Radial Regge trajectories for higher $\psi(nS)$ and $\psi(nD)$ states

A.M.Badalian

State Research Center, Institute of Theoretical and Experimental Physics, Moscow, 117218 Russia[*]

(Dated: December 9, 2010)

The masses of $\psi((n+1)\,^3S_1)$ and $\psi(n\,^3D_1)$ are calculated using the relativistic string Hamiltonian with "linear+gluon-exchange" potential. They occur in the range 4.5–5.8 GeV, in particular, $M(3D) = 4.54$ GeV, $M(5S) = 4.79$ GeV, $M(4D) = 4.85$ GeV are calculated with accuracy ~ 50 MeV. For higher charmonium states linear Regge trajectories: $M^2(nS) = M^2(\psi(4.42)) + 2.91$ GeV^2 $(n-4)$ $(n \geq 4)$ and $M^2(nD) = (4.54^2 + 2.88(n-3))$ GeV^2 ($n \geq 3$) are obtained only for higher charmonium states. They have a slope two times larger than that of light mesons and give a good description of calculated masses. These masses are compared to enhancements in some recent e^+e^- experiments.

I. INTRODUCTION

Observation of higher charmonium states is very important for theory, first of all, to understand the $c\bar{c}$ dynamics at large distances. At present only the $\psi(4415)$ resonance, discovered long ago in 1976 [1], is well established; its mass, $M(4415) = 4421 \pm 7$ MeV, is now known with a good accuracy [2, 3]. However, even for this resonance there is an uncertainty in the value of its dielectron width [3]. The analysis of most precise BES data on the ratio $R = \sigma(e^+e^- \to \text{hadrons})\,/\sigma(e^+e^- \to \mu^+\mu^-)$ in [4] has given $\Gamma_{ee}(\psi(4415)) = 0.37 \pm 0.14$ KeV, while in [5] from the same experimental R values four different $\Gamma_{ee}(\psi(4415))$, in the range 0.45–0.78 KeV, have been extracted in different fits. Meanwhile precise knowledge of dielectron widths of higher charmonium states may give an important information on the $S-D$ mixing and different decays.

Therefore in our paper we concentrate on the masses for the higher nS and nD charmonium states. Although the resonances, like $\psi(3D)$, $\psi(5S)$, and $\psi(4D)$, are not well

[*]E-mail;badalian@itep.ru

established yet, several enhancements in the range 4.5–5.0 GeV were observed in a number of recent e^+e^- experiments: in $e^+e^- \to D^0\bar{D}^{*-}\pi^+$, $D^{*+}D^{*-}$, $D_s^{*+}D_s^{*-}$ [6], $e^+e^- \to \Lambda_c^+\Lambda_c^-$ [7] of the Belle Collab., and also in the BaBar data on $e^+e^- \to \pi^+\pi^- J/\psi$ [8], $e^+e^- \to D^*\bar{D}^*$, $e^+e^- \to D\bar{D}^*$ via the initial state state radiation [9]. These enhancements have been analysed in [10, 11], where they are interpreted as the $\psi(3D)$, $\psi(5S)$, and $\psi(4D)$ vector charmonium states, and their masses and total widths were extracted from fits to experimental data.

Here we consider only conventional $c\bar{c}$ mesons in the framework of relativistic string picture. We perform calculations of two kinds: with a universal linear + gluon-exchange (GE) potential [12] and also for purely linear potential when GE interaction is taken as a perturbation. We shall show that linear confining potential dominates in $c\bar{c}$ dynamics at large distances, thus simplifying an analysis for several reasons.

Firstly, at large distances GE potential is small as compared to confining term. Its typical contribution to the energy excitation $E(nl)$ $(3 \leq n \leq 8)$ is of order 150 MeV, while a contribution from linear potential is $\sim 1.5 - 2.2$ GeV. Therefore the masses of higher states weakly depend on the parameters of GE potential, which may be very much different even in QCD motivated models [13, 14].

Secondly, higher states have large sizes and their hyperfine and fine-structure splittings are small, so that their masses practically coincide with the centroid masses. Thus one escapes uncertainties coming from parameters of spin-dependent potentials [15].

