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Effective theory for deformed nuclei

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Abstract

Techniques from effective field theory are applied to nuclear rotation. This approach exploits the spontaneous breaking of rotational symmetry and the separation of scale between low-energy Nambu-Goldstone rotational modes and high-energy vibrational and nucleonic degrees of freedom. A power counting is established and the Hamiltonian is constructed at next-to-leading order.

Keywords: effective theory, collective excitations, deformed nuclei

1. Introduction

Collective states are the low-lying excitations of heavy nuclei. One of the hallmarks of deformed nuclei are rotational bands of the form

$$E(l) \approx A[l(l+1) - I_0(I_0+1)] \ . \tag{1}$$

Here, $l \geq I_0$ denotes the angular momentum, A is a constant determined by fit to data, and I_0 is the spin of the band head under consideration. Rotational states are the lowest-lying excitations in open-shell heavy nuclei. In rotational even-even nuclei in the rare-earth region, for instance, the $l = 2$ state of the $(I_0 = 0)$ ground-state band has an excitation energy of about 80 keV while the vibrational $I_0 = 2$ band head is at about 1 MeV of excitation energy. Our understanding of nuclear rotation is based on the ground-breaking papers by Bohr [1], and by Bohr and Mottelson [2]. Here, collective nuclear excitations are modeled in terms of quadrupole vibrations of the nuclear surface. Rotational nuclei are intrinsically deformed, i.e. they exhibit a mostly axially symmetric static deformation in the co-rotating intrinsic reference frame, giving rise to rotational bands of the form of Eq. (1). The Bohr-Mottelson model has been extended to include nuclei with tri-axial deformation within the asymmetric rotor model [3]. Rotational nuclear states are also described within the variable moment of inertia model [7], and the

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general collective model [4, 5, 6]. The interacting boson model [8], an algebraic model based on s-wave and d-wave bosons, is also widely employed for the description of collective nuclear phenomena. While these models are very appealing due to their physical motivation and due to their mathematical beauty, it is difficult to systematically extend them, to gauge their limitations, or to compute results with reliable error estimates. Furthermore, it is non-trivial to keep generalizations of collective models computationally tractable [9, 10]. This is due to the difficulties posed by the linear realization of rotational symmetry.

The present paper attempts to overcome these limitations. It proposes a model-independent description of rotational nuclei that is based on an effective theory (EFT). It treats even-even, odd mass and odd-odd nuclei on an equal footing. In recent years, EFTs enjoyed considerable popularity and success in low-energy nuclear structure. Examples are the application of pion-less EFT to few-body systems [11], dilute Fermi gases with repulsive [12] and attractive [13, 14, 15] interactions, nuclear interactions based on chiral EFT [16, 17, 18], and the description of halo and cluster states within an EFT [19, 20]. Effective field theories exploit a separation of scale, and provide us with a model independent description of physical phenomena based on observables [21, 22]. They often exhibit an impressive efficiency as highlighted by analytical results and economical means of calculations.

Within an effective theory, one proceeds as follows. First, the relevant low-energy degrees of freedom have to be identified. For even-even nuclei in the rare earth region, for instance, the spins and parities of low-lying states can be explained in terms of quadrupole degrees of freedom. Second, the relevant symmetries (and pattern of spontaneous symmetry breaking) have to be identified. For atomic nuclei, the Hamiltonian is invariant under rotations. Furthermore, the concept of an intrinsically deformed ground state corresponds to the spontaneous breaking of rotational symmetry. (See Bohr's Nobel lecture [23], or Mottelson's lectures in Les Houches [24].) Indeed, the rotational spectrum (1) consists of the low-lying excitations associated with the spontaneous breakdown of rotational symmetry. In an infinite system, the corresponding excitations are massless Nambu-Goldstone bosons. In a finite system such as the atomic nucleus, there is – strictly speaking – no spontaneous symmetry breaking and thus no Nambu-Goldstone boson. However, as we will see below, the Nambu-Goldstone modes generate the low-energetic discrete excitations (1) upon the quantization of the finite system [25]. Third, we have to identify (and exploit) a separation of scales and introduce a power counting. In the case of deformed even-even nuclei in the rare earth region, the separation of scales is the separation between, e.g. the rotational $J^\pi = 2^+$ state (at several tens of keV of energy) and the low-lying vibrational 2^+ state (at about 1 MeV of excitation energy). Below the vibrational threshold, the physics can be described purely within Nambu-Goldstone modes. Above the threshold, the vibrations have explicitly to be taken into account, and one might also consider the coupling to even higher energetic nucleonic degrees of freedom (with a corresponding energy scale of a few to several MeV). For even-even nuclei, the phenomenological models [2, 4, 5, 6, 8] practically include the physics of the Nambu-Goldstone modes and the vibrational degrees of freedom. However, the models do not further exploit the separation of scales, and they do not present a power counting that would allow for a systematic extension.

We want to apply the tools of effective field theory to nuclear rotation. The concept of spontaneous symmetry breaking is central to this approach. The ground breaking papers by Weinberg [26], by Coleman, Wess and Zumino [27], and by Callan, Coleman, Wess and Zumino [28] describe the construction of low-energy Lagrangians in the presence of spontaneous symmetry breaking. Leutwyler applied and extended this approach to non-relativistic Lagrangians [29], and excellent description of the approach can be found in Refs. [30, 31, 32, 22, 33]. As we will see below, the Nambu-Goldstone fields employed in the spontaneous breakdown of rotational symmetry only depend on time (and not on space). This is due the finite size of the atomic nucleus. Thus, we do not deal with a field theory but rather with quantum mechanics. For this reason, this approach is an effective theory (and not an EFT). Note that nuclear rotation and the spontaneous breakdown of rotational symmetry has been addressed within a field theoretical approach by Fujikawa and Ui [34]. These authors succeeded in linking nuclear rotation to the Higgs mechanism, but they did not pursue the systematic construction of low-energy Lagrangians.

There are – of course – microscopic approaches to nuclear rotation (see the recent review [35] and references therein), and the microscopic computation of the parameters of collective models is a long-standing [36] and interesting problem [37, 38, 39, 40]. The aim of the present paper is not to extract a collective model from an underlying microscopic Hamiltonian, but rather to construct a low-energy collective Hamiltonian within an effective theory. Apart from symmetry principles, no details of the microscopic Hamiltonian are needed for this task.

This paper is organized as follows. Section 2 presents the derivation of the low-energy Lagrangians and Hamiltonians that govern the dynamics of axially deformed nuclei, i.e. the spontaneous breakdown of $SO(3)$ to $SO(2)$ is the central concept. In Sect. 3 we consider the physics of Nambu-Goldstone modes that result from the symmetry breaking. In Sect. 4 we study the coupling of vibrations to the Nambu-Goldstone modes. Fermions and Nambu-Goldstone modes are coupled in Sect. 5. The main results of this paper are summarized in Sect. 6. Some technical details are presented in the Appendix.

2. Construction of low-energy Lagrangians

Let us consider a system with a continuous symmetry such as invariance under rotations. In this case, the ground state has definite angular momentum. Spontaneous symmetry breaking happens when an *arbitrarily small* symmetry-breaking perturbation yields a deformed ground state, i.e. a state with no definite angular momentum (see, e.g., Ref. [41]). Clearly, spontaneous symmetry breaking can only take place in infinite systems as only these can exhibit gap-less excitations.

Ferromagnets and antiferromagnets are well known examples for the spontaneous breaking of rotational symmetry. In these system, the ground state exhibits a finite magnetization and a long-range order of staggered magnetization, respectively. Thus, the ground state does not exhibit the full rotational symmetry, but is only invariant under rotations around the axis of magnetization. This is the spontaneous breaking of rotational symmetry down to axial symmetry. In the case of the ferromagnet, ground states with different orientations of the magnetization are inequivalent. The effective field theory for magnets has been derived by Leutwyler [29].

In systems with a finite number of degrees of freedom, such as atomic nuclei, spontaneous symmetry breaking does not occur in a strict sense since arbitrarily small perturbations do not lead to a deformation of the ground state. However, for nuclei in the rare earth region, perturbations of the size of a few tens of keV can mix states with different angular momenta. This is a particularly low energy scale compared to other excitations, and it is two orders of magnitude smaller than the nucleon separation energy. In a technical sense, spontaneous symmetry breaking does not take place because differently oriented ground states of intrinsically deformed nuclei are unitary equivalent to each other. A superposition (i.e. the projection) of these states creates states with good angular momentum that are invariant under the full rotation group [25]. This phenomenon is well established in mean field calculations (see, e.g., Refs. [42, 43, 44]). However, the small energy scale necessary to induce a symmetry breaking justifies and motivates us to apply the ideas of spontaneous symmetry breaking to this case.

For axially deformed nuclei, the rotational symmetry is spontaneously broken. The ground state $\psi^{(0)}$ is only invariant under operations of the subgroup $\mathcal{H} = SO(2)$, i.e.

$$h\psi^{(0)} = \psi^{(0)} \quad \text{for } h \in \mathcal{H} \ , \tag{2}$$

but it is not invariant under general operations of the full rotation group $\mathcal{G} = SO(3)$ of the Hamiltonian. Any rotation $g \in \mathcal{G}$ can be decomposed as $g = \tilde{g}h$ with $\tilde{g} \in \mathcal{G}$ and $h \in \mathcal{H}$. Two rotations $g = \tilde{g}h$ and $g' = \tilde{g}h'$ with $h, h' \in \mathcal{H}$ that differ from each other by an element of the subgroup $\mathcal{H}$ yield the same state when applied to the ground state $\psi^{(0)}$. Thus, such group elements must be identified, and they form an equivalence class. This equivalence class is the coset $\mathcal{G}/\mathcal{H}$. The coset $SO(3)/SO(2)$ can thus be used to describe the low-energy degrees of freedom which change the orientation of the ground state $\psi^{(0)}$. The corresponding degrees of freedom are Nambu-Goldstone modes. In infinite systems, these modes have zero mass and thus zero energy in the limit of vanishing momenta. We will see below that the Nambu-Goldstone modes of deformed nuclei generate rotational bands upon quantization.

