Correlation Widths in Quantum–Chaotic Scattering

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Abstract

An important parameter to characterize the scattering matrix S for quantum–chaotic scattering is the width $\Gamma_{\rm corr}$ of the S–matrix autocorrelation function. We show that the "Weisskopf estimate" $d/(2\pi)\sum_c T_c$ (where d is the mean resonance spacing, T_c with $0 \leq T_c \leq 1$ the "transmission coefficient" in channel c and where the sum runs over all channels) provides a very good approximation to $\Gamma_{\rm corr}$ even when the number of channels is small. That same conclusion applies also to the cross–section correlation function.

1. Purpose

Quantum–chaotic scattering is an ubiquitous phenomenon. It occurs whenever Schrödinger waves are scattered by a system with chaotic intrinsic dynamics. Examples are the passage of electrons through disordered mesoscopic samples, and compound–nucleus scattering. Moreover, quantum–chaotic scattering is simulated when electromagnetic waves of sufficiently low frequency are transmitted through a microwave cavity with the shape of a classically chaotic billiard. In all these cases, chaotic scattering is due to the numerous quasibound states of the system that appear as resonances in the scattering process and that obey random–matrix statistics.

The generic approach to quantum–chaotic scattering [1] is based upon a random–matrix model for the resonances and, thus, for the scattering matrix $S_{ab}(E)$, a function of energy E where a, b denote the open channels. Within that approach, the energy correlation function of the scattering matrix (the ensemble average $\langle S_{ab}(E-\varepsilon/2)S^*_{cd}(E+\varepsilon/2)\rangle$) can be worked out analytically [2] as a function of the energy difference ε, and approximate expressions for the cross–section correlation function are also available [3, 4, 5]. The correlation width of the cross section turns out to be rather close to that of the scattering matrix in all cases [5]. That is why we focus attention on the S–matrix correlation function in what follows.

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The determination of the correlation function requires the numerical evaluation of analytical expressions that are somewhat complicated. To gain an orientation of what to expect in a given situation, a simple approximate expression for the width $\Gamma_{\rm corr}$ of the S–matrix correlation function (and, by implication, of the cross section) would be helpful. A commonly used approximation is the "Weisskopf estimate" [6]. It has for example been applied to resonance spectra obtained from microwave experiments on quantum chaotic scattering [5, 7]. The measurements were performed in the regimes of isolated and weakly overlapping resonances and the associated S matrix comprised two dominant scattering channels and a large number of weakly coupled ones. A motivation for the present paper was to test the accuracy of the Weisskopf estimate under such conditions. We will demonstrate that it provides a good approximation for $\Gamma_{\rm corr}$ not only in the regime of strongly overlapping resonances. For simplicity and without loss of generality we confine ourselves to the case where the average S–matrix is diagonal, $\langle S_{ab}\rangle = \langle S_{aa}\rangle\delta_{ab}$. The unitarity deficit of the average S–matrix is then measured by the transmission coefficients $T_c = 1 - |\langle S_{cc}\rangle|^2$. These obey $0 \leq T_c \leq 1$ for all c.

Naively, one might consider two alternatives for estimating $\Gamma_{\rm corr}$. (i) The Weisskopf estimate expresses the total average resonance width in terms of the mean resonance spacing d of the resonances and of the transmission coefficients T_c,

$$\Gamma_{\rm W} = \frac{d}{2\pi}\sum_c T_c \ . \tag{1}$$

The sum in Eq. (1) runs over the open channels.

(ii) The "Moldauer–Simonius sum rule" [8, 9] gives the following expression for the mean distance $(1/2)\langle\Gamma_\mu\rangle$ of the poles of the scattering matrix (labeled by a running index μ) from the real energy axis.

$$\langle\Gamma_\mu\rangle = -\frac{d}{2\pi}\sum_c \ln[1 - T_c] \ . \tag{2}$$

For the case of unitary symmetry, the sum rule (2) has been derived rigorously [10]. There is no reason to doubt that the sum rule (2) holds also in the orthogonal case although a proof exists only in fragmentary form [11].

