

On the critical condition in gravitational shock wave collision and heavy ion collisions

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Abstract

The paper derived a critical condition for matter equilibration in heavy ion collisions using a holographic approach. A gravitational shock waves with infinite transverse extension is used to model infinite nucleus with an intrinsic saturation scale. We constructed the trapped surface in the collision of two asymmetric shock waves and formulated a critical condition for matter equilibration in collision of nucleus with the same energy but different saturation scale. We expressed the critical energy as a function of two saturation scales and found it is insensitive to the softer saturation scale. To understand the origin of the critical condition, we computed the Next to Leading Order stress tensor in the boundary field theory due to the interaction of the nucleus and found the critical condition corresponds to the breaking down of the perturbative expansion. We indeed expect non-perturbative effects be needed to describe black hole formation.

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1 Introduction

The AdS/CFT correspondence is conjectured as a duality between weakly coupled gravity theory and strongly coupled $\mathcal{N}=4$ Super Yang-Mills theory in the limit of large N_c and strong coupling [1, 3, 2]. Its applications to strongly coupled Quark Gluon Plasma (sQGP) have revealed many novel features of the strongly coupled medium such as very low viscosity [4], absence of jets, Mach cone formation and other hydrodynamical phenomena, see e.g. review [5]. While static and near equilibrium properties have been extensively studied in this context, deriving corrections to hydrodynamics, the out-of-equilibrium aspects of the strongly coupled gauge theory remains less understood. One of the main challenges in heavy ion collisions remains the understanding of early equilibration of matter produced in the collisions.

Recently there have been several attempts to model the initial non-equilibrium stage of the collision, including [6] and [7]. Our paper [8] provided relatively simple description of black hole formation, due to elastic membrane falling under its own weight. In this case the ultimate equilibration is always assured, and it happens as a gradual propagation of the equilibration boundary in the scale space (along the 5-th holographic dimension), from the ultraviolet (UV) toward the infrared (IR) direction.

According to principles of AdS/CFT, due to large N_c limit all issues have to be understood in terms of classical gravity problem. Thermal equilibration of matter and early entropy production is in this setting dual to the formation of a (black hole) horizon, trapping some amount of information from the distant observer, where our world is. This mechanism not only is able to provide some lower bound on the amount of *entropy production* in the collision, but it also provides qualitative “yes” or “no” answer if the information trapping does or does not happen, as a function of given initial condition of the problem. Thus one of the interesting unexpected features of the problem are some rapid transition into a new regime, as a function of e.g. collision energy, density of the colliding objects or (not discussed in this work) the impact parameter of the collisions.

The relation between the trapped surface at the collision moment and the lower bound on the entropy production has been introduced by Gubser, Pufu and Yarom [9], who have considered collision of ultrarelativistic small black holes in AdS₅. It can be viewed as a collision of gravitational shock waves, having near-zero longitudinal width but possessing a certain profile in 3 transverse coordinates x^2, x^3 and the holographic coordinate z . Mathematically, the trapped surface was related to certain function which satisfies the Laplace eqn inside it, plus certain nontrivial boundary conditions on that surface. Furthermore, if

a solution to those conditions is found, the trapped surface area gives (the lower bound to) the entropy production in the collisions. Technically construction of the trapped surface closely follows early works in flat space background [14, 15, 16, 17]. The specific problem addressed in that work [9] was central collision of two point black holes. Among its important conclusions was e.g. a prediction of the entropy dependence on the (CM) collision energy $S(E) \sim E^{2/3}$.

This approach has been then generalized to the non-central collisions. We found [11] that trapped surface formation is not possible beyond certain critical impact parameter, depending on the collision energy. Furthermore, the disappearance of the trapped surface happens suddenly, as a 1-st order transition. An intriguing observation, also pointed out in our paper [11], is that phenomenologically the multiplicity of the produced particles (the entropy) per participant nucleon in ultrarelativistic collisions at RHIC also changes rapidly between “non-thermal” peripheral and “thermal” more central collisions. The specific results about the trapped surface were later confirmed in [10, 12]. Like it has been the case in flat space, the value of the critical impact parameter can be understood as a bound on the angular momentum for the shock wave pair at a given center of mass to form a AdS-Kerr black hole.

All the above-mentioned works have been using a shock wave arising from a point source in the bulk (small black holes). The size of the colliding nuclei were thus incorporated via the distance of those objects from the boundary along the holographic coordinate z . However, as emphasized in [11], this is an oversimplification of the problem. The transverse extension of the colliding objects in x^2, x^3 can be introduced independently of the profile in the holographic z direction. The latter, due to very basic features of the AdS/CFT correspondence, should be ascribed instead to the *intrinsic scale* variable, in the sense of the renormalization group, describing its microscopic structure. In the collision of ordinary objects it would be interatomic scale, for high energy QCD the holographic coordinate z at which the colliding objects are before the collisions should represent the typical scale of their wave function, known as the “*saturation scale*”. This scale affects the typical “equilibration time” and other properties of the problem, and it cannot thus be set arbitrarily.

The model with the simplest possible geometry, proposed for this purpose in [11], is thus the *wall* shock waves, which are infinite and homogeneous in 2 transverse spatial dimensions. The extension in z of the trapped region has been found for the collision of such wall shock waves. It has been done for the simplest case of a symmetric collision, in which both colliding walls are the same. We start this paper discussing a more general

case, in which two colliding walls are not the same. Physically, one may think of two colliding objects made of different materials with different densities, which are modeled by their different “saturation scales” z_1, z_2 . The question we will answer is the precise critical condition on their values z_1, z_2 beyond which the trapped surface *is* formed.

Perhaps the reader may wonder why are we interested in such a question. It is clear that one of the most important variable is the energy (rapidity) of the colliding objects: the black holes can only be formed if it is large enough. However, let us also remind the reader that in heavy ion collisions the energy per nucleon is not the only important variable: for example rapid equilibration and hydrodynamical behavior experimentally observed at RHIC for collisions of two heavy ions such as AuAu, are indeed *not* observed say for deuteron-Au collisions at the same rapidity of the colliding nuclei. Similarly, we find that two walls, made of sufficiently different materials, can also collide *without* classical equilibration and entropy formation, at the same energy at which the symmetric walls would produce the trapped surface.

Another issue, to be addressed in section 4, deals with the difficult problem of finding the gravitational solution for the non-zero time, in the future quadrant of the time-longitudinal coordinates. In flat Minkowski space-time this is a long-standing problem of the general relativity, which for gamma factor of the order of few units has been recently studied numerically [18]. However, it remains unsolved for very ultrarelativistic collisions even in this case, with some important partial results reported in [13, 15, 16, 17]. The problem gets even more complicated in the curved 5-dimensional space AdS_5 needed for current applications, see e.g. [19, 20, 23]. The “Next-to-Leading Order” (NLO) effect we will discuss are the “debris” produced in the shock wave collision, the gravitons radiated perturbatively. We will compute the NLO correction to the metric, and read the corresponding stress tensor on the dual field theory on the boundary: such “early time” stress tensor plays an important role in the theory of heavy ion collisions, as it provides the initial conditions for the standard hydrodynamical treatment. We will follow most closely the work by Taliotis[23] in the settings: the main difference is that our source is localized in the holographic direction z , instead of the transverse directions as in [23].

2 Wall on wall shock wave collision

We start with the wall shock wave model proposed in [11]. The metric of a single shock wave moving in direction x^+ is given by:

$$ds^2 = L^2 \frac{-dx^+ dx^- + dx_\perp^2 + \phi(z, x^+) dx^{+2} + dz^2}{z^2} \quad (1)$$

The shock wave profile $\phi(z)$ satisfies the following equation:

$$(\partial_z^2 - \frac{3}{z}\partial_z)\phi(z, x^+) = -16\pi G_5 \mu \frac{z_0^3}{L^3} \delta(x^+) \delta(z - z_0) \quad (2)$$

with J_{++} is the source, which has infinite extension in the directions of x_\perp , thus the name wall shock wave. The parameter z_0 is interpreted as the saturation scale. The solution to (2) is given by:

$$\phi(z, x^+) = 4\pi G_5 \mu \frac{z_0^4}{L^3} \delta(x^+) \begin{cases} \frac{z^4}{z_0^4} & z \leq z_0 \\ 1 & z > z_0 \end{cases} \quad (3)$$

The stress tensor follows from (3) reads:

$$T_{++} = \mu \delta(x^+) \quad (4)$$

Now consider the collision of two shock waves, as a model of heavy ion collisions. The metric of the shock waves before collision is given by:

$$ds^2 = L^2 \frac{-dx^+ dx^- + dx_\perp^2 + \phi_1(x^+, z) dx^{+2} + \phi_2(x^-, z) dx^{-2} + dz^2}{z^2} \quad (5)$$