Also we assume here that hadronic shifts of higher resonances due to open channel(s) are not large, being of the same order as for low-lying states, which are typically $\simeq 40$ MeV [16, 17], and only for $X(3872)$ a hadronic shift is larger, ~ 70 MeV, due to specifically strong coupling of the $P-$wave charmonium state to the $S-$wave threshold. In this respect the situation in charmonium differs from that of light mesons, where hadronic shifts of radial excitations are large and a creation of virtual quark-antiquark pairs should be taken into account [18]. Hence, we can perform calculations in single-channel approximation, estimating an accuracy of our calculations as ± 50 MeV.

We use here the relativistic string Hamiltonian (RSH) [19–21], which describes light, heavy-light mesons, and heavy quarkonia in a universal way, only via such fundamental parameters as string tension and the pole (current) mass of the c quark. For low-lying states it is also important to fix the value of the vector strong coupling at large distances – α_{crit} (the freezing constant), but for high excitations different choice of $\alpha_{\rm crit}$ gives small uncertainty in their masses, ~ 20 MeV.

At this point we would like to underline that widely used spinless Salpeter equation (SSE) appears to be a particular case of RSH with the only restriction. If in constituent potential models the $c-$ quark mass is taken as a fitting parameter, in our approach in SSE the c-quark mass has to be equal to its pole mass. At present the pole mass of the c quark is defined with a good accuracy: $m = m_c(\text{pole}) = 1.40 \pm 0.07$ GeV [2]. It is of interest that the masses of higher charmonium states appear to be very sensitive to accepted value of m(pole). We also show that if GE potential is considered as a perturbation, then the masses $\tilde{M}(nl)$ $(n \geq 3)$ coincide with exact solutions of RSH (or SSE) with an accuracy $\simeq 2\%$.

Moreover, in "linear" approximation the masses are shown to be defined by simple analytical expressions.

We do not consider here non-conventional charmonium resonances, in particular, those which occur near thresholds, since they may be calculated only within two-(many-)channel approach [16, 17].

II. THE MASSES $M(nS)$ AND $M(nD)$

Although RSH was derived for an arbitrary $q_1\bar{q}_2$ meson [19, 20], in case of heavy quarkonia it has more simple form, because so-called string and self-energy corrections are small and can be neglected [21]:

$$H = \omega(nl) + \frac{m^2}{\omega(nl)} + \frac{\boldsymbol{p}^2}{\omega(nl)} + V_{\rm B}(r), \quad H\varphi_{nl} = M(nl)\varphi_{nl}. \tag{1}$$

We use here einbein approximation (EA) [20, 21], when the mass $M(nl) \equiv M_{\rm cog}(nl)$ is defined as

$$M(nl) = \omega(nl) + \frac{m^2}{\omega(nl)} + E_{nl}(\omega(nl)). \tag{2}$$

This mass formula does not contain any overall (fitting) constant and depends on the pole mass of the c quark m, which is defined via the current mass of the c quark and now known with an accuracy ~ 70 MeV [2]; in our paper we take $m = 1.40$ GeV.

In (2) a variable $\omega(nl)$ is the averaged kinetic energy of the c quark for a given nl state, which plays a role of a constituent quark mass, being different for different states:

$$\omega(nl) = \langle\sqrt{\boldsymbol{p}^2 + m^2}\rangle_{nl}. \tag{3}$$

In (2) $E_{nl}(\omega(nl))$ is the excitation energy of a given state nl; its depends on static potential used. Here we take "linear + GE" potential $V_{\rm B}(r)$ as in [12, 21],

$$V_{\rm B}(r) = \sigma r - \frac{4\alpha_{\rm B}(r)}{3r}. \tag{4}$$

For low-lying states both linear and GE terms are important and to calculate E_{nl}, $\omega(nl)$ one needs to solve two equations in consistent way: firstly, the equation (1) and also the equation for $\omega(nl)$:

$$\omega(nl)^2 = m^2 + \omega(nl)^2 \frac{\partial E_{nl}}{\partial \omega(nl)} \tag{5}$$

For higher states confining potential dominates and due to this fact exact solutions of (1), (5) and the masses $\tilde{M}(nl)$, calculated for linear potential with GE potential taken as a correction, coincide with an accuracy better 2% (see Tables II, III).