The Nambu-Goldstone modes parameterize the coset $SO(3)/SO(2)$, and we need to work out the basic expressions from which rotationally invariant Lagrangians can be constructed. This problem was solved for the spontaneous breaking of chiral symmetry by Weinberg [26], and for general Lie groups by Coleman, Wess and Zumino [27], and by Callan, Coleman, Wess and Zumino [28]. For detailed reviews of this matter, the reader is refered to Refs. [30, 31, 32, 33]. In this Section, we closely follow [30].

Let us parameterize a rotation $r \in SO(3)$ by the three Euler angles α, β, and γ as [45]

$$r(\alpha, \beta, \gamma) = e^{-i\alpha\hat{J}_z} e^{-i\beta\hat{J}_y} e^{-i\gamma\hat{J}_z} \ . \tag{3}$$

Here, $\hat{J}_k$, $k = x, y, z$ denote the appropriate components of the angular momentum operator (i.e. the generators of the Lie group $SO(3)$). These operators fulfill the usual commutation relations

$$[\hat{J}_j, \hat{J}_k] = i \sum_l \varepsilon_{jkl} \hat{J}_l \ . \tag{4}$$

Let us assume that the subgroup $\mathcal{H} = SO(2)$ consists of the rotations $h(\gamma) = \exp(-i\gamma\hat{J}_z)$, i.e. its generator is $\hat{J}_z$. This implies that the ground state $\psi^{(0)}$ has a finite expectation value in the z-direction. Thus, any two rotations $r(\alpha, \beta, \gamma)$ that differ by the Euler angle γ from each other yield the same state when acting on the ground state. The coset $SO(3)/SO(2)$ thus consists of the rotations

$$g(\alpha, \beta) = e^{-i\alpha\hat{J}_z} e^{-i\beta\hat{J}_y} \ , \tag{5}$$

and the Euler angles α and β are the degrees of freedom of the Nambu-Goldstone modes.

The dynamics of the Nambu-Goldstone modes is determined by the time derivative $\partial_t g(\alpha,\beta)$. It is simpler to compute

$$g^{-1}(\alpha,\beta)\partial_t g(\alpha,\beta) \equiv iE_x\hat{J}_x + iE_y\hat{J}_y + iE_z\hat{J}_z \ , \tag{6}$$

as this is an element of the Lie algebra of the group $\mathcal{G} = SO(3)$. This defines the functions E_k, $k = x, y, z$.

We want to construct Lagrangians that are invariant under rotations and therefore need to study the transformation properties of the functions E_x, E_y, and E_z. Let us act with a rotation $r \in \mathcal{G}$ onto a rotation $g \in \mathcal{G}/\mathcal{H}$. This yields rg which again can be decomposed into a product of two rotations $\tilde{g} \in \mathcal{G}/\mathcal{H}$ and $h \in \mathcal{H}$

$$rg = \tilde{g}(g,r)h(g,r) \ . \tag{7}$$

Solving for $\tilde{g}$ yields

$$\tilde{g}(g,r) = rgh^{-1}(g,r) \ . \tag{8}$$

The rotations $\tilde{g}$ and h are nonlinear (and complicated) functions of the three Euler angles that define r and the two Nambu-Goldstone modes that define g. We have

$$\begin{aligned} g^{-1}\partial_t g &= (rg)^{-1}\partial_t(rg) \\ &= h^{-1}(\tilde{g}\partial_t\tilde{g})h + h^{-1}\partial_t h \ . \end{aligned} \tag{9}$$

In this derivation we used that r is time independent, and we employed Eq. (7). We solve for $(\tilde{g}\partial_t\tilde{g})$ and find

$$\tilde{g}^{-1}\partial_t\tilde{g} = h(g^{-1}\partial_t g)h^{-1} - (\partial_t h)h^{-1} \ . \tag{10}$$

Similar to Eq. (6) we again have

$$\tilde{g}^{-1}\partial_t\tilde{g} \equiv i\tilde{E}_x\hat{J}_x + i\tilde{E}_y\hat{J}_y + i\tilde{E}_z\hat{J}_z \ , \tag{11}$$

as this is an element of the Lie algebra of $SO(3)$. We employ Eq. (11) and Eq. (6) on the left and right-hand side of Eq. (10), respectively, and observe that $h\hat{J}_{x,y}h^{-1}$ is a linear combination of $J_{x,y}$ while $h\hat{J}_z h^{-1} = \hat{J}_z$. Note that the term $(\partial_t h)h^{-1}$ on the right-hand side of Eq. (10) is also proportional to $\hat{J}_z$. Thus,

$$\tilde{E}_x\hat{J}_x + \tilde{E}_y\hat{J}_y = E_x\, h\hat{J}_x h^{-1} + E_y\, h\hat{J}_y h^{-1} \ , \tag{12}$$

$$\tilde{E}_z\hat{J}_z = E_z\hat{J}_z - i(\partial_t h)h^{-1} \ . \tag{13}$$

These equations show that under a general rotation r, the functions E_x and E_y transform as the x and y components of a vector under the rotation h around the z-axis, while E_z transforms as a gauge field. Indeed,

$$h(\partial_t - iE_z\hat{J}_z)h^{-1} = \partial_t - i\tilde{E}_z\hat{J}_z \ . \tag{14}$$

Let us further illuminate these derivations. We express the rotation h as $h = \exp\left(-i\gamma(g,r)\hat{J}_z\right)$. Here the angle $\gamma(g,r)$ is a (complicated) function of the Nambu-Goldstone modes α and β of the rotation g and the three angles that parameterize the rotation r. We employ the transformation (12) and find

$$\begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} , \tag{15}$$

while Eq. (13) yields

$$\tilde{E}_z = E_z - \dot{\gamma} \ . \tag{16}$$

Note that Eq. (15) implies that

$$E_{\pm 1} \equiv E_x \mp iE_y \tag{17}$$

transforms as the upper and lower components of a vector (a spherical tensor of degree one), i.e.

$$e^{i\gamma \hat{J}_z} E_{\pm 1} = e^{\pm i\gamma} E_{\pm 1} \ . \tag{18}$$

A Lagrangian consisting of combinations of the functions E_x, and E_y that are formally invariant under the subgroup $\mathcal{H} = SO(2)$ will thus be invariant under general rotations $r \in \mathcal{G} = SO(3)$ [30]. This is all we need for the construction of Lagrangians involving the Nambu-Goldstone modes. Equation (16) shows that the function E_z is not invariant under a rotation, but merely changes by a total derivative. The function E_z will be important in cases where time reversal invariance is broken by the ground state (i.e. for nuclei with a finite ground-state spin).

Let us also consider the presence of a field ψ that describes physics at a higher energy scale. The key is [30] to rewrite

$$\psi \equiv g(\alpha, \beta)\phi \ . \tag{19}$$

Here, g is the element (5) of the coset $\mathcal{G}/\mathcal{H} = SO(3)/SO(2)$, and ϕ is defined in terms of ψ and g. Due to the particular choice (19), a rotation $r \in \mathcal{G}$ transforms $\psi \to \tilde{\psi}$ as

$$\tilde{\psi} \equiv r\psi = rg\phi = \tilde{g}h\phi \ . \tag{20}$$

Here, we used Eq. (7). By definition (compare to Eq. (19)), we must also have $\tilde{\psi} \equiv \tilde{g}\tilde{\phi}$. The comparison with Eq. (20) shows that under a rotation, the field ϕ transforms as $\phi \to \tilde{\phi}$ with

$$\tilde{\phi} \equiv h\phi \ . \tag{21}$$

The time derivative of the field

$$\partial_t \tilde{\phi} = (\partial_t h)\phi + h\partial_t \phi \tag{22}$$

thus transforms as a gauge field. Employing Eq. (13) we have

$$h\left(\partial_t - iE_z\hat{J}_z\right)\phi = \left(\partial_t - i\tilde{E}_z\hat{J}_z\right)\tilde{\phi} \ . \tag{23}$$

Thus, the covariant derivative

$$D_t \equiv \partial_t - iE_z J_z \ , \tag{24}$$

when acting onto ϕ, transforms properly under rotations and can be employed in the construction of rotationally invariant Lagrangians for the Nambu-Goldstone modes and the field ψ. A Lagrangian consisting of of E_x, E_y, ϕ, and $D_t\phi$ that is formally invariant under rotations of the subgroup $\mathcal{H} = SO(2)$ will be invariant under the full action of the group $\mathcal{G} = SO(3)$. We now recognize the advantage of employing a nonlinear realization of the rotational symmetry. While the derivation of the basic building blocks E_x, E_y, ϕ, and $D_t\phi$ and their transformation properties are somewhat more complicated than for the well-known linear representations, the construction of rotationally invariant Lagrangians can be achieved by constructing Lagrangians that (at first sight) only appear to exhibit axial symmetry. This will make the resulting effective theory computationally tractable. This is in contrast to algebraic models which are linear representations of the rotational symmetry and employ dynamical symmetries to remain computationally feasible [9].

