

# Efimov Trimers : Analytic Solution Via Separable Potentials

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## Abstract

The exact dynamical equation for Efimov trimers in the short range limit of dimer resonances is derived via Yamaguchi separable potentials. This equation which overcomes the non-uniqueness problem of zero-range potentials, closely resembles one derived recently by Gogolin et al, with no further assumptions, and hence enjoys all the benefits of the latter results.

Keywords: Efimov states, separable potentials, dimer resonances, effective range, short-range limit.

## 1 Introduction

Although Efimov states were proposed nearly four decades ago [1], they have come into prominence only in recent times following the experimental detection of a possible candidate in ultra cold atoms [2]. On the other hand, their mathematical background dates back even earlier through the work of Danilov [2], on the basis of a seminal paper by Skorniakov and Ter-Martirosian [3] for the 3-body bound states of as many identical bosons, under conditions of near resonance for dimers with "zero range" forces. In view of the obvious importance of such states, their theoretical foundations too have been the subject of renewed scrutiny from different angles. In particular, Gogolin et al [5](GME), following a method of Jona-Lasino et al [6], have found an interesting analytical solution to the spectrum of Efimov

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states for 3 identical bosons, using the language of creation and annihilation operators in field theory. In the process they have also confirmed Danilov's observation [2] on the inadequacy of the zero range approximation [3] to produce a unique Efimov spectrum without additional input (as pointed out by Gribov [2]), and remedied this defect with a certain parameter present in their formalism which comes nearest to the more familiar *effective range* theory in nuclear physics.

In this paper we attempt an elementary derivation of the Efimov effect, in the more orthodox language of nuclear physics via quantum mechanics, using Yamaguchi separable potentials [7] which had once been employed for the solution of the bound state 3-body problem [8]. The reason for this old-fashioned choice is that the system under study strictly conserves the number of particles (no pair creation / annihilation effects!) which not only allows one to dispense with the more advanced language of field theory but also has a built-in "effective range" feature which automatically incorporates an additional length dimension on this account (the  $R^*$  term of GME [5]), without the need for a fresh insertion by hand. And since the Yamaguchi separable method was originally conceived in momentum space for the 3-body problem[8], the *single particle* structure of the GME [5] equations in momentum space is automatically reproduced without extra efforts. As an extra bonus, we shall also find a detailed correspondence of the Yamaguchi formalism [7] to its GME counterpart [5]. In the next section we recall the basic results of the 2- and 3-body wave functions for identical particles under pairwise separable potentials [8,9],and show how they automatically reduce to the GME equations [5], and in so doing, overcome the non-uniqueness problem of the zero-range approximation [2,3].

## 2 1D Trimers via Separable Potentials

We now give in barest outline a 3-body formalism [8] with *separable* potentials [7], using 3 identical bosons with no internal d.o.f.s., in a more or less similar notation to the GME formalism [5]. To that end we start with the essentials of a 2-body system directly in momentum ( $p$ ) space, as originally employed [8], where a positive coupling constant  $\lambda$  indicates an attractive interaction, and the units are such that  $m = \hbar = 1$ .

## 2.1 The 2-body amplitude on- and off- shell

The wave function  $\psi$  for a 2-body scattering state in the c.m. frame satisfies the S.equation [7,8]

$$(p_1^2 + p_2^2 - E)\psi(q) = \frac{\lambda}{(2\pi)^3} \int d^3q' g(q)g(q')\psi(q'); \quad E = k^2 + i\epsilon; \quad 2q = p_1 - p_2 \quad (2.1)$$

leading to the off-shell scattering amplitude defined by

$$\psi_k(q) = \delta^3(\mathbf{q} - \mathbf{k}) + \frac{a(q, k)}{(2\pi)^3(q^2 - k^2 - i\epsilon)}$$

which works out as

$$\begin{aligned} a(q, k) &= \frac{g(p)g(k)}{4\pi(\lambda^{-1} - h(k))} \\ h(k) &= \frac{1}{(2\pi)^3} \int d^3q g^2(q)/(q^2 - k^2 - i\epsilon) \end{aligned} \quad (2.2)$$

where  $k^2 > 0$  and the function  $g(q)$  is taken as  $1/(\beta^2 + q^2)$  [7]. Then on integration over  $q$  the on-shell scattering amplitude  $a(k, k) = \sin \delta e^{i\delta}/k$  works out as an "exact" effective range formula

$$k \cot \delta = 4\pi\lambda^{-1}(\beta^2 + k^2)^2 - \beta/2 + \frac{k^2}{2\beta} \equiv -1/a + \frac{1}{2}r_0k^2 \quad (2.3)$$

from which the set  $\lambda; \beta$  is expressible in terms of  $1/a; r_0$  or vice versa. This result is ready for employment in the corresponding trimer equations that follow next.