In "linear" approximation (with only linear potential) the excitation energy $E_0(nl)$ is given by the expression:

$$E_0(nl) = \left(\frac{\sigma^2}{\omega_0(nl)}\right)^{1/3} \zeta_{nl}, \tag{6}$$

while from (5) the equation for $\omega_0(nl)$ is

$$\omega_0(nl)^2 = m^2 + \frac{1}{3}(\sigma\omega_0(nl))^{2/3}\zeta_{nl}. \tag{7}$$

From (6) and (7) one can see that $E_0(nl)$ and $\omega_0(nl)$ are expressed via the string tension σ and the Airy numbers ζ_{nl}. It is also important that $\omega_0(nl)$ depends on the $c-$ quark pole mass, being proportional m. Through our paper the conventional values $\sigma = 0.18$ GeV^2 and $m \equiv m_c(\text{pole}) = 1.40$ GeV are taken. The Airy numbers for $n = 1, ..., 8$ $(l = 0.2)$ are given in Appendix.

The equation (7) (with $m \neq 0$) easily reduces to the Cardano equation, from which $\omega_0(nl)$ is obtained in analytical form:

$$\omega_0^{2/3}(nl) = \left(\frac{m^2}{2}\right)^{1/3} \left\{ \left(1 + \sqrt{1 - \left(\frac{2\sigma}{27m^2}\right)^2 \zeta_{nl}^3}\right)^{1/3} + \left(1 - \sqrt{1 - \left(\frac{2\sigma}{27m^2}\right)^2 \zeta_{nl}^3}\right)^{1/3} \right\} \tag{8}$$

From this equation it follows that

$$\omega_0 = \frac{m}{\sqrt{2}} \left\{ \left(1 + \sqrt{1 - \left(\frac{2\sigma}{27m^2} \right)^2 \zeta_{nl}^3} \right)^{1/3} + \left(1 - \sqrt{1 - \left(\frac{2\sigma}{27m^2} \right)^2 \zeta_{nl}^3} \right)^{1/3} \right\}^{3/2} . \tag{9}$$

In linear approximation the kinetic energies $\omega_0(nl)$ have several characteristic features (see Table 1):

1. They differ for the states with different quantum numbers nl, increasing for larger radial excitations: from 1.73 GeV for the $4S$ state to $\omega_0(7S) = 1.94$ GeV.

2. For $n \geq 3$ $\omega_0(nD)$ and $\omega_0((n+1)S)$ almost coincide and due to this property the masses of these states are degenerated for linear potential – a difference between them is ≤ 5 MeV.

3. The masses $\omega_0(nl)$ are proportional to the c-quark pole mass.

4. The values of $\omega_0(nl)$ do not practically depend on GE interaction, coinciding with exact $\omega(nl)$ for $n \geq 3$ with an accuracy better 3% (see Table VI in Appendix).

A growth of $\omega_0(nl)$ for larger n is an important feature of "a constituent" mass in relativistic string approach. Due to this property, the r.m.s. of higher charmonium states are not very large, changing from 1.4 fm for the $4S$ state to 2.0 fm for the $7S$ state (these radii are given in Appendix). Therefore one can expect that higher charmonium resonances exist and can manifest themselves in different e^+e^- processes, if their leptonic widths are not small.

In Appendix (Table VI) the values of $\omega_0(nl)$ are compared to "exact" $\omega(nl)$ calculated for SSE :

$$\{2\sqrt{\boldsymbol{p}^2 + m^2} + V_{\rm B}(r)\}\varphi_{nl} = M(nl)\varphi_{nl}, \tag{10}$$

with the same "linear+GE" potential (4); their values coincide with an accuracy better 3%, i.e. for higher excitations $\omega(nl)$ appears to be independent of GE potential used.