The shock wave profiles solve the following equations:

$$(\partial_z^2 - \frac{3}{z}\partial_z)\phi_1(x^+, z) = -16\pi G_5 \mu_1 \frac{z_1^3}{L^3} \delta(x^+) \delta(z - z_1) \quad (6)$$

$$(\partial_z^2 - \frac{3}{z}\partial_z)\phi_2(x^-, z) = -16\pi G_5 \mu_2 \frac{z_2^3}{L^3} \delta(x^-) \delta(z - z_2) \quad (7)$$

Note we have absorbed the delta function into the definition of the shock wave profiles as in [23]. The dual stress tensor reads:

$$\begin{aligned} T_{++} &= \mu_1 \delta(x^+) \\ T_{--} &= \mu_2 \delta(x^-) \end{aligned} \quad (8)$$

The superposition of two shock waves (5), solves the Einstein equation in the region with $\theta(x^+)\theta(x^-) = 0$. (Here $\theta(x)$ is the Heaviside step function, 1 for positive and 0 for negative argument.) The shock waves only interact and modifies the metric in the future quadrant $\theta(x^+)\theta(x^-) > 0$.

As explained in the Introduction, our colliding walls are dual to “nuclei” of infinite size, so the concept of impact parameter does not exist. Instead, we have also chosen two nucleus to have the same energy, but different saturation scales. (To be specific, we demand $z_1 > z_2$.) Although our shock waves are sourced by the delta functions, they have finite size in the z direction, decreasing both into the UV and the IR. Other types of shock waves used to model relativistic nucleus include [19, 20, 21, 22].

These finite extension of the shock waves in z explains why the trapped surface can be found also in a finite interval in z , we will call upper and lower positions of the trapped surface z_a, z_b .

The entropy lower bound, dual to the “area” of the trapped surface is given by:

$$S = \frac{2A}{4G_5} = \frac{\int \sqrt{g} dz d^2 x_\perp}{2G_5}$$

$$s \equiv \frac{S}{\int d^2 x_\perp} = \frac{L^3}{4G_5} \left(\frac{1}{z_a^2} - \frac{1}{z_b^2} \right) \quad (9)$$

3 Critical condition for trapped surface formation

In this section, we will construct the trapped surface associated with the collision of two shock waves. Let us for the completeness recall the mathematical basis defining the trapped surface. The equations are produced by required vanishing of the so called “expansion” combination: loosely speaking it means that two massless particles moving along two nearby null geodesics on this surface neither approach each other nor diverge. It can be shown to correspond to a relatively simple problem a la electrostatic solution in a cavity (the Laplacian with given sources) with zero boundary condition on the surface, complemented by additional nontrivial condition for the magnitude of the field derivatives at the surface itself. Following [9, 11, 10], the master equation for trapped surface is given by:

$$z^2 \Psi_i'' - z \Psi_i' - 3 \Psi_i = -16\pi G_5 \mu_i z_i^4 \delta(z - z_i)$$

$$\Psi_i(z_a) = \Psi_i(z_b) = 0$$

$$\Psi_1'(z_a) \Psi_2'(z_a) \frac{z_a^2}{L^2} = \Psi_1'(z_b) \Psi_2'(z_b) \frac{z_b^2}{L^2} = 4 \quad (10)$$

with $i = 1, 2$. The trapped surface for wall-on-wall shock wave collision is just $z_a < z < z_b$.

The first two equations can be solved as:

$$\Psi_i(z) = \begin{cases} C_i \left(\frac{z^3}{z_a^3} - \frac{z_a}{z} \right) & z < z_i \\ D_i \left(\frac{z^3}{z_b^3} - \frac{z_b}{z} \right) & z > z_i \end{cases} \quad (11)$$

with

$$C_i = -4\pi G_5 \mu_i \frac{\left(\frac{z_i^4}{z_b^4} - 1 \right) z_b}{\frac{z_b^4 - z_a^4}{z_a^3 z_b^3}} \quad (12)$$

$$D_i = -4\pi G_5 \mu_i \frac{\left(\frac{z_i^4}{z_a^4} - 1 \right) z_a}{\frac{z_b^4 - z_a^4}{z_a^3 z_b^3}} \quad (13)$$

We can always apply a longitudinal boost such that both shock waves have the same energy density $\sqrt{\mu_1 \mu_2}$. Then the third equation in (10) leads to

$$C_1 C_2 = D_1 D_2 = \frac{L^2}{4} \quad (14)$$

Let us consider the case $z_1 = z_2 \equiv z_0$ first. (14) leads to:

$$z_a + z_b = \frac{8\pi G_5 \sqrt{E_1 E_2}}{L} = A_1 \quad (15)$$

$$\frac{(z_a + z_b)^2 - 3z_a z_b}{(z_a z_b)^3} = \frac{L^3}{z_0^4} = \frac{1}{A_2} \quad (16)$$

in which two appearing combinations of parameters are for brevity called A_1, A_2 . The resulting cubic eqn

$$(z_a z_b)^3 + 3A_2(z_a z_b) - A_1 A_2 = 0 \quad (17)$$

can be solved by Cardano formula. The explicit solution is not illustrative and is not showed here. We note, however the solution has to satisfy the inequality $4z_a z_b \leq (z_a + z_b)^2 = A_1^2$, which gives rise to the following constraint:

$$\frac{2\pi^2}{N_c^2} \mu z_0^3 \geq 1 \quad (18)$$

where we have used $G_5 = \frac{\pi L^3}{2N_c^2}$. (18) is the critical condition for trapped surface formation in a symmetric collision of gravitational shock waves.

When $z_1 > z_2$, we define $\frac{z_1^4}{z_a^2 z_b^2} = \lambda_1$, $\frac{z_2^4}{z_a^2 z_b^2} = \lambda_2$, (14) can be simplified to:

$$\begin{cases} \left(\frac{z_a}{z_b}\right)^2 + \left(\frac{z_b}{z_a}\right)^2 + 1 = \frac{\lambda_1 + \lambda_2 + 1}{\lambda_1 \lambda_2} \\ (z_a z_b)^3 \frac{\frac{z_a + z_b}{z_b + z_a}}{\left(\left(\frac{z_a}{z_b}\right)^2 + \left(\frac{z_b}{z_a}\right)^2\right)^2} = \frac{L^2}{(8\pi G_5 \mu)^2 (1 - \lambda_1 \lambda_2)} \end{cases} \quad (19)$$

where the first equation follows from $\frac{C_1 C_2}{D_1 D_2} = 1$ and the second equation can be obtained from $C_1 C_2 = \frac{L^2}{4}$. The first equation can be used to give $\frac{z_a}{z_b} + \frac{z_b}{z_a} = \sqrt{\frac{(\lambda_1 + 1)(\lambda_2 + 1)}{\lambda_1 \lambda_2}}$. Combining this with the second equation, we can express μ as a function of λ_1 and λ_2 , which in terms of variable $F = \lambda_1 \lambda_2$ and $r = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{(z_1^4 + z_2^4)^2}{z_1^4 z_2^4}$ reads:

$$\frac{(8\pi G_5 \mu)^2 (z_1 z_2)^3}{L^6} = \frac{F^{3/4}}{1 - F} \left(\frac{\sqrt{rF} + 1}{F} + 1 \right)^{1/2} \left(\frac{\sqrt{rF} + 1}{F} - 3 \right) \quad (20)$$

Note that r depends on the degree of the asymmetry of the collision, for $z_1 = z_2$ one has $r = 4$. Let us thus fix r and study the r.h.s. of the previous equation as a function of the other variable F , to be called $A(r, F)$. From the second equation of (19), we know $F < 1$ and by definition $F > 0$. For a given r , we have the following limits: as $F \rightarrow 0$, $A \rightarrow F^{-\frac{3}{4}}$ and as $F \rightarrow 1$, $A \rightarrow \frac{1}{1-F} (\sqrt{r} + 2)^{\frac{1}{2}} (\sqrt{r} - 2)$. Unless $r = 4$ (the symmetric case), in both limits the function tends to positive infinity. Therefore a minimum must exist at certain $F = F_{min}$, which gives rise to the critical condition we are looking for. Fig.1 contains a plot of A as a function of F at several r .