## 2.2 3-Body Equation in one-dimensional form

The trimer wave function  $\Psi$  satisfies an S.equation of the form[8,9]

$$D(E)\Psi(q, p) = \sum_i \lambda g(q_i) \int dq'_i \Psi(q'_i, p_i) d^3q'_i \quad D(E) \equiv p_1^2 + p_2^2 + p_3^2 - E \quad (2.4)$$

where  $q_1 \equiv (p_2 - p_3)/2$ , and cyclical permutations. This equation yields an effectively 2-body structure of the form [8,9]

$$D(E)\Psi(q, p) \equiv \sum_i g(q_i)\phi(p_i); \quad D(E) = q_i^2 + 3p_i^2/4 - E \quad (2.5)$$

where  $\phi$  is a one-dimensional function of the indicated arguments. On plugging (2.5) back into the basic form (2.4), one arrives at an explicit equation for the  $\phi$  function :

$$[\lambda^{-1} - h(k_1)]\phi(p) = "2" \int d^3 p' g(p' + p/2)g(p + p'/2)\phi(p') \quad (2.6)$$

where the integration variable on the rhs has been so adjusted as to make the argument of the  $\phi$ -function equal to the integration variable, and the factor of "2" in front indicates that the two "cross" terms give equal contributions. And  $h(k_1)$  is the same function of  $k_1 = \sqrt{-3p^2/4 + E}$  as  $h(k)$  is of  $k = \sqrt{E}$  in eq (2.2), except that for the bound trimer state,  $E = -\alpha^2 < 0$ .

Before proceeding further, a few words of comparison of (2.6) with eq. (4) of GME [5] are in order. The function  $\phi(p)$  (which had been named the "spectator function" in ref [8]), corresponds to the symbol  $\beta_K$  of GME, which may also be called the relative wave function of a particle of momentum  $p$  wrt the pair of the other two. As to the significance of the various terms in (2.6), the term  $h(k_1)\phi(p)$  on the lhs of our (2.6) has a precise counterpart in the term  $AK, k$  of the GME eq(4); and the integral term on the rhs of (2.6) matches, (complete with the factor of "2"), with the GME term  $2Ak - K/2, -k/2 - 3K/4$ , where the arguments correspond to those of our  $g$ - functions on the rhs of (2.6) *before* the shifts in the integration variables were effected. Finally the role of the length scale  $R^*$  in GME [5], is automatically incorporated in the Yamaguchi form factor  $g(q)$  [7] inasmuch as the exact effective range formula (2.3) goes beyond the scattering length formalism of ref [3] which in turn had led Danilov [2] to the conclusion of non-uniqueness of that approximation. As we shall see below, Eq (7) of GME is fully reproduced when (2.6) is subjected to the short range limit, which also generates a correction term that plays the role of  $R^*$  [5].

### 3 GME Equation: Result and Conclusion

To obtain the short range limit of Eq(2.6), one needs the following steps. First, Eq(2.3) yields, up to terms of order  $k^2$ , the "effective range" relation

$$\begin{aligned} \frac{4\pi\beta^4}{\lambda} &= -\frac{1}{a} - k^2 R_0 + \beta/2 \\ -R_0 &= r_0/2 - 3/2\beta + 2/a\beta^2 \end{aligned} \quad (3.1)$$

which expresses  $\lambda^{-1}$  in terms of  $a; R_0$ . Next, this expression for  $\lambda^{-1}$  must be substituted in (2.6) carefully enough so that the short range limit  $\beta \rightarrow \infty$  emerges smoothly. To that end, the term  $\beta/2$  in (3.1) which is large in this limit, must get cancelled by a corresponding term in the function  $h(k_1)$ , so that it makes sense to combine these terms together before other operations are performed. The result of this step is to produce a combination which yields simply

$$\beta/2 - 4\pi\beta^4 h(k_1) = \int \frac{d^3\beta^4 g^2(q)(3p^2/4 + \alpha^2)}{(2\pi^2 q^2(q^2 + 3p^2/4))} \rightarrow \sqrt{3p^2/4 + \alpha^2} \quad (3.2)$$

the last step being taken in the short range limit. The rest is straightforward: Just multiply both sides of (2.6) by  $4\pi\beta^4$  after substituting for  $\lambda^{-1}$  from (3.1), and take the limit of large  $\beta$ , noting that  $\beta^2 \times g$  approaches unity. The final result is

$$[-1/a + R_0(3p^2/4 + \alpha^2) + \sqrt{3p^2/4 + \alpha^2}]\phi(p) = \pi^{-2} \int \frac{d^3p'\phi(p')}{p^2 + p'^2 + p.p'} \quad (3.3)$$

which on azimuthal integration and a transformation  $p\phi(p) = \psi(p)$  yields the GME equation (7) [5], with the identification  $R_0 \Leftrightarrow R^*$ .

To conclude, we have obtained a 1D equation via Yamaguchi separable potentials which in the short range limit closely resembles the GME [5] equation under identical conditions for dimer resonances. Since there already exists a vast literature on this subject, including educational ones [10], we need hardly comment further, except to claim that all the benefits of the GME results, subsequent to their eq (7), also accrue to the present formalism.

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