The SSE (10) may be considered as a particular case of the RSH, in which a string correction is neglected as in (1). It can be derived from RSH, if the extremum condition is put as $\frac{\partial H}{\partial \omega} = 0$ [22]. On the other hand, EA follows from RSH, if the extremum condition is

TABLE I: The kinetic energies $\omega_0(nl)$ ($l = 0, 2$) (in GeV) from (8) for linear potential with $\sigma = 0.18$ GeV^2 ($m = 1.40$ GeV).

nS	$\omega_0(nS)$	nD	$\omega_0(nD)$
1S	1.512	-	-
2S	1.598	1D	1.606
3S	1.669	2D	1.674
4S	1.732	3D	1.736
5S	1.789	4D	1.793
6S	1.884	5D	1.847
7S	1.896	6D	1.898
8S	1.945	7D	1.947

put on the mass (2) as $\frac{\partial M(nl)}{\partial \omega} = 0$ [20, 22]. Here we mostly use EA, because in this approach the wave functions (w.f.) with $l = 0$ are finite near the origin, while for SSE the $S-$wave solutions diverge.

In Tables II, III "exact" solutions of SSE, denoted as $M(nl)$, are compared to approximate masses $\tilde{M}(nl)$:

$$\tilde{M}(nl) = M_0(nl) + < V_{GE} >_{nl}, \tag{11}$$

where $M_0(nl)$ is a solution of (1) with only linear potential. The masses $M_0(nl)$ for $l = 0, 2$ and the matrix elements (m.e.) $< V_{GE} >_{nl}$ are given in Appendix, Tables VII and VIII. The GE contribution to $\tilde{M}(nl)$ is negative, with much smaller magnitude ($\sim$ 150 MeV) than that for linear potential, which is $\sim 1.5 - 2.0$ GeV. However, this GE correction, $\sim 10\%$, is important to improve an agreement with known experimental masses.

From Tables II, III one can see that the differences between $M(nl)$ for SSE and $\tilde{M}(nl)$, $\simeq 40$ MeV, lie within accuracy of our calculations. Thus our calculations (in single-channel approximation) show that

1. The value $M(3S) = 4.09$ GeV is by 50 MeV larger than experimental number, $M(\psi(4040) = 4.04$ GeV, and this difference between them agrees with the value of hadronic shift for this resonance, ~ 40 MeV, predicted in [16].

TABLE II: The masses $\tilde{M}(nS)$ (11) and exact solutions $M(nS)$ for SSE (in GeV) ($\sigma = 0.18\ GeV^2$, $\alpha_{\rm crit} = 0.54$)

State	$M(nS)$ for SSE	$\tilde{M}(nl)$	experiment
-	m=1.41 GeV	m=1.40 GeV	-
1S	3.07	3.068	3.067
2S	3.67	3.663	3.67(4)
3S	4.09	4.099	4.040
4S	4.45	4.464	4.421
5S	4.75	4.792	4.78[a]
6S	5.04	5.087	5.09[a]
7S	5.31	5.365	5.44[a]
8S	-	5.630	5.91[a]

[a] This number is taken from the fit to experimental data [10].

2. For $\psi(4415)$ a smaller hadronic shift, ~ 30 MeV, follows.

3. Calculated $M(5S) = 4.79$ GeV agrees with the prediction of $M(\psi(5S)) = 4.78 - 4.82$ GeV from [10], [11], where this mass has been extracted from fits to experimental cross sections for different e^+e^- processes [6–9]. Such a coincidence takes also place for the $M(6S) = 5.09$ GeV.

4. On the contrary, our masses for the $7S$, $8S$ charmonium states: $M(7S) = 5.365$ GeV and $M(8S) = 5.63$ GeV, are by 80 MeV and ~ 300 MeV smaller than those from [10] (see Table II). ,

5. For the $D-$wave states $M(3D) = 4.54$ GeV and $M(4D) = 4.86$ GeV are obtained; the value of $M(4D)$ agrees with $M(4D) \sim 4.87$ GeV from [10] (see Table III). For higher $5D$ and $6D$ our values are smaller, by 160 MeV and 250 MeV, respectively, than in [10].