All that remains is to actually compute the functions E_k, $k = x, y, z$. We use the Baker-Campbell-Hausdorff formula and compute the left-hand-side of Eq. (6) directly from the expression (5). This yields

$$\begin{aligned} E_x &= \dot\alpha \sin\beta \ , \\ E_y &= -\dot\beta \ , \\ E_z &= -\dot\alpha\cos\beta \ . \end{aligned} \tag{25}$$

From a practical point of view, these expressions are the main result of this Section. Note that the components E_k can be viewed as angular velocities.

The invariance under rotation implies that angular momentum is a conserved quantity. To become more familiar with the nonlinear realization of rotational symmetry, we compute this conserved quantity via Noether's theorem. Under infinitesimal rotations around the k axis ($k = x, y, z$) by an angle $\delta\omega_k$, the Euler angles α and β change by the infinitesimal amounts $\delta\alpha$, and $\delta\beta$, respectively. One finds (see Appendix C for details)

$$\left(\begin{array}{c} \delta\alpha \\ \delta\beta \end{array}\right) = \hat M \left(\begin{array}{c} \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{array}\right) \ , \tag{26}$$

with

$$\hat M = \left(\begin{array}{ccc} -\cot\beta\cos\alpha & -\cot\beta\sin\alpha & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{array}\right) \ . \tag{27}$$

The Lagrangian $L(\dot\alpha, \dot\beta, \beta)$ of the Nambu-Goldstone modes is invariant under rotations. We apply Noether's theorem and find the conserved quantities (see Appendix D for details)

$$\left(\begin{array}{c} Q_x \\ Q_y \\ Q_z \end{array}\right) = \hat M^T \left(\begin{array}{c} p_\alpha \\ p_\beta \end{array}\right) \ . \tag{28}$$

Here,

$$p_\alpha = \frac{\partial L}{\partial\dot\alpha} \ , \tag{29}$$

$$p_\beta = \frac{\partial L}{\partial\dot\beta} \tag{30}$$

are the momenta conjugate to α and β, respectively. Thus, we have

$$\begin{aligned} Q_x &= -p_\beta \sin\alpha - p_\alpha \cot\beta \cos\alpha \ , \\ Q_y &= p_\beta \cos\alpha - p_\alpha \cot\beta \sin\alpha \ , \\ Q_z &= p_\alpha \ , \end{aligned} \tag{31}$$

and the total angular momentum squared is

$$Q^2 \equiv Q_x^2 + Q_y^2 + Q_z^2 = p_\beta^2 + \frac{p_\alpha^2}{\sin^2\beta} \ . \tag{32}$$

Clearly, Q^2 is the squared angular momentum of a rotor, and we will see in Sect. 3 that the leading order Hamiltonian of the Nambu-Goldstone modes is proportional to this quantity.

In what follows, we will continue to employ the Euler angles α and β as the Nambu-Goldstone modes. This is a convenient but arbitrary choice. The algebraic transformation laws derived in this subsection are independent of this particular choice as it only depends on the pattern of the symmetry breaking $SO(3) \to SO(2)$. It is only the explicit expressions (25) for the functions E_x, E_y, and E_z, respectively, that depend on this parameterization of the coset.

Let us briefly contrast the effective theory based on nonlinear realizations with the earlier phenomenological approaches. The Bohr Hamiltonian [1] and its generalizations [6] model the collective states of atomic nuclei by surface vibrations. In leading order, these are quadrupole phonons. Within this approach, one transforms from the laboratory coordinate system to the co-rotating (or body fixed) coordinate system. Thereby one introduces three Euler angles and the two moments of inertia that identify an axially deformed nucleus as the relevant degrees of freedom. Within the effective theory presented in this paper, different variables are chosen. In the case of axially deformed nuclei, only two Euler angles are relevant for the description of low-energy excitations. As we will see, the nonlinear realization naturally corresponds to the spontaneously broken rotational symmetry and facilitates the introduction of a power counting.

3. Nambu-Goldstone modes

3.1. Even-even nuclei

Let us assume that the ground state is invariant under time reversal (as is the case for even-even nuclei). The simplest Lagrangian is second order in the time derivative, and below we will see that this Lagrangian is indeed of leading order. In leading order (LO), the Lagrangian of the Nambu-Goldstone modes thus is

$$L_{\rm LO}^{(ee)} = \frac{C_0}{2}\left(E_x^2 + E_y^2\right) = \frac{C_0}{2}\left(\dot\beta^2 + \dot\alpha^2 \sin^2\beta\right) \ . \tag{33}$$

Here C_0 is a low-energy constant to be determined by fit to data. A Legendre transformation yields the Hamiltonian

$$H = \frac{p_\beta^2}{2C_0} + \frac{p_\alpha^2}{2C_0 \sin^2\beta} \ . \tag{34}$$

Here, we employed the canonical momenta $p_\beta = \partial L/\partial\dot\beta$ and $p_\alpha = \partial L/\partial\dot\alpha$. The Hamiltonian (34) is obviously proportional to the squared angular momentum (32). The quantization in curvilinear coordinates is well known [46], see Appendix A for details. This yields the Hamiltonian (we use units where $\hbar = 1$)

$$\hat{H} = -\frac{1}{2C_0}\left(\partial_\beta^2 + \cot\beta\partial_\beta + \frac{1}{\sin^2\beta}\partial_\alpha^2\right) . \tag{35}$$

Comparison with the classical Hamiltonian (34) thus yields

$$\begin{aligned} p_\beta^2 &= -\frac{1}{\sin\beta}\partial_\theta \sin\beta\partial_\beta \ , \\ p_\alpha &= -i\partial_\alpha \ . \end{aligned} \tag{36}$$

The eigenfunctions of the Hamiltonian (35) are spherical harmonics, i.e.

$$\hat{H}Y_{lm}(\beta,\alpha) = \frac{l(l+1)}{2C_0}Y_{lm}(\beta,\alpha) \ , \tag{37}$$

and $l = 0, 1, 2, \ldots$ thus yields a rotational spectrum. Note that the low-energy constant C_0 is, within the collective model, associated with the moment of inertia.

For even-even nuclei, only even values of l are permitted for the ground-state band. This can be understood as follows. The ground state of even-even nuclei is not only invariant under rotations $h \in \mathcal{H} = SO(2)$ but also under discrete operations such as parity, or a rotation about π around an axis perpendicular to axial symmetry axis. Thus, we must augment the invariant subgroup $\mathcal{H}$ by these discrete operations, and this modifies the coset accordingly. As a result, the angles (β, α) and $(\pi - \beta, \pi + \alpha)$ have to be identified, and this limits the values of l to even numbers [2].

3.2. Odd-mass and odd-odd nuclei

Odd nuclei have a half-integer ground-state spin, while odd-odd nuclei can also exhibit a nonzero integer ground-state spin. Thus, the ground state is not invariant under time reversal. This modifies the low-energy effective Lagrangian as terms that are not invariant under time reversal need to be considered. Such terms consist of only one time derivative, and we need to consider the functions (25). The transformation properties (12) and (13) show that the functions E_x, E_y, and E_z are not invariant under rotations. However, the function E_z only changes by a total derivative, see Eq. (16). Thus, the action changes by an irrelevant phase. In quantum field theory, such a function is known as a Wess-Zumino term. In our case, the Wess-Zumino term $L_{\rm WZ}$ is

$$L_{\rm WZ} \equiv qE_z = -q\dot\alpha\cos\beta \ . \tag{38}$$

Recall that a low-energy Lagrangian can be understood as resulting from integrating out high-energy fermion modes in a more fundamental Lagrangian [21]. In the case that the considered fermion system consists of an odd number of fermions, or has a finite ground-state spin, the resulting low-energy Lagrangian must reflect this behavior. Thus, the appearance of a corresponding symmetry-breaking term is unavoidable. For details, the reader is refered to Refs. [47, 48]. In Sect. 5, we will couple fermions to the Nambu-Goldstone modes and see that the relevant term is indeed proportional to E_z, as stated in Eq. (38).

Let us consider the transformation properties of the Wess-Zumino term. Under a rotation by $\delta\omega_k$ around the k axis ($k = x, y, z$), the Wess-Zumino Lagrangian $L_{\rm WZ}$ changes by (see Appendix D for details)

$$\delta L_{\rm WZ} = q\left(\delta\omega_x\partial_t\left(\frac{\cos\alpha}{\sin\beta}\right) + \delta\omega_y\partial_t\left(\frac{\sin\alpha}{\sin\beta}\right)\right) , \tag{39}$$

and this is obviously a total derivative. The application of Noether's theorem yields the conserved quantities

$$\begin{aligned} Q_x &= -\frac{\cos\alpha}{\sin\beta}q - p_\beta\sin\alpha - p_\alpha\cot\beta\cos\alpha , \\ Q_y &= -\frac{\sin\alpha}{\sin\beta}q + p_\beta\cos\alpha - p_\alpha\cot\beta\sin\alpha , \\ Q_z &= p_\alpha , \end{aligned} \tag{40}$$

which are the components of the angular momentum. The total angular momentum squared is

$$Q^2 \equiv Q_x^2 + Q_y^2 + Q_z^2 = p_\beta^2 + \frac{1}{\sin^2\beta}\left(p_\alpha + q\cos\beta\right)^2 + q^2 . \tag{41}$$

With the Wess-Zumino term added, the leading order Lagrangian becomes

$$L_{\rm LO} = L_{\rm LO}^{(ee)} + L_{\rm WZ} = \frac{C_0}{2}\left(\dot\beta^2 + \dot\alpha^2\sin^2\beta\right) - q\dot\alpha\cos\beta . \tag{42}$$

