Duality violations in τ hadronic spectral moments

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Evidence is presented for the necessity of including duality violations in a consistent description of spectral function moments employed in the precision determination of α_s from τ decay. A physically motivated *ansatz* for duality violations in the spectral functions enables us to perform fits to spectral moments employing both pinched and unpinched weights. We describe our analysis strategy and provide some preliminary findings. Final numerical results await completion of an ongoing re-determination of the ALEPH covariance matrices incorporating correlations due to the unfolding procedure which are absent from the currently posted versions. To what extent this issue affects existing analyses and our own work will require further study.

1. Introduction

Hadronic decays of the τ lepton provide a particularly clean environment for the study of low-energy QCD. The mass of the τ (which is the only lepton heavy enough to decay into hadrons) is large enough that perturbative QCD should provide a good description of the inclusive decay process, but not so large that nonperturbative effects can be completely neglected.

Since the seminal work of Ref. [1], it has been recognised that the perturbative series supplemented with the Operator Product Expansion (OPE) provides a solid framework for describing these decays. The main achievement of this program is a precision determination of the QCD coupling α_s. The determination of α_s at different scales provides a highly nontrivial test of the theory. Hadronic τ decays also yield values for vacuum condensates [2], low-energy constants [3], and the CKM matrix element $|V_{us}|$ [4].

Unfolded high-statistics spectral functions extracted from the final LEP data are available from ALEPH [2] and OPAL [5]. On the theory side, the $\mathcal{O}(\alpha_s^4)$ term of the perturbative contribution has been calculated recently [6]. This has triggered several reanalyses of hadronic τ decays from the ALEPH and OPAL spectral functions. While the average value of α_s from τ data is competitive with the current world average, the values obtained by different groups are barely compatible. After the evolution to the Z mass, a sample of results obtained by various groups is

$$\alpha_s(m_Z^2) = \begin{cases} 0.1202\,(6)_{\rm exp}(18)_{\rm th} & [6]\ , \\ 0.1212\,(5)_{\rm exp}(9)_{\rm th} & [7]\ , \\ 0.1180\,(4)_{\rm exp}(7)_{\rm th} & [8]\ , \\ 0.1187\,(6)_{\rm exp}(15)_{\rm th} & [9]\ . \end{cases} \qquad (1)$$

Furthermore, there is some tension between the

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result of Ref. [7] and recent lattice determinations; *e.g.*, the analysis of Ref. [10] gives

$$\alpha_s(m_Z^2) = 0.1184 \pm 0.0006. \tag{2}$$

Even though much of the variation in Eq. (1) results from differences in the prescriptions used for resumming the perturbative series (CIPT versus FOPT [11,8,12]), at the current level of precision nonperturbative effects thus far neglected may become significant.

One can think of several different, but not completely independent, sources of systematic uncertainties in the theoretical description of hadronic τ decays. Examples are the truncation in powers of α_s and/or variation in the choice of resummation prescription for the perturbative series [8,12], unchecked assumptions regarding the truncation in dimension of the OPE [9], and contributions from duality violations (DVs) [13,14]. In this work we address the latter. The OPE is believed to be an asymptotic expansion at best, and it breaks down on the Minkowski axis. This lack of convergence is related to the presence of DVs, and indeed, our preliminary results confirm previous observations on the presence of nonnegligible DVs for certain weighted spectral integrals [16]. A full reanalysis including all systematic effects noted above will be presented in a forthcoming article.

2. Theoretical framework

We start from the total nonstrange branching ratio $R_\tau^{S=0}$ defined as

$$R_\tau^{S=0} = \frac{\Gamma[\tau \to \text{hadrons}^{S=0}\,\nu_\tau]}{\Gamma[\tau \to e^-\bar{\nu}_e\,\nu_\tau]} = R_\tau^V + R_\tau^A\,. \tag{3}$$

With vector (V) and axial-vector (A) currents $J_\mu^V(x) = \bar{u}\gamma_\mu d(x)$ and $J_\mu^A(x) = \bar{u}\gamma_\mu\gamma_5 d(x)$, the $J = 0$, 1 parts of the V and A current two-point functions, $\Pi_{V,A}^{(J)}$, are defined by

$$\begin{aligned}\Pi_{\mu\nu}^{V,A}(q) &= i\int d^4x\, e^{iqx}\langle 0|T\{J_\mu^{V,A}(x)J_\nu^{V,A}(0)^\dagger\}|0\rangle \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_{V,A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{V,A}^{(0)}(q^2)\,. \end{aligned}\tag{4}$$