6. For purely linear potential the spacings $\delta_{n+1,n} = M_0(nD) - M_0((n+1)S)$ are small, $\sim 15 \pm 5$ MeV (see Tables VII, VIII), i.e., these levels are degenerated. However, due

TABLE III: The masses $M(nD)$ for SSE and $\tilde{M}(nl)$ (11) (in GeV) with $\sigma = 0.18$ GeV^2

nD	$M(nD)$ for SSE $m = 1.41$ GeV	$\tilde{M}(nD)$ $m = 1.40$ GeV	experiment
1D	3.80	3.80	3.77
2D	4.18	4.192	4.16
3D	4.51	4.543	4.55[a]
4D	4.81	4.854	4.87[a]
5D	5.09	5.143	5.30[a]
6D	5.35	5.413	5.66[a]
7D	5.62	5.669	-

[a] See the footnote to Table 2.

to GE potential these mass differences increase, so that $\tilde{M}(3D) - \tilde{M}(4S) = 80$ MeV and $\tilde{M}(6D) - \tilde{M}(7S) = 50$ MeV.

Here in our analysis of high charmonium excitations we do not use flattening potential, introduced for light mesons to take indirectly into account a creation of virtual $q\bar{q}$ pairs (q is a light quark) [18]. Such flattening of confining potential was useful for light mesons, which have large hadronic shifts. The situation in charmonium is supposed to be different, because for higher states the c-quark kinetic energy increases, being $\sim 1.7 - 1.9$ GeV, and one can expect that their hadronic shifts are not large (≤ 40 MeV) and their overlapping integrals, which describe different decay modes, are smaller than those for low-lying resonances.

In [23] the masses of higher charmonium states have been calculated with the use of a static potential, which contains a large number of additional parameters and large overall constant, while the value of the string tension is relatively small. Nevertheless calculated in [23] masses of the nS $(n = 5, 6)$ and nD $(n = 3, 4, 5)$ charmonium states coincide with our predictions within ± 50 MeV, while in [23] $M(\psi(6D) = 6.03$ GeV is by 260 MeV larger than in our calculations.

At this point we would like to stress that with the use of RSH all calculated masses do not contain a fitting constant and totally defined only by $\sigma = 0.18$ GeV^2, $m(\text{pole}) = 1.40$ GeV, while a choice of the freezing value of the strong coupling α_{crit} is not very important.

III. RADIAL REGGE TRAJECTORIES FOR THE nS AND nD STATES

The Regge trajectories, orbital and radial, are usually studied in light mesons and now it remains unclear whether a regime of linear trajectories takes place for the charmonium family or not. In [24] it was assumed that linear Regge trajectories describe charmonium states with different quantum numbers with an accuracy ~ 100 MeV, while the slopes were defined fitting the masses of low-lying (well-established) charmonium states.

Here from our dynamical calculations of the $M(nS)$ and $M(nD)$ it follows that linear Regge trajectories take place only for higher charmonium states.

The radial Regge trajectory can be presented as:

$$M^2(nl) = \mu_l^2 + \Omega_l\ n, \tag{12}$$

where μ_l and the slope Ω_l are supposed to be constants. In classical string picture for massless quarks $\Omega_l = 4\pi\sigma = 2.26$ GeV^2 ($\sigma = 0.18$ GeV^2), however, for light mesons the values of Ω_l have appeared to be smaller, $1.3 - 1.6$ GeV^2, because of large hadronic shifts [18].