The Legendre transformation yields the classical Hamiltonian

$$H_{\rm LO} = \frac{p_\beta^2}{2C_0} + \frac{\left(p_\alpha + q\cos\beta\right)^2}{2C_0\sin^2\beta} . \tag{43}$$

The comparison with the angular momentum (41) shows that $H_{\rm LO} = (Q^2 - q^2)/(2C_0)$. We employ the quantization (36), and obtain

$$\hat H_{\rm LO} = -\frac{1}{2C_0\sin\beta}\partial_\beta\sin\beta\partial_\beta + \frac{1}{2C_0\sin^2\beta}\left(-i\partial_\alpha + q\cos\beta\right)^2 . \tag{44}$$

The eigenfunctions of this Hamiltonian are products $e^{-i\alpha m}d^l_{mq}(\beta)$. Here d^l_{mq} denotes the "little" Wigner d function [45], i.e. the Wigner D function is defined as $D^l_{mq}(\alpha,\beta,\gamma) \equiv e^{-im\alpha}d^l_{mq}(\beta)e^{-iq\gamma}$. Details are presented in the Appendix E. Thus,

$$\hat H_{\rm LO}\, d^l_{mq}(\beta)e^{-i\alpha m} = E_{\rm LO}(q,l)d^l_{mq}(\beta)e^{-i\alpha m} , \tag{45}$$

and the eigenvalues are

$$\begin{aligned} E_{\rm LO}(q,l) &= \frac{l(l+1) - q^2}{2C_0} \\ &= \frac{|q|}{2C_0} + \frac{l(l+1) - |q|(|q|+1)}{2C_0} . \end{aligned} \tag{46}$$

Here, $q \neq 0$ must be integer or half integer, while l formally assumes the values $l = |q|, |q|+1, |q|+2, \ldots$, and $m = -l, -l+1, \ldots l$. (For $q = 0$, l only assumes even values.)

Apart from the irrelevant constant $|q|/(2C_0)$, the comparison of Eq. (46) with Eq. (1) shows that we have to identify the low-energy constant q with the ground-state spin I_0 of the nucleus under consideration, i.e. $|q| = I_0$. Thus, the spin I_0 of the ground-state and the spacing between the lowest two states in the rotational spectrum fix the low-energy constants q and C_0. This unified description of odd and even nuclei within an effective theory is very encouraging. The effective Hamiltonian (44) yields the correct low-energy description of axially deformed nuclei. The case of even-even nuclei (and odd-odd nuclei with zero ground-state spin) is particularly simple as $q = I_0 = 0$, while odd-mass nuclei or odd-odd nuclei with nonzero ground-state spin I_0 have a more complicated Hamiltonian due to $q = I_0 > 0$.

3.3. Next-to-leading order

We need to establish a power counting. Let us denote the low-energy scale associated with the Nambu-Goldstone modes as ξ. Thus, the leading-order energy (46) scales as $E_{\rm LO} \sim \xi$, and we get the following estimates

$$\begin{aligned} C_0 &\sim \xi^{-1} , \\ p_\beta \sim p_\alpha \sim q &\sim \xi^0 , \\ \dot\beta \sim \dot\alpha \sim E_{x,y,z} &\sim \xi . \end{aligned} \tag{47}$$

The estimate in the second line of Eq (47) is due to the quantization of angular momentum (recall that $\hbar = 1$), and the estimate in the third line of Eq. (47) follows from the usual relationship between velocities and momenta. It is clear that the Wess-Zumino term scales as $L_{\rm WZ} \sim \xi$, and is indeed of leading order.

The Nambu-Goldstone modes of energy $\sim \xi$ are well separated from the high energy (breakdown) scale $\Omega \gg \xi$ which is associated with degrees of freedom (vibrational, pairing or single-particle degrees of freedom) we have omitted from our theory. In an effective theory, these omitted terms manifest themselves through interactions between the Nambu-Goldstone modes. At next-to-leading order (NLO), higher derivatives of the Nambu-Goldstone modes appear [30, 29], and one new term enters the Lagrangian

$$L_{\rm NLO} = L_{\rm LO} + \frac{C_2}{4}\left(E_x^2 + E_y^2\right)^2 . \tag{48}$$

Here, C_2 is the corresponding low-energy constant. The ratio C_2/C_0 has units of energy^{-2}, and is assumed to be due to omitted physics at the breakdown scale. Thus, the dimensional analysis yields

$$\frac{C_2}{C_0} \sim \Omega^{-2} , \tag{49}$$

and

$$\frac{C_2}{C_0}\left(E_x^2 + E_y^2\right) \sim \left(\frac{\xi}{\Omega}\right)^2 \ll 1 . \tag{50}$$

The last equation indicates that the next-to-leading-order correction is suppressed by two powers of ξ/Ω compared to the leading-order terms.

For the computation of the Hamiltonian at next-to-leading order, we employ the conjugate momenta

$$p_\beta \equiv \frac{\partial L_{\rm NLO}}{\partial \dot\beta} = \left(C_0 + C_2\left(E_x^2 + E_y^2\right)\right)\dot\beta \ , \tag{51}$$

$$p_\alpha \equiv \frac{\partial L_{\rm NLO}}{\partial \dot\alpha} = \left(C_0 + C_2\left(E_x^2 + E_y^2\right)\right)\dot\alpha \sin^2\beta - q\cos\beta \ , \tag{52}$$

and apply Eq. (50) consistently in the Legendre transform. This yields the Hamiltonian at next-to-leading order

$$H_{\rm NLO} = \left(1 - \frac{C_2}{C_0^2} H_{\rm LO}\right) H_{\rm LO} \ . \tag{53}$$

Here, $H_{\rm LO}$ is the leading-order Hamiltonian (43). Thus, the spectrum of the Nambu-Goldstone modes becomes

$$E_{\rm NLO} = \left(1 - \frac{C_2}{C_0^2} E_{\rm LO}\right) E_{\rm LO} \ , \tag{54}$$

where $E_{\rm LO}$ is given in Eq. (46). Thus, the spectrum at next-to-leading order is a polynomial of degree two in the right-hand side of Eq. (1). This is exactly as given by Bohr and Mottelson [2]. The low-energy constant C_2 can be determined by fitting the spacing between the rotational state with $l = I_0 + 4$ and $l = I_0$. For deformed rare earth nuclei, the ratio $\xi/\Omega \approx 10^{-1}$, and this explains the high quality of many rotational bands, i.e. the next-to-leading-order contribution is a small correction for angular momenta l with $l \ll \Omega/\xi$. Higher corrections at next-to-next-to-leading order include terms of the form

$$C_4\left(E_x^2 + E_y^2\right)^3 \ , \tag{55}$$

and the dimensional analysis yields $C_4/C_2 \sim \Omega^{-2}$. Thus, another factor $(\xi/\Omega)^2 \approx 10^{-2}$ separates these contributions from the contributions at next-to-leading order. However for rotational states with high angular momentum $l \approx \mathcal{O}(\Omega/\xi)$, the description in terms of the Nambu-Goldstone modes must break down, as higher order terms become large. In this case, one needs to include degrees of freedom hat are higher in energy such as vibrations, pairing effects, and nucleonic excitations. This is the subject of the next two sections.

Note finally that the results presented in this section can also be obtained by simpler means. An alternative derivation is given in Appendix B.

4. Coupling of quadrupole bosons to the Nambu-Goldstone modes

In this Section, we couple higher-energetic phonon degrees of freedom to the Nambu-Goldstone modes. The appropriate phonons are quadrupole vibrations, as evident from the spins and parities of low-energy states. This results from the observation that spectra of even-even nuclei exhibit $I_0 = 2$ band heads at the energy scale Ω. In this section we couple the quadrupole phonons (whose ground state spontaneously breaks the rotational symmetry) to the Nambu-Goldstone modes.

4.1. Quadrupole phonons

We consider a quadrupole field ψ (spin-two boson) with complex components ψ_μ, $\mu = -2, \ldots, 2$. Invariance under time reversal implies

$$\psi^*_{-\mu} = (-1)^\mu \psi_\mu \ , \tag{56}$$

such that we deal with five degrees of freedom.

As discussed in Section 2, we parameterize the phonon as $\psi = g\phi$ with $g \in \mathcal{G}/\mathcal{H} = SO(3)/SO(2)$. By assumption, the ground state $\phi^{(0)}$ spontaneously breaks rotational symmetry, but remains invariant under the operations of the subgroup $\mathcal{H} = SO(2)$ of the rotation group $SO(3)$. Again, we choose $\hat{J}_z$ as the generator of the subgroup $\mathcal{H}$. Consequently, the ground-state expectation value of the field ϕ fulfills

$$\langle \phi^{(0)} \rangle_\mu = v \delta^0_\mu \ , \tag{57}$$

with $v \neq 0$, and is obviously invariant under rotations around the (arbitrarily chosen) z axis.