The corresponding spectral functions $\rho_{V,A}^{(J)} = \frac{1}{\pi}\,\mathrm{Im}\,\Pi_{V,A}^{(J)}$ can be extracted from the differential distribution $dR_\tau^{V,A}/dq^2$, which is experimentally available [2,5]. Explicitly, with $s = q^2$,

$$\begin{aligned}\frac{dR_\tau^{V,A}}{ds} &= 12\pi^2 S_{\rm EW}|V_{ud}|^2\frac{1}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2 \times \\ &\left[\left(1+2\frac{s}{m_\tau^2}\right)\rho_{V,A}^{(1+0)}(s) - 2\frac{s}{m_\tau^2}\rho_{V,A}^{(0)}(s)\right]\,. \end{aligned}\tag{5}$$

Except for the pion-pole contribution to $\rho_A^{(0)}(s)$, the $J = 0$ part of the spectral function is numerically negligible. The combination $\Pi_{V,A}(s) \equiv \Pi_{V,A}^{(1+0)}(s)$ is free of kinematic singularities and analytic in the complex s-plane cut along the positive real axis. Cauchy's theorem thus yields the finite-energy sum rule (FESR)[2]

$$\begin{aligned}R_{V,A}^{[w]}(s_0) &= 12\pi^2\int_0^{s_0}\frac{ds}{s_0}\,w(s/s_0)\,\rho_{V,A}(s) \\ &= 6\pi i\oint_{|s|=s_0}\frac{ds}{s_0}\,w(s/s_0)\,\Pi_{V,A}(s)\,, \end{aligned}\tag{6}$$

valid for an arbitrary analytic weight $w(z)$.

For $s_0 \gg \Lambda_{\rm QCD}^2$, one can expect $\Pi_{V,A}$ to be well approximated by the OPE, *i.e.*,

$$\Pi_{V,A}(s) = \Pi_{V,A}^{\rm OPE}(s) + \Delta_{V,A}(s)\,, \tag{7}$$

with $\Delta_{V,A}(s)$ a small (but in general nonzero) correction accounting for the presence of DVs.

Assuming DVs to vanish sufficiently fast for $|s| \to \infty$ in the whole complex plane, one can show that the correction from $\Delta_{V,A}(s)$ to the contour integral in Eq. (6) can also be written as [15]

$$\mathcal{D}_{V,A}^{[w]}(s_0) = -\,12\pi\int_{s_0}^{\infty}\frac{ds}{s_0}\,w(s/s_0)\,\mathrm{Im}\,\Delta_{V,A}(s)\,. \tag{8}$$

FESRs thus provide constraints on $\mathrm{Im}\,\Delta_{V,A}(s)$ beyond those from the region $s_0 < s < m_\tau^2$ obtained by fitting to the experimental spectral function. In standard τ-decay-based determinations of α_s, $\Delta_{V,A}(s)$ is typically neglected, with at most a check on the self-consistency of this

[2] In the remainder the factor of $S_{\rm EW}|V_{ud}|^2$ is absorbed into $\Pi_{V,A}(s)$.

assumption [9].[3] It is our aim to quantitatively determine the impact of DVs on the values for the QCD parameters accessible through such fits.

Since at present no theory for DVs exists, we adopt the following *ansatz* for the asymptotic form of the DVs as a working hypothesis [15]

$$\frac{1}{\pi}\mathrm{Im}\,\Delta_{V,A}(s) \underset{\mathrm{large\,s}}{\longrightarrow} \kappa_{V,A}e^{-\gamma_{V,A}s}\sin(\alpha_{V,A}+\beta_{V,A}s)\,, \tag{9}$$

reflecting the presumed asymptotic character of the OPE expansion. Model studies suggest the exponential decay, originating from the finite width of the resonances. The oscillatory behaviour arises naturally in a spectral function with resonances distributed with some periodicity as, for instance, on the daughter trajectories in Regge theory [13,14,15]. With the (eight) parameters $\kappa_{V,A}$, $\gamma_{V,A}$, $\alpha_{V,A}$, and $\beta_{V,A}$ we expect the *ansatz* of Eq. (9) to capture the generic features of the DVs, thus avoiding any more specific models. This *ansatz* is certainly more general, and hence more likely to be realistic, than that used in analyses which entirely neglect DV contributions (which correspond to the special case $\kappa_V = \kappa_A = 0$).

