UdeM-GPP-TH-10-194

Diagrammatic Analysis of Charmless Three-Body B Decays

Nicolas Rey-Le Lorier [a,1], Maxime Imbeault [b,2], and David London [a,3]

a: *Physique des Particules, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, QC, Canada H3C 3J7*

b: *Département de physique, Cégep de Baie-Comeau, 537 boulevard Blanche, Baie-Comeau, QC, Canada G5C 2B2*

(December 6, 2010)

Abstract

We express the amplitudes for charmless three-body B decays in terms of diagrams. As in two-body decays, we neglect annihilation-type contributions. When this is done, many of the exact, purely isospin-based results are modified, leading to new tests of the standard model. In addition, we show that, contrary to what was thought previously, it is possible to cleanly extract weak-phase information from three-body decays, and we discuss methods for $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$ and $\pi\pi\pi$.

[1]nicolas.rey-le.lorier@umontreal.ca

[2]imbeault.maxime@gmail.com

[3]london@lps.umontreal.ca

1 Introduction

The B-factories BaBar and Belle ran for over ten years, and made an enormous number of measurements of observables in B decays. For the most part, these decays were of the form $B \to M_1 M_2$ (M_i is a meson), as these are most accessible experimentally. Nevertheless, there have still been some probes of three-body $B \to M_1 M_2 M_3$ decays. To be specific, experiments have made measurements of (or obtained upper limits on) the branching ratios and indirect (mixing-induced) CP asymmetries of many of the decay modes in $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, $\pi\pi\pi$ [1].

Things are similar on the theory side. The vast majority of theoretical analyses involve two-body B decays. This is in part due to the relative angular momentum of the final-state particles. For example. consider $B_d^0 \to \pi^+\pi^-$. Because there are two particles in the final state, it has a fixed value of l (in this case $l = 0$). On the other hand, in the decay $B_d^0 \to K_S \pi^+\pi^-$, the $\pi^+\pi^-$ can have even or odd relative angular momentum. This makes it much more difficult to find clean predictions of the standard model (SM) to compare with experimental measurements. This is a general property of three-body decays.

Still, there have been some theoretical analyses of CP-conserving observables in three-body $B \to K\pi\pi$, $KK\bar{K}$ decays [2, 3, 4, 5]. In general, these studies examined the isospin decomposition of the decay amplitudes, and symmetry relations among them. The analyses were carried out using isospin amplitudes.

In this paper, we examine the amplitudes of the three-body charmless decays $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, $\pi\pi\pi$ using diagrams. The aim of this is as follows. As has been shown in Ref. [6], the amplitudes for two-body B decays can be expressed in terms of 9 diagrams. However, 3 of these – the annihilation-type diagrams – are expected to be quite a bit smaller than the others, and can be neglected, to a good approximation. This same procedure can be applied to three-body decays. When one does this, new features appear. A given set of three-body decays (e.g. $B \to K\pi\pi$) contains a number of different transitions (e.g. $B^+ \to K^+\pi^+\pi^-$, $B_d^0 \to K^+\pi^0\pi^-$, etc.). There are exact relations among the amplitudes for these specific decays. However, when one neglects certain diagrams, these relations can be modified, and this can lead to new effects. For example, some linear combinations of the isospin amplitudes vanish for certain decays. Also, there are additional tests of the SM. In some cases, it is even possible to obtain clean information about the CP-violating phases.

In Sec. 2, we present the diagrams describing $B \to M_1 M_2 M_3$ processes. The decays $B \to K\pi\pi$, $B \to KK\bar{K}$, $B \to K\bar{K}\pi$ and $B \to \pi\pi\pi$ are discussed in Secs. 3, 4, 5 and 6, respectively. In all cases, we give the expressions for the decay amplitudes in terms of diagrams, and examine the prospects for the clean extraction of weak-phase information. Other subjects related to the particular decays are also discussed: resonances and penguin dominance in $B \to K\pi\pi$ (Sec. 3), penguin dominance and isospin amplitudes in $B \to KK\bar{K}$ (Sec. 4), and T dominance in $B \to K\bar{K}\pi$ (Sec. 5).

We conclude in Sec. 7.

2 Diagrams

It has been shown in Ref. [6] that the amplitudes for two-body B decays can be expressed in terms of 9 diagrams: the color-favored and color-suppressed tree amplitudes T and C, the gluonic-penguin amplitudes P_{tc} and P_{uc}, the color-favored and color-suppressed electroweak-penguin (EWP) amplitudes P_{EW} and P_{EW}^C, the annihilation amplitude A, the exchange amplitude E, and the penguin-annihilation amplitude PA. These last three all involve the interaction of the spectator quark, and are expected to be much smaller than the other diagrams. It is standard to neglect them.

For the three-body decays considered in this paper, we adopt a similar procedure. That is, we neglect all annihilation-type diagrams, and express all amplitudes in terms of tree, penguin, and EWP diagrams. We assume isospin invariance, but not flavor SU(3) symmetry. (It is straightforward to modify our analysis by imposing SU(3).) The diagrams are shown in Fig. 1. A few words of explanation. These diagrams are for the decay $B \to \pi\pi\pi$. There are changes of notation for the other decays:

- For $\bar{b} \to \bar{d}$ transitions ($B \to K\bar{K}\pi$, $\pi\pi\pi$), the diagrams are written without primes; for $\bar{b} \to \bar{s}$ transitions ($B \to K\pi\pi$, $KK\bar{K}$), they are written with primes.

- In all diagrams, it is necessary to "pop" a quark pair from the vacuum. It is assumed that this pair is $u\bar{u}$ or $d\bar{d}$ ($\equiv q\bar{q}$); if the popped pair is $s\bar{s}$, the diagram is written with an additional subscript "s." Thus, for $B \to K\bar{K}\pi$, $KK\bar{K}$, in the penguin or EWP diagrams with a popped $q\bar{q}$ pair, the virtual particle decays to $s\bar{s}$; if the popped quark pair is $s\bar{s}$ (so the diagram is written with an additional subscript "s"), the virtual particle decays to $q\bar{q}$.

- The subscript "1" indicates that the popped quark pair is between two (non-spectator) final-state quarks; the subscript "2" indicates that the popped quark pair is between two final-state quarks including the spectator.

In principle, one can also include the gluonic-penguin diagrams in which the popped quark pair is between the pair of quarks produced by the gluon. This corresponds to the case where the virtual spin-1 gluon decays to two spin-0 mesons (with relative angular momentum $l = 1$). In order to account for the color imbalance, additional gluons must be exchanged. Although this can take place at low energy, it will still suppress these diagrams somewhat, and so we do not include them here. (Note: their inclusion does not change any of our conclusions.)

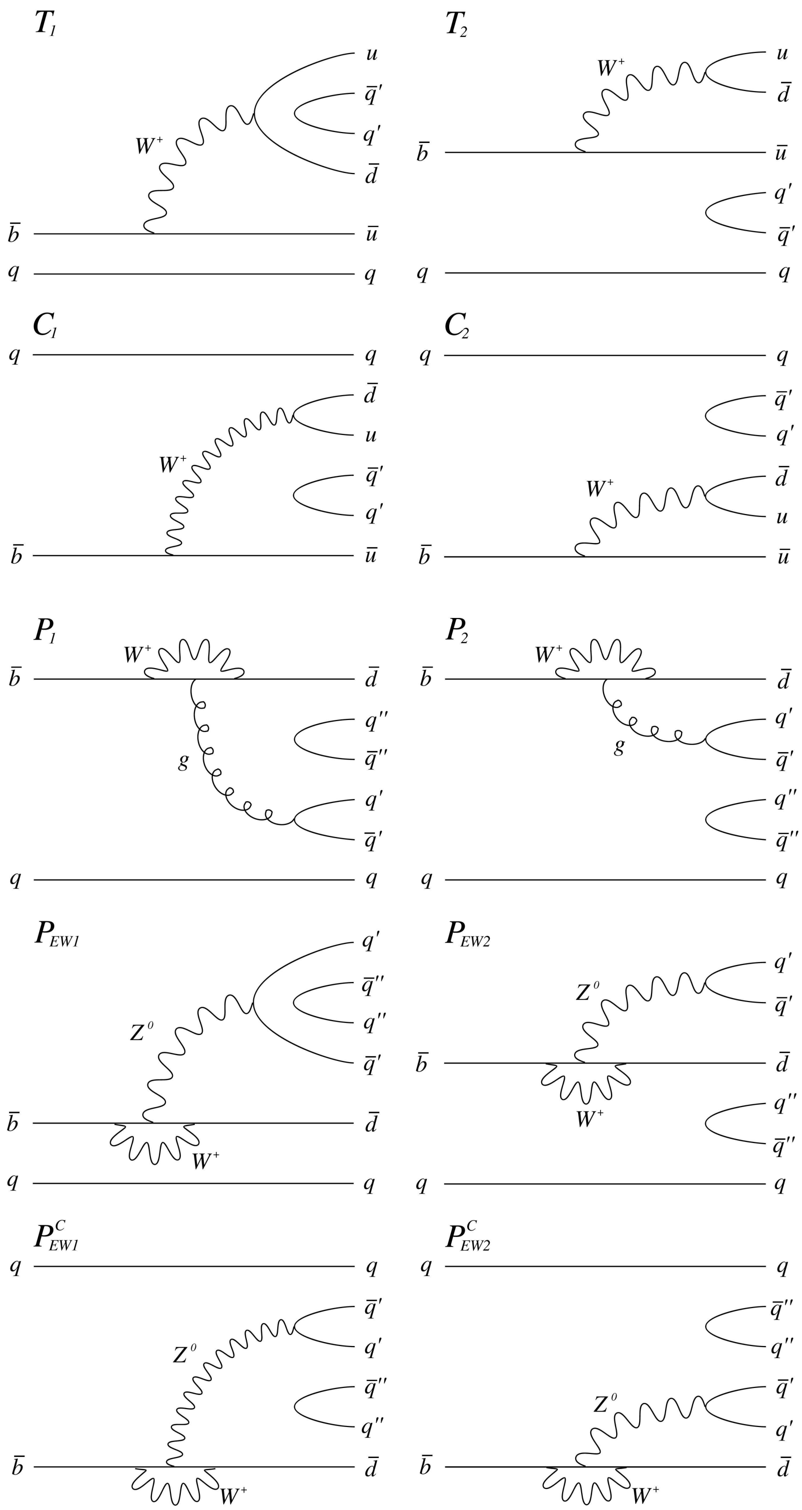

Figure 1: Diagrams contributing to $B \to \pi\pi\pi$.

3 $B \to K\pi\pi$ Decays

We begin with $B \to K\pi\pi$ decays, a $\bar{b} \to \bar{s}$ transition. There are six processes: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^0\pi^0\pi^0$. In all of these, the overall wavefunction of the final $\pi\pi$ pair must be symmetrized with respect to the exchange of these two particles. There are two possibilities. If the relative angular momentum is even (odd), the isospin state must be symmetric (antisymmetric). We refer to these two cases as $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$, and discuss them in turn.