The extremum of $A(r, F)$ is found to be the roots of the following equation:

$$-3F - 4rF^2 + 3\sqrt{rF} + 3F^3 + 13F^2 - 6F\sqrt{rF} + 3F^2\sqrt{rF} + 3 = 0 \quad (21)$$

It is not difficult to locate the minimum of $A(r, F)$ numerically, which gives rise to a critical condition for the collision energy:

$$\frac{4\pi^2}{N_c^2} \mu (z_1 z_2)^{3/2} \geq \sqrt{G(r)} \quad (22)$$

where we have used $G_5 = \frac{\pi L^3}{2N_c^2}$. $G(r)$ is the minimum of $A(r, F)$ at a given r . For $r = 4$ ($z_1 = z_2$), $A(r, F)$ has a minimum at $F \rightarrow 1$: $G(r) = 4$. We recover the critical condition for the symmetric collision (18). For general $r > 4$, we find $G(r)$ numerically and as $\frac{z_1}{z_2}$ grows, $\sqrt{G(r)}$ has a power like asymptotics $\sqrt{G(r)} \sim \left(\frac{z_1}{z_2}\right)^{3/2}$. Fig.2 shows a the power law dependence of $G(r)$ on $\frac{z_1}{z_2}$. The power measured by the slope in the log-log plot is approximately 1.5.

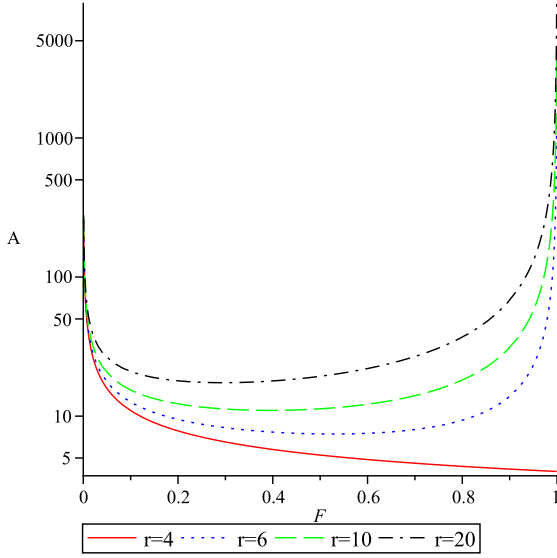


Figure 1: A as a function of F . A minimum always exists in $0 < F < 1$ for $r > 4$. In the extreme case $r = 4$, the minimum locates at $F = 1$

The asymptotic power law behavior of $G(r)$ can be obtained analytically. We note the root of (21) corresponding to the minimum of $A(r, F)$ goes to zero as $r \rightarrow \infty$. As the result, (21) simplifies to $-4rF^2 + 3\sqrt{rF} = 0$, which is solved by

$$F = \left(\frac{3}{4}\right)^{2/3} r^{-1/3} + \dots \quad (23)$$

where \dots denotes subleading terms. Substituting the root to $A(r, F)$, we obtain $G(r) = r^{3/4} + \dots$. Combined with the definition of r , we indeed have:

$$\sqrt{G(r)} = \left(\frac{z_1}{z_2}\right)^{3/2} + \dots \quad (24)$$

In the limit $r \gg 1$ ($z_1 \gg z_2$), the critical condition simplifies to

$$\frac{16\pi^2}{N_c^2} \mu z_2^3 \geq 1 \quad (25)$$

We would like to point out the non-uniqueness of the trapped surface, as first remarked by Eardley and Giddings [15], the unusual boundary value problem defining the trapped surface could have multiple solutions. We will see it is indeed the case in our wall-on-wall collision.^{#1}.

^{#1}Apart from this, there is also the foliation dependence of the trapped surface, which we do not discuss

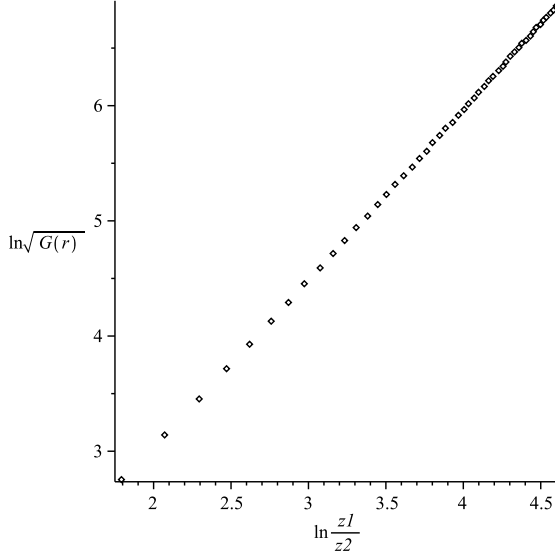


Figure 2: A log-log plot of $\sqrt{G(r)}$ versus $\frac{z_1}{z_2}$

Suppose we have the energy of the shock wave well above the critical value, i.e. $A_0 \equiv \frac{4\pi^2}{N_c^2} \mu (z_1 z_2)^{3/2} \gg G(r)$. We know from the previous analysis that

$$\begin{aligned} A(r, F) &\rightarrow F^{-3/4} \quad \text{as } F \rightarrow 0 \\ A(r, F) &\rightarrow \frac{1}{1-F} (\sqrt{r} + 2)^{1/2} (\sqrt{r} - 2) \quad \text{as } F \rightarrow 1 \end{aligned}$$

This allows two solutions $F = A_0^{-4/3} + \dots$ and $F = 1 - \frac{(\sqrt{r}+2)^{1/2}(\sqrt{r}-2)}{A_0} + \dots$. Without explicit solution of the trapped surface, we can compare the area of two corresponding trapped surface, which is related to the entropy production per transverse area[11]:

$$\begin{aligned} s &= \frac{N_c^2}{2\pi} \left(\frac{1}{z_a^2} - \frac{1}{z_b^2} \right) \\ &= (2\pi N_c^2 \mu)^{1/3} A_0^{-1/3} F^{-1/4} \left(1 + \sqrt{rF} - 3F \right)^{1/2} \end{aligned} \quad (26)$$

With the former solution, we have $s = (2\pi N_c^2 \mu)^{1/3} + \dots$, while the latter solution gives rise to $s = (2\pi N_c^2 \mu)^{1/3} A_0^{-1/3} (\sqrt{r} - 2)^{1/2} + \dots$. In the limit $A_0 \rightarrow \infty$, the former trapped surface has a much greater area than the latter. Therefore we choose the former as the “outermost” trapped surface. This branch of solution is precisely the one used in [24] for a comparison of sourced shock wave and source-free shock wave.

A view of the trapped surface formation is included in Fig.3. We note the trapped surface only starts to appear when the wave fronts of the shocks are separated by finite

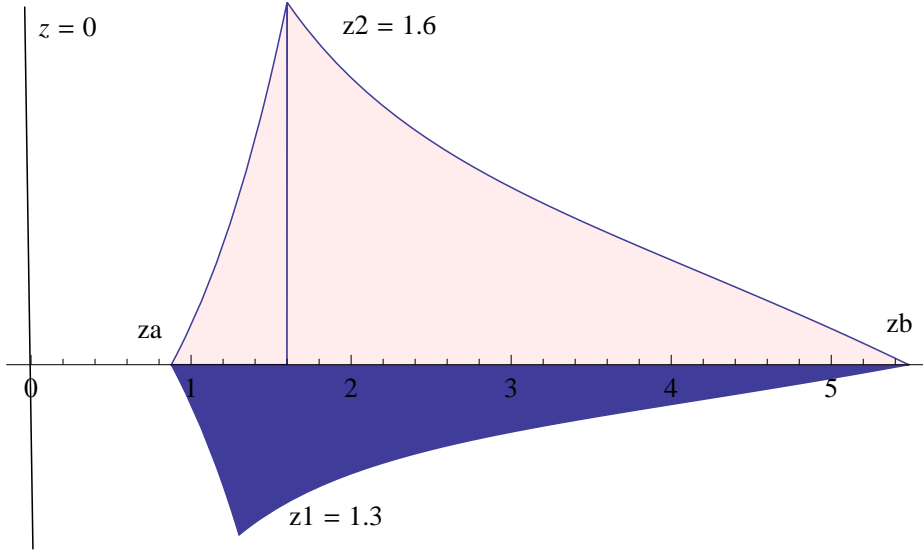


Figure 3: (color online) A view of the outer-most trapped surface formation in a wall-on-wall collision with two sources at different depths. The pink and blue area indicate the growth of the trapped surface Ψ in the bulk. The sources of the shock waves lie at $\frac{z_1}{L} = 1.3$ and $\frac{z_2}{L} = 1.6$, and the energy density is fixed by $\frac{(8\pi G_5 \mu)^2 (z_1 z_2)^3}{L^3} = 20$. The trapped surface at the collision point is bounded by z_a and z_b

distance, unlike the situation in point shock wave collision, where the trapped surface starts to grow even for infinitely separated shock waves. Furthermore, the origin of the critical condition in wall-on-wall collision is different from that of point shock wave collision : The latter can be understood as a constraint on angular momentum for the shock wave pair to form a AdS-Kerr black hole. The energy dependence of the critical impact parameter obeys an asymptotic power law, with the power extracted numerically in [11] to be 0.37, and argued by Gubser, Pufu and Yarom to be $1/3$ [10]. The power $1/3$ was confirmed later in a detailed numerical analysis[12]. The critical condition for wall-on-wall collision (22) has no analogy here, as the shock wave pair does not have an obvious angular momentum. In the next section, we will compute the NLO stress tensor in the dual field theory, which will help us to understand the origin of the critical condition.