The width $\Gamma_{\rm W}$ in Eq. (1) and the double average pole distance $\langle\Gamma_\mu\rangle$ as given by Eq. (2) agree whenever $T_c \ll 1$ for all c. In general, however, the values of both quantities differ widely. For instance, for the case of a single channel with $T \approx 1$, Eq. (1) yields $\Gamma_{\rm W} \approx d/(2\pi)$ while Eq. (2) yields $\langle\Gamma_\mu\rangle \gg d/(2\pi)$. An identification of $\Gamma_{\rm W}$ (of $\langle\Gamma_\mu\rangle$) with the correlation width $\Gamma_{\rm corr}$ would suggest that we deal with isolated (with strongly overlapping) resonances, respectively. It is known [12, 13, 14] that Eq. (2) fails when any of the T_c is close to unity, and a comparison of the values of $\Gamma_{\rm corr}$ given in the figures below with Eq. (2) confirms that fact. We ascribe that failure of the Moldauer–Simonius sum rule to the fact that the fluctuation properties of the scattering matrix depend not only on the location of the poles of S but also on the values of the residues. Little is actually known about the latter [15, 16].

That leaves us with Eq. (1) as the only viable alternative. We recall the conditions under which Eq. (1) is obtained [6]. One uses a time–dependent description and considers a scattering system with constant resonance spacing d_0 coupled to a number of channels. The frequency with which a typical wave function of the system approaches the entrance of a given channel c is d_0/h, the probability with which the system escapes into that channel is given by T_c, the partial width for decay into channel c is accordingly $d_0 T_c/(2\pi)$. Summing over all channels and postulating that the result applies also to systems that do not have constant resonance spacing d_0, one replaces d_0 by the actual mean resonance spacing d and arrives at Eq. (1). The argument being semiclassical, one expects Eq. (1) to give an approximate expression for the average resonance decay width in the case of many channels or, more precisely, for $\sum_c T_c \gg 1$.

The argument just given leaves open the question how $\Gamma_{\rm W}$ relates to the correlation width $\Gamma_{\rm corr}$. In Ericson's work [17] the identity of $\Gamma_{\rm W}$ and of $\Gamma_{\rm corr}$ was postulated for $\sum_c T_c \gg 1$. A proof for that assertion became available with the work of Ref. [18]. There it was shown that an expansion of the S–matrix correlation function derived in Ref. [2] in inverse powers of $\sum_c T_c$ yields as the leading term a Lorentzian with width $\Gamma_{\rm W}$. This implies $\Gamma_{\rm corr} = \Gamma_{\rm W}$ for $\sum_c T_c \gg 1$. In the present paper we investigate how much $\Gamma_{\rm W}$ and $\Gamma_{\rm corr}$ differ outside the Ericson regime $\sum_c T_c \gg 1$. We do so using the analytical results of Ref. [2].

2. Approach

Starting point is the expression (see the review [1])

$$S_{ab}(E) = \delta_{ab} - 2i\pi \sum_{\mu\nu} W_{a\mu}[D^{-1}(E)]_{\mu\nu} W_{\nu b} \tag{3}$$

for the element $S_{ab}(E)$ of the scattering matrix connecting channels a and b, with

$$D_{\mu\nu}(E) = E\delta_{\mu\nu} - H_{\mu\nu} + i\pi \sum_c W_{\mu c} W_{c\nu} \ . \tag{4}$$

Here E is the energy. The real and symmetric matrix H with elements $H_{\mu\nu}$ and $\mu, \nu = 1, \ldots, N$ is a member of the Gaussian orthogonal ensemble of random matrices (GOE). The elements $H_{\mu\nu}$ are Gaussian–distributed random variables with zero mean values and second moments given by $\langle H_{\mu\nu} H_{\rho\sigma} \rangle = (\lambda^2/N)[\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}]$. The matrix H represents N quasibound levels and their mutual interaction. The parameter λ has the dimension energy and defines the average level spacing d of the eigenvalues of H. In the center of the GOE spectrum we have $d = \pi\lambda/N$. The parameter d defines the energy scale so that both E and $\Gamma_{\rm corr}$ are expressed in units of d. The real matrix elements $W_{c\mu}$ couple the space of quasibound levels to Λ channels labelled $a, b, c, \ldots$. In the cases considered in the present work the amplitudes for the passage from an intrinsic state to a scattering channel coincide with those for the reverse process, that is $W_{\nu c} = W_{c\nu}$. Without loss of generality we assume that $\sum_\mu W_{a\mu} W_{b\mu} = N v_a^2 \delta_{ab}$.