We first consider $I_{\pi\pi}^{sym}$, i.e. $I = (0,2)$. The final state has $I = \frac{1}{2}$, $\frac{3}{2}$, or $\frac{5}{2}$. The B-meson has $I = \frac{1}{2}$ and the weak Hamiltonian has $\Delta I = 0$ or 1. The final state with $I = \frac{5}{2}$ cannot be reached. So there are three different ways of getting to the final state. Given that there are six decays, this means that there should be three relations among their amplitudes. This conclusion is an exact result; the relations can be found by applying the Wigner-Eckart theorem:

$$\begin{aligned}
A(B^+ \to K^0\pi^+\pi^0)_{sym} &= -A(B_d^0 \to K^+\pi^0\pi^-)_{sym} ~, \\
\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{sym} &= A(B_d^0 \to K^0\pi^+\pi^-)_{sym} + \sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{sym} ~, \\
\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{sym} &= A(B^+ \to K^+\pi^+\pi^-)_{sym} + \sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{sym} ~.
\end{aligned} \tag{1}$$

These relations were first given (implicitly) in Ref. [2]. The subscript 'sym' indicates that the $\pi\pi$ isospin state is symmetrized.

In terms of diagrams, the amplitudes are given by

$$\begin{aligned}
\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{sym} &= -T_1' e^{i\gamma} - C_2' e^{i\gamma} + P_{EW2}' + P_{EW1}'^C ~, \\
A(B_d^0 \to K^0\pi^+\pi^-)_{sym} &= -T_1' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}_{uc}' e^{i\gamma} + \tilde{P}_{tc}' \\
&\qquad + \frac{1}{3}P_{EW1}' + \frac{2}{3}P_{EW1}'^C - \frac{1}{3}P_{EW2}'^C ~, \\
\sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{sym} &= C_1' e^{i\gamma} - C_2' e^{i\gamma} + \tilde{P}_{uc}' e^{i\gamma} - \tilde{P}_{tc}' \\
&\qquad - \frac{1}{3}P_{EW1}' + P_{EW2}' + \frac{1}{3}P_{EW1}'^C + \frac{1}{3}P_{EW2}'^C ~, \\
A(B^+ \to K^+\pi^+\pi^-)_{sym} &= -T_2' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}_{uc}' e^{i\gamma} + \tilde{P}_{tc}' \\
&\qquad + \frac{1}{3}P_{EW1}' - \frac{1}{3}P_{EW1}'^C + \frac{2}{3}P_{EW2}'^C ~, \\
\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{sym} &= T_1' e^{i\gamma} + T_2' e^{i\gamma} + C_1' e^{i\gamma} + C_2' e^{i\gamma} + \tilde{P}_{uc}' e^{i\gamma} - \tilde{P}_{tc}' \\
&\qquad - \frac{1}{3}P_{EW1}' - P_{EW2}' - \frac{2}{3}P_{EW1}'^C - \frac{2}{3}P_{EW2}'^C ~, \\
\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{sym} &= T_1' e^{i\gamma} + C_2' e^{i\gamma} - P_{EW2}' - P_{EW1}'^C ~,
\end{aligned} \tag{2}$$

where $\tilde{P}' \equiv P_1' + P_2'$. (Note: all amplitudes have been multiplied by $\sqrt{2}$.) Above we have explicitly written the weak-phase dependence (including the minus sign from

$V_{tb}^* V_{ts}$ [$\tilde{P}'_{tc}$ and EWP's]), while the diagrams contain strong phases. (The phase information in the Cabibbo-Kobayashi-Maskawa quark mixing matrix is conventionally parametrized in terms of the unitarity triangle, in which the interior (CP-violating) angles are known as α, β and γ [7].) It is straightforward to verify that the three relations of Eq. (1) are reproduced. Thus, in this case, there is no difference between the exact and diagrammatic amplitude relations.

We now turn to $I_{\pi\pi}^{anti}$, i.e. $I = 1$. Here there are four processes: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$ (one cannot antisymmetrize a $\pi^0\pi^0$ state). The final state has $I = \frac{1}{2}$ or $\frac{3}{2}$, so there are still three different paths to get to the final state. We therefore expect one relation among the four amplitudes. Ref. [2] notes that it is similar to that in $B \to \pi K$:

$$\begin{aligned} \sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{anti} + A(B^+ \to K^0\pi^+\pi^0)_{anti} = \\ \sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-)_{anti} + A(B_d^0 \to K^+\pi^0\pi^-)_{anti} \ , \end{aligned} \tag{3}$$

where the subscript '*anti*' indicates that the $\pi\pi$ isospin state is antisymmetrized.

Writing the amplitudes in terms of diagrams is a bit more complicated because antisymmetrization is involved. Depending on the order of the pions, there might be an extra minus sign. To account for this, we use the following prescription:

- All diagrams with the pions in order of decreasing charge from top to bottom are unmodified; all diagrams with the pions in order of increasing charge from top to bottom get an additional factor of -1.

This requires that diagrams always be drawn the same way. For example, the spectator quark for all tree diagrams should always appear in the same place (e.g. at the bottom of the diagram), and the decay products of the neutral bosons in penguin and EWP diagrams should always appear in the same order (e.g. quark on top, antiquark on the bottom).

With this rule, the amplitudes take the form[4]

$$\begin{aligned} \sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{anti} &= -T'_1 e^{i\gamma} - C'_2 e^{i\gamma} - 2\tilde{P}'_{uc} e^{i\gamma} + 2\tilde{P}'_{tc} \\ &\qquad - P'_{EW2} - \frac{1}{3}P'^C_{EW1} + \frac{2}{3}P'^C_{EW2} \ , \\ A(B_d^0 \to K^0\pi^+\pi^-)_{anti} &= -T'_1 e^{i\gamma} - C'_1 e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} \\ &\qquad + P'_{EW1} - \frac{2}{3}P'^C_{EW1} + \frac{1}{3}P'^C_{EW2} \ , \\ A(B^+ \to K^+\pi^+\pi^-)_{anti} &= T'_2 e^{i\gamma} - C'_1 e^{i\gamma} + \tilde{P}'_{uc} e^{i\gamma} - \tilde{P}'_{tc} \\ &\qquad + P'_{EW1} - \frac{1}{3}P'^C_{EW1} + \frac{2}{3}P'^C_{EW2} \ , \end{aligned}$$

[4]Note: even though the diagrams of Eq. (4) have the same names as those of Eq. (2), they are not the same diagrams. That is, in general, they have different sizes.

$$\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{anti} = T_1' e^{i\gamma} + 2T_2' e^{i\gamma} - C_2' e^{i\gamma} + 2\tilde{P}_{uc}' e^{i\gamma} - 2\tilde{P}_{tc}' - P_{EW2}' + \frac{1}{3} P_{EW1}'^C + \frac{4}{3} P_{EW2}'^C . \quad (4)$$

(As above, all amplitudes have been multiplied by $\sqrt{2}$.) The relation of Eq. (3) is reproduced. Therefore, there is no difference between the exact and diagrammatic amplitude relations in the antisymmetric case.

As detailed above, there are two cases in $B \to K\pi\pi$, $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$. In subsequent subsections, we will examine the individual consequences of these two scenarios. This assumes that it is possible to experimentally distinguish $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$. Now, it has been shown in Ref. [8] that, by performing a Dalitz-plot analysis, one can differentiate experimentally all the final isospin states in $D \to \pi\pi\pi$. Since $\pi\pi\pi$ is more complicated than $K\pi\pi$ (from an isospin point of view), it should be possible to adapt the Dalitz-plot technique to distinguish the symmetric and antisymmetric contributions of $B \to K\pi\pi$. This is our assumption in this paper.

For the decays examined in other sections, whenever there are identical particles in the final state (e.g. $B \to KK\bar{K}$, $B \to \pi\pi\pi$), it is necessary to differentiate different symmetry combinations of the final-state particles. Again, we assume that the method of Ref. [8] can be adapted to these decays.

3.1 Resonances

It is possible that the B decays to an intermediate on-shell $M_1 M_2$ state, which then subsequently decays to $K\pi\pi$. Examples of such resonances are $M_1M_2 = K\rho$, $K^*\pi$, $Kf_0(980)$. The question now is: how does the diagrammatic analysis presented above jibe with resonant decays?

The key observation is that the diagrams which describe $B \to M_1M_2 \to K\pi\pi$ are a subset of those used for $B \to K\pi\pi$ (Fig. 1). Thus, any conclusions derived from the use of non-resonant diagrams will also apply to resonant decays. To be specific, let us consider the resonances in turn.

Consider first $M_1M_2 = K\rho$. The four decays are $B^+ \to K^+\rho^0$, $B^+ \to K^0\rho^+$, $B_d^0 \to K^0\rho^0$, $B_d^0 \to K^+\rho^-$, whose amplitudes take the form

$$\begin{aligned}
\sqrt{2}A(B^+ \to K^+\rho^0) &= -T_V' e^{i\gamma} - C_P' e^{i\gamma} - P_{uc,V}' e^{i\gamma} + P_{tc,V}' + P_{EW,P}' + \frac{2}{3} P_{EW,V}'^C , \\
A(B^+ \to K^0\rho^+) &= P_{uc,V}' e^{i\gamma} - P_{tc,V}' + \frac{1}{3} P_{EW,V}'^C , \\
\sqrt{2}A(B_d^0 \to K^0\rho^0) &= -C_P' e^{i\gamma} + P_{uc,V}' e^{i\gamma} - P_{tc,V}' + P_{EW,P}' + \frac{1}{3} P_{EW,V}'^C , \\
A(B_d^0 \to K^+\rho^-) &= -T_V' e^{i\gamma} - P_{uc,V}' e^{i\gamma} + P_{tc,V}' + \frac{2}{3} P_{EW,V}'^C , \qquad (5)
\end{aligned}$$

where the subscript P or V indicates which final-state meson [pseudoscalar (K) or vector (ρ)] contains the spectator quark of the B meson [9]. The relation among the amplitudes is

$$\sqrt{2}A(B^+ \to K^+\rho^0) + A(B^+ \to K^0\rho^+) = \\ \sqrt{2}A(B_d^0 \to K^0\rho^0) + A(B_d^0 \to K^+\rho^-) \ . \quad (6)$$

Given that $\rho^0 \to \pi^+\pi^-$, $\rho^+ \to \pi^+\pi^0$ and $\rho^- \to \pi^0\pi^-$, this reproduces Eq. (3), which is the relation for the antisymmetric $\pi\pi$ isospin state. This makes sense, since the ρ decays to $(\pi\pi)_{anti}$.

Consider now $M_1M_2 = Kf_0(980)$. There are two decays: $B^+ \to K^+f_0(980)$ and $B_d^0 \to K^0f_0(980)$. It is straightforward to show that there is no relation between the two amplitudes. However, the $f_0(980)$ decays to a pion pair in a symmetric isospin state, with $A(f_0 \to (\pi^+\pi^-)_{sym}) = -\sqrt{2}A(f_0 \to \pi^0\pi^0)$. This leads to

$$\begin{aligned} A(B_d^0 \to K^0\pi^+\pi^-) + \sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0) &= 0 \ , \\ A(B^+ \to K^+\pi^+\pi^-) + \sqrt{2}A(B^+ \to K^+\pi^0\pi^0) &= 0 \ . \end{aligned} \quad (7)$$

Given that $A(B^+ \to K^0\pi^+\pi^0) = A(B_d^0 \to K^+\pi^0\pi^-) = A(B^+ \to K^0\pi^+\pi^0) = 0$, this reproduces Eq. (1), which are the relations for the symmetric $\pi\pi$ isospin state.