With our description of $\mathrm{Im}\,\Delta_{V,A}(s)$, the experimental V and A spectral functions constrain the DV parameters, provided the asymptotic behaviour of Eq. (9) has set in for $s > s_{min}$ with $s_{\mathrm{min}} < m_\tau^2$. Above s_{min} one then has

$$\rho_{V,A}(s) = \theta\,(s - s_{\mathrm{min}})\left[\frac{Nc}{12\pi^2}\,[1+\hat{\rho}(s)] + \kappa_{V,A}e^{-\gamma_{V,A}s}\sin(\alpha_{V,A}+\beta_{V,A}s)\right]. \tag{10}$$

The function $\hat{\rho}(s)$ contains the perturbative corrections up to and including $\mathcal{O}(\alpha_s^4)$ as well as condensate contributions. The latter were shown to be numerically irrelevant for $s_{\mathrm{min}} \gtrsim 1.1\ \mathrm{GeV}^2$ [15]. In Ref. [15], fits to ALEPH V and A spectral functions employing Eq. (10) have been carried out, testing the onset of the asymptotic behaviour of Eq. (9) in real data. The results show the model description of $\mathrm{Im}\,\Delta_{V,A}(s)$ to be reasonable, and certainly compatible with ALEPH data. Similar tests confirm its compatibility with the OPAL data.

[3] An exception is Ref. [7], which discusses the relevance of this term, though only for the combined $V + A$ correlator, and concludes that its contribution lies within the error bars. On the basis of the analysis of Ref. [15] and the present study we believe this conclusion needs to be reconsidered.

3. Strategy of our analysis

With the OPE plus DV representation for the relevant correlators, Eq. (8) takes the form

$$R^{[w]}_{V,A}(s_0) = 6\pi i \oint_{|s|=s_0} \frac{ds}{s_0}\,w(s/s_0)\,\Pi^{\mathrm{OPE}}_{V,A}(s) + \mathcal{D}^{[w]}_{V,A}(s_0)\,. \tag{11}$$

The LH side is to be evaluated using $\rho^{(J)}_{V,A}$ extracted from the LEP experimental data, while the RH side contains the various OPE parameters as well as the parameters of the DV *ansatz*. Because of the high quality of the spectral data, use of a range of s_0 with $s_0 > s_{min}$ and a set of weight functions $w_i(x)$ allow fits to obtain both OPE and DV parameters.

Different weight functions $w(x)$ emphasise different terms of the OPE of $\Pi_{V,A}(s)$. For want of a good description of DVs, existing analyses have been restricted to the case of so-called pinched weights (polynomials in $x = s/s_0$ having zeros at $s = s_0$ which suppress contributions to the RHS of Eq. (6) from the vicinity of the timelike point on the contour $|s| = s_0$, where DVs are expected to be largest). Polynomials of higher degree, which provide more "pinching," are sensitive to higher order terms in the OPE. In some analyses, for practical reasons, the OPE has been truncated at dimensions below that for which non-α_s-suppressed contributions are in principle present. This can introduce an uncontrolled systematic uncertainty [9,15].

The three main features of our strategy are as follows. First, because we include DV contributions, our sum rules need not be restricted to pinched weights only. Low-degree unpinched weights, sensitive to only a small number of condensates (modulo α_s-suppressed logarithmic corrections) are employed together with pinched ones. Second, Eqs. (8) and (10) allow simultaneous fits to both the spectral function and the moments $R^{[w]}_{V,A}(s_0)$. Third, following Ref. [9], we do not restrict ourselves only to $s_0 = m_\tau^2$. The use of a range of s_0 values allows for a better use of the available data, facilitating the separation of condensate contributions of different dimension, and providing a consistency check on the fits from which $\alpha_s(m_\tau^2)$ and the condensates are extracted. Values for all fitted parameters should be independent, within errors, of the precise set of s_0 and weight functions used in their determination. We emphasise that this check has never been performed in full before.[4]

In our framework, in order to determine the various DV parameters, one must perform a separate analysis of V and A channels. With the eight parameters of Eq. (9) under control, one can then also analyse $V + A$ or $V - A$. This allows us to consider the first Weinberg sum rule. The fact that our fits satisfy this constraint provides a consistency check on the V and A fits.