We have to account for the fact that the Nambu-Goldstone modes result from the symmetry breaking of the quadrupole phonon ϕ. Thus, we parameterize the five components of $\psi = g\phi$ in terms of the two Nambu-Goldstone modes α and β, and three non-Nambu-Goldstone modes as

$$\phi = \begin{pmatrix} \phi_2 \\ 0 \\ \phi_0 \\ 0 \\ \phi_2^* \end{pmatrix} . \tag{58}$$

The three non-Nambu-Goldstone modes are parameterized by a complex component ϕ_2 and the real component ϕ_0. The choice (58) for the non-Nambu-Goldstone mode is appropriate [30] as ϕ is "independent" of the Nambu-Goldstone modes, i.e. it is orthogonal to an infinitesimal rotation of the ground state induced by the generators $\hat{J}_k$, $k = x, y$ that do not belong to the Lie algebra of the subgroup $\mathcal{H} = SO(2)$

$$\phi^\dagger (\hat{J}_k \langle \phi^{(0)} \rangle) = 0 \quad \text{with } k = x, y. \tag{59}$$

The "building blocks" for rotationally invariant Lagrangians are $D_t\varphi$, $D_t\phi_2$, $D_t\phi_{-2}$, E_1, E_{-1}, φ_0, ϕ_2, and $\phi_{-2} = \phi_2^*$. Any Lagrangian in these quantities that is formally invariant under $SO(2)$ will indeed exhibit full rotational invariance because of the nonlinear realization of the symmetry.

4.2. Transformation properties

Recall the transformation properties and the conserved quantities of our effective theory. Under a rotation r, the field ϕ transforms as

$$\phi \to h(\gamma)\phi = e^{-i\gamma \hat{J}_z} \phi \ . \tag{60}$$

Here $\gamma = \gamma(\alpha, \beta, r)$ is a complicated angle of the rotation r and the Nambu-Goldstone modes. Note that the component ϕ_0 is invariant under rotations. For infinitesimal rotations by angles $\delta\omega_k$ around the k-axis, we have (see Appendix C for details)

$$\gamma = \frac{\cos\alpha}{\sin\beta}\delta\omega_x + \frac{\sin\alpha}{\sin\beta}\delta\omega_y \ . \tag{61}$$

Note that $\gamma = 0$ for a rotation around the z-axis. We formally apply Noether's theorem to the Lagrangian $L(\dot\alpha, \dot\beta, \beta, \dot\phi, \phi)$ and find the conserved quantities

$$\begin{aligned} Q_x &= -2\frac{\cos\alpha}{\sin\beta} l_2 - p_\beta \sin\alpha - p_\alpha \cot\beta\cos\alpha \ , \\ Q_y &= -2\frac{\sin\alpha}{\sin\beta} l_2 + p_\beta \cos\alpha - p_\alpha \cot\beta\sin\alpha \ , \\ Q_z &= p_\alpha \ . \end{aligned} \tag{62}$$

Here, we decomposed $\phi_2 = \phi_{2r} + i\phi_{2i}$ into its real and imaginary part, respectively, denoted the momentum conjugate to ϕ_{2r} (ϕ_{2i}) by p_{2r} (p_{2i}), and employed the angular momentum

$$l_2 \equiv (\phi_{2r}p_{2i} - \phi_{2i}p_{2r}) \ . \tag{63}$$

The total angular momentum squared is

$$\begin{aligned} Q^2 &= p_\beta^2 + \frac{1}{\sin^2\beta}\left(p_\alpha^2 + 4p_\alpha l_2\cos\beta + 4l_2^2\right) \\ &= p_\beta^2 + \frac{1}{\sin^2\beta}\left(p_\alpha^2 + 2l_2\cos\beta\right)^2 + (2l_2)^2 \ . \end{aligned} \tag{64}$$

4.3. Power counting

Let us consider the power counting. There are two possibly distinct breakdown scales for our effective theory. First, the restoration of rotational symmetry at high excitation energies signals the breakdown of the effective theory. This scale results from the scales ξ and Ω that describe the quadrupole phonon $\psi = g\phi$. Second, additional degrees of freedom enter at an energy scale $\Lambda \gg \Omega$, and their effect also needs to be considered. (We neglect, for instance, collective phonons of higher multipolarity, pairing, and single-particle degrees of freedom.) We will focus here primarily on the restoration of rotational symmetry. The effects from not included high-lying degrees of freedom will be neglected by formally sending $\Lambda \to \infty$.

The field (58) spontaneously breaks the rotational symmetry, and we separate the vacuum expectation value v from the fluctuating contribution φ_0 as

$$\varphi_0 \equiv \phi_0 - v \ . \tag{65}$$

We have the following scaling relations

$$\begin{aligned} v \sim \phi_0 &\sim \xi^{-1/2} \ , \\ \varphi_0 \sim \phi_2 &\sim \Omega^{-1/2} \ , \\ D_t\varphi_0 \sim D_t\phi_2 &\sim \Omega^{1/2} \ , \\ \dot\varphi_0 = \dot\phi_0 \sim \dot\phi_2 &\sim \Omega^{1/2} \ , \end{aligned} \tag{66}$$

in addition to the relations (47). The finite vacuum expectation value v must clearly be associated with the low-energy scale ξ, while the fluctuations of the field ϕ are associated with the high-energy scale Ω, and are necessarily much smaller than the vacuum expectation value. Indeed, we have $\varphi_0/v \sim \phi_2/v \sim (\xi/\Omega)^{1/2} \ll 1$.

Let us briefly discuss the case of a finite breakdown scale $\Lambda \gg \Omega$. Consider as an example the term $C\dot{\phi}_2^4$. Here, C has dimensions of energy^{-1}, and it must scale as $C \sim \Lambda^{-1}$. Thus, the contribution $C\dot{\phi}_2^4 \sim \Omega^2/\Lambda$ is of next-to-leading order, i.e. it corrects the leading-order term that scale as Ω. By sending $\Lambda \to \infty$, we neglect such terms. However, it is clear that a full-fledged effective theory needs to deal with them at some point in the power counting. The importance of such terms depends on how the ratio ξ/Ω that governs the validity of the spontaneous symmetry breaking compares to the ratio Ω/Λ that governs the relevance of neglected degrees of freedom.

The construction of low-energy Lagrangians in the presence of a spontaneously broken symmetry has been discussed in Section 2, with a focus on kinetic terms. We also need to discuss the form of admissible potentials $V(\phi)$. A rotationally invariant potential $V(\phi)$ consists of expressions involving ϕ that are formally invariant under rotations around the z axis. Furthermore, the potential must exhibit spontaneous symmetry breaking. We expand $V = V_{\rm LO} + V_{\rm NLO} + \ldots$. In leading order we have

$$V_{\rm LO}(\phi) = \frac{\omega_0^2}{2}(\phi_0 - v)^2 + \frac{\omega_2^2}{4}|\phi_2|^2 \ . \tag{67}$$

The low energy constants must scale as $\omega_0 \sim \omega_2 \sim \Omega$ to yield the ground-state expectation value $\langle V_{\rm LO}\rangle \sim \Omega$.

For the construction of next-to-leading order potential terms, we need to determine the breakdown scale for our effective theory. With increasing energy, the fluctuations φ_0 grow in size, and the effective theory breaks down once $\varphi_0 \approx v$, since this implies a restoration of the spherical symmetry. Likewise, large excitation energies correspond to a large amplitude $|\phi_2| \approx v$, again resulting in a restoration of the spherical symmetry and in the breakdown of the effective theory. The minimum kinetic energy of a large-amplitude field ϕ is $v^{-2} \sim \xi$, while its potential energy is $\langle V_{\rm LO}\rangle \sim \Omega^2/\xi \gg \Omega$. We will see that the power counting has to employ the kinetic energy scale.

Let us make a polynomial expansion of the potential

$$V = V_{\rm LO} + \sum_{k+2l>2} v_{kl}\varphi_0^k|\phi_2|^{2l} \ . \tag{68}$$

Here, k, l are integers, and it is understood that only terms with $k + 2l > 2$ are being summed over. (The leading order terms are $k + 2l = 2$.) Clearly, the potential exhibits spontaneous symmetry breaking and is formally invariant under $SO(2)$. Due to the nonlinear realization of the $SO(3)$ symmetry, it is also invariant under $SO(3)$.

Note that $v_{kl}\varphi_0^k|\phi_2|^{2l}/V_{\rm LO}$ is dimensionless. This implies that v_{kl}/Ω^2 has dimension of energy$^{l-1+k/2}$. For the power counting we have to assume that this energy scale has to be identified with the kinetic energy scale. Thus, we assume

$$v_{kl} \sim \Omega^2\xi^{l-1+k/2} \ , \tag{69}$$

and find

$$v_{kl}\varphi_0^k|\phi_2|^{2l} \sim \Omega\left(\frac{\xi}{\Omega}\right)^{l-1+k/2} \ . \tag{70}$$

This establishes the power counting for the potential. Clearly, for large amplitudes $\varphi_0, |\phi_2| \sim \xi^{-1/2}$ each potential term of the sum (68) is of order Ω, signaling the break down of the effective theory.

Our power counting is valid for well deformed nuclei that are "rigid" rotors , i.e. nuclei for which $\omega_0, \omega_2 \gg \xi$. Some deformed nuclei are not in this category. So-called γ-soft nuclei, for instance, do not exhibit a separation of scale between excitations within a rotational band (energy scale ξ) and the vibration with frequency ω_2. For these nuclei, the power counting is different.

4.4. Even-even nuclei

Let us consider the case where the symmetry-breaking ground state is invariant under time reversal. The following kinetic terms with two time derivatives are invariant under rotations

$$\begin{aligned} (D_t\varphi_0)^2 &= \dot{\varphi}_0^2 \\ D_t\phi_2 D_t\phi_{-2} &= \left|\dot{\phi}_2\right|^2 + 4\mathrm{Im}\left(\dot{\phi}_2\phi_2^*\right)E_z + 4|\phi_2|^2E_z^2 \\ E_x^2 + E_y^2 &= \dot{\beta}^2 + \dot{\alpha}^2\sin^2\beta \ . \end{aligned} \tag{71}$$

Note that, in leading order, the covariant derivative is simply the usual time derivative.