Here we consider the masses of the centers of gravity and define the Regge trajectories for a given l, when from (12) the spacing between squared masses:

$$\Delta_{n+1,n} = \tilde{M}^2((n+1)l) - \tilde{M}^2(nl) = \Omega_l \tag{13}$$

has to be a constant Ω_l. Taking from [2] experimental values of the c.o.g. masses for $J/\psi - \eta_c(1S)$, $\psi(3686) - \eta_c(2S)$, $\psi(4040)$, $\psi(4415)$ one obtains that the spacing $\Delta_{21} = 4.07$ GeV^2, while $\Delta_{32} = 2.84$ GeV^2 is significantly smaller, and $\Delta_{43} = 3.22$ GeV^2 is by $\sim 15\%$ larger than Δ_{32}. A decrease of Δ_{32} possibly occurs due to hadronic shift of the $\psi(4040)$ resonance, which is ~ 50 MeV. If one takes unshifted masses from Table II: $M(3S) = 4.099$ GeV and $M(4S) = 4.464$ GeV, then $\Delta_{32} = 3.32$ GeV^2 and $\Delta_{43} = 3.13$ GeV^2 become close to each other, still being larger than Ω_S for higher states (calculated $\Delta_{n+1,n}$ are given in Table IV).

The numbers from Table 4 show that $M(nS)$ with $4 \leq n \leq 8$ can be described by linear (radial) Regge trajectory with the slope

$$\Omega_S = 2.91 \text{ GeV}^2, \tag{14}$$

which is a constant with a good accuracy. From here

TABLE IV: The differences $\Delta_{(n+1),n}$ (in GeV^2) between squared masses $\tilde{M}^2(nl)$ for the nS states

Δ_{43}	3.22
Δ_{54}	3.04
Δ_{65}	2.91
Δ_{76}	2.91
Δ_{87}	2.91

$$M^2(nS) = M^2(4.21) + 2.91 \text{ GeV}^2(n-4). \tag{15}$$

For the masses $M(nD)$ the slope Ω_D slightly decreases changing from $\Delta_{43} = 3.13$ GeV^2 to a smaller value, $\Delta_{76} = 2.84$ GeV^2 (see masses from Table III). Therefore for the nD states their masses are described by linear Regge trajectory with worse accuracy than for the nS excitations, giving

$$\Omega_D = (2.88 \pm 0.04)\ GeV^2, \tag{16}$$

and

$$M^2(nD) = M^2(4.54) + \Omega_D\ (n-3). \tag{17}$$

In [24] charmonium states with different quantum numbers, including low-lying states, were described by linear Regge trajectories with μ_l^2 and the slopes Ω_J, defined from fits to known experimental masses. For the masses of $\psi(nS)$ and $\psi(nD)$ the slope $\Omega_S = \Omega_D = 3.2$ GeV^2 was obtained, which is only 10% larger than that in our dynamical calculations, while the values of $\mu_S = 2.6$ GeV and $\mu_D = 3.31$ GeV in [24] are taken as fitting parameters. In our calculations linear Regge trajectories can be applied only to higher charmonium states and μ_l $(l = 0, 2)$ is equal to experimental mass.

Moreover, for the slopes Ω_S, Ω_D approximate analytical expressions can easily be derived. If in (2), (6) one takes an averaged $\bar{\omega}_0 = \bar{\omega}_S = \bar{\omega}_D$ for a kinetic energies with $n \geq 4$, then the slope

$$\Omega_l = \left(\frac{\sigma^2}{\bar{\omega}_0}\right)^{1/3} (\zeta_{(n+1)l} - \zeta_{nl}) \left\{ \left(\frac{\sigma^2}{\bar{\omega}_0}\right)^{1/3} (\zeta_{(n+1)l} + \zeta_{nl}) + 2\bar{\omega}_0 + \frac{2m^2}{\bar{\omega}_0} \right\} \tag{18}$$

is fully defined by $\bar{\omega}_0$ and the Airy numbers. From (18), taking $\bar{\omega}_0 \simeq 1.84$ GeV (ζ_{nS}, ζ_{nD} are given in Table V), one obtains $\Omega_S \simeq \Omega_D \simeq 2.90$ GeV^2 in good agreement with "exact" number in (14), (16).

It is important to stress that in charmonium the slopes Ω_S, Ω_D have appeared to be two times larger than those for light mesons [18].