The parameters v_a^2 define the mean strength of the coupling to channel a. Since H is random, the S–matrix is a matrix–valued random process that depends on E. All moments and correlation functions of $S(E)$ (defined by averaging over the GOE with the energy at or close to the center of the GOE spectrum) depend only on the average S–matrix elements $\langle S_{ab} \rangle$, on the transmission coefficients T_c, and on energy differences. The latter are expressed in units of d. With $x_a = \pi^2 v_a^2/d$ we have

$$\langle S_{ab} \rangle = \frac{1 - x_a}{1 + x_a} \delta_{ab} \ , \quad T_a = \frac{4 x_a}{(1 + x_a)^2} \ . \tag{5}$$

In Ref. [2], the autocorrelation and cross–correlation functions of the elements of the S–matrix are given in terms of these parameters. They are worked out for fixed Λ in the limit $N \to \infty$. We do not repeat the analytical expressions here. These contain a threefold integration over real variables. We make use of a simplification of these integrals in terms of variable transformations first introduced in Ref. [18] and summarized in the Appendix of Ref. [5]. For a given set of transmission coefficients $T_1, T_2, \ldots, T_\Lambda$ the resulting formula for the S–matrix autocorrelation function $\langle S(E - \varepsilon/2) S^*(E + \varepsilon/2) \rangle$ is evaluated numerically as a function of ε/d. The full width at half maximum of that function yields $\Gamma_{\rm corr}/d$.

3. Results

According to the Weisskopf estimate in Eq. (1), the correlation width $\Gamma_{\rm corr}$ should depend only on the number of channels Λ and on the sum $T = \sum_c T_c$ of the transmission coefficients. To test that assertion, we have for fixed values of Λ and of T with $0 < T < \Lambda$ calculated $\Gamma_{\rm corr}$ for several sets of parameters $\{T_1, T_2, \ldots, T_\Lambda\}$. These are subject to the constraints $\sum_c T_c = T$ and $0 < T_c \leq 1$ and were obtained with the help of a random–number generator. The number of sets chosen was typically 25. For each such set we have determined $\Gamma_{\rm corr}/d$. In some of our calculations we have increased the number of sets to 100 without a noticeable change of the results.

We display our results in Figure 1. For several values of T given in the panels we plot the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ versus Λ. Each set of parameter values $T_1, T_2, \ldots, T_\Lambda$ corresponds to a dot in the plot. We observe that the points scatter about a mean value that is close to but slightly below unity. For fixed T (fixed Λ), the width of the distribution decreases with increasing Λ (T, respectively).

Figures 2 and 3 serve to quantify these statements. In a plot similar to that of Fig. 1, Fig. 2 shows the average of the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ taken over the 25 realizations for each of the given values of Λ. The average values tend to lie a couple of percent below unity even for large values of Λ and deviate up to 10 percent from unity for $T \approx 1$ and $\Lambda < 10$. A significant scatter of the average values by more than 10 per cent occurs only for $\Lambda < 10$ and for $T < 8$ or so.

Figure 3 shows similarly the root mean square (rms) deviation of the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ from unity (more precisely: the square root of the mean square deviation of the ratios from unity). Again, the rms deviations are very small unless

$T < 8$ where they increase with decreasing T and Λ but do not exceed a few percent. Due to the condition $T \leq \Lambda$ there are only a few data points at our disposal in that region of largest deviations. Accordingly, for a more thorough test of the Weisskopf estimate we also considered cases where $\Lambda \leq 8$ is fixed and T is varied. In Fig. 4 we show the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$, with all transmission coefficients chosen equal, while in Figs. 5 and 6 the values of the transmission coefficients were obtained with a random-number generator as above. Again the deviations from unity are less than 10 per cent. Furthermore, as T approaches Λ the average of $\Gamma_{\rm corr}/\Gamma_W$ takes a value slightly above unity as is also observed in Fig. 2 for comparable values of Λ and T. For values of $T << \Lambda$, however, the ratio is slightly below unity for randomly obtained transmission coeffecients, whereas it is always above unity for equally chosen ones. In the first case for a given value of Λ all except a few transmission coeffcients are very small. This corresponds to the typical situation e.g. in the experiments with microwave billiards [5].

We conclude that the Weisskopf estimate (1) constitutes a very good approximation to $\Gamma_{\rm corr}$ for practically all values of $\sum_c T_c$. Maximal deviations that occur for small values of Λ and T do not exceed the 10 percent range.

Acknowledgement

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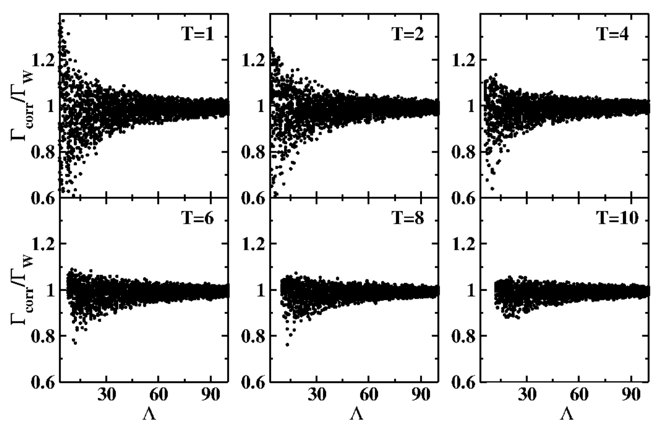

Figure 1: The ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ (dots) versus the number Λ of channels for several values of $T = \sum_c T_c$ as indicated in the panels. For each value of Λ and of T, 25 sets of the parameters $T_1, T_2, \ldots, T_\Lambda$ were randomly chosen. Each such set corresponds to a dot in the plot.