Finally, consider $M_1M_2 = K^*\pi$. The four decays are $B^+ \to K^{*0}\pi^+$, $B^+ \to K^{*+}\pi^0$, $B_d^0 \to K^{*+}\pi^-$, $B_d^0 \to K^{*0}\pi^0$. The amplitudes are [9]

$$\begin{aligned} A(B^+ \to K^{*0}\pi^+) &= P'_{uc,P}e^{i\gamma} - P'_{tc,P} + \frac{1}{3}P'^C_{EW,P} \ , \\ \sqrt{2}A(B^+ \to K^{*+}\pi^0) &= -T'_Pe^{i\gamma} - C'_Ve^{i\gamma} - P'_{uc,P}e^{i\gamma} + P'_{tc,P} + P'_{EW,V} + \frac{2}{3}P'^C_{EW,P} \ , \\ A(B_d^0 \to K^{*+}\pi^-) &= -T'_Pe^{i\gamma} - P'_{uc,P}e^{i\gamma} + P'_{tc,P} + \frac{2}{3}P'^C_{EW,P} \ , \\ \sqrt{2}A(B_d^0 \to K^{*0}\pi^0) &= -C'_Ve^{i\gamma} + P'_{uc,P}e^{i\gamma} - P'_{tc,P} + P'_{EW,V} + \frac{1}{3}P'^C_{EW,P} \ . \end{aligned} \quad (8)$$

The relation among the amplitudes is

$$A(B^+ \to K^{*0}\pi^+) + \sqrt{2}A(B^+ \to K^{*+}\pi^0) = \\ A(B_d^0 \to K^{*+}\pi^-) + \sqrt{2}A(B_d^0 \to K^{*0}\pi^0) \ . \quad (9)$$

Now, the K^* decays to $K\pi$, and both charge assignments are allowed:

$$\begin{aligned} K^{*+} &\to \sqrt{1/3}\,K^+\pi^0 - \sqrt{2/3}\,K^0\pi^+ \ , \\ K^{*0} &\to \sqrt{2/3}\,K^+\pi^- - \sqrt{1/3}\,K^0\pi^0 \ . \end{aligned} \quad (10)$$

There are therefore several $K^*\pi$ contributions to a particular $K\pi\pi$ final state. However, one never reproduces the relations in Eqs. (1) or (3). This reflects the fact that this resonance contributes to both $(\pi\pi)_{sym}$ and $(\pi\pi)_{anti}$.

3.2 Penguin Dominance

In general, the dominant contribution to $\bar{b} \to \bar{s}$ transitions comes from the penguin amplitude. In Ref. [4], Gronau and Rosner explore the consequences for $B \to K\pi\pi$ decays of assuming penguin dominance and neglecting all other contributions. They note that, in this limit, the amplitudes must respect isospin reflection (i.e. $u \leftrightarrow d$), which implies that

$$\begin{aligned} A(B^+ \to K^+\pi^+\pi^-) &= A(B_d^0 \to K^0\pi^+\pi^-) \ , \\ A(B^+ \to K^0\pi^+\pi^0) &= A(B_d^0 \to K^+\pi^0\pi^-) \ , \\ A(B_d^0 \to K^0\pi^0\pi^0) &= A(B^+ \to K^+\pi^0\pi^0) \ , \end{aligned} \tag{11}$$

up to possible relative signs. They find that, on the whole, the data respect these relations.

The expression of the amplitudes in terms of diagrams allows us to go beyond these results. Assuming the method of Ref. [8] can be adapted to distinguish $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$, it is possible to consider the two cases separately, under the condition that only the diagram $\tilde{P}'_{tc}$ is retained in the amplitudes.

In the symmetric scenario, we have the following predictions:

$$\begin{aligned} A(B^+ \to K^0\pi^+\pi^0) &= A(B_d^0 \to K^+\pi^0\pi^-) = 0 \ , \\ A(B^+ \to K^+\pi^+\pi^-) = A(B_d^0 \to K^0\pi^+\pi^-) & \\ &= -\sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0) = -\sqrt{2}A(B^+ \to K^+\pi^0\pi^0) \ . \end{aligned} \tag{12}$$

And in the antisymmetric scenario, we have

$$\begin{aligned} A(B_d^0 \to K^0\pi^0\pi^0) &= A(B^+ \to K^+\pi^0\pi^0) = 0 \ , \\ A(B^+ \to K^0\pi^+\pi^0) = -A(B_d^0 \to K^+\pi^0\pi^-) & \\ &= -\sqrt{2}A(B^+ \to K^+\pi^+\pi^-) = \sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-) \ . \end{aligned} \tag{13}$$

These are further tests of the SM which can be made once the symmetric and antisymmetric scenarios are experimentally distinguished.

3.3 Weak-Phase Information

Since the expressions for the decay amplitudes include the weak phase γ, it is natural to ask whether γ can be extracted from measurements of $B \to K\pi\pi$ decays. The answer is 'yes' if the number of unknown theoretical parameters in the amplitudes is less than or equal to the number of observables. In performing this comparison, we examine separately the $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$ scenarios.

Consider first the $I_{\pi\pi}^{sym}$ case. Here there are six $B \to K\pi\pi$ decays. On the other hand, the first relation in Eq. (1) shows that the amplitudes for $B^+ \to K^0\pi^+\pi^0$ and $B_d^0 \to K^+\pi^0\pi^-$ are equal (up to a sign), so that there are only five independent decays. These yield 12 observables: the branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^0\pi^0\pi^0$, and the indirect CP asymmetries[5] of $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^0\pi^0\pi^0$. However, the indirect CP asymmetry of $B_d^0 \to K^0\pi^0\pi^0$ will be very difficult, if not impossible, to measure. Thus, there are essentially 11 observables in $I_{\pi\pi}^{sym}$ $B \to K\pi\pi$ decays.

For the case of $I_{\pi\pi}^{anti}$, there are four decays, yielding 9 observables: the branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, and the indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$. Since this is fewer than above, we conclude that the $I_{\pi\pi}^{sym}$ scenario is the more promising for extracting γ.

The six $I_{\pi\pi}^{sym}$ amplitudes are given in Eq. (2). Although there are a large number of diagrams in these amplitudes, they can be combined into a smaller number of effective diagrams:

$$
\begin{aligned}
\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{sym} &= -T'_a e^{i\gamma} - T'_b e^{i\gamma} + P'_{EW,a} + P'_{EW,b} ~, \\
A(B_d^0 \to K^0\pi^+\pi^-)_{sym} &= -T'_a e^{i\gamma} - P'_a e^{i\gamma} + P'_b ~, \\
\sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{sym} &= -T'_b e^{i\gamma} + P'_a e^{i\gamma} - P'_b + P'_{EW,a} + P'_{EW,b} ~, \\
A(B^+ \to K^+\pi^+\pi^-)_{sym} &= -P'_a e^{i\gamma} + P'_b - P'_{EW,a} ~, \\
\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{sym} &= T'_a e^{i\gamma} + T'_b e^{i\gamma} + P'_a e^{i\gamma} - P'_b - P'_{EW,b} ~, \\
\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{sym} &= T'_a e^{i\gamma} + T'_b e^{i\gamma} - P'_{EW,a} - P'_{EW,b} ~,
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
T'_a &\equiv T'_1 - T'_2 ~, \\
T'_b &\equiv C'_2 + T'_2 ~, \\
P'_a &\equiv \tilde{P}'_{uc} + T'_2 + C'_1 ~, \\
P'_b &\equiv \tilde{P}'_{tc} + \frac{1}{3}P'_{EW1} + \frac{2}{3}P'^C_{EW1} - \frac{1}{3}P'^C_{EW2} ~, \\
P'_{EW,a} &\equiv P'^C_{EW1} - P'^C_{EW2} ~, \\
P'_{EW,b} &\equiv P'_{EW2} + P'^C_{EW2} ~.
\end{aligned} \tag{15}
$$

The amplitudes can therefore be written in terms of 6 effective diagrams. This corresponds to 12 theoretical parameters[6]: 6 magnitudes of diagrams, 5 relative (strong) phases, and γ. However, as noted above, there are only 11 experimental

[5]The indirect CP asymmetry depends on the CP of the final state, and a-priori $K^0\pi^+\pi^-$ and $K^0\pi^0\pi^0$ are mixtures of CP + and CP −. However, the separation of symmetric and antisymmetric $\pi\pi$ states also separates the final-state CP, so that the $I_{\pi\pi}^{sym}$ and $I_{\pi\pi}^{anti}$ states have well-defined CP's.

[6]In fact, there is another theoretical parameter – the phase of B_d^0-$\bar{B}_d^0$ mixing, β, enters in the expression for the indirect CP asymmetry. However, the value for β can be taken from the indirect CP asymmetry in $B_d^0 \to J/\psi K_S$ [7].

observables. Therefore, in order to extract weak-phase information (γ), one requires additional input.

A previous analysis made an attempt in this direction. In 2003, Deshpande, Sinha and Sinha (DSS) wrote schematic expressions for the symmetric $B \to K\pi\pi$ amplitudes, including tree and EWP contributions [10]. Now, in $B \to \pi K$ decays, it was shown that, under flavor SU(3) symmetry, the EWP diagrams are proportional to the tree diagrams (apart from their weak phases) [11]. DSS assumed that the EWP and tree contributions to $B^+ \to K^0\pi^+\pi^0$ are related in the same way. This gives the additional input, and allows the measurement of γ. Unfortunately, it was subsequently noted that the assumed EWP-tree relation in $K\pi\pi$ does not hold [12], so that γ cannot be extracted. This is the present situation.

In fact, the situation can be remedied. Referring to the $B_d^0 \to K^0\pi^+\pi^0$ amplitude in Eq. (2), DSS made the assumption that $T_1' + C_2'$ is related to $P'_{EW2} + P'^C_{EW1}$, and was shown not to be true. We agree with this. However, there are other EWP-tree relations which do hold, and their inclusion does allow the extraction of γ. The full derivation is rather complicated, and so we present this in a separate paper [13].

Finally, we note that there is another method for obtaining γ from $B \to K\pi\pi$ decays. In two-body $\bar{b} \to \bar{s}$ B decays, the diagrams are expected to obey the approximate hierarchy [6]

$$\begin{aligned} 1 &: P'_{tc} ~, \\ \bar{\lambda} &: T', P'_{EW} ~, \\ \bar{\lambda}^2 &: C', P'_{uc}, P'^C_{EW} ~, \end{aligned} \tag{16}$$

where $\bar{\lambda} \simeq 0.2$. If the three-body decay diagrams obey a similar hierarchy, one can neglect C_1', C_2', $\tilde{P}'_{uc}$, P'^C_{EW1}, P'^C_{EW2}, and incur only a $\sim 5\%$ theoretical error. But if these diagrams are neglected, then two of the effective diagrams vanish: $P'_{EW,a} \to 0$ and $T_b' - P_a' \to 0$ [Eq. (15)]. In this case, the amplitudes can be written in terms of 4 effective diagrams, corresponding to 8 theoretical parameters: 4 magnitudes of diagrams, 3 relative (strong) phases, and γ. Given that there are 11 experimental observables, the weak phase γ can be extracted[7].

The downside of this method is that it is difficult to test the assumption that certain diagrams are negligible. Indeed, the presence of resonances may change the hierarchy. In light of this, the theoretical error is uncertain, and this must be addressed if this method is used.