For calculated masses a spacing between neighbouring radial excitations, $M((n+1)l) - M(nl)$, is large, being ~ 300 MeV for the nS states and ~ 270 MeV for the nD states. On the contrary, the mass difference $M(nD) - M((n+1)S)$ is smaller, decreasing from ~ 120 MeV for low-lying states to $\sim 80 - 60$ MeV for large n. Evidently, that for such small spacings the $S-D$ mixing has to be important. Then the $S-D$ mixing strongly affects dielectron widths of vector charmonium states. As shown in [21], higher nS states have large dielectron widths, ~ 1 KeV, which are by two orders larger than $\Gamma_{ee}(nD)$ for purely $D-$wave states, e.g. $\Gamma_{ee}(1D) \simeq 15$ eV and ~ 40 eV for the $4D$ state. Therefore purely nD resonances with such small dielectron widths cannot be observed in the e^+e^- experiments, while they may be seen, if due to the $S-D$ mixing, their dielectron widths are of the same order as those of the nS states.

IV. CONCLUSIONS

Our calculations of higher charmonium states were performed in single-channel approximation when the $c\bar{c}$ dynamics at large distances can be studied in detail. With the use of RSH we have obtained that

1. The $5S-8S$ and $3D-7D$ states occur in the range 4.5-5.8 GeV and the spacing between neighbouring radial excitations is of the order of 250-300 MeV for $n \geq 4$.

2. The mass differences between $M(nD)$ and $M((n+1)S)$ are rather small, decreasing from ~ 80 MeV for $n=4$ to ~ 50 MeV for $n=7$. The important point is that for purely linear potential these levels are degenerated (their mass difference is $\simeq 15$ MeV), while due to GE interaction a spacing $M(nD) - M((n+1)S$ increases.

3. The masses of radial excitations, $M(nS)$ and $M(nD)$ with $n \geq 3$, are described with good accuracy by linear Regge trajectories with the slope $\Omega_S = 2.91$ GeV^2 and $\Omega_D = 2.88$ GeV^2.

4. The masses of high excitations in charmonium are mostly defined by linear confining potential and at the same time they depend on the pole mass of a $c-$quark. Here $m(\text{pole}) = 1.40$ GeV is used.

5. Higher nD resonances can be observed in experiments only if their dielectron widths are of the same order as those for the nS states, which happens due to the $S-D$ mixing.

6. We predict the following masses: $M(3D) = 4.54$ GeV, $M(5S) = 4.79$ GeV, $M(4D) = 4.85$ GeV, $M(6S) = 5.09$ GeV, $M(5D) = 5.14$ GeV, $M(7S) = 5.365$ GeV, $M(6D) = 5.41$ GeV, $M(8S) = 5.63$ GeV, and $M(7D) = 5.67$ GeV. An accuracy of our calculations is estimated to be ~ 20 MeV, if hadronic shifts are neglected.

These characteristic features of the $c\bar{c}$ dynamics at large distances can be tested by future experiments in which the masses and dielectron widths of higher charmonium resonances have to be measured with precision accuracy.

Acknowledgements.

I am grateful to Prof. Yu.A.Simonov for useful discussions. This work was supported by the RFFI Grant 09-02-00629a.

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Appendix

The matrix elements, $M_0(nl)$, and the Airy numbers

Firstly, we give the Airy numbers for the nS and nD states.

TABLE V: The Airy numbers ζ_{nl} for $l = 0, 2$.

nS state	ζ_{nS}	nD state	ζ_{nD}
1S	2.338107	-	-
2S	4.087949	1D	4.24818
3S	5.520560	2D	5.62971
4S	6.786708	3D	6.86889
5S	7.944134	4D	8.00981
6S	9.022651	5D	9.07700
7S	10.040174	6D	10.08646
8S	11.008524	7D	11.04874

Using the Airy numbers, one can calculate the kinetic energies $\omega_0(nl)$ (9) as well as excitation energies $E_0(nS)$ and $E_0(nD)$ (6) for purely linear potential. In Table VI $\omega_0(nS)$ and $\omega_0(nD)$ for linear potentil and also "exact" $\omega(nS)$, calculated for SSE, are given.

