \frac{C}{n^{\frac{5}{2}}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} |X_{jk}^{(\alpha)}|^3 \quad (5.35)$$

Assumption (5.1) now yields

$$\frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} |X_{jk}^{(\alpha)}|^3 |[\mathbf{P}^{(4)}]_{kj}| \leq C\tau_n. \quad (5.36)$$

Similar we get the bounds for  $\nu = 5, 6, 7$

$$\frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} |X_{jk}^{(\alpha)}|^3 |[\mathbf{P}^{(\nu)}]_{kj}| \leq C\tau_n. \quad (5.37)$$

and

$$|L_\mu| \leq C\tau_n, \quad \mu = 1, \dots, 5. \quad (5.38)$$

The bound of the quantity

$$\widehat{L}_\mu = \sum_{j=1}^{p_{\alpha-1}} \sum_{k=1}^{p_\alpha} \mathbf{E} X_{j,k}^{(\alpha)} \left[ \frac{\partial \mathbf{Q}_\nu}{\partial X_{jk}^{(\alpha)}} \right]_{kj}. \quad (5.39)$$

is similar. Thus, the Lemma is proved.  $\square$

**Lemma 5.7.** *Under the conditions of Theorem 1.1 we have*

$$\sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} X_{jk}^{(\nu)} [\mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}]_{kj} = \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} \left[ \frac{\partial \mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}} \right]_{kj} + \varepsilon_n(z)$$

and

$$\begin{aligned} & \sum_{j=1}^{p_{m-\nu}} \sum_{k=1}^{p_{m-\nu+1}} \mathbf{E} X_{j,k}^{(m-\nu+1)} [\mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}]_{j+p_{\nu-1},k} \\ &= \sum_{j=1}^{p_{m-\nu}} \sum_{k=1}^{p_{m-\nu+1}} \mathbf{E} \left[ \frac{\partial \mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}} \right]_{j+p_{\nu-1},k} + \varepsilon_n(z), \end{aligned}$$

where  $|\varepsilon_n(z)| \leq \frac{C\tau_n}{v^4}$ .

*Proof.* We apply Taylor's formula twice,

$$\mathbf{E} \xi f(\xi) = f'(0) \mathbf{E} \xi^2 + \mathbf{E} \xi^3 f''(\theta \xi),$$

and

$$f'(0) = \mathbf{E} f'(\xi) + \mathbf{E} \xi f''(\theta \xi) \quad (5.40)$$

where  $\theta$  denotes uniformly distributed r.v. on the unit interval which is independent of  $\xi$ . After simple calculations we get

$$\begin{aligned} \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} X_{jk}^{(\nu)} [\mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}]_{kj} &= \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} \left[ \frac{\partial \mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)}} \right]_{kj} \\ &+ \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E} (X_{jk}^{(\nu)} + X_{jk}^{(\nu)3}) \left[ \frac{\partial^2 \mathbf{V}_{\nu+1,m} \mathbf{JRV}_{1,m-\nu+1}}{\partial X_{jk}^{(\nu)2}} (\theta_{jk}^{(\nu)} X_{jk}^{(\nu)}) \right]_{kj}. \end{aligned}$$

Using the results of Lemma 5.6, we conclude the proof.  $\square$

## References

- [1] Alexeev, N.; Götze, F.; Tikhomirov, A. N. *Asymptotic distribution of singular values of powers of random matrices*, Lithuanian math. J. vol. 50, No 2, P. 121-132
- [2] Alexeev, N.; Götze, F.; Tikhomirov, A. N. *On the singular spectrum of powers and products of random matrices*, *Doklady Mathematics*, vol. 82, N 1, 2010, pp.505-507
- [3] Alexeev, N.; Götze, F.; Tikhomirov, A. N. *On the asymptotic distribution of singular values of product of large random matrices. Method of moments*, *Manuscript in preparing*

- [4] Banica, T. Belinschi, S. Capitaine, M. and Collins B. *Free Bessel Laws* Preprint. arXiv:0710.5931
- [5] Bai, Z. D. and Silverstein, J. *Spectral analysis of large dimensional random matrices*. Mathematics Monograph Series **2**, Sciences Press, Beijing 2006
- [6] H. Bercovici, and J.-C. Wang *Limit theorems for free multiplicative convolutions*. Trans. Americ. MATH. Soc., vol. 360, N 11, (2008), pp. 6089–6102.
- [7] Girko, V. L. *Spectral theory of random matrices*. (Russian) Uspekhi Mat. Nauk **40** (1985), no. 1(241), 67–106.
- [8] Götze, F.; Tikhomirov, A. N. *The rate of convergence for spectra of GUE and LUE matrix ensembles*. Cent. Eur. J. Math. **3**, no. 4, (2005), 666–704
- [9] Götze, F. and Tikhomirov, A. N. *Rate of convergence to the semi-circular law*. Probab. Theory Relat. Fields **127** (2003), 228–276
- [10] Götze, F.; Tikhomirov, A. N. Rate of convergence in probability to the Marchenko-Pastur law. Bernoulli **10**, no. 3, (2004), 503–548.
- [11] Marchenko and V., Pastur, L. *The eigenvalue distribution in some ensembles of random matrices*. Math.USSR Sbornik, **1** (1967), 457-483
- [12] Oravecz F. *On the powers of Voiculescu's circular element*. Studia Math. **145** (2001), no. 1, 85–95.
- [13] Mingo, J. A. and Speicher, R. *Sharp Bounds for Sums Associated to Graphs of Matrices* Preprint. arXiv:0909.4277
- [14] Michel Talagrand . *A new look at independence* Annals of Probab. **24**, No. 1, (1996), 1–34
- [15] Vershynin, R. *Spectral norm of products of random and deterministic matrices* <http://arXiv:0812.2432> 1–34, (2008)
- [16] Conway, John B. *Functions of one complex variable I*. Springer–Verlag, Berlin 1995 - 2nd ed., 316 pp.