4 $B \to KK\bar{K}$ Decays

We now turn to $B \to KK\bar{K}$ decays, also a $\bar{b} \to \bar{s}$ transition. The four processes are: $B^+ \to K^+K^+K^-$, $B^+ \to K^+K^0\bar{K}^0$, $B_d^0 \to K^+K^0K^-$, $B_d^0 \to K^0K^0\bar{K}^0$. Here

[7]This technique does not work when the $\pi\pi$ pair is in an antisymmetric state of isospin. In this case, there are still more theoretical unknowns than observables, so that γ cannot be extracted.

the overall wavefunction of the final KK pair must be symmetrized. If the relative angular momentum is even, the isospin state must be symmetric ($I = 1$); if it is odd, the isospin state must be antisymmetric ($I = 0$).

For the symmetric case, the final state has $I = \frac{1}{2}$ or $\frac{3}{2}$, so there are three different ways of reaching it. There should therefore be one relation among the four decay amplitudes. From the Wigner-Eckart theorem, it is

$$\begin{aligned} & A(B^+ \to K^+K^+K^-)_{sym} + \sqrt{2}A(B^+ \to K^+K^0\bar{K}^0)_{sym} = \\ & \qquad \sqrt{2}A(B_d^0 \to K^+K^0K^-)_{sym} + A(B_d^0 \to K^0K^0\bar{K}^0)_{sym} \ . \end{aligned} \tag{17}$$

In terms of diagrams, the amplitudes are given by

$$\begin{aligned} A(B^+ \to K^+K^+K^-)_{sym} &= -T'_{2,s}e^{i\gamma} - C'_{1,s}e^{i\gamma} - \hat{P}'_{uc}e^{i\gamma} + \hat{P}'_{tc} \\ & \qquad + \frac{2}{3}P'_{EW1,s} - \frac{1}{3}P'_{EW1} + \frac{2}{3}P'^{C}_{EW2,s} - \frac{1}{3}P'^{C}_{EW1} \ , \\ \sqrt{2}A(B^+ \to K^+K^0\bar{K}^0)_{sym} &= \hat{P}'_{uc}e^{i\gamma} - \hat{P}'_{tc} \\ & \qquad + \frac{1}{3}P'_{EW1,s} + \frac{1}{3}P'_{EW1} + \frac{1}{3}P'^{C}_{EW2,s} + \frac{1}{3}P'^{C}_{EW1} \ , \\ \sqrt{2}A(B_d^0 \to K^+K^0K^-)_{sym} &= -T'_{2,s}e^{i\gamma} - C'_{1,s}e^{i\gamma} - \hat{P}'_{uc}e^{i\gamma} + \hat{P}'_{tc} \\ & \qquad + \frac{2}{3}P'_{EW1,s} - \frac{1}{3}P'_{EW1} + \frac{2}{3}P'^{C}_{EW2,s} - \frac{1}{3}P'^{C}_{EW1} \ , \\ A(B_d^0 \to K^0K^0\bar{K}^0)_{sym} &= \hat{P}'_{uc}e^{i\gamma} - \hat{P}'_{tc} \\ & \qquad + \frac{1}{3}P'_{EW1,s} + \frac{1}{3}P'_{EW1} + \frac{1}{3}P'^{C}_{EW2,s} + \frac{1}{3}P'^{C}_{EW1} \ , \end{aligned} \tag{18}$$

where $\hat{P}' \equiv P'_{2,s} + P'_1$. It is straightforward to verify that the relation of Eq. (17) is reproduced. On the other hand, one sees that there are, in fact, two relations:

$$\begin{aligned} A(B^+ \to K^+K^+K^-)_{sym} &= \sqrt{2}A(B_d^0 \to K^+K^0K^-)_{sym} \ , \\ \sqrt{2}A(B^+ \to K^+K^0\bar{K}^0)_{sym} &= A(B_d^0 \to K^0K^0\bar{K}^0)_{sym} \ . \end{aligned} \tag{19}$$

What's happening is the following. Eq. (17) is exact. However, when annihilation-type diagrams are neglected – as is done in our diagrammatic expressions of amplitudes – then one finds the two relations above. This is an example of how one can go beyond the exact relations if certain negligible diagrams are dropped.

In the antisymmetric case, there are only two decays: $B^+ \to K^+K^0\bar{K}^0$ and $B_d^0 \to K^+K^0K^-$. $A(B^+ \to K^+K^+K^-)$ and $A(B_d^0 \to K^0K^0\bar{K}^0)$ vanish because there is no way of antisymmetrizing the K^+K^+ or K^0K^0 pair. Here the final state has $I = \frac{1}{2}$, and there are two different ways of reaching it. We therefore expect no relation between the amplitudes.

In order to write the amplitudes in terms of diagrams, we have to antisymmetrize the K^+-K^0 state. As was done for $K\pi\pi$, we adopt the following rule: all diagrams with the K^+-K^0 in order of decreasing charge from top to bottom are unmodified; all diagrams with the K^+-K^0 in order of increasing charge from top to bottom get an additional factor of -1. The amplitudes (multiplied by $\sqrt{2}$) are then given by

$$\begin{aligned}
\sqrt{2}A(B^+ \to K^+ K^0 \bar{K}^0)_{anti} &= -\hat{P}'_{uc} e^{i\gamma} + \hat{P}'_{tc} \\
&\quad - \frac{1}{3} P'_{EW1,s} - \frac{1}{3} P'_{EW1} + \frac{1}{3} P'^C_{EW2,s} + \frac{1}{3} P'^C_{EW1} \ , \\
\sqrt{2}A(B_d^0 \to K^+ K^0 K^-)_{anti} &= -T'_{2,s} e^{i\gamma} + C'_{1,s} e^{i\gamma} - \hat{P}'_{uc} e^{i\gamma} + \hat{P}'_{tc} \\
&\quad + \frac{2}{3} P'_{EW1,s} - \frac{1}{3} P'_{EW1} - \frac{2}{3} P'^C_{EW2,s} + \frac{1}{3} P'^C_{EW1} \ .
\end{aligned} \qquad (20)$$

As expected, there is no relation between these two amplitudes.

4.1 Penguin Dominance

Assuming penguin dominance, Gronau and Rosner find that isospin reflection implies the following equalities [4]:

$$\begin{aligned}
A(B^+ \to K^+ K^+ K^-) &= -A(B_d^0 \to K^0 K^0 \bar{K}^0) \ , \\
A(B^+ \to K^+ K^0 \bar{K}^0) &= -A(B_d^0 \to K^+ K^0 K^-) \ .
\end{aligned} \qquad (21)$$

Assuming the symmetric and antisymmetric isospin states can be experimentally distinguished, it is possible to go beyond these predictions. In the symmetric scenario, if only $\hat{P}'_{tc}$ is retained, we predict

$$\begin{aligned}
& A(B^+ \to K^+ K^+ K^-) = -A(B_d^0 \to K^0 K^0 \bar{K}^0) \\
& \qquad = -\sqrt{2}A(B^+ \to K^+ K^0 \bar{K}^0) = \sqrt{2}A(B_d^0 \to K^+ K^0 K^-) \ .
\end{aligned} \qquad (22)$$

(Note: the relations given in Eq. (19) actually hold for all diagrams, not just $\hat{P}'_{tc}$.) In the antisymmetric scenario, we have only $A(B^+ \to K^+ K^0 \bar{K}^0) = A(B_d^0 \to K^+ K^0 K^-)$. As with $K\pi\pi$ decays, these are further tests of the SM which can be performed if the symmetric and antisymmetric cases can be distinguished.

4.2 Isospin Amplitudes

In Ref. [3], Gronau and Rosner (GR) write the amplitudes for $B \to KK\bar{K}$ decays in terms of isospin amplitudes. It is instructive to compare this with the diagrammatic description.

As described above, there are five independent isospin amplitudes, denoted by $A_{\Delta I}^{I(KK),I_f} \equiv \langle I(KK), I_f | \, \Delta I \left| \frac{1}{2} \right\rangle$, where $I(KK)$ is the isospin of the KK pair $[I(KK) = 1 \ (0)$ is symmetric (antisymmetric)], I_f is the isospin of the final state, and the weak Hamiltonian has $\Delta I = 0$ or 1. They are listed as $A_0^{0,\frac{1}{2}}$, $A_0^{1,\frac{1}{2}}$, $A_1^{0,\frac{1}{2}}$, $A_1^{1,\frac{1}{2}}$, $A_1^{1,\frac{3}{2}}$.

As noted by GR, the $B \to KK\bar{K}$ amplitudes depend on the kaons' momenta. The amplitudes for $B^+ \to K^+ K^0 \bar{K}^0$ and $B_d^0 \to K^+ K^0 K^-$ take different values when the K^+ and K^0 momenta are exchanged. Thus, GR obtain expressions for six decay amplitudes in terms of the five isospin amplitudes:

$$
\begin{aligned}
A(B^+ \to K^+ K^+ K^-)_{p_1 p_2 p_3} &= 2A_0^{1,\frac{1}{2}} - 2A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~, \\
A(B_d^0 \to K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3} &= -2A_0^{1,\frac{1}{2}} - 2A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~, \\
A(B^+ \to K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} &= A_0^{0,\frac{1}{2}} - A_0^{1,\frac{1}{2}} - A_1^{0,\frac{1}{2}} + A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~, \\
A(B^+ \to K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3} &= -A_0^{0,\frac{1}{2}} - A_0^{1,\frac{1}{2}} + A_1^{0,\frac{1}{2}} + A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~, \\
A(B_d^0 \to K^+ K^0 K^-)_{p_1 p_2 p_3} &= A_0^{0,\frac{1}{2}} + A_0^{1,\frac{1}{2}} + A_1^{0,\frac{1}{2}} + A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~, \\
A(B_d^0 \to K^+ K^0 K^-)_{p_2 p_1 p_3} &= -A_0^{0,\frac{1}{2}} + A_0^{1,\frac{1}{2}} - A_1^{0,\frac{1}{2}} + A_1^{1,\frac{1}{2}} + A_1^{1,\frac{3}{2}} ~. \qquad (23)
\end{aligned}
$$

The above amplitudes are related to those of Eqs. (18) and (20) as follows:

$$
\begin{aligned}
A(B^+ \to K^+ K^+ K^-)_{sym} &= A(B^+ \to K^+ K^+ K^-)_{p_1 p_2 p_3} ~, \\
A(B_d^0 \to K^0 K^0 \bar{K}^0)_{sym} &= A(B_d^0 \to K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3} ~, \\
\sqrt{2} A(B^+ \to K^+ K^0 \bar{K}^0)_{sym} &= \\
& A(B^+ \to K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} + A(B^+ \to K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3} ~, \\
\sqrt{2} A(B_d^0 \to K^+ K^0 K^-)_{sym} &= \\
& A(B_d^0 \to K^+ K^0 K^-)_{p_1 p_2 p_3} + A(B_d^0 \to K^+ K^0 K^-)_{p_2 p_1 p_3} ~, \\
\sqrt{2} A(B^+ \to K^+ K^0 \bar{K}^0)_{anti} &= \\
& A(B^+ \to K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} - A(B^+ \to K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3} ~, \\
\sqrt{2} A(B_d^0 \to K^+ K^0 K^-)_{anti} &= \\
& A(B_d^0 \to K^+ K^0 K^-)_{p_1 p_2 p_3} - A(B_d^0 \to K^+ K^0 K^-)_{p_2 p_1 p_3} ~. \qquad (24)
\end{aligned}
$$

Now, because there are six decay amplitudes, but only five isospin amplitudes, there must be a relation between the decay amplitudes. GR give this relation as

$$
\begin{aligned}
& A(B^+ \to K^+ K^+ K^-)_{p_1 p_2 p_3} + A(B^+ \to K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} \\
& \qquad\qquad + A(B^+ \to K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3} = \\
& A(B_d^0 \to K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3} + A(B_d^0 \to K^+ K^0 K^-)_{p_1 p_2 p_3} \\
& \qquad\qquad + A(B_d^0 \to K^+ K^0 K^-)_{p_2 p_1 p_3} = 3A_1^{1,\frac{3}{2}} ~. \qquad (25)
\end{aligned}
$$

This is the same as the relation in Eq. (17). However, when one expresses the amplitudes in terms of diagrams, there are, in fact, two relations instead of one [Eq. (19)]. This implies that

$$
A_1^{1,\frac{1}{2}} = -\frac{1}{4} A_1^{1,\frac{3}{2}} ~, \qquad (26)
$$

so that there are really four independent isospin amplitudes instead of five. As described above, the extra relation is a consequence of neglecting the annihilation-type diagrams. In other words, the above relation among isospin amplitudes is a good approximation, and could not have been deduced without performing a diagrammatic analysis.