4 The NLO stress tensor after the collision

The Einstein equation in the presence of the cosmological constant can be written as:

$$R_{\mu\nu} - 4g_{\mu\nu} = -8\pi G_5 S_{\mu\nu} \quad (27)$$

where $S_{\mu\nu} = J_{\mu\nu} - \frac{1}{3}Jg_{\mu\nu}$. We have set the AdS radius $L = 1$. Since the relevant scales are completely fixed by μ and z_i , we expect L will be absent in the final result of the stress tensor. The equation (27) to the first order in the amplitude of the shock wave is:

$$R_{\mu\nu}^{(1)} - 4g_{\mu\nu}^{(1)} = -8\pi G_5 S_{\mu\nu}^{(1)} = -8\pi G_5 (J_{\mu\nu}^{(1)} - \frac{1}{3}J^{(1)}g_{\mu\nu}^{(0)}) \quad (28)$$

where the upper index denotes the order of the quantity with respect to the amplitude of the shock wave. e.g. $J_{\mu\nu}^{(1)}$ is just the source of the shock waves, and $g_{\mu\nu}^{(0)}$ is the pure AdS metric. Before the collision the two moving shock waves are the solution to this equation.

The NLO metric involves the interaction of the shock waves, it appears after they pass through each other (which in the limit of very large rapidity of the shocks happen instantaneously.) The equation now takes the form:

$$R_{\mu\nu}^{(2)} + R_{\mu\nu}^{(1,1)} - 4g_{\mu\nu}^{(2)} = -8\pi G_5 S_{\mu\nu}^{(2)} = -8\pi G_5 (J_{\mu\nu}^{(2)} - J^{(2)}g_{\mu\nu}^{(0)} - J^{(1)}g_{\mu\nu}^{(1)}) \quad (29)$$

Our contracted source of order k is always defined as $J^{(k)} = J_{\mu\nu}^{(k)}g^{\mu\nu(0)}$ and the k -th order Ricci tensor $R_{\mu\nu}^{(k)}$ is due to k -th order metric correction only. In (29), $R_{\mu\nu}^{(1,1)}$ denotes the Ricci tensor quadratic in the first order metric (the unmodified shock wave), which is effectively the source due to the interaction of the shock wave fields. The contribution to the second order source $J_{\mu\nu}^{(2)}$ is due to the interaction between the source of one shock wave and the field of the other shock wave, i.e. the source of the shock wave deviates from its path in the presence of the other shock wave.

Since our wall shock wave has trivial dependence on transverse coordinates, the problem gets simplified a lot. The sources can still move in z , and the determination of their trajectory after the collision is subtle. In general it depends on the equation of state of the extended source itself. We will assume the action due to the shock wave source is of Nambu-Goto type, which is proportional to the invariant area of the extended source.

We can choose the worldvolume coordinates of the source of the shock wave as λ, x_1, x_2 , with λ to be specified later. The trajectory of the wall source can be specified by $x^+ = X^+(\lambda)$, $x^- = X^-(\lambda)$, $z = Z(\lambda)$. Suppose the covariant source $J^{\mu\nu}$ due to one the shock wave sources is parametrized by

$$J^{\mu\nu(2)} = \# \int d\lambda u^\mu u^\nu \delta(x^+ - X^+(\lambda)) \delta(x^- - X^-(\lambda)) \delta(z - Z(\lambda)) \quad (30)$$

with $w^\mu = \frac{dx^\mu}{d\lambda}$ $\mu = +, -, z$. $\#$ can be some function of x^+, x^-, z .

We work out this contribution in the appendix. The LO and NLO sources are given by:

$$\begin{aligned}
8\pi G_5 J_{++}^{(1)} &= -\frac{1}{2} \nabla^2 \phi_1 \\
8\pi G_5 J_{--}^{(1)} &= -\frac{1}{2} \nabla^2 \phi_2 \\
8\pi G_5 J_{++}^{(2)} &= \frac{1}{2} \left(\int \phi_2 dx^- \partial_+ \nabla^2 \phi_1 + \frac{1}{2} \int dx^- \int dx^- \partial_z \phi_2 \partial_z \nabla^2 \phi_1 \right) \\
8\pi G_5 J_{--}^{(2)} &= \frac{1}{2} \left(\int \phi_1 dx^+ \partial_- \nabla^2 \phi_2 + \frac{1}{2} \int dx^+ \int dx^+ \partial_z \phi_1 \partial_z \nabla^2 \phi_2 \right) \\
8\pi G_5 J_{+-}^{(2)} &= \frac{1}{2} (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla^2 \phi_1) \\
8\pi G_5 J_{+z}^{(2)} &= \frac{1}{2} \int \partial_z \phi_2 dx^- \nabla^2 \phi_1 \\
8\pi G_5 J_{-z}^{(2)} &= \frac{1}{2} \int \partial_z \phi_1 dx^- \nabla^2 \phi_2
\end{aligned} \tag{31}$$

We can check the following relations to the NLO:

$$(\nabla^\mu J_{\mu\nu})^{(2)} = \nabla^{\mu(0)} J_{\mu\nu}^{(2)} + \nabla^{\mu(1)} J_{\mu\nu}^{(1)} = 0 \tag{32}$$

$$(g^{\mu\nu} J_{\mu\nu})^{(2)} = g^{\mu\nu(0)} J_{\mu\nu}^{(2)} + g^{\mu\nu(1)} J_{\mu\nu}^{(1)} = 0 \tag{33}$$

The first one is the conservation of the source, which is a necessary condition for the consistency of Einstein equation. The second traceless condition allows us to simplify the RHS of (29). Moving The Ricci tensor quadratic in the first order metric $R_{\mu\nu}^{(1,1)}$ to the RHS and noting the tracelessness of $J^{(2)} = J^{(1)} = 0$, we obtain the reshuffled Einstein equation for the second order corrections only

$$R_{\mu\nu}^{(2)} - 4g_{\mu\nu}^{(2)} = -8\pi G_5 J_{\mu\nu}^{(2)} - R_{\mu\nu}^{(1,1)} = -8\pi G_5 \bar{J}_{\mu\nu}^{(2)} \tag{34}$$

where we have defined the effective source $\bar{J}_{\mu\nu}^{(2)} = J_{\mu\nu}^{(2)} + \frac{1}{8\pi G_5} R_{\mu\nu}^{(1,1)}$

One can now work out $R_{\mu\nu}^{(1,1)}$ for the case of wall shock waves and we obtain the effective source as:

$$\begin{aligned}
8\pi G_5 \bar{J}_{++}^{(2)} &= \frac{1}{2} \int dx^- \phi_2 \partial_+ \nabla^2 \phi_1 - \frac{1}{4} \int dx^- \int dx^- \partial_z \phi_2 \partial_z \nabla^2 \phi_1 \\
8\pi G_5 \bar{J}_{--}^{(2)} &= \frac{1}{2} \int dx^+ \phi_1 \partial_- \nabla^2 \phi_2 - \frac{1}{4} \int dx^+ \int dx^+ \partial_z \phi_1 \partial_z \nabla^2 \phi_2 \\
8\pi G_5 \bar{J}_{+-}^{(2)} &= \frac{1}{2} (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla^2 \phi_1) - \partial_+ \phi_1 \partial_- \phi_2 + \partial_z \phi_1 \partial_z \phi_2 - \frac{1}{z} (\phi_1 \partial_z \phi_2 + \phi_2 \partial_z \phi_1) \\
8\pi G_5 \bar{J}_{+z}^{(2)} &= \frac{1}{2} \int dx^- \partial_z \phi_2 \nabla^2 \phi_1 - \partial_+ \phi_1 \partial_z \phi_2 \\
8\pi G_5 \bar{J}_{-z}^{(2)} &= \frac{1}{2} \int dx^+ \partial_z \phi_1 \nabla^2 \phi_2 - \partial_- \phi_2 \partial_z \phi_1 \\
8\pi G_5 \bar{J}_{\perp\perp}^{(2)} &= \frac{2}{z} (\phi_1 \partial_z \phi_2 + \phi_2 \partial_z \phi_1) \\
8\pi G_5 \bar{J}_{zz}^{(2)} &= -2(\phi_2 \partial_z^2 \phi_1 + \phi_1 \partial_z^2 \phi_2) + \frac{2}{z} (\phi_2 \partial_z \phi_1 + \phi_1 \partial_z \phi_2) - 2\partial_z \phi_1 \partial_z \phi_2
\end{aligned}$$