A comment is in order regarding an important technical issue. In real life, data for the spectral functions are available in the form of binned histograms and the spectral integrals in Eq. (6) are approximated by a finite sum over bins. Even in the absence of correlations in the data, two different moments $R^{[w]}_{V,A}(s_1)$ and $R^{[w]}_{V,A}(s_2)$ with $s_2 > s_1$ will be correlated since they share contributions from the bins k with $s_k < s_1$. This correlation is stronger if s_2 and s_1 are closer to each other. In practice, if one chooses adjacent bins, the correlation can be well over 90%. Such strong correlations may generate very small eigenvalues in the correlation matrix for the moments $R^{[w]}_{V,A}(s_i)$. This renders the task of constructing reliable fits and a trustworthy treatment of uncertainties rather involved. Nevertheless, strategies to deal with such difficulties exist [17,18], and can be used to study the quality of the fits.

4. Evidence for duality violations

To illustrate our argument, let us concentrate on fits using the weights $w_1(x) = 1$, $w_2(x) = 1-x$, and $w_3(x) = (1 - x)^2$. Since w_1 is unpinched, its moments $R^{[w_1]}_{V,A}$ are expected to be more sensitive to DVs. The weights w_2 and w_3 are singly and doubly pinched, respectively. The results discussed in this section come from fits using 52 values of s_0 ranging from 1.5125 GeV^2 up to 2.7875 GeV^2. In the perturbative contribution we employ the Fixed Order prescription for the RG resummation.

In the fits using $w_1(x)$ the parameters are $\alpha_s(m_\tau^2)$ and the eight DV parameters $\kappa_{V,A}$, $\gamma_{V,A}$, $\alpha_{V,A}$ and $\beta_{V,A}$. (This moment is sensitive to the gluon condensate only through tiny logarithmic corrections.) A comparison of the fit results for $w_1(x)$ with the experimental moments from ALEPH data is shown in Fig. 1. The main feature of this figure is that a model in which DVs are neglected ($\kappa_{V,A} = 0$, red line in Fig. 1) cannot account for the data. The results obtained with DVs (blue line), on the other hand, compare very well with the data. For the A channel this is true right up to the kinematic endpoint. For the V channel, the agreement is less good above 2.8 GeV^2. Here one should bear in mind that (1) high-statistics measurements of 4π modes from BaBar suggest the ALEPH V spectral function may be overestimated towards the end of the spectrum [19], and (2) the OPE plus DV fit to the OPAL V channel data displays no such problem. This point will have to be clarified in the future.

Results for V channel fits using the w_2 and w_3 moments are shown in Fig. 2. In these fits, the value of the gluon condensate is fitted as well. Additionally, w_3 allows for a fit of the $D = 6$ condensate. The fits with DVs (blue lines) are in all cases excellent. For w_2, the fit including DVs is superior to that without. While for w_3 both fits are of excellent quality, we recall that the fit without DVs requires $\langle \alpha_s G^2 \rangle$, which can only be determined in a fit which includes DVs as input.

From such fits, values for α_s, the condensates

[4]In Ref. [9], a window of s_0 values *was* used. The restriction to $s_0 > 2$ GeV^2, however, imposed by requiring the neglect of DVs to be self-consistent, made it impossible to fit the gluon condensate $\langle \alpha_s G^2 \rangle$, which was therefore taken as external input. The strong anticorrelation found between the fitted α_s and input $\langle \alpha_s G^2 \rangle$ creates a systematic uncertainty which can only be avoided in a framework like ours, which includes DVs, and allows also $\langle \alpha_s G^2 \rangle$ to be obtained through a fit to data.

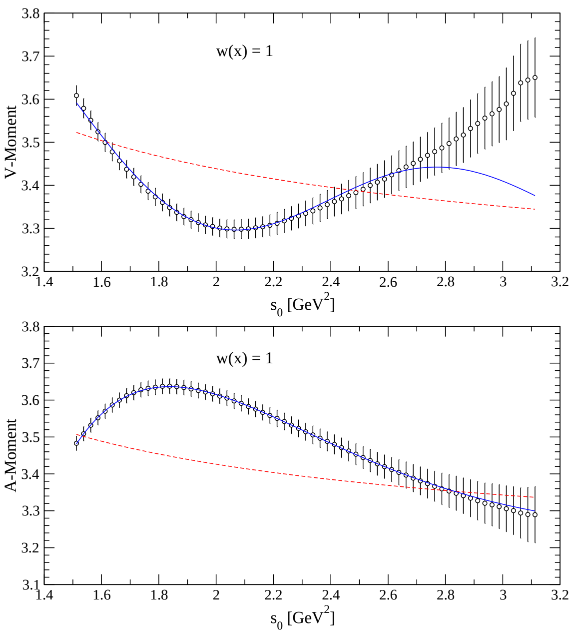

Figure 1. Fits to the $w_1(x) = 1$ spectral integrals for the V channel (top) and A channel (bottom). The blue (solid) lines show fits including DVs whereas the red (dashed) lines represent the model without DVs ($\kappa_{V,A} = 0$).