It is straightforward to express the remaining isospin amplitudes in terms of diagrams:

$$
\begin{aligned}
A_0^{1,\frac{1}{2}} &= \frac{1}{4}\left[-T'_{2,s}e^{i\gamma} - C'_{1,s}e^{i\gamma} - 2\hat{P}'_{uc}e^{i\gamma} + 2\hat{P}'_{tc} \right. \\
&\qquad \left. + \frac{1}{3}P'_{EW1,s} - \frac{2}{3}P'_{EW1} + \frac{1}{3}P'^{C}_{EW2,s} - \frac{2}{3}P'^{C}_{EW1}\right] , \\
A_1^{1,\frac{3}{2}} &= \frac{1}{3}\left[-T'_{2,s}e^{i\gamma} - C'_{1,s}e^{i\gamma} + P'_{EW1,s} + P'^{C}_{EW2,s}\right] , \\
A_0^{0,\frac{1}{2}} &= \frac{1}{4}\left[-T'_{2,s}e^{i\gamma} + C'_{1,s}e^{i\gamma} - 2\hat{P}'_{uc}e^{i\gamma} + 2\hat{P}'_{tc} \right. \\
&\qquad \left. + \frac{1}{3}P'_{EW1,s} - \frac{2}{3}P'_{EW1} - \frac{1}{3}P'^{C}_{EW2,s} + \frac{2}{3}P'^{C}_{EW1}\right] , \\
A_1^{0,\frac{1}{2}} &= \frac{1}{4}\left[-T'_{2,s}e^{i\gamma} + C'_{1,s}e^{i\gamma} + P'_{EW1,s} - P'^{C}_{EW2,s}\right] , \qquad (27)
\end{aligned}
$$

(Recall that, despite their having the same name, the diagrams which contribute to the $A_{\{0,1\}}^{1,\{\frac{1}{2},\frac{3}{2}\}}$ and $A_{\{0,1\}}^{0,\frac{1}{2}}$ isospin amplitudes are not the same – they can have different sizes.) In the limit of penguin dominance, $A_1^{1,\frac{3}{2}}$ and $A_0^{1,\frac{1}{2}}$ vanish. This is consistent with what is found in the previous subsection.

4.3 Weak-Phase Information

As was the case for $B \to K\pi\pi$ decays, the amplitudes contain the weak phase γ, and so one wonders if it can be measured in $B \to KK\bar{K}$ decays. Here the answer is 'perhaps'.

When the isospin state of the KK pair is symmetric, there are four decays. However, due to the equality relations in Eq. (19), two of these have the same amplitudes as the other two. There are therefore 6 observables: the branching ratios and the direct and indirect CP asymmetries of $B_d^0 \to K^+K^0K^-$, $B_d^0 \to K^0K^0\bar{K}^0$. In the antisymmetric scenario, there are 5 observables: the branching ratios and direct CP asymmetries of $B^+ \to K^+K^0\bar{K}^0$, $B_d^0 \to K^+K^0K^-$, and the indirect CP asymmetry of $B_d^0 \to K^+K^0K^-$. (As with $B \to K\pi\pi$, the separation of symmetric and antisymmetric KK states fixes the CP of the final state for the indirect CP asymmetries.)

However, in either case, the amplitudes [Eqs. (18) and (20)] are written in terms of 4 effective diagrams, corresponding to 8 theoretical parameters: 4 magnitudes of diagrams, 3 relative (strong) phases, and γ. This is larger than the number of observables, and so the weak phase γ cannot be extracted from $B \to KK\bar{K}$ decays.

The best that one can do is to assume the hierarchy of Eq. (16), and neglect all C', $\hat{P}'_{uc}$ and P'^{C}_{EW} diagrams. This reduces the number of effective diagrams to 3, which corresponds to 6 theoretical parameters. This is equal to the number of observables in the symmetric case, so that γ can be extracted here, albeit with discrete ambiguities. And, as described above, the theoretical error is uncertain.

5 $B \to K\bar{K}\pi$ Decays

We now consider $B \to K\bar{K}\pi$ decays, which are a $\bar{b} \to \bar{d}$ transition. Here there are seven processes: $B^+ \to K^+K^-\pi^+$, $B^+ \to K^+\bar{K}^0\pi^0$, $B^+ \to K^0\bar{K}^0\pi^+$, $B_d^0 \to K^+K^-\pi^0$, $B_d^0 \to K^+\bar{K}^0\pi^-$, $B_d^0 \to K^0\bar{K}^0\pi^0$, $B_d^0 \to K^0K^-\pi^+$. There are no identical particles in the final state, so here we do not have the problem of distinguishing symmetric and antisymmetric isospin states.

In $B \to K\bar{K}\pi$, the final state has $I=0$, $I=1$ (twice) or $I=2$. The weak Hamiltonian has $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$, so there are six paths to the final state. This implies that there is one relation among the seven decay amplitudes. It is

$$
\begin{aligned}
\sqrt{2}A(B_d^0 \to K^+K^-\pi^0) &+ A(B_d^0 \to K^0K^-\pi^+) - A(B^+ \to K^+K^-\pi^+) \\
&+ \sqrt{2}A(B_d^0 \to K^0\bar{K}^0\pi^0) + A(B_d^0 \to K^+\bar{K}^0\pi^-) \\
&- A(B^+ \to K^0\bar{K}^0\pi^+) - \sqrt{2}A(B^+ \to K^+\bar{K}^0\pi^0) = 0 \ . \qquad (28)
\end{aligned}
$$

In terms of diagrams, the amplitudes are given by

$$
\begin{aligned}
A(B^+ \to K^+K^-\pi^+) &= [T_{2,s} + C_{1,s} + P_{a;uc}]\, e^{-i\alpha} \\
&\quad - P_{a;tc} + \frac{1}{3}P_{EW1} - \frac{2}{3}P_{EW1,s} + \frac{1}{3}P_{EW1}^{C} - \frac{2}{3}P_{EW2,s}^{C} \ , \\
\sqrt{2}A(B^+ \to K^+\bar{K}^0\pi^0) &= [T_{1,s} + C_{2,s} - P_{a;uc} + P_{b;uc}]\, e^{-i\alpha} \\
&\quad + P_{a;tc} - P_{b;tc} - P_{EW2,s} - \frac{1}{3}P_{EW1}^{C} - \frac{2}{3}P_{EW1,s}^{C} + \frac{1}{3}P_{EW2}^{C} - \frac{1}{3}P_{EW2,s}^{C} \ , \\
A(B^+ \to K^0\bar{K}^0\pi^+) &= -P_{b;uc}e^{-i\alpha} \\
&\quad + P_{b;tc} - \frac{1}{3}P_{EW1} - \frac{1}{3}P_{EW1,s} - \frac{1}{3}P_{EW1,s}^{C} - \frac{1}{3}P_{EW2}^{C} \ , \\
\sqrt{2}A(B_d^0 \to K^+K^-\pi^0) &= C_{1,s}e^{-i\alpha} + \frac{1}{3}P_{EW1} - \frac{2}{3}P_{EW1,s} \ , \\
A(B_d^0 \to K^+\bar{K}^0\pi^-) &= [T_{1,s} + P_{b;uc}]\, e^{-i\alpha} - P_{b;tc} - \frac{2}{3}P_{EW1,s}^{C} + \frac{1}{3}P_{EW2}^{C} \ , \\
\sqrt{2}A(B_d^0 \to K^0\bar{K}^0\pi^0) &= [C_{2,s} - P_{a;uc} - P_{b;uc}]\, e^{-i\alpha} \qquad (29) \\
&\quad + P_{a;tc} + P_{b;tc} - \frac{1}{3}P_{EW1} - \frac{1}{3}P_{EW1,s} - P_{EW2,s}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{3}P^C_{EW1} - \frac{1}{3}P^C_{EW1,s} - \frac{1}{3}P^C_{EW2} - \frac{1}{3}P^C_{EW2,s} \ , \\
A(B_d^0 \to K^0 K^- \pi^+) \ = \ & [T_{2,s} + P_{a;uc}]\, e^{-i\alpha} - P_{a;tc} + \frac{1}{3}P^C_{EW1} - \frac{2}{3}P^C_{EW2,s} \ ,
\end{aligned}
$$

where $P_a \equiv P_1 + P_{2,s}$, $P_b \equiv P_{1,s} + P_2$, and all amplitudes have been multiplied by $e^{i\beta}$. With these expressions, the relation of Eq. (28) is reproduced.

However, there are, in fact, two relations:

$$
\begin{aligned}
\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) + A(B_d^0 \to K^0 K^- \pi^+) \ &= \ A(B^+ \to K^+ K^- \pi^+) \ , \\
\sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0) + A(B_d^0 \to K^+ \bar{K}^0 \pi^-) & \\
= \ A(B^+ \to K^0 \bar{K}^0 \pi^+) + \sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) \ . & \qquad (30)
\end{aligned}
$$

As was the case in $B \to KK\bar{K}$ decays, the (justified) neglect of certain annihilation-type diagrams breaks the relation in Eq. (28) into two.

5.1 T Dominance

In two-body B decays, T is the dominant diagram in $\bar{b} \to \bar{d}$ transitions. Assuming this also holds in three-body B decays, we have the following predictions:

$$
\begin{aligned}
A(B^+ \to K^+ K^- \pi^+) \ &= \ A(B_d^0 \to K^0 K^- \pi^+) \ , \\
\sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) \ &= \ A(B_d^0 \to K^+ \bar{K}^0 \pi^-) \ , \\
A(B^+ \to K^0 \bar{K}^0 \pi^+) = A(B_d^0 \to K^+ K^- \pi^0) &= A(B_d^0 \to K^0 \bar{K}^0 \pi^0) \simeq 0 \ . \qquad (31)
\end{aligned}
$$

These are tests of the SM which can be carried out once these decays are measured.

