Dynamical Symmetries of Dirac Hamiltonian

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Several dynamical symmetries of the Dirac Hamiltonian are reviewed in a systematic manner and the conditions under which such symmetries hold. These includes relativistic spin and orbital angular momentum symmetries, $SO(4) \times SU_\sigma(2)$ symmetry for Dirac Hydrogen atom, $SO(3) \times SU_\sigma(2)$ symmetry for the relativistic simple harmonic oscillator. The energy spectrum in each case is calculated from group-theoretic considerations.

I. INTRODUCTION

As is well known symmetries play an important role in the progress of physics. Symmetries are of two types: (i) Geometric, the well known example of which are space-time symmetries, e.g. rotation (ii) Dynamical, where the underlying dynamics manifest some symmetry. This is well illustrated by the Hydrogen atom in Schrodinger theory, where the energy levels show n^2 degeneracy. The $(2l+1)$ degeneracy with respect to magnetic quantum number is due to rotational symmetry of the potential $-\frac{\alpha}{r}$, where α is fine structure constant. For one energy value E, there are $\sum_{l=0}^{n-1}(2l+1) = n^2$ different possible eigen-functions; such a degeneracy occurs only for $1/r$ potential of force. Usually a degeneracy is associated with a symmetry and it is known [1] that in this case there is an external symmetry $\left[\vec{R}, H\right] = 0$,with $H = \frac{\vec{p}^2}{2m} - \frac{\alpha}{r}$, and $\vec{R} = \frac{1}{2m}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \frac{\alpha}{r}\vec{r}$. The operator $\vec{R}$ in Quantum Mechanics corresponds to the Lenz's vector in the classical Kepler problem, where the bound orbits close on themselves and do not precess. The orbital angular momentum $\vec{L}$ and $\vec{K} = \sqrt{\frac{-m}{2H}}\vec{R}$ ($\vec{K}$ is hermitian when acting on eigenstates of H with-negative energy eigenvalues- the ones in which we are interested; $\sqrt{\frac{-m}{2H}}$ commutes with $\vec{R}$ and $\vec{L}$) generate $SO(4)$ algebra, which is isomorphic to $SU_M(2) \times SU_N(2)$ generated by $\vec{M} = \frac{\vec{L}+\vec{K}}{2}$, $\vec{N} = \frac{\vec{L}-\vec{K}}{2}$. This symmetry leads to n^2 degeneracy.

Similarly for the non- relativistic harmonic oscillator, the energy spectrum shows degeneracies in addition to those which arise due to rotational invariance. As is well known, the energy spectrum $E_N = \frac{1}{2}\hbar w(2N+3)$ depends on the quantum number $N = 2n+l$, where $n \geq 0$ is the radial quantum

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number and l is the orbital angular momentum. Thus all states with $l = N, N-2, ...0$ or 1 have the same energy. These degeneracies are produced by an $SU(3)$ dynamical symmetry [2].

Obviously the above symmetries are broken in the relativistic quantum mechanics, where a spin 1/2 particle satisfies the Dirac equation. As is well known this is because the spin-orbit coupling leads to splitting of the energy levels [3]. The question then naturally arises, under which conditions the larger symmetries are obtained in the Dirac Hamiltonian. This has been answered for harmonic oscillator in [4] and following the method of [4], for the Hydrogen atom in [5]. The purpose of this paper is to review the previous work mentioned above and to discuss the symmetries of the Dirac Hamiltonian in a systematic way, using the Dirac algebra and for harmonic Oscillator $SU(3)$ algebra generated in the Gell-Mann basis [6]. The energy spectrum in each case has also been calculated from group-theoretic considerations.

II. DYNAMICAL SPIN AND ORBITAL ANGULAR MOMENTUM SYMMETRY FOR THE DIRAC HAMILTONIAN

The Dirac Hamiltonian is given by

$$H = \vec{\alpha} \cdot \vec{p} + V_V\ (\vec{r}) + \beta m \tag{1}$$