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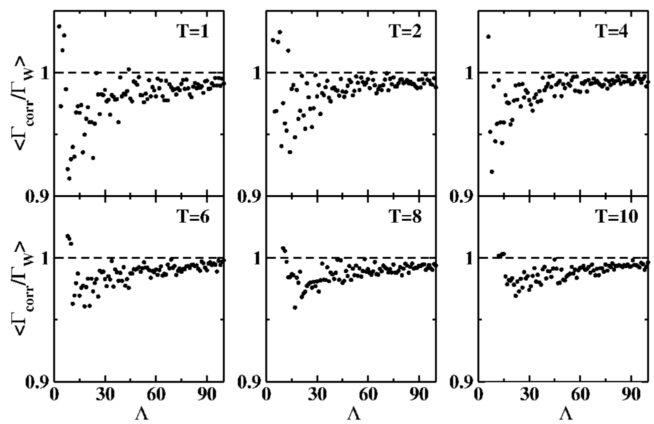

Figure 2: Average of the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ (dots) over 25 sets of transmission coefficients versus the number Λ of channels for several values of T as indicated in the panels.

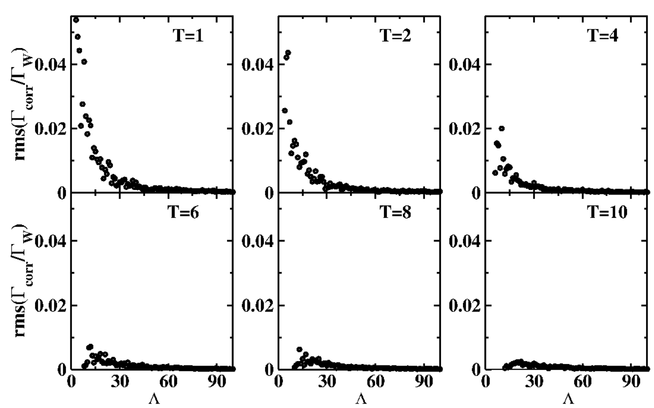

Figure 3: Same as Fig. 2 but for the rms deviation from unity.

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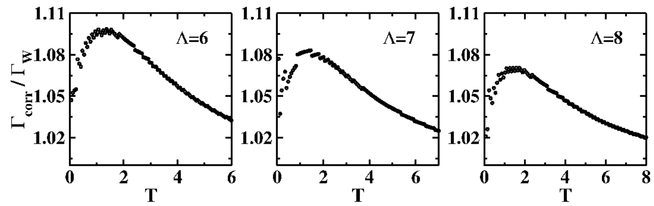

Figure 4: Ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ (dots) for Λ equivalent transmission coefficients versus T for several numbers Λ of channels as indicated in the panels.

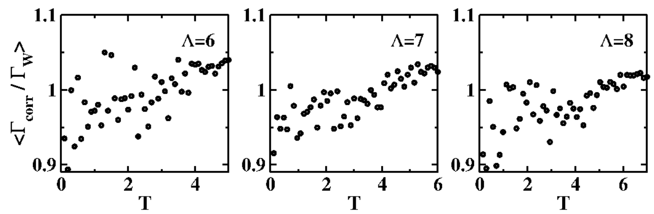

Figure 5: Average of the ratio $\Gamma_{\rm corr}/\Gamma_{\rm W}$ (dots) over 25 sets of transmission coefficients versus T for several numbers Λ of channels as indicated in the panels.

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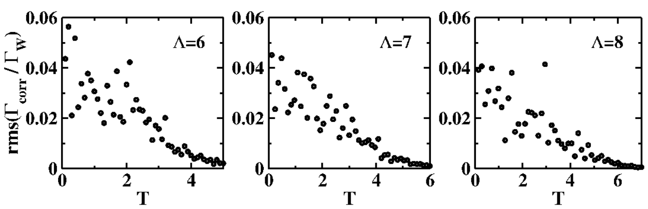

Figure 6: Same as Fig. 5 but for the rms deviation from unity.