5.2 Weak-Phase Information

There are seven $B \to K\bar{K}\pi$ decays, which yield 16 observables: the branching ratios and direct CP asymmetries of $B^+ \to K^+ K^- \pi^+$, $B^+ \to K^+ \bar{K}^0 \pi^0$, $B^+ \to K^0 \bar{K}^0 \pi^+$, $B_d^0 \to K^+ K^- \pi^0$, $B_d^0 \to K^+ \bar{K}^0 \pi^-$, $B_d^0 \to K^0 \bar{K}^0 \pi^0$, $B_d^0 \to K^0 K^- \pi^+$, and the indirect CP asymmetries of $B_d^0 \to K^+ K^- \pi^0$, $B_d^0 \to K^0 \bar{K}^0 \pi^0$.

The $B \to K\bar{K}\pi$ amplitudes in Eq. (29) can be written in terms of 10 effective diagrams:

$$
\begin{aligned}
A(B^+ \to K^+ K^- \pi^+) \ &= \ [D_1 + D_3] e^{-i\alpha} + D_2 + D_4 \ , \\
\sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) \ &= \ D_9 e^{-i\alpha} + D_{10} \ , \\
A(B^+ \to K^0 \bar{K}^0 \pi^+) \ &= \ D_7 e^{-i\alpha} + D_8 \ , \\
\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) \ &= \ D_1 e^{-i\alpha} + D_2 \ , \\
A(B_d^0 \to K^+ \bar{K}^0 \pi^-) \ &= \ D_5 e^{-i\alpha} + D_6 \ , \\
\sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0) \ &= \ [-D_5 + D_7 + D_9] e^{-i\alpha} - D_6 + D_8 + D_{10} \ , \\
A(B_d^0 \to K^0 K^- \pi^+) \ &= \ D_3 e^{-i\alpha} + D_4 \ , \qquad (32)
\end{aligned}
$$

where

$$\begin{aligned}
D_1 &\equiv C_{1,s} \ , \\
D_2 &\equiv \frac{1}{3}P_{EW1} - \frac{2}{3}P_{EW1,s} \ , \\
D_3 &\equiv T_{2,s} + P_{a;uc} \ , \\
D_4 &\equiv -P_{a;tc} + \frac{1}{3}P^C_{EW1} - \frac{2}{3}P^C_{EW2,s} \ , \\
D_5 &\equiv T_{1,s} + P_{b;uc} \ , \\
D_6 &\equiv -P_{b;tc} + \frac{1}{3}P^C_{EW2} - \frac{2}{3}P^C_{EW1,s} \ , \\
D_7 &\equiv -P_{b;uc} \ , \\
D_8 &\equiv P_{b;tc} - \frac{1}{3}P_{EW1} - \frac{1}{3}P_{EW1,s} - \frac{1}{3}P^C_{EW2} - \frac{1}{3}P^C_{EW1,s} \ , \\
D_9 &\equiv T_{1,s} + C_{2,s} - P_{a;uc} + P_{b;uc} \ , \\
D_{10} &\equiv P_{a;tc} - P_{b;tc} - P_{EW2,s} - \frac{1}{3}P^C_{EW1} - \frac{2}{3}P^C_{EW1,s} + \frac{1}{3}P^C_{EW2} - \frac{1}{3}P^C_{EW2,s} \ . \qquad (33)
\end{aligned}$$

This corresponds to 20 theoretical parameters: 10 magnitudes of diagrams, 9 relative (strong) phases, and α. With only 16 observables, α cannot be extracted.

We therefore need additional input. Fortunately, we have some. In two-body $\bar{b} \to \bar{d}$ B decays, the diagrams obey the approximate hierarchy [6]

$$\begin{aligned}
1 &: T \ , \\
\bar{\lambda} &: C, P_{tc}, P_{uc} \ , \\
\bar{\lambda}^2 &: P_{EW} \ , \\
\bar{\lambda}^3 &: P^C_{EW} \ . \qquad (34)
\end{aligned}$$

Assuming that the three-body decay diagrams obey a similar hierarchy, all EWP diagrams can be neglected, leading to an error of only $\sim 5\%$. In this limit, we have $D_2 = 0$, $D_8 = -D_6$, and $D_{10} = -D_4 + D_6$. So the number of independent diagrams is reduced to 7, i.e. 14 theoretical parameters[8]. Thus, by measuring the observables in $B \to K\bar{K}\pi$ decays, weak-phase information can be obtained. In fact, not all 16 observables are necessary. Experimentally, this is not easy, but it is at least theoretically possible.

6 $B \to \pi\pi\pi$ Decays

Finally, we examine $B \to \pi\pi\pi$ decays, also a $\bar{b} \to \bar{d}$ transition. There are four processes: $B^0_d \to \pi^0\pi^0\pi^0$, $B^+ \to \pi^+\pi^0\pi^0$, $B^+ \to \pi^-\pi^+\pi^+$, $B^0_d \to \pi^+\pi^0\pi^-$. In

[8] We assume that, for the indirect CP asymmetries, the CP of the final state can be fixed as for the decays in previous sections. Otherwise there are 2 additional theoretical parameters.

contrast to the other decays, here the final state includes three identical particles under isospin, so that the six permutations of these particles (the group S_3) must be considered. Numbering the particles 1, 2, 3, the six possible orders are 123, 132, 312, 321, 231, 213. Under S_3, there are six possibilities for the isospin state of the three π's: a totally symmetric state $|S\rangle$, a totally antisymmetric state $|A\rangle$, or one of four mixed states $|M_i\rangle$ ($i =$ 1-4). These can be defined as

$$\begin{aligned}
|S\rangle &\equiv \frac{1}{\sqrt{6}}\left(|123\rangle + |132\rangle + |312\rangle + |321\rangle + |231\rangle + |213\rangle\right) ~, \\
|M_1\rangle &\equiv \frac{1}{\sqrt{12}}\left(2\,|123\rangle + 2\,|132\rangle - |312\rangle - |321\rangle - |231\rangle - |213\rangle\right) ~, \\
|M_2\rangle &\equiv \frac{1}{\sqrt{4}}\left(|312\rangle - |321\rangle - |231\rangle + |213\rangle\right) ~, \\
|M_3\rangle &\equiv \frac{1}{\sqrt{4}}\left(-\,|312\rangle - |321\rangle + |231\rangle + |213\rangle\right) ~, \\
|M_4\rangle &\equiv \frac{1}{\sqrt{12}}\left(2\,|123\rangle - 2\,|132\rangle - |312\rangle + |321\rangle - |231\rangle + |213\rangle\right) ~, \\
|A\rangle &\equiv \frac{1}{\sqrt{6}}\left(|123\rangle - |132\rangle + |312\rangle - |321\rangle + |231\rangle - |213\rangle\right) ~.
\end{aligned} \tag{35}$$

This choice of mixed states implies that two truly identical particles go in positions 2 and 3. Under the exchange $2 \leftrightarrow 3$, $|M_1\rangle$ and $|M_2\rangle$ are symmetric, while $|M_3\rangle$ and $|M_4\rangle$ are antisymmetric.

For the four $B \to \pi\pi\pi$ decays, we have:

1. $B_d^0 \to \pi^0\pi^0\pi^0$: all final-state particles are the same, which means $|123\rangle = |132\rangle = |312\rangle = |321\rangle = |231\rangle = |213\rangle$. In this case, only the state $|S\rangle$ is allowed.

2. $B^+ \to \pi^+\pi^0\pi^0$: particle 1 is π^+, particles 2 and 3 are π^0. Thus, $|123\rangle = |132\rangle$, $|312\rangle = |213\rangle$, $|231\rangle = |321\rangle$. This implies that each of $|M_3\rangle$, $|M_4\rangle$, $|A\rangle$ is not allowed.

3. $B^+ \to \pi^-\pi^+\pi^+$: particle 1 is π^-, particles 2 and 3 are π^+. Thus, $|123\rangle = |132\rangle$, $|312\rangle = |213\rangle$, $|231\rangle = |321\rangle$. This implies that each of $|M_3\rangle$, $|M_4\rangle$, $|A\rangle$ is not allowed.

4. $B_d^0 \to \pi^+\pi^0\pi^-$: we choose the order such that particle 1 is π^+, particle 2 is π^0, particle 3 is π^-. All six states are allowed.

The amplitude for a decay with two truly identical particles has an extra factor of $1/\sqrt{2}$; with three truly identical particles, the factor is $1/\sqrt{6}$.

The six elements of S_3 are: I (identity), P_{12} (exchanges particles 1 and 2), P_{13} (exchanges particles 1 and 3), P_{23} (exchanges particles 2 and 3), P_{cyclic} (cyclic permutation of particle numbers, i.e. $1 \to 2$, $2 \to 3$, $3 \to 1$), $P_{anticyclic}$ (anticyclic permutation of particle numbers, i.e. $1 \to 3$, $2 \to 1$, $3 \to 2$). Under the group transformations, $|S\rangle \to |S\rangle$ and $|A\rangle \to \pm|A\rangle$. It is easy to see that $|M_1\rangle$ and $|M_3\rangle$ transform among themselves. Writing

$$|M_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \qquad |M_3\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \tag{36}$$

we can represent each group element by a 2×2 matrix:

$$\begin{gathered} I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ P_{12} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ , \ P_{13} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \ , \\ P_{23} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ P_{cyclic} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \ , \ P_{anticyclic} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \ . \end{gathered} \tag{37}$$

Similarly, if we write

$$|M_2\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \qquad |M_4\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \tag{38}$$

the S_3 matrices take the same form, showing that $|M_2\rangle$ and $|M_4\rangle$ also transform among themselves.

The above allows us to express the amplitudes for all $B \to \pi\pi\pi$ decays in terms of diagrams. We begin with some general comments about diagrams. As an example, consider T_1. In principle, there are six possibilities, T_1^{ijk}, in which the final-state pions i, j, k run from top to bottom of the diagram in all permutations. Suppose that we want the expression for the amplitude of $B \to \pi_1\pi_2\pi_3$ in a particular $|S_3\rangle$ state, and suppose that the diagram T_1^{ijk} contributes to the decay. For $|S_3\rangle = |S\rangle$, we define T_1^S:

$$T_1^S \equiv \frac{1}{\sqrt{6}} \left(T_1^{123} + T_1^{132} + T_1^{312} + T_1^{321} + T_1^{231} + T_1^{213} \right) \ . \tag{39}$$

Each T_1^{ijk} leads to T_1^S in the amplitude. For $|S_3\rangle = |A\rangle$, we have

$$T_1^A \equiv \frac{1}{\sqrt{6}} \left(T_1^{123} - T_1^{132} + T_1^{312} - T_1^{321} + T_1^{231} - T_1^{213} \right) \ . \tag{40}$$

Again, each T_1^{ijk} leads to T_1^A in the amplitude, with a coefficient of 1 (-1) if ijk is in cyclic (anticyclic) order.