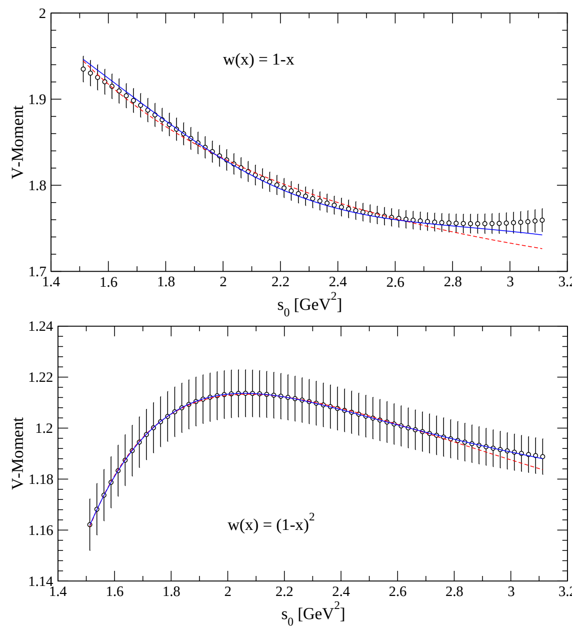

Figure 2. Fits to the V channel w_2 (top) and w_3 (bottom) spectral moments. The blue (solid) lines show fits including DVs whereas the red (dashed) lines represent the model without DVs ($\kappa_V = 0$).

and the DV parameters can in principle be extracted. For the time being, we have not carried through the final version of this analysis because of an inconsistency we uncovered in the correlation matrix publicly available from ALEPH [20]. This problem is explained in the next section.

5. Correlations in the ALEPH spectral function data

In order to study the uncertainties associated with our fits, a Monte Carlo generator of toy data sets was built based on ALEPH's spectra and correlations [20].[5] The unfolded ALEPH data are quite correlated, as can be seen from the upper panel of Fig. 3, whose data is much less scattered than the plotted errors would naively suggest. The toy data samples exhibit points that are much more scattered than the original data, as can be seen in Fig. 3, which shows the real data together with a representative toy sample (both plotted with the same errors). The strong correlations in the upper panel are not present in the toy sample, which points to a problem in the posted correlation matrix. According to the authors of the original analysis, the publicly available data do not contain correlations due to unfolding [21].

The effect of the missing correlations seems to be small for the doubly or triply pinched weights used in recent analyses based on the publicly available ALEPH spectral data.[6] More pronounced effects, however, may be expected for FESRs based on unpinched weights, which are required for a reliable exploration of DVs. A precise quantification awaits the completion of a reanalysis of the ALEPH covariance matrices.

[5] We thank A. Höcker for this suggestion.

[6] An exploratory reanalysis by the authors of Ref. [7] based on one exclusive channel ($\tau^- \to \pi^-\pi^0\nu_\tau$) suggests that this is indeed the case [21].

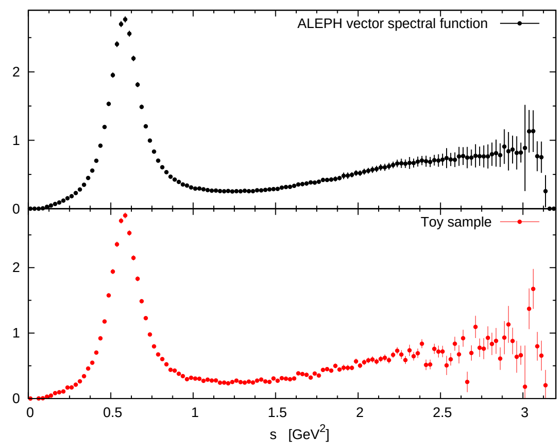

Figure 3. ALEPH data [20] for the vector spectral function (top) and a MC toy data sample based on the ALEPH correlation matrix (bottom).

Acknowledgements

We thank C. Bernard, S. Descotes-Genon, M. Martínez, S. Menke and R. Miquel for discussions and M. Davier, A. Höcker, B. Malaescu, and Z. Zhang for correspondence. This work was supported in part by the Spanish Ministry (grants CICYT-FEDER FPA2007-60323, FPA2008-01430, CPAN CSD2007-00042), by the Catalan Government (grant SGR2009-00894), EU Contract MRTN-CT-2006-035482, NERSC (Canada), and the US Department of Energy.

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