From here on, we can use the method developed in [25, 26] to compute the stress tensor on the boundary field theory to the NLO. The procedure is to first obtain the reshuffled source s_{mn} defined as

$$s_{mn}^{(2)} = \bar{J}_{mn}^{(2)} - \int_0^z \left(\bar{J}_{zm,n}^{(2)} + \bar{J}_{zn,m}^{(2)} \right) dz + \frac{1}{2} h_{,m,n} + \frac{1}{2z} \eta_{mn} h_{,z} \quad (35)$$

with

$$h = \frac{1}{3} \int_0^z dz \cdot z \left(\bar{J}_{zz}^{(2)} - \eta^{mn} \bar{J}_{mn}^{(2)} + 2 \int_0^z dz \left(-\eta^{mn} \bar{J}_{zm,n}^{(2)} \right) \right) \quad (36)$$

In our particular case, h is given by:

$$8\pi G_5 h = -4\phi_1 \phi_2 + 4 \int dz \cdot z \int dz \frac{1}{z} \partial_z \phi_1 \partial_z \phi_2 \quad (37)$$

The reshuffled source takes the following form:

$$\begin{aligned}
8\pi G_5 s_{++} &= \frac{1}{2} \int dx^- \phi_2 \partial_+ \nabla^2 \phi_1 + \frac{1}{4} \int dx^- \int dx^- \partial_z \phi_2 \partial_z \nabla^2 \phi_1 \\
&\quad - \int dx^- \int dz \partial_z \phi_2 \partial_+ \nabla^2 \phi_1 - 2 \int dz \phi_2 \partial_z \partial_+^2 \phi_1 + 2 \int dz \cdot z \int dz \frac{1}{z} \partial_+^2 \partial_z \phi_1 \partial_z \phi_2 \\
8\pi G_5 s_{--} &= \frac{1}{2} \int dx^+ \phi_1 \partial_- \nabla^2 \phi_2 + \frac{1}{4} \int dx^+ \int dx^+ \partial_z \phi_1 \partial_z \nabla^2 \phi_2 \\
&\quad - \int dx^+ \int dz \partial_z \phi_1 \partial_- \nabla^2 \phi_2 - 2 \int dz \phi_1 \partial_z \partial_-^2 \phi_2 + 2 \int dz \cdot z \int dz \frac{1}{z} \partial_-^2 \partial_z \phi_2 \partial_z \phi_1 \\
8\pi G_5 s_{+-} &= \partial_z \phi_1 \partial_z \phi_2 - 2\partial_+ \phi_1 \partial_- \phi_2 + \frac{1}{2} \int dz (\phi_1 \partial_z \nabla^2 \phi_2 + \phi_2 \partial_z \nabla^2 \phi_1) \\
&\quad + 2 \int dz \cdot z \int dz \frac{1}{z} \partial_+ \partial_z \phi_1 \partial_- \partial_z \phi_2 - \int dz \frac{1}{z} \partial_z \phi_1 \partial_z \phi_2 \\
8\pi G_5 s_{\perp\perp} &= 2 \int dz \frac{1}{z} \partial_z \phi_1 \partial_z \phi_2
\end{aligned} \quad (38)$$

Separating the derivatives of x^+ , x^- and derivative of z , we obtain:

$$\begin{aligned}
8\pi G_5 s_{++} &= \frac{1}{2} \bar{\phi}_2 \nabla^2 \bar{\phi}_1 \theta(x^-) \delta'(x^+) + \frac{1}{4} \bar{\phi}_2' \nabla^2 \bar{\phi}_1' x^- \theta(x^-) \delta(x^+) - 2 \int dz \bar{\phi}_1' \bar{\phi}_2 \delta''(x^+) \delta(x^-) \\
&+ 2 \int dz \cdot z \int dz \frac{1}{z} \bar{\phi}_1' \bar{\phi}_2' \delta''(x^+) \delta(x^-) - \int dz \bar{\phi}_2' \nabla^2 \bar{\phi}_1 \theta(x^-) \delta'(x^+) \\
8\pi G_5 s_{--} &= \frac{1}{2} \bar{\phi}_1 \nabla^2 \bar{\phi}_2 \theta(x^+) \delta'(x^-) + \frac{1}{4} \bar{\phi}_1' \nabla^2 \bar{\phi}_2' x^+ \theta(x^+) \delta(x^-) - 2 \int dz \bar{\phi}_2' \bar{\phi}_1 \delta''(x^-) \delta(x^+) \\
&+ 2 \int dz \cdot z \int dz \frac{1}{z} \bar{\phi}_1' \bar{\phi}_2' \delta''(x^-) \delta(x^+) - \int dz \bar{\phi}_1' \nabla^2 \bar{\phi}_2 \theta(x^+) \delta'(x^-) \\
8\pi G_5 s_{+-} &= \frac{1}{2} \int dz (\bar{\phi}_1 \nabla^2 \bar{\phi}_2' + \bar{\phi}_2 \nabla^2 \bar{\phi}_1') \delta(x^+) \delta(x^-) + \bar{\phi}_1' \bar{\phi}_2' \delta(x^+) \delta(x^-) \\
&- 2 \bar{\phi}_1 \bar{\phi}_2 \delta'(x^+) \delta'(x^-) + 2 \int dz \cdot z \int dz \frac{1}{z} \bar{\phi}_1' \bar{\phi}_2' \delta'(x^+) \delta'(x^-) - \int dz \frac{1}{z} \bar{\phi}_1' \bar{\phi}_2' \delta(x^+) \delta(x^-) \\
8\pi G_5 s_{\perp\perp} &= 2 \int dz \frac{1}{z} \bar{\phi}_1' \bar{\phi}_2' \delta(x^+) \delta(x^-)
\end{aligned}$$

where $\phi_1(x^+, z) = \bar{\phi}_1(z) \delta(x^+)$ and $\phi_2(x^-, z) = \bar{\phi}_2(z) \delta(x^-)$. In (39), all primes are ordinary derivatives. The explicit forms of $\bar{\phi}_1$ and $\bar{\phi}_2$ are given by:

$$\bar{\phi}_1 = 4\pi G_5 \mu_1 \frac{z_1^4 z^4 - (z^4 - z_1^4) \theta(z - z_1)}{L^3 z_1^4} \quad (39)$$

$$\bar{\phi}_2 = 4\pi G_5 \mu_2 \frac{z_2^4 z^4 - (z^4 - z_2^4) \theta(z - z_2)}{L^3 z_2^4} \quad (40)$$

The reshuffled source (39) will be convoluted with a bulk to boundary propagator. Such propagator has been built in various applications of AdS/CFT, e.g.[27, 20, 26, 28] We will use a slightly different propagator from the above. The propagator takes the following form in the lightcone coordinates:

$$P_R = \frac{\theta(x^+ - x^{+'} + x^- - x^{-'})}{2\pi} \left[\frac{\delta'''(z - w)^{1/2}}{8w^3} + \frac{3\delta''(z - w)}{8w^4} + \frac{3\delta'(z - w)}{8w^5} \right] \quad (41)$$

with $w = \sqrt{(x^+ - x^{+'})(x^- - x^{-'}) - (\vec{x}_\perp - \vec{x}'_\perp)^2}$.