For the mixed states, one has to take into account the fact that, under group transformations, there is $|M_1\rangle$-$|M_3\rangle$ and $|M_2\rangle$-$|M_4\rangle$ mixing. In order to illustrate how this is done, we focus first on the M_1/M_3 sector. We define

$$\begin{aligned} T_1^{M_1} &\equiv \frac{1}{\sqrt{12}} \left(2T_1^{123} + 2T_1^{132} - T_1^{312} - T_1^{321} - T_1^{231} - T_1^{213} \right) \ , \\ T_1^{M_3} &\equiv \frac{1}{\sqrt{4}} \left(-T_1^{312} - T_1^{321} + T_1^{231} + T_1^{213} \right) \ . \end{aligned} \tag{41}$$

Suppose $|S_3\rangle = |M_1\rangle$. The contribution to the amplitude of $B \to \pi_1\pi_2\pi_3$ is $[M \times (T_1^{M_1}, T_1^{M_3})^T]_{upper\ component}$, where M is the matrix representing the S_3 group element which transforms ijk to 123 [Eq. (37)]. In general, this is a combination of $T_1^{M_1}$ and $T_1^{M_3}$ (though the $T_1^{M_3}$ component can be zero if $M = I$ or P_{23}). Factors of -1 for each $\bar{u}$ and $1/\sqrt{2}$ for each π^0 must also be included. If $|S_3\rangle = |M_3\rangle$, the contribution to the amplitude is $[M \times (T_1^{M_1}, T_1^{M_3})^T]_{lower\ component}$. This can be applied analogously to the M_2/M_4 sector, where we define

$$
\begin{aligned}
T_1^{M_2} &\equiv \frac{1}{\sqrt{4}}\left(T_1^{312} - T_1^{321} - T_1^{231} + T_1^{213}\right) , \\
T_1^{M_4} &\equiv \frac{1}{\sqrt{12}}\left(2T_1^{123} - 2T_1^{132} - T_1^{312} + T_1^{321} - T_1^{231} + T_1^{213}\right) .
\end{aligned} \tag{42}
$$

The entire procedure holds for all diagrams[9].

With these rules, we can now work out the amplitudes for all decays. We begin first with $|S_3\rangle = |S\rangle$. The amplitudes are

$$
\begin{aligned}
\frac{2}{\sqrt{3}}A(B_d^0 \to \pi^0\pi^0\pi^0)_{|S\rangle} &= -\left[C_1^S - C_2^S + P_{uc}^S\right]e^{-i\alpha} \\
&\quad + \left[P_{tc}^S + \frac{1}{3}P_{EW1}^S - P_{EW2}^S - \frac{1}{3}P_{EW1}^{C,S} - \frac{1}{3}P_{EW2}^{C,S}\right] , \\
\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|S\rangle} &= -\left[T_2^S + C_1^S + P_{uc}^S\right]e^{-i\alpha} \\
&\quad + \left[P_{tc}^S + \frac{1}{3}P_{EW1}^S - \frac{1}{3}P_{EW1}^{C,S} + \frac{2}{3}P_{EW2}^{C,S}\right] , \\
\frac{1}{\sqrt{2}}A(B^+ \to \pi^-\pi^+\pi^+)_{|S\rangle} &= \left[T_2^S + C_1^S + P_{uc}^S\right]e^{-i\alpha} \\
&\quad - \left[P_{tc}^S + \frac{1}{3}P_{EW1}^S - \frac{1}{3}P_{EW1}^{C,S} + \frac{2}{3}P_{EW2}^{C,S}\right] , \\
\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|S\rangle} &= \left[C_1^S - C_2^S + P_{uc}^S\right]e^{-i\alpha} \\
&\quad - \left[P_{tc}^S + \frac{1}{3}P_{EW1}^S - P_{EW2}^S - \frac{1}{3}P_{EW1}^{C,S} - \frac{1}{3}P_{EW2}^{C,S}\right] ,
\end{aligned} \tag{43}
$$

where $P \equiv P_1 + P_2$ and all amplitudes have been multiplied by $e^{i\beta}$.

For the M_1/M_3 sector, the amplitudes are

$$
\begin{aligned}
\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|M_1\rangle} &= \Bigg[\frac{3}{2}T_1^{M_1} - \frac{\sqrt{3}}{2}T_1^{M_3} - T_2^{M_1} - C_1^{M_1} + \frac{3}{2}C_2^{M_1} - \frac{\sqrt{3}}{2}C_2^{M_3} \\
&\quad - P_{uc}^{M_1} + \sqrt{3}P_{uc}^{M_3}\Bigg]e^{-i\alpha} + \Bigg[P_{tc}^{M_1} - \sqrt{3}P_{tc}^{M_3} - \frac{1}{6}P_{EW1}^{M_1} - \frac{1}{2\sqrt{3}}P_{EW1}^{M_3}
\end{aligned}
$$

[9] When applied to the decays in the previous sections, this method produces the same amplitude decomposition as when we used the simple rule of adding a minus sign to diagrams in which the identical particles are exchanged (e.g. in $B \to K\pi\pi$ or $KK\bar{K}$).

$$
\begin{aligned}
& + \sqrt{3}P^{M_3}_{EW2} - \frac{1}{3}P^{C,M_1}_{EW1} - \frac{2}{\sqrt{3}}P^{C,M_3}_{EW1} - \frac{5}{6}P^{C,M_1}_{EW2} - \frac{1}{2\sqrt{3}}P^{C,M_3}_{EW2}\Bigg] \ , \\
\sqrt{2}A(B^+ \to \pi^-\pi^+\pi^+)_{|M_1\rangle} \ = \ & \Big[-T^{M_1}_2 + \sqrt{3}T^{M_3}_2 - C^{M_1}_1 - \sqrt{3}C^{M_3}_1 \\
& - P^{M_1}_{uc} + \sqrt{3}P^{M_3}_{uc}\Big] e^{-i\alpha} + \Bigg[P^{M_1}_{tc} - \sqrt{3}P^{M_3}_{tc} + \frac{4}{3}P^{M_1}_{EW1} - \frac{2}{\sqrt{3}}P^{M_3}_{EW1} \\
& - \frac{1}{3}P^{C,M_1}_{EW1} + \frac{1}{\sqrt{3}}P^{C,M_3}_{EW1} + \frac{2}{3}P^{C,M_1}_{EW2} - \frac{2}{\sqrt{3}}P^{C,M_3}_{EW2}\Bigg] \ , \\
6\sqrt{2}A(B^0_d \to \pi^+\pi^0\pi^-)_{|M_1\rangle} \ = \ & \Big[9T^{M_1}_1 - 3\sqrt{3}T^{M_3}_1 - 3C^{M_1}_1 + 3\sqrt{3}C^{M_3}_1 + 3C^{M_1}_2 \\
& - 3\sqrt{3}C^{M_3}_2 - 3P^{M_1}_{uc} + 3\sqrt{3}P^{M_3}_{uc}\Big] e^{-i\alpha} + \Big[3P^{M_1}_{tc} - 3\sqrt{3}P^{M_3}_{tc} \\
& - 5P^{M_1}_{EW1} + \sqrt{3}P^{M_3}_{EW1} - 3P^{M_1}_{EW2} + 3\sqrt{3}P^{M_3}_{EW2} \\
& - P^{C,M_1}_{EW1} - 5\sqrt{3}P^{C,M_3}_{EW1} - P^{C,M_1}_{EW2} + \sqrt{3}P^{C,M_3}_{EW2}\Big] \ , \\
2\sqrt{6}A(B^0_d \to \pi^+\pi^0\pi^-)_{|M_3\rangle} \ = \ & \Big[-3T^{M_1}_1 + \sqrt{3}T^{M_3}_1 - 4\sqrt{3}T^{M_3}_2 + 3C^{M_1}_1 + \sqrt{3}C^{M_3}_1 \\
& + 3C^{M_1}_2 + \sqrt{3}C^{M_3}_2 + 3P^{M_1}_{uc} - 3\sqrt{3}P^{M_3}_{uc}\Big] e^{-i\alpha} + \Big[-3P^{M_1}_{tc} + 3\sqrt{3}P^{M_3}_{tc} \\
& - P^{M_1}_{EW1} + \sqrt{3}P^{M_3}_{EW1} + \ 3P^{M_1}_{EW2} + \sqrt{3}P^{M_3}_{EW2} \\
& + P^{C,M_1}_{EW1} + \sqrt{3}P^{C,M_3}_{EW1} - 5P^{C,M_1}_{EW2} + \sqrt{3}P^{C,M_3}_{EW2}\Big] \ . \qquad (44)
\end{aligned}
$$

For the M_2/M_4 sector, the amplitudes are

$$
\begin{aligned}
\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|M_2\rangle} \ = \ & \Bigg[\frac{3}{2}T^{M_2}_1 - \frac{\sqrt{3}}{2}T^{M_4}_1 - T^{M_2}_2 - C^{M_2}_1 + \frac{3}{2}C^{M_2}_2 - \frac{\sqrt{3}}{2}C^{M_4}_2 \\
& - P^{M_2}_{uc} + \sqrt{3}P^{M_4}_{uc}\Big] e^{-i\alpha} + \Bigg[P^{M_2}_{tc} - \sqrt{3}P^{M_4}_{tc} - \frac{1}{6}P^{M_2}_{EW1} - \frac{1}{2\sqrt{3}}P^{M_4}_{EW1} \\
& + \sqrt{3}P^{M_4}_{EW2} - \frac{1}{3}P^{C,M_2}_{EW1} - \frac{2}{\sqrt{3}}P^{C,M_4}_{EW1} - \frac{5}{6}P^{C,M_2}_{EW2} - \frac{1}{2\sqrt{3}}P^{C,M_4}_{EW2}\Bigg] \ , \\
\sqrt{2}A(B^+ \to \pi^-\pi^+\pi^+)_{|M_2\rangle} \ = \ & \Big[-T^{M_2}_2 + \sqrt{3}T^{M_4}_2 - C^{M_2}_1 - \sqrt{3}C^{M_4}_1 \\
& - P^{M_2}_{uc} + \sqrt{3}P^{M_4}_{uc}\Big] e^{-i\alpha} + \Bigg[P^{M_2}_{tc} - \sqrt{3}P^{M_4}_{tc} + \frac{4}{3}P^{M_2}_{EW1} - \frac{2}{\sqrt{3}}P^{M_4}_{EW1} \\
& - \frac{1}{3}P^{C,M_2}_{EW1} + \frac{1}{\sqrt{3}}P^{C,M_4}_{EW1} + \frac{2}{3}P^{C,M_2}_{EW2} - \frac{2}{\sqrt{3}}P^{C,M_4}_{EW2}\Bigg] \ , \\
6\sqrt{2}A(B^0_d \to \pi^+\pi^0\pi^-)_{|M_2\rangle} \ = \ & \Big[9T^{M_2}_1 - 3\sqrt{3}T^{M_4}_1 - 3C^{M_2}_1 + 3\sqrt{3}C^{M_4}_1 + 3C^{M_2}_2 \\
& - 3\sqrt{3}C^{M_4}_2 - 3P^{M_2}_{uc} + 3\sqrt{3}P^{M_4}_{uc}\Big] e^{-i\alpha} + \Big[3P^{M_2}_{tc} - 3\sqrt{3}P^{M_4}_{tc} \\
& - 5P^{M_2}_{EW1} + \sqrt{3}P^{M_4}_{EW1} - 3P^{M_2}_{EW2} + 3\sqrt{3}P^{M_4}_{EW2} \\
& - P^{C,M_2}_{EW1} - 5\sqrt{3}P^{C,M_4}_{EW1} - P^{C,M_2}_{EW2} + \sqrt{3}P^{C,M_4}_{EW2}\Big] \ ,
\end{aligned}
$$

$$
\begin{aligned}
2\sqrt{6}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_4\rangle} \;\; = \;\; & \Big[-3T_1^{M_2} + \sqrt{3}T_1^{M_4} - 4\sqrt{3}T_2^{M_4} + 3C_1^{M_2} + \sqrt{3}C_1^{M_4} \\
& + 3C_2^{M_2} + \sqrt{3}C_2^{M_4} + 3P_{uc}^{M_2} - 3\sqrt{3}P_{uc}^{M_4}\Big]\, e^{-i\alpha} + \Big[-3P_{tc}^{M_2} + 3\sqrt{3}P_{tc}^{M_4} \\
& - P_{EW1}^{M_2} + \sqrt{3}P_{EW1}^{M_4} + \ 3P_{EW2}^{M_2} + \sqrt{3}P_{EW2}^{M_4} \\
& + P_{EW1}^{C,M_2} + \sqrt{3}P_{EW1}^{C,M_4} - 5P_{EW2}^{C,M_2} + \sqrt{3}P_{EW2}^{C,M_4}\Big] \ . \qquad (45)
\end{aligned}
$$