4.4.1. Leading order

In leading order ($\mathcal{O}(\Omega)$), the Lagrangian is

$$L_{\mathrm{LO}} = \frac{1}{2}\dot{\varphi_0}^2 + \left|\dot{\phi}_2\right|^2 - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}|\phi_2|^2 \ . \tag{72}$$

This Lagrangian describes harmonic vibrations of the quadrupole degrees of freedom. The Legendre transform yields the Hamiltonian

$$H_{\mathrm{LO}} = \frac{1}{2}p_0^2 + \frac{1}{4}\left(p_{2r}^2 + p_{2i}^2\right) + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{\omega_2^2}{4}\left(\phi_{2r}^2 + \phi_{2i}^2\right) \ . \tag{73}$$

Here, ϕ_{2r} and ϕ_{2i} are real variables that denote the real and imaginary parts of ϕ_2, respectively, i. e.

$$\phi_2 = \phi_{2r} + i\phi_{2i} \ . \tag{74}$$

The momenta are defined as

$$p_0 \equiv \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\varphi}_0} \ , \quad p_{2r} \equiv \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\phi}_{2r}} \ , \quad p_{2i} \equiv \frac{\partial L_{\mathrm{LO}}}{\partial\dot{\phi}_{2i}} \ . \tag{75}$$

One might wonder whether the most general leading-order Lagrangian should not have dimensionless low-energy constants in front of its kinetic terms. Note, however, that any such constants could be absorbed by a redefinition of the variables φ_0 and ϕ_2, and a rescaling of the oscillator frequencies ω_0 and ω_2.

The leading order part of the Hamiltonian (73) describes an axially symmetric harmonic oscillator in three dimensions. The canonical quantization rules $p_0 \to -i\partial_{\varphi_0}$,

$p_{2r} \to -i\partial_{\phi_{2r}}$, and $p_{2i} \to -i\partial_{\phi_{2i}}$ yields the Hamiltonian operator $\hat{H}_{\rm LO}$. It is of advantage to seek not the Cartesian eigenstates but rather eigenstates which reflect the axial symmetry. To his purpose, we write $\phi_2 = \varphi_2 e^{i\gamma}$ and obtain the leading-order Hamiltonian

$$\hat{H}_{\rm LO} = -\frac{1}{2}\partial^2_{\varphi_0} + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{1}{4}\left(-\partial^2_{\varphi_2} - \frac{1}{\varphi_2}\partial_{\varphi_2} + \frac{\hat{l}_2^2}{\varphi_2^2} + \omega_2^2\varphi_2^2\right) . \tag{76}$$

Here,

$$\hat{l}_2 = -i\partial_\gamma \tag{77}$$

is the operator corresponding to the classical expression (63). The eigenstates and eigenenergies fulfill

$$\hat{H}_{\rm LO}|n_0 n_2 l_2\rangle = E(n_0, n_2, l_2)|n_0 n_2 l_2\rangle . \tag{78}$$

The energies are

$$E(n_0, n_2, l_2) = \omega_0\left(n_0 + \frac{1}{2}\right) + \frac{\omega_2}{2}(2n_2 + |l_2| + 1) . \tag{79}$$

Here, $n_0 = 0, 1, 2, \ldots$ denotes the quantum number in the φ_0-coordinate (i.e. excitation along the symmetry axis), $n_2 = 0, 1, 2, \ldots$ denotes the radial quantum number associated with the radius $\varphi_2 = (\phi_{2r}^2 + \phi_{2i}^2)^{1/2}$, while $l_2 = 0, \pm 1, \pm 2, \ldots$ denotes azimuthal quantum number, i.e. the quantum number of the operator $\hat{l}_2$. Clearly, the energies (79) are of order Ω.

4.4.2. Next-to-leading order

In next-to-leading order ($\mathcal{O}(\xi)$), the Lagrangian is

$$L_{\rm NLO} = L_{\rm LO} + \frac{v^2}{2}\left(E_x^2 + E_y^2\right) - 4E_z{\rm Im}\left(\dot{\phi}_2\phi_2^*\right) + R_{\rm NLO} \tag{80}$$

with

$$\begin{aligned} R_{\rm NLO} &= \left(Z_{00}\varphi_0^2 + Z_{02}|\phi_2|^2\right)\dot{\varphi}_2^2 + \left(Z_{20}\varphi_0^2 + Z_{22}|\phi_2|^2\right)|D_t\phi_2|^2 \\ &- \sum_{k+2l=3,4} v_{kl}\varphi_0^k|\phi_2|^{2l} . \end{aligned} \tag{81}$$

The terms of the Lagrangian (80) describe the rotations, and the coupling between rotations and vibrations, while anharmonic corrections to the vibrations are encoded in the remainder (81). It is assumed that the "wave function renormalizations" $Z_{00}, Z_{20}, Z_{02}, Z_{22}$ are of order ξ. Under this assumption, the remainder $R_{\rm NLO}$ merely adds perturbative corrections to the vibrations. As we are mainly interested in the lowest energetic vibrational states (the band heads) and the modifications of the rotational bands due to the vibrational coupling, we do not consider these anharmonicities. A Legendre transformation yields the Hamiltonian

$$H_{\rm NLO} = H_{\rm LO} + \frac{1}{2v^2}\left(p_\beta^2 + \frac{1}{\sin^2\beta}\left(p_\alpha - 2l_2\cos\beta\right)^2\right) . \tag{82}$$

The comparison with the angular momentum (64) shows that

$$H_{\rm NLO} = H_{\rm LO} + \frac{1}{2v^2}\left(Q^2 - 4l_2^2\right) \ . \tag{83}$$

Note that the identification of the squared vacuum expectation value v^2 with the moment of inertia is arbitrary but convenient. We could also have employed the low-energy constant C_0 and avoided any reference to v. The vacuum expectation value v is not an observable, and the most general Lagrangian can be expressed in terms of the variable φ_0 instead of $\phi_0 = v + \varphi_0$.

Let us turn to the quantization of the Hamiltonian at next-to-leading order. From the simple form of the classical Hamiltonian (83) we expect a rotational band upon each vibrational state, with a spin of the band-head equal to $2l_2$. It is instructive to derive this anticipated result. For the quantization of the Nambu-Goldstone modes we use Eq. (36). This yields the Hamiltonian operator

$$\hat{H}_{\rm NLO} \ = \ \hat{H}_{\rm LO} - \frac{1}{2v^2}\frac{1}{\sin\beta}\partial_\beta\left(\sin\beta\partial_\beta\right) + \frac{1}{2v^2}\frac{1}{\sin^2\beta}\left(-i\partial_\alpha - 2\hat{l}_2\cos\beta\right)^2 \ . \tag{84}$$

The eigenfunctions of $\hat{H}_{\rm NLO} - H_{\rm LO}$ are again given in terms of the Wigner d function as $e^{-im\alpha}d^l_{ml_2}(\beta)$. The energies at order ξ thus are

$$\begin{aligned} E(n_0,n_2,l_2,l) \ &= \ \omega_0\left(n_0 + \frac{1}{2}\right) + \frac{\omega_2}{2}(2n_2 + |l_2| + 1) \\ &+ \ \frac{1}{2v^2}\left(l(l+1) - (2l_2)^2\right) \ . \end{aligned} \tag{85}$$

Here, the angular momentum l is an integer with $l \geq 2|l_2|$ and depends on the l_2 value of the vibrational state with quantum numbers (n_0, n, l_2) under consideration. The spectrum (85) is the well known rotational-vibrational spectrum: upon each vibrational state, there is a rotational band. Note that the moment of inertia is, at this order of the effective theory, equal for each vibrational state.

The low-energy constants of the effective theory are the vacuum expectation value v (which fixes the moment of inertia), and the frequencies ω_0 and ω_2 (which determine the spacing of the ground state to so-called β band with quantum numbers $(n_0 = 1, n_2 = 0, l_2 = 0)$ and the so-called γ band with quantum numbers $(n_0 = 0, n_2 = 0, l_2 = \pm 1)$, respectively. If we are only interested in these three bands (as is often the case), there is no need to determine the low-energy constants of the anharmonic corrections (81).

4.4.3. Next-to-next-to-leading order

At next-to-next-to-leading order (N^{2}LO) ($\mathcal{O}(\xi^2/\Omega)$), new terms enter. There will be (relatively uninteresting) terms $R_{\rm N^2LO}$ that only depend on the vibrational degrees of freedom. More interesting are the additional terms that couple rotations and vibrations. These are

$$\begin{aligned} \Delta L_{\rm N^2LO} \ &= \ 4|\phi_2|^2 E_z^2 \\ &+ \ D_0\left(E_x^2 + E_y^2\right)\varphi_0^2 + F_0\left(E_x^2 + E_y^2\right)\dot{\varphi}_0^2 \\ &+ \ D_1\varphi_0\left(\phi_2 E_{-1}^2 + \phi_{-2}E_{+1}^2\right) + F_1\dot{\varphi}_0\left(E_{+1}^2 D_t\phi_{-2} + E_{-1}^2 D_t\phi_2\right) \\ &+ \ D_2\left(E_x^2 + E_y^2\right)|\phi_2|^2 + F_2\left(E_x^2 + E_y^2\right)|D_t\phi_2|^2 \ . \end{aligned} \tag{86}$$

Here, D_0, D_1, and D_2 are dimensionless low-energy constants of order one, while F_1, F_2, and F_3 are expected to be of order Ω^{-2}. Many of these terms are not considered in the generalizations [4, 5, 6] of the Bohr Hamiltonian.