As seen from Table VI, for linear potential the kinetic energies $\omega_0((n+1)S)$ and $\omega_0(nD)$ practically coincide for all n, while "exact" $\omega(nS)$, calculated for SSE, differ from $\omega_0(nS)$ only by $\leq 3\%$. It means that $\omega(nS)$ weakly depends on GE potential taken and only for low-lying states a difference between them is $\sim 6\%$.

Knowing $\omega_0(nl)$ one can define the excitation energy $E_0(nl)$, the total mass $\tilde{M}(nl)$, and also the w.f. at the origin for a given state nl. The excitation energies $E_0(nS)$ and $E_0(nD)$ are given in Tables VII, VIII together with r.m.s. $\sqrt{<r^2>_{nl}}$ and m.e. $< V_{GE}(r) >_{nl}$.

For the $D-$wave states a contribution from GE potential is smaller (see $< V_{GE}(\mathrm{r}) >_{\mathrm{nD}}$ in Tables VI, VII); due to this fact the mass differences $\tilde{M}(nD) - \tilde{M}((n+1)S)$ increase.

In our calculations of $\tilde{M}(nl) = M_0(nl) + < V_{GE} >_{nl}$ for higher states the strong coupling $\alpha_B(r) = \alpha_{\mathrm{crit}} =$ constant was taken, i.e., in the GE potential $V_{GE} = -\frac{4\alpha_{\mathrm{crit}}}{3r}$ the asymptotic

TABLE VI: The averaged kinetic energies $\omega_0(nS)$, $\omega_0(nD)$ (in GeV) for linear potential and $\omega(nl)$ from SSE ($\sigma = 0.18$ GeV^2).

state	$\omega_0(nS)$	$\omega(nS)$	state	$\omega_0(nD)$
1S	1.512	1.60	-	-
2S	1.598	1.66	1D	1.606
3S	1.669	1.73	2D	1.674
4S	1.732	1.78	3D	1.736
5S	1.789	1.84	4D	1.793
6S	1.844	1.88	5D	1.847
7S	1.896	1.94	6D	1.898
8S	1.9449	-	7D	1.9470

freedom behavior of the strong coupling was neglected , since it gives negligible correction for high excitations. Here the value $\alpha_{\rm crit} = 0.54$ and $< V_{GE} >_{nl} = -0.72 < r^{-1} >_{nl}$ are used.

TABLE VII: The values $E_0(nS)$, $M_0(nS)$, the m.e. $< V_{GE}(r) >_{nS}$ (in GeV), and $\sqrt{<r^2>_{nS}}$ (in fm) for linear potential with $\sigma = 0.18$ GeV^2, $m = 1.40$ GeV.

nS state	$E_0(nS)$	$M_0(nS)$	$< V_{GE}(r) >_{nS}$	$\sqrt{<r^2>_{nS}}$
1S	0.6494	3.458	- 0.390	0.519
2S	1.1147	3.939	- 0.276	0.891
3S	1.4837	4.327	- 0.228	1.186
4S	1.8016	4.665	-0.201	1.440
5S	2.0861	4.971	-0.179	1.667
6S	2.3454	5.252	-0.165	1.875
7S	2.5861	5.516	-0.151	2.067
8S	2.8115	5.764	-0.134	2.250

TABLE VIII: The values $E_0(nD)$, $M_0(nD)$, the m.e. $< V_{GE}(r) >_{nD}$ (in GeV), and $\sqrt{<r^2>_{nD}}$ (in fm) for linear potential with $\sigma = 0.18$ GeV^2, $m = 1.40$ GeV.

nD state	$E_0(nD)$	$M_0(nD)$	$< V_{GE}(r) >_{nD}$	$\sqrt{<r^2>_{nD}}$
1D	1.5645	3.983	-0.183	0.879
2D	1.5114	4.356	- 0.164	1.179
3D	1.8219	4.687	- 0.144	1.436
4D	2.1017	4.988	- 0.134	1.664
5D	2.3584	5.267	- 0.124	1.872
6D	2.5970	5.528	- 0.115	2.065
7D	2.8208	5.775	- 0.106	2.246