Finally, for $|S_3\rangle = |A\rangle$, we have

$$
\begin{aligned}
\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|A\rangle} \;\; = \;\; & \Big[2T_1^A - 2T_2^A - C_1^A - C_2^A - 3P_{uc}^A\Big]\, e^{-i\alpha} \\
& + \Big[3P_{tc}^A + P_{EW1}^A - \ P_{EW2}^A - P_{EW1}^{C,A} - P_{EW2}^{C,A}\Big] \ . \qquad (46)
\end{aligned}
$$

Now, the final state has isospin $1 \otimes 1 \otimes 1 = 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$. Given that the B-meson has $I = \frac{1}{2}$ and the weak Hamiltonian has $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$, there are 9 paths to the final state. We therefore expect four relations among the 13 decay amplitudes. This is indeed what is found:

$$
\begin{aligned}
\sqrt{2}A(B_d^0 \to \pi^0\pi^0\pi^0)_{|S\rangle} &= -\sqrt{3}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|S\rangle} \ , \\
2A(B^+ \to \pi^+\pi^0\pi^0)_{|S\rangle} &= -A(B^+ \to \pi^-\pi^+\pi^+)_{|S\rangle} \ , \\
\frac{3}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_1\rangle} + \frac{\sqrt{3}}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_3\rangle} \;\; &= \\
& A(B^+ \to \pi^+\pi^0\pi^0)_{|M_1\rangle} - A(B^+ \to \pi^-\pi^+\pi^+)_{|M_1\rangle} \ , \\
\frac{3}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_2\rangle} + \frac{\sqrt{3}}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_4\rangle} \;\; &= \\
& A(B^+ \to \pi^+\pi^0\pi^0)_{|M_2\rangle} - A(B^+ \to \pi^-\pi^+\pi^+)_{|M_2\rangle} \ . \qquad (47)
\end{aligned}
$$

These relations can also be found using the Wigner-Eckart theorem.

In passing, we note that, within the SM, the final state with $I = 3$ is unreachable. This then provides a test of the SM. If it is possible to distinguish the various isospin final states, as is done in Ref. [8], one can look for a state with $I = 3$. If one is observed, this will be a smoking-gun signal of new physics.

6.1 Weak-Phase Information

It is not at all clear whether or not the six S_3 states in $B \to \pi\pi\pi$ can be experimentally separated. However, if it can be done (e.g. by using the method of Ref. [8]), it may be possible to extract clean information about weak phases. (Note: by measuring the S_3 states, one fixes the CP of the final states, which makes the indirect CP asymmetries well-defined.)

Consider $|S_3\rangle = |A\rangle$. Here there is one decay, which yields three observables: the branching ratio, the direct CP asymmetry, and the indirect CP asymmetry of $B_d^0 \to \pi^+\pi^0\pi^-|_{|A\rangle}$. The amplitude is expressed in terms of two effective diagrams:

$A(B_d^0 \to \pi^+\pi^0\pi^-)_{|A\rangle} = D_1 e^{-i\alpha} + D_2$, which has four theoretical parameters – the magnitudes of $D_{1,2}$, the relative strong phase, and α. Since the number of theoretical unknowns is greater than the number of observables, one cannot obtain α. Things are similar for $|S_3\rangle = |S\rangle$. Due to the first two relations in Eq. (47), there are only two independent decays, yielding 5 observables. However, there are 8 theoretical parameters, so that, once again, α cannot be extracted.

Things are different for the case of mixed states. Consider the M_1/M_3 sector. There are four decays: (1) $B^+ \to \pi^+\pi^0\pi^0|_{|M_1\rangle}$, (2) $B^+ \to \pi^-\pi^+\pi^+|_{|M_1\rangle}$, (3) $B_d^0 \to \pi^+\pi^0\pi^-|_{|M_1\rangle}$, (4) $B_d^0 \to \pi^+\pi^0\pi^-|_{|M_3\rangle}$. These yield 10 observables: 4 branching ratios, 4 direct CP asymmetries, and 2 indirect CP asymmetries (of $B_d^0 \to \pi^+\pi^0\pi^-|_{|S_3\rangle}$, $S_3 = M_1, M_3$). The four decay amplitudes all have the form $D_{1,i}e^{-i\alpha} + D_{2,i}$, i = 1-4. The $D_{1,i}$ are related to one another by the third relation in Eq. (47), as are the $D_{2,i}$. The amplitudes are thus a function of 6 effective diagrams, resulting in 12 theoretical parameters: 6 magnitudes, 5 relative strong phases, and α. Since the number of theoretical unknowns exceeds the number of observables, α cannot be extracted. However, as discussed in Sec. 5.2, all EWP diagrams can be neglected, to a good approximation. In this case, all the $D_{2,i}$ are proportional to $P_{tc}^{M_1} - \sqrt{3}P_{tc}^{M_3}$. There are thus only 4 effective diagrams, which yield 8 theoretical parameters. Now the number of theoretical unknowns is smaller than the number of observables, so that α can be obtained from a fit to the data. (It is not even necessary to measure all 10 observables. A difficult-to-obtain quantity, such as the direct CP asymmetry in $B^+ \to \pi^+\pi^0\pi^0|_{|M_1\rangle}$, can be omitted.) A similar method holds for the M_2/M_4 sector. The error on α can be reduced by comparing the two values found.

Now, it must be conceded that the above analysis is quite theoretical – it is far from certain that this can be carried out experimentally. Still, it is interesting to see that, in principle, clean weak-phase information can be obtained from $B \to \pi\pi\pi$, or, more generally, from $B \to M_1M_2M_3$ decays.

7 Conclusions

In this paper, we have expressed the amplitudes for $B \to M_1M_2M_3$ decays (M_i is a pseudoscalar meson) in terms of diagrams, concentrating on the charmless final states $K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$ and $\pi\pi\pi$. The diagrams are similar to those used in two-body decays: the color-favored and color-suppressed tree amplitudes T and C, the gluonic-penguin amplitudes P_{tc} and P_{uc}, and the color-favored and color-suppressed electroweak-penguin (EWP) amplitudes P_{EW} and P_{EW}^C. Here, because the final state has three particles, there are two types of each diagram, which we call T_1, T_2, C_1, C_2, etc.

The main advantage of a diagrammatic analysis is that the approximate relative sizes of the diagrams can be estimated. For example, there are annihilation- and exchange-type diagrams which contribute to these decays. However, these are expected to be negligible, and are not included in our analysis. Previous studies of three-body decays were carried out using isospin amplitudes, and gave exact results. On the other hand, the (justified) neglect of annihilation-type diagrams can modify these results, and can lead to interesting new effects.

As an example, consider $B \to KK\bar{K}$, which consists of four decays. For the case where the two K's are in a symmetric isospin state, the Wigner-Eckart theorem gives a single relation among the four amplitudes. However, when the amplitudes are written in terms of the non-negligible diagrams, it is found that this relation actually consists of two equalities, and this leads to new tests of the standard model. In the same vein, $B \to KK\bar{K}$ decays can be written in terms of five isospin amplitudes. The diagrammatic analysis shows that, in fact, only four of these are independent – two of the isospin amplitudes are proportional to one another.

Another consequence of the diagrammatic analysis has to do with weak phases. The CP of a three-particle final state is not fixed, because the relative angular momenta are unknown (i.e. they can be even or odd). For this reason, in the past it was thought that it is not possible to cleanly extract weak-phase information from three-body B decays. In this paper, we demonstrate that this is not true. Using the diagrams, we show that it is possible to cleanly measure the weak phases in some decays, assuming that it is experimentally possible to distinguish different symmetry combinations of the final-state particles. We explicitly give methods for $K\bar{K}\pi$ and $\pi\pi\pi$, and note that the the procedure for $K\pi\pi$ is presented separately. Ways of cleanly extracting the CP phases from other three-body decays will surely be suggested.

There are thus a number of interesting measurements that can be carried out with $B \to M_1 M_2 M_3$. LHCb is running at present, and the super-B factories will run in the future. Hopefully, these machines will provide interesting data on three-body B decays.

Acknowledgments: We thank M. Gronau, J. Rosner, R. Sinha, R. MacKenzie and A. Soffer for helpful communications, and A. Datta for collaboration in the beginning stages of this project. This work was financially supported by NSERC of Canada and FQRNT of Québec.

References

[1] E. Barberio *et al.* [Heavy Flavor Averaging Group], arXiv:0808.1297 [hep-ex], and online update at `http://www.slac.stanford.edu/xorg/hfag`

[2] H. J. Lipkin, Y. Nir, H. R. Quinn and A. Snyder, Phys. Rev. D **44**, 1454 (1991).

[3] M. Gronau and J. L. Rosner, Phys. Lett. B **564**, 90 (2003) [arXiv:hep-ph/0304178].

[4] M. Gronau and J. L. Rosner, Phys. Rev. D **72**, 094031 (2005) [arXiv:hep-ph/0509155].

[5] H. Y. Cheng, C. K. Chua and A. Soni, Phys. Rev. D **72**, 094003 (2005) [arXiv:hep-ph/0506268], Phys. Rev. D **76**, 094006 (2007) [arXiv:0704.1049 [hep-ph]].

[6] M. Gronau, O. F. Hernandez, D. London and J. L. Rosner, Phys. Rev. D **50**, 4529 (1994), Phys. Rev. D **52**, 6374 (1995).

[7] C. Amsler *et al.* [Particle Data Group], Phys. Lett. B **667**, 1 (2008).

[8] M. Gaspero, B. Meadows, K. Mishra and A. Soffer, Phys. Rev. D **78**, 014015 (2008) [arXiv:0805.4050 [hep-ph]].

[9] See, for example, C. W. Chiang and D. London, Mod. Phys. Lett. A **24**, 1983 (2009) [arXiv:0904.2235 [hep-ph]].

[10] N. G. Deshpande, N. Sinha and R. Sinha, Phys. Rev. Lett. **90**, 061802 (2003) [arXiv:hep-ph/0207257].

[11] M. Neubert and J. L. Rosner, Phys. Lett. B **441**, 403 (1998) [arXiv:hep-ph/9808493], Phys. Lett. B **441**, 403 (1998) [arXiv:hep-ph/9808493]; M. Gronau, D. Pirjol and T. M. Yan, Phys. Rev. D **60**, 034021 (1999) [Erratum-ibid. D **69**, 119901 (2004)] [arXiv:hep-ph/9810482]; M. Imbeault, A. L. Lemerle, V. Page and D. London, Phys. Rev. Lett. **92**, 081801 (2004) [arXiv:hep-ph/0309061].

[12] M. Gronau, Phys. Rev. Lett. **91**, 139101 (2003) [arXiv:hep-ph/0305144].

[13] M. Imbeault, N. Rey-Le Lorier and D. London, arXiv:1011.4973 [hep-ph].