The details of the propagator are included in the appendix. Since we are dealing with wall sources, which do not depend on x'_\perp , we can perform the integral with respect to the transverse coordinate x'_\perp . By repeated use of integration by parts, we end up with a concise form:

$$\begin{aligned}
\int d^2 x'_\perp P_R &= 2\pi \int_0^\infty dx'_\perp \cdot x'_\perp P_R = \theta(x^+ - x^{+'} + x^- - x^{-'}) \times \\
&\left[\frac{\delta'(z - \sqrt{(x^+ - x^{+'})(x^- - x^{-'})})}{8(x^+ - x^{+'})(x^- - x^{-'})^{3/2}} + \frac{\delta''(z - \sqrt{(x^+ - x^{+'})(x^- - x^{-'})})}{8(x^+ - x^{+'})(x^- - x^{-'})} \right] \quad (42)
\end{aligned}$$

The final task is to convolute the source (39) with the integrated propagator (42). Due to the presence of the delta function, the integration in z is trivial. We are only left with integration of x^+ and x^- . Completing the integrals, we obtain as the final results:

$$\begin{aligned}
T_{++}^{NLO} &= \frac{8\pi^2 G_5 \mu_1 \mu_2}{N_c^2} \left[-x^{-2} \theta(z_2 - \tau) \theta(\tau) + \frac{x^- z_2^3}{2x^+} \delta(z_2 - \tau) \right] \\
T_{--}^{NLO} &= \frac{8\pi^2 G_5 \mu_1 \mu_2}{N_c^2} \left[-x^{+2} \theta(z_2 - \tau) \theta(\tau) + \frac{x^+ z_2^3}{2x^-} \delta(z_2 - \tau) \right] \\
T_{+-}^{NLO} &= \frac{1}{2} T_{\perp\perp}^{NLO} = \frac{8\pi^2 G_5 \mu_1 \mu_2}{N_c^2} \left[2\tau^2 \theta(z_2 - \tau) \theta(\tau) - \frac{z_2^3}{2} \delta(z_2 - \tau) \right] \quad (43)
\end{aligned}$$

where $\tau = \sqrt{x^+ x^-}$ is the proper time. This is the main result of this chapter.

(Few technical comments on the derivation: We have also used the distributional relations in the final results $\delta^{(n)}(x) f(x) = (-1)^n f^{(n)}(x) \delta(x)$. In doing this, we have treated τ and $\frac{x^+}{x^-}$ as separate variables. It is however necessary to keep in mind one subtlety. We have assumed the source has a series expansion near the boundary $z = 0$, in the derivation of the propagator. As our source contains delta functions and Heaviside theta functions, (43) is obtained with a particular representation of them and the limit is taken in the final results.)

Several more general comments on the result, the NLO stress tensor (43), are in order:

i) The NLO stress tensor is conserved and traceless $\partial^m T_{mn}^{NLO} = 0$, $\eta^{mn} T_{mn}^{NLO} = 0$. The presence of the delta function is necessary for the conservation relation. As conjectured in [24], The limit $z_2 \rightarrow \infty$ of our results should recover the NLO stress tensor in the collision of sourceless shock wave [19, 20]. We can see it is indeed the case as $\theta(z_2 - \tau) = 1$, $\delta(z_2 - \tau) = 0$.

ii) (43) is essentially boost invariant. This is a special property of the NLO stress tensor, which is symmetric with respect to the exchange of the two shock waves. Since we are colliding asymmetric nucleus, we expect higher order correction will violate boost invariance.

iii) It is interesting to note that the NLO stress tensor does not depend on z_1 . It suggests the NLO stress tensor for collision of two nucleus with different saturation scales does not feel the softer saturation scale $\frac{1}{z_1}$.

iv) The appearance of the Heaviside theta function is of particular interest. It encodes information on thermalization. As the LO stress tensor $T_{++}^{LO} = \mu_1 \delta(x^+)$, $T_{--}^{LO} = \mu_2 \delta(x^-)$ has a simple interpretation as nucleus moving on the lightcone. The NLO stress tensor (43) tells us matter created in the collision is only nonvanishing when $0 < \tau < z_2$. At time $t > z_2$, matter created in the collision separates into two pieces $z_2 < x_3 < t$ and $-t < x_3 < -z_2$.

Presumably higher order correction is needed to fill the gap. This also suggest the NLO result is insufficient to provide an initial condition for hydrodynamics.

v) Comparing the normalization of the delta functions in the LO and NLO stress tensor, we conclude the perturbation should break down when $\mu \lesssim 8\pi G_5 \mu^2 z_2^3$, which is precisely the critical condition (25).

Therefore the field theory interpretation of the thermalization condition is understood as the breaking down of perturbative treatment. Presumably the combined effect of all the gravitons should be included in further evolution of the trapped surface, from its position at time zero discussed at the beginning of the paper.

Alternatively, we can take a bulk point of view: The perturbation breaks down when the sources of the shock wave, originally moving at constant radial position, deviate significantly in the radial direction. Specially, we have worked this out in the Appendix A. With $\mu_1 = \mu_2 = \mu$, the sources of shock wave gain velocities

$$\begin{aligned} u_1^z &= \int dx^- \frac{\partial_z \phi_2}{2} \Big|_{z=z_1} = 0 \\ u_2^z &= \int dx^+ \frac{\partial_z \phi_1}{2} \Big|_{z=z_2} = \frac{8\pi^2 \mu}{N_c^2} z_2^3 \theta(x^+) \end{aligned} \quad (44)$$

after the collision. Due to the special profile of the shock waves, the source deeper in the bulk does not shift its path in the NLO computation. The perturbation breaks down when $u_2^z \lesssim 1$, which again is consistent with the critical condition (25) when $\frac{z_1}{z_2} \gg 1$.

The improved understanding of the critical condition leads to the following prediction: In the collision of two nucleus with the same energy density but different saturation scale, the thermalization condition is insensitive to the softer saturation scale. The energy density has to exceed certain critical value set by the harder saturation scale as (25) in order to reach thermalization.

5 Discussion

In this work, we have constructed the trapped surface in a wall-on-wall collision, which is used to model collisions of nucleus with different saturation scales. We have derived a critical condition for matter equilibration in nucleus collisions. The condition (22) is set by the saturation scales of both nucleus. When critical energy scales as the ratio of the saturation scales approximately by a power law, with the power 3/2. The approximate power law indicates the critical energy is insensitive to the softer saturation scale. We have

also observed a non-uniqueness of the trapped surface when the energy density is beyond the critical value. The outer-most trapped surface is selected for an estimate of the entropy production.

We have computed the NLO stress tensor on the boundary. The result turns out to be independent on the soft saturation scale $1/z_1$. Based on the NLO results, we propose the critical condition corresponds to the breaking down of the perturbation, i.e. when the LO and NLO correction become comparable. The criterion reproduces the critical condition (25). On the other hand, the critical condition is also understood in terms of bulk physics. The breaking down of the perturbation is encoded in the condition when the sources of the shock wave gain significant deviation in its velocity after the collision. This also leads to the correct critical condition (25). While in the NLO, no dependence on z_1 is observed, it must show up beyond NLO, as the source deeper in the bulk will also deviate from its original path, giving rise to correction to (25). It is tempting to see how this shows up in higher order computation.

Finally we stress the physics of critical condition for matter equilibration is very different from the counterpart in the collision of point source shock wave. In terms of the gravity dual, the latter originates from the constraint on the angular momentum possessed by a pair of black holes in order for the merging to be possible. The critical condition for wall shock wave collision can be understood as the breaking down of the perturbative calculation. In the dual field theory, it manifests as a constraint on the collision energy for given parton saturation scale.

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A NLO source in shock wave collision

A.1 Point source shock wave in AdS_3

It is helpful to look at collision of point shock wave in AdS_3 first. The LO metric is given by:

$$ds^2 = -\frac{-dx^+dx^- + dz^2 + \phi_1 dx^{+2} + \phi_2 dx^{-2}}{z^2} \quad (45)$$

The shock wave profiles ϕ_1 and ϕ_2 are normalized as:

$$\nabla^2 \phi_1 = -16\pi G_5 \delta(x^+) \delta(z - z_1) \quad (46)$$

$$\nabla^2 \phi_2 = -16\pi G_5 \delta(x^-) \delta(z - z_2) \quad (47)$$

where $\nabla^2 = \partial_z^2 - \frac{1}{z}\partial_z$ is a Laplacian operator. The NLO source arises from the deviation of the path of one shock wave source in the presence of the other. For point source, the null geodesic equation is given by:

$$\frac{du^\mu}{d\lambda} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta = 0 \quad (48)$$

For source of shock wave 1 before the collision, x^- can be chosen as the affine parameter λ , thus $u^- = 1$, $u^+ = u^z = 0$. After the collision, the geodesic to the first order in the shock wave amplitude is as follows:

$$\frac{du^+}{d\lambda} = \partial_- \phi_2 \quad (49)$$

$$\frac{du^z}{d\lambda} = \frac{1}{z}u^+ + \frac{z^2}{2}\partial_z\left(\frac{\phi_2}{z^2}\right) \quad (50)$$

$$\frac{du^-}{d\lambda} = \frac{2}{z}u^z \quad (51)$$

Assuming $\lambda = x^-$ holds after the collision, we find from (49) and (50) that $u^+ = \phi_2$ and $u^z = \int dx^- \frac{\partial_z \phi_2}{2}$. However we see it contradicts (51) as $\frac{du^-}{dx^-} = 0$. This indicates that x^- is no longer a good affine parameter, but to the order we are interested, $u^+ = \phi_2$ and $u^z = \int \frac{\partial_z \phi_2}{2} dx^-$ remains valid, as correction will be of higher order. Integrating once, we further obtain $x^+ = \int dx^- \phi_2$ and $z = z_1 + \int dx^- \int dx^- \frac{\partial_z \phi_2}{2}$.