Let us briefly discuss the quantization procedure. As is well known, the quantization of a classical Hamiltonian is only without ambiguity in flat space, i.e. when the metric exhibits no curvature. For a metric with nonzero curvature, one can think of the system as a constrained system, i.e. the motion is constrained to a (curved) hypersurface in a higher-dimensional space. Kaplan, Maitra, and Heller [49] showed that the quantization of such a constrained system depends on the exact nature of the constraints, i.e. on details regarding the implementation of the constraining forces which – in the limit of infinite forces – confine the motion to the hypersurface. These authors point out that the *physical* situation (i.e. the implementation of the constraints in a limiting process) resolves the ambiguity. Our quantization follows this rule. The leading order vibrational Hamiltonian is quantized without ambiguity. When the rotations are coupled to the vibrations, we know that – in the limit of infinite frequencies ω_0 and ω_2 – the physics of Nambu-Goldstone bosons as presented in Sect. 3 must result. Thus, we employ the previously derived quantization rules. When considering terms beyond next-to-leading order, the quantization rules remain unchanged.

5. Coupling of fermions to Nambu-Goldstone modes

For deformed odd-mass nuclei, it is not sufficient to simply include terms that are first order in the time derivative (such as Wess-Zumino terms) into the Lagrangian. Such nuclei oft exhibit several low-lying band heads with rotational bands upon them. In this case, fermionic degrees of freedom must be included in the description, and we need to coupled nucleon fields N to Nambu-Goldstone modes. As discussed in Section 2, we write

$$N = g\chi \, , \tag{87}$$

where $g \in \mathcal{H} = SO(2)$ as given in Eq. (3). The most general rotationally invariant Lagrangian consists of combination of χ, $D_t\chi$, and the functions E_x and E_y that is formally invariant under rotations around the z-axis. In lowest order, we have the Lagrangian [50]

$$L = \frac{C_0}{2}\left(E_x^2 + E_y^2\right) + \chi^\dagger \hat{T}\chi + \chi^\dagger\left(i\partial_t + E_z\hat{J}_z\right)\chi + C_1\chi^\dagger\left(E_x\hat{J}_x + E_y\hat{J}_y\right)\chi \, . \tag{88}$$

Here, $\hat{T}$ is the operator of the kinetic energy (or the single-particle energy) for the fermion. The Lagrangian (88) reminds us of the particle-rotor model [2, 6]. The choice of the spin q of the fermion field is determined by the nucleus under consideration. On could, of course, identify χ with a two-spinor $\chi(\vec{r})$ (Then, we would also need to integrate over space in the Lagrangian.) However, for heavy nuclei, it might be more adequate to identify χ with the shell-model orbital (n, l, j, τ_z) (denoting radial quantum number, orbital angular momentum, total spin, and isospin projection, respectively) that is most relevant for the open-shell nucleus under consideration.

It is insightful to consider the limit of a spin-q fermion with components $\chi = (\chi_q, \vec{0})^T$. Let us assume that χ_q is the ground state of the operator $\hat{T}$, with $\hat{T} \sim \Lambda \gg \xi$. In this case, the low-energy Lagrangian (88) for the Nambu-Goldstone modes becomes (we use $|\chi_q|^2 = 1$)

$$L = \frac{C_0}{2}\left(E_x^2 + E_y^2\right) + qE_z + \langle\chi|\hat{T}|\chi\rangle \ . \tag{89}$$

Apart from the irrelevant constant $\langle\chi|\hat{T}|\chi\rangle$, this Lagrangian has the same form as the Lagrangian (42) with the Wess-Zumino term. This example shows how the Wess-Zumino term arises from high-energy fermionic degrees of freedom. The detailed discussion of fermions coupled to Nambu-Goldstone modes is beyond the scope of this paper. However, the techniques we developed in this paper seem to be applicable to this case, too. For a nucleus with a finite ground-state spin, one would need to couple the Nambu-Goldstone modes to quadrupole bosons and single-particle degrees of freedom.

6. Summary

This paper developed an effective theory for deformed nuclei. The approach exploits the spontaneous breaking of the rotational symmetry and is based on the separation of scale between low-energetic Nambu-Goldstone rotational modes, higher energetic quadrupole modes, and single-particle degrees of freedom at much higher energies. The nonlinear realization of the rotation group is key to the effective theory, and a power counting is established. We derived the Lagrangian (and Hamiltonian) for the Nambu-Goldstone modes and the for the quadrupole bosons coupled to Nambu-Goldstone modes in next-to-leading order. At this order in the power counting, well known results from phenomenological models were rederived in a model-independent way. More interesting phenomena are expected at next-to-next-to-leading order, as the effective theory predicts the appearance of terms that are not employed in the phenomenological models. The effective theory treats nuclei with finite spins in their ground states (odd-odd and odd-mass nuclei) on equal footing to nuclei with zero ground-states spins (even-even nuclei).

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Appendix A. Legendre Transformation

Let the Lagrangian depend on velocities $\partial_t\vec{x}$ ($\vec{x} = (x_1, \ldots, x_N)$) as follows

$$L = \frac{1}{2}\left(\partial_t\vec{x}\right)^T \hat{G}\partial_t\vec{x} + \vec{k}^T\partial_t\vec{x} \ . \tag{A.1}$$

Here $\hat{G}$ denotes the symmetric mass matrix, ^{T} denotes the transpose, and $\vec{k}$ is a constant vector. Then, the Legendre transformation

$$p_j = \frac{\partial L}{\partial \dot{x}_j} \tag{A.2}$$

yields the Hamiltonian function

$$H = \frac{1}{2}\left(\vec{p} - \vec{k}\right)^T \hat{G}^{-1}\left(\vec{p} - \vec{k}\right) \; . \tag{A.3}$$

For the quantization of the Nambu-Goldstone modes (α, β), we follow [46] and write the Lagrangian as a quadratic form

$$L = \frac{1}{2}(\dot{\beta}, \dot{\alpha})\, \hat{G} \left(\begin{array}{c} \dot{\beta} \\ \dot{\alpha} \end{array} \right) \; . \tag{A.4}$$

This defines the (2×2) matrix of the metric $\hat{G}$. The Hamiltonian becomes (let us use units where $\hbar = 1$)

$$\hat{H} \;\; = \;\; \frac{-1}{2\sqrt{\det \hat{G}}}(\partial_\beta, \partial_\alpha)\hat{G}^{-1}\sqrt{\det \hat{G}} \left(\begin{array}{c} \partial_\beta \\ \partial_\alpha \end{array} \right) \; . \tag{A.5}$$

For an in-depth discussion of the quantization of constrained systems, the reader is refered to Ref. [49].

Appendix B. Nambu-Goldstone modes revisited

The derivations presented in Sect. 2 are more formal than necessary if one is only interested in the physics of Nambu-Goldstone modes. Here, we follow Leutwyler [29] for a quicker derivation. The dynamics of the Nambu-Goldstone modes is determined by the coset $SO(3)/SO(2)$ which is isomorphic to the two-sphere S^2. Thus, the Nambu-Goldstone modes parameterize the two-sphere, and we can choose the parameterization

$$\vec{n}(\alpha, \beta) = \left(\begin{array}{c} \cos\alpha \sin\beta \\ \sin\alpha \sin\beta \\ \cos\beta \end{array} \right) \; . \tag{B.1}$$

Here, α and β are time-dependent variables. The simplest low-energy Lagrangian that is invariant under time reversal is

$$L \;\; = \;\; \frac{C_0}{2}\left(\partial_t \vec{n}\right) \cdot \left(\partial_t \vec{n}\right) = \frac{C_0}{2}\left(\dot{\beta}^2 + \dot{\alpha}^2 \sin^2\beta\right) \; . \tag{B.2}$$

This is Eq. (33). One can pursue this direction further and also construct the Wess-Zumino term. Details are given in Ref. [29].

Appendix C. Transformation properties under rotations

Let

$$g(\alpha, \beta) = e^{-i\alpha \hat{J}_z} e^{-i\beta \hat{J}_y} \tag{C.1}$$

denote an element of the coset $SO(3)/SO(2)$ and

$$r(\alpha, \beta, \gamma) = e^{-i\alpha \hat{J}_z} e^{-i\beta \hat{J}_y} e^{-i\gamma \hat{J}_z} \tag{C.2}$$

be a general rotation parameterized in terms of the three Euler angles. Let

$$h(\gamma) = e^{-i\gamma \hat{J}_z} \tag{C.3}$$

be a rotation of the subgroup $\mathcal{H} = SO(2)$.