The covariant source due to shock wave 1 has the general form:

$$J^{\mu\nu} = \# u^\mu u^\nu \delta(x^+ - X^+(x^-)) \delta(z - Z(x^-)) \quad (52)$$

where $X^+(x^-)$ and $Z(x^-)$ specifies the trajectory of the point source. $\#$ can be some function of x^+ , x^- and z . Writing the LO covariant stress tensor is simply:

$$8\pi G_5 J^{--(1)} = -2z^4 \nabla^2 \phi_1 \quad (53)$$

$$8\pi G_5 J^{++(1)} = -2z^2 \nabla^2 \phi_2 \quad (54)$$

The NLO source comes from the correction to u^μ and x^μ . Adding the contributions from two shock waves, we obtain:

$$\begin{aligned} 8\pi G_5 J^{--(2)} &= 2z^4 \left(\int dx^- \phi_2 \partial_+ \nabla^2 \phi_1 + \frac{1}{2} \int dx^- \int dx^- \partial_z \phi_2 \partial_z \nabla^2 \phi_1 \right) \\ 8\pi G_5 J^{++(2)} &= 2z^4 \left(\int dx^+ \phi_1 \partial_- \nabla^2 \phi_2 + \frac{1}{2} \int dx^+ \int dx^+ \partial_z \phi_1 \partial_z \nabla^2 \phi_2 \right) \\ 8\pi G_5 J^{+- (2)} &= -2z^4 (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla^2 \phi_1) \\ 8\pi G_5 J^{-z(2)} &= -2z^4 \cdot \frac{1}{2} \int dx^- \partial_z \phi_2 \nabla^2 \phi_1 \\ 8\pi G_5 J^{+z(2)} &= -2z^4 \cdot \frac{1}{2} \int dx^+ \partial_z \phi_1 \nabla^2 \phi_2 \end{aligned} \quad (55)$$

The conservation of the source to the second order can be checked $(\nabla_\mu T^{\mu\nu})^{(2)} = \partial_\mu T^{\mu\nu(2)} + \Gamma_{\mu\lambda}^{\mu(0)} T^{\lambda\nu(2)} + \Gamma_{\mu\lambda}^{\nu(0)} T^{\mu\lambda(2)} + \Gamma_{\mu\lambda}^{\mu(1)} T^{\lambda\nu(1)} + \Gamma_{\mu\lambda}^{\nu(1)} T^{\mu\lambda(1)} = 0$.

A.2 Wall source shock wave in AdS_5

Now we look at wall shock wave in AdS_5 . The LO metric is given by:

$$ds^2 = \frac{-dx^+ dx^- + dx_\perp^2 + dz^2 + \phi_1 dx^{+2} + \phi_2 dx^{-2}}{z^2} \quad (56)$$

The shock wave profiles ϕ_1 and ϕ_2 are normalized as

$$\nabla^2 \phi_1 = -16\pi G_5 \delta(x^+) \delta(z - z_1) \quad (57)$$

$$\nabla^2 \phi_2 = -16\pi G_5 \delta(x^-) \delta(z - z_2) \quad (58)$$

The Laplacian operator becomes $\nabla = \partial_z^2 - \frac{3}{z} \partial_z$ due to the additional transverse directions. Being different from the point source, the trajectory of the source is specified by $X^\mu(\sigma)$ with σ the worldvolume parameters. The induced metric is given by:

$$h_{\alpha\beta} = \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\nu}{\partial \sigma^\beta} g_{\mu\nu} = \begin{pmatrix} \frac{-u^+ u^- + u^{z^2}}{z^2} & \\ & \frac{1}{z^2} \\ & & \frac{1}{z^2} \end{pmatrix} \quad (59)$$

with $u^\pm = \frac{dx^\pm}{d\lambda}$ and $u^z = \frac{dz}{d\lambda}$. Assuming the action of the shock wave depends on $\det h$ only, then the trajectory can be effectively determined by considering a point source in the metric

$$ds^2 = \frac{-dx^+ dx^- + dz^2 + \phi_1 dx^{+2} + \phi_2 dx^{-2}}{z^6} \quad (60)$$

Working out the geodesic deviation, we find surprisingly that the trajectory of the wall source is the same as point source in AdS_3 . As a result, the LO and NLO source are given by:

$$8\pi G_5 J^{--(1)} = -2z^4 \nabla^2 \phi_1 \quad (61)$$

$$8\pi G_5 J^{++(1)} = -2z^2 \nabla^2 \phi_2 \quad (62)$$

$$8\pi G_5 J^{--(2)} = 2z^4 \left(\int dx^- \phi_2 \partial_+ \nabla^2 \phi_1 + \frac{1}{2} \int dx^- \int dx^- \partial_z \phi_2 \partial_z \nabla^2 \phi_1 \right) \quad (63)$$

$$8\pi G_5 J^{++(2)} = 2z^4 \left(\int dx^+ \phi_1 \partial_- \nabla^2 \phi_2 + \frac{1}{2} \int dx^+ \int dx^+ \partial_z \phi_1 \partial_z \nabla^2 \phi_2 \right) \quad (64)$$

$$8\pi G_5 J^{+- (2)} = -2z^4 (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla^2 \phi_1) \quad (65)$$

$$8\pi G_5 J^{-z(2)} = -2z^4 \cdot \frac{1}{2} \int dx^- \partial_z \phi_2 \nabla^2 \phi_1 \quad (66)$$

$$8\pi G_5 J^{+z(2)} = -2z^4 \cdot \frac{1}{2} \int dx^+ \partial_z \phi_1 \nabla^2 \phi_2 \quad (67)$$

While (61) has the same functional form as (55), they are different in the Laplacian operator. We can check (61) is again conserved and the Christoffels involving the additional directions are accounted for the difference in the Laplacian operators.

With some care, we can obtain the NLO contravariant source, which include contribution from both LO and NLO covariant sources. The result is shown in (31) in the main text.

B The bulk to boundary propagator

In this appendix, we want to build a propagator, which produces the stress tensor on the boundary field theory when convoluted with the bulk source. We start with a bulk to bulk propagator for massive scalar defined as follows:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) G - m^2 G = \frac{1}{\sqrt{-g}} \delta^{(d+1)}(x - x') \quad (68)$$

The metric is the Poincare patch of AdS_{d+1} . Using the Fourier transform: $\tilde{G}(\omega, k, z) = \int G(t, x, z) e^{-i\omega(t-t') + i\vec{k}(\vec{x}-\vec{x}'')}$, (68) takes the following explicit form:

$$z^2(\omega^2 - k^2)\tilde{G} + z^2\partial_z^2\tilde{G} + (1-d)z\partial_z\tilde{G} = m^2\tilde{G} = z'^{d+1}\delta(z - z') \quad (69)$$

The boundary condition to impose is that $\tilde{G} \rightarrow 0$ as $z \rightarrow 0$ and \tilde{G} is outgoing as $z \rightarrow \infty$. The solution to (69) is found to be

$$\tilde{G} = -(zz')^{\frac{d}{2}} I_{\Delta}(\sqrt{k^2 - \omega^2}z_{<}) K_{\Delta}(\sqrt{k^2 - \omega^2}z_{>}) \quad (70)$$

where $\Delta = \frac{\sqrt{d^2+4m^2}}{2}$ and $z_{>} = \max\{z, z'\}$, $z_{<} = \min\{z, z'\}$.