The product rg is again a rotation, and we have

$$r(\alpha_2, \beta_2, \gamma_2) g(\alpha, \beta) = r(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \ . \tag{C.4}$$

Expressions for $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ in terms of the other angles are well known [45]

$$\begin{aligned} \cot(\tilde{\alpha} - \alpha_2) &= \cos\beta_2 \cot(\alpha + \gamma_2) + \cot\beta \frac{\sin\beta_2}{\sin(\alpha + \gamma_2)} \ , \\ \cos\tilde{\beta} &= \cos\beta\cos\beta_2 - \sin\beta\sin\beta_2\cos(\alpha + \gamma_2) \ , \\ \cot\tilde{\gamma} &= \cos\beta\cot(\alpha + \gamma_2) + \cot\beta_2 \frac{\sin\beta}{\sin(\alpha + \gamma_2)} \ . \end{aligned} \tag{C.5}$$

Due to the definitions (C.1)-(C.3), we also have

$$r(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = g(\tilde{\alpha}, \tilde{\beta}) h(\tilde{\gamma}) \ . \tag{C.6}$$

Under a rotation $r(\alpha_2, \beta_2, \gamma_2)$ the element g of the coset transforms as

$$g = g(\alpha, \beta) \to \tilde{g} \equiv g(\tilde{\alpha}, \tilde{\beta}) \ , \tag{C.7}$$

and $\tilde{g} = \tilde{g}(g, r)$ depends on the Nambu-Goldstone modes (α, β) that parameterize g and the rotation angles $(\alpha_2, \beta_2, \gamma_2)$ of r. The rotation thus maps the Nambu-Goldstone modes (α, β) into $(\tilde{\alpha}, \tilde{\beta})$, and one needs to employ the transformations (C.5) to obtain explicit results. These transformation laws are complicated, but we do not need them for the construction of Lagrangians that are invariant under rotations (as shown in Subsection 2). They do, however, enter the derivation of the conserved quantities, i.e. the components of the angular momentum. For this purpose, we need to know the transformation laws for rotations by infinitesimal angles $\delta\omega_k$ around the $k = x, y, z$ axes. The corresponding rotations are $r(-\pi/2, \delta\omega_x, \pi/2)$, $r(0, \delta\omega_y, 0)$, and $r(\delta\omega_z, 0, 0)$ respectively. Employing the transformation laws (C.5) we find

$$\begin{pmatrix} \delta\alpha \\ \delta\beta \end{pmatrix} = \hat{M} \begin{pmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{pmatrix} \ , \tag{C.8}$$

with

$$\hat{M} = \begin{pmatrix} -\cot\beta\cos\alpha & -\cot\beta\sin\alpha & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{pmatrix} \ . \tag{C.9}$$

This is Eq. (26).

We can repeat these considerations for the quadrupole phonons. As shown in Sect. 2, under rotations $r(\alpha_2, \beta_2, \gamma_2)$, the quadrupole modes ϕ transform as

$$\phi \to h(\tilde{\gamma})\phi = e^{-i\tilde{\gamma}\hat{J}_z}\phi \ . \tag{C.10}$$

According to the transformation laws (C.5), under infinitesimal rotations by $\delta\omega$ around the x, y, and z axis, the angle $\tilde{\gamma}$ becomes $\tilde{\gamma} = \delta\omega \frac{\cos\alpha}{\sin\beta}$, $\tilde{\gamma} = \delta\omega \frac{\sin\alpha}{\sin\beta}$, and $\tilde{\gamma} = 0$ respectively. The application of Eq. (C.10) shows that the quadrupole fields (with $\phi_2 = \phi_{2r} + i\phi_{2i}$) transform as

$$\begin{pmatrix} \delta\phi_{2r} \\ \delta\phi_{2i} \\ \delta\varphi_0 \end{pmatrix} = \hat{N} \begin{pmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{pmatrix} , \tag{C.11}$$

where

$$\hat{N} \equiv \begin{pmatrix} +2\frac{\cos\alpha}{\sin\beta}\phi_{2i} & +2\frac{\sin\alpha}{\sin\beta}\phi_{2i} & 0 \\ -2\frac{\cos\alpha}{\sin\beta}\phi_{2r} & -2\frac{\sin\alpha}{\sin\beta}\phi_{2r} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{C.12}$$

As expected for the nonlinear realization, φ_0 is invariant under rotations. We can use these transformation laws to apply Noether's theorem.

Appendix D. Application of Noether's theorem

Let us recall Noether's theorem for our purposes. Consider a Lagrangian L of coordinates q_ν and velocities $\dot{q}_\nu$, $\nu = 1, \ldots, N$. Let us also consider a coordinate transformation that depends on parameters ω_k, $k = 1, \ldots, K$. For infinitesimal small arguments $\delta\omega_k$ the coordinates change by

$$\delta q_\nu = \sum_{k=1}^{K} \hat{M}_{\nu k} \delta\omega_k . \tag{D.1}$$

The coordinate transformation changes the Lagrangian by the amount

$$\delta L = \sum_{\nu=1}^{N} \left(\frac{\partial L}{\partial \dot{q}_\nu} \delta\dot{q}_\nu + \frac{\partial L}{\partial q_\nu} \delta q_\nu \right) . \tag{D.2}$$

We employ the equations of motions

$$\frac{\partial L}{\partial q_\nu} = \partial_t \frac{\partial L}{\partial \dot{q}_\nu} , \quad \nu = 1, \ldots, N , \tag{D.3}$$

and find

$$\delta L = \partial_t \sum_{\nu=1}^{N} \frac{\partial L}{\partial \dot{q}_\nu} \delta q_\nu . \tag{D.4}$$

Let us consider the case that only one parameter ω_k is varied. Thus, the Lagrangian changes by

$$\delta L = \delta\omega_k \partial_t \sum_{\nu=1}^{N} \frac{\partial L}{\partial \dot{q}_\nu} \hat{M}_{\nu k} . \tag{D.5}$$

In our case, the transformations are rotations. In the case that the ground state is invariant under time reversal, we have $\delta L = 0$ for arbitrary $\delta\omega_k$. Thus,

$$Q_k \equiv \sum_{\nu=1}^{N} \frac{\partial L}{\partial \dot{q}_\nu} \hat{M}_{\nu k} = \sum_{\nu=1}^{N} p_\nu \hat{M}_{\nu k} \tag{D.6}$$

is a conserved quantity (i.e. $\partial_t Q_k = 0$). Here, p_ν is the momentum conjugate to q_ν. We insert the matrix $\hat{M}$ into Eq. (D.6) and obtain the three components of the angular momentum (31).

We can repeat these considerations when quadrupole phonons are coupled to the Nambu-Goldstone modes. The application of Noether's theorem yields

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \hat{M}^T \begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} + \hat{N}^T \begin{pmatrix} p_{2r} \\ p_{2i} \\ p_0 \end{pmatrix} . \tag{D.7}$$

Thus ($l_2 \equiv \phi_{2r} p_{2i} - \phi_{2i} p_{2r}$),

$$\begin{aligned} Q_x &= -2\frac{\cos\alpha}{\sin\beta} l_2 - p_\beta \sin\alpha - p_\alpha \cot\beta \cos\alpha , \\ Q_y &= -2\frac{\sin\alpha}{\sin\beta} l_2 + p_\beta \cos\alpha - p_\alpha \cot\beta \sin\alpha , \\ Q_z &= p_\alpha . \end{aligned} \tag{D.8}$$

This is Eq. (62).

In the case that the ground state breaks time reversal symmetry, the Lagrangian is not invariant under infinitesimal rotations but changes by the amount (39) due to the Wess-Zumino term. Thus,

$$\delta L = \delta L_{\rm WZ} = q \left(\delta\omega_x \partial_t \left(\frac{\cos\alpha}{\sin\beta} \right) + \delta\omega_y \partial_t \left(\frac{\sin\alpha}{\sin\beta} \right) \right) . \tag{D.9}$$

This change clearly is a total time derivative, and equating the expressions (D.5) and (D.9) for $k = x, y, z$ yields the conserved quantities (40).

Appendix E. Solution of differential equation

We discuss the diagonalization of the Hamiltonians (44) and (84). The Nambu-Goldstone modes are essentially governed by the Hamiltonian

$$\hat{H} = -\frac{1}{\sin\beta} \partial_\beta \sin\beta \partial_\beta + \frac{1}{\sin^2\beta} \left(-i\partial_\alpha + q\cos\beta \right)^2 . \tag{E.1}$$

For the eigenfunction $f^l_{mq}(\alpha, \beta)$, we make the ansatz

$$f^l_{mq}(\alpha, \beta) = g^l_{mq}(\beta) e^{-im\alpha} . \tag{E.2}$$

Here l is a quantum number that needs to be determined. This yields the eigenvalue equation

$$\left\{ -\frac{1}{\sin\beta} \partial_\beta \sin\beta \partial_\beta + \frac{1}{\sin^2\beta} \left(m - q\cos\beta \right)^2 \right\} g_{mq}(\beta) = E(l, m, q) g_{mq}(\beta) . \tag{E.3}$$

We expand the square, rewrite $\cos^2\beta = 1 - \sin^2\beta$ and find

$$\left\{-\frac{1}{\sin\beta}\partial_\beta \sin\beta\partial_\beta + \frac{1}{\sin^2\beta}\left(m^2 - 2mq\cos\beta + q^2\right)\right\} g^l_{mq}(\beta) = \left(E(l,m,q) + q^2\right) g^l_{mq}(\beta) \ . \tag{E.4}$$

For the eigenfunctions of this differential operator, we recall that the Wigner D functions

$$D^l_{mq}(\alpha,\beta,\gamma) \equiv e^{-im\alpha} d^l_{mq}(\beta) e^{-iq\gamma} \tag{E.5}$$

(and the functions $d^l_{mq}(\beta)$ themselves) solve the differential equation [45]

$$\left\{-\frac{1}{\sin\beta}\partial_\beta \left(\sin\beta\partial_\beta\right) + \frac{1}{\sin^2\beta}\left(m^2 - 2mq\cos\beta + q^2\right)\right\} D^l_{mq}(\alpha,\beta,\gamma) = l(l+1) D^l_{mq}(\alpha,\beta,\gamma) \ . \tag{E.6}$$

Thus, we have to identify the eigenfunctions as $g^l_{mq}(\beta) = d^l_{mq}(\beta)$, and the eigenvalue is $E(l,m,q) = l(l+1) - q^2$. It is well known [45] that for $q = 0$, the Wigner function $D^l_{m0}(\alpha,\beta,\gamma)$ is proportional to the spherical harmonics $Y_{l-m}(\beta,\alpha)$.

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