The inverse Fourier transform gives the bulk to bulk propagator:

$$G(t, x, z) = -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^d} \int I_{\Delta}(\sqrt{k^2 - \omega^2}z_{<}) K_{\Delta}(\sqrt{k^2 - \omega^2}z_{>}) e^{i\omega(t-t') - i\vec{k}(\vec{x}-\vec{x}'')} d\omega d^{d-1}k \quad (71)$$

Note there are two branch cuts on the real axis $(-\infty, -k)$ and (k, ∞) . The retarded propagator can be obtained if we take the integration contour of ω slightly below the real axis: $\omega \rightarrow \omega - i\epsilon$. We can push the integration contour to wrap around the two branch cuts, so that all the contributions come from two sides of the branch cuts.

$$\begin{aligned} G(t, x, z, z') &= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^d} \theta(t-t') \left(\int_{-\infty}^{-k} d\omega + \int_k^{\infty} d\omega \right) \int d^{d-1}k e^{i\omega(t-t') - i\vec{k}(\vec{x}-\vec{x}'')} \times \\ &\left[K_{\Delta}(-i\sqrt{k^2 - \omega^2}z_{>}) I_{\Delta}(-i\sqrt{\omega^2 - k^2}z_{<}) - K_{\Delta}(i\sqrt{k^2 - \omega^2}z_{>}) I_{\Delta}(i\sqrt{\omega^2 - k^2}z_{<}) \right] \\ &= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^d} \theta(t-t') \int_k^{\infty} d\omega \int d^{d-1}k 2\pi J_{\Delta}(\sqrt{\omega^2 - k^2}z_{>}) J_{\Delta}(\sqrt{\omega^2 - k^2}z_{<}) \times \\ &\sin \omega(t-t') e^{-i\vec{k}(\vec{x}-\vec{x}'')} \end{aligned} \quad (72)$$

Doing the angular integration for the spatial momentum k , we obtain:

$$\begin{aligned}
G(t, x, z, z') &= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}\theta(t-t')\int_k^\infty d\omega\int k^{d-2}dkd\Omega^{d-2}\sin\omega(t-t')e^{ikr\cos\theta}\times \\
&J_\Delta(\sqrt{\omega^2-k^2}z_>)J_\Delta(\sqrt{\omega^2-k^2}z_<) \\
&= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}\theta(t-t')\int_k^\infty d\omega\int k^{d-2}dkd\theta(\sin\theta)^{d-3}d\Omega^{d-2}\sin\omega(t-t')e^{ikr\cos\theta}\times \\
&J_\Delta(\sqrt{\omega^2-k^2}z_>)J_\Delta(\sqrt{\omega^2-k^2}z_<) \\
&= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}\frac{2^{\frac{d-1}{2}}\pi^{\frac{d-1}{2}}}{r^{\frac{d-3}{2}}}\theta(t-t'-r)\int_k^\infty d\omega\int dk k^{\frac{d-1}{2}}J_{\frac{d-3}{2}}(kr)\sin\omega(t-t')\times \\
&J_\Delta(\sqrt{\omega^2-k^2}z_>)J_\Delta(\sqrt{\omega^2-k^2}z_<) \tag{73}
\end{aligned}$$

We have defined $r = |x - x'|$. Writing $\beta = \sqrt{\omega^2 - k^2}$ allows us to do the k -integral[30]:

$$\begin{aligned}
G(t, x, z, z') &= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}\frac{2^{\frac{d-1}{2}}\pi^{\frac{d-1}{2}}}{r^{\frac{d-3}{2}}}\theta(t-t')\int\frac{\sin\sqrt{\beta^2+k^2}(t-t')}{\sqrt{\beta^2+k^2}}\beta d\beta k^{\frac{d-1}{2}}dk\times \\
&J_\Delta(\beta z_>)J_\Delta(\beta z_<)J_{\frac{d-3}{2}}(kr) \\
&= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}2^{\frac{d-2}{2}}\pi^{\frac{d}{2}}\theta(t-t'-r)\int J_\Delta(\beta z_>)J_\Delta(\beta z_<)\beta^{\frac{d}{2}}J_{-\frac{d-2}{2}}(\beta w)w^{-\frac{d-2}{2}} \tag{74}
\end{aligned}$$

where $w = \sqrt{(t-t')^2 - r^2}$. The final integration of β can also be done[30], we end up with

$$\begin{aligned}
G(t-t', x-x', z, z') &= -\frac{(zz')^{\frac{d}{2}}}{(2\pi)^{d-1}}2^{\frac{d-2}{2}}\pi^{\frac{d}{2}}\theta(t-t'-r)\times \\
&\begin{cases} \sqrt{\frac{2}{\pi^3}}(zz')^{-\frac{d}{2}}(\sinh u)^{-\frac{d-1}{2}}\sin\left[-\frac{d}{2}+1-\Delta\right]\pi e^{-i\frac{d-1}{2}\pi}Q_{\Delta-\frac{1}{2}}^{\frac{d-1}{2}}(\cosh u) & w > z_> + z_< \\ \frac{1}{\sqrt{2\pi}}(zz')^{-\frac{d}{2}}(\sin v)^{-\frac{d-1}{2}}P_{\Delta-\frac{1}{2}}^{\frac{d-1}{2}}(\cos v) & z_> - z_< < w < z_> + z_< \\ 0 & \text{otherwise} \end{cases} \tag{75}
\end{aligned}$$

(75) is in agreement with early results on bulk to bulk propagator [29, 23]. However there is a non-analyticity at $\sqrt{t^2 - r^2} = z_> + z_<$, which is missing in (75). We choose to start with (74) in building the bulk to boundary propagator.

The relevant Green's function $G^b(t-t', x-x', z, z')$ is given by:

$$\frac{z^2}{2}(-\partial_t^2 + \partial_x^2 + \partial_z^2)G^b + \frac{z}{2}\partial_z G^b - 4G^b = \delta(z-z')\delta^d(x-x') \tag{76}$$

The metric perturbation in the axial gauge h_{mn} is related to the reshuffled source s_{mn} by:

$$h(t, x, z) = \int dz' dt' d^3 x' s(t', x', z') G^b(t - t', x - x', z, z') \quad (77)$$

We have suppressed the tensor indices in h_{mn} and s_{mn} . G^b is related to the bulk to bulk propagator by:

$$G^b = \frac{2}{z^2 z'^3} G|_{\Delta=2, d=4} \quad (78)$$

Let us suppose the source adopts the following expansion near the boundary.

$$s(t', x', z') = \sum_n s_n(t', x') z'^n \quad (79)$$

We can perform the integrations first in z' and then in β to obtain:

$$h(t, x, z) = -\frac{1}{2\pi} \int dt' d^3 x' \sum_n w^{n-6} z^2 \frac{n(n-2)(n-4)}{8} F\left(\frac{1-n}{2}, \frac{3-n}{2}; 3; \frac{z^2}{w^2}\right) s_n(t', x') \quad (80)$$

We are interested in the coefficient of z^2 , which encodes the boundary stress tensor. Note $\lim_{z \rightarrow 0} F\left(\frac{1-n}{2}, \frac{3-n}{2}; 3; \frac{z^2}{w^2}\right) \rightarrow 1$. The coefficient is given by:

$$\begin{aligned} & -\frac{1}{2\pi} \int dt' d^3 x' \sum_n w^{n-6} \frac{n(n-2)(n-4)}{8} s_n(t', x') \\ &= -\frac{1}{2\pi} \int dt' d^3 x' \sum_n w^{n-6} \frac{n(n-2)(n-4)}{8} \frac{1}{n!} \partial_z^n s(t', x', z')|_{z'=0} \\ &= -\frac{1}{2\pi} \int dt' d^3 x' dz' \sum_n w^{n-6} \frac{n(n-2)(n-4)}{8} \frac{(-1)^n}{n!} s(t', x', z') \delta^{(n)}(z') \quad (81) \end{aligned}$$

We can sum the n -series and obtain as our bulk to boundary propagator

$$\begin{aligned} P^R &= -\frac{\theta(t-t' - |x-x'|)}{2\pi} \sum_n w^{n-6} \frac{n(n-2)(n-4)}{8} \frac{(-1)^n}{n!} \delta^{(n)}(z') \\ &= -\frac{\theta(t-t' - |x-x'|)}{2\pi} \sum_n w^{n-6} \frac{n(n-1)(n-2) - 3n(n-1) + 3n}{8} \frac{(-1)^n}{n!} \delta^{(n)}(z') \\ &= \frac{\theta(t-t' - |x-x'|)}{2\pi} \left[\frac{w^{-3}}{8} \delta'''(z' - w) + \frac{3w^{-4}}{8} \delta''(z' - w) + \frac{3w^{-5}}{8} \delta'(z' - w) \right] \quad (82) \end{aligned}$$

We can further use the property of delta function to replace $\theta(t - t' - |x - x'|)$ by $\theta(t - t')$:

$$P^R = \frac{\theta(t - t')}{2\pi} \left[-\frac{w^{-3}}{8} \delta'''(z' - w) - \frac{3w^{-4}}{8} \delta''(z' - w) - \frac{3w^{-5}}{8} \delta'(z' - w) \right] \quad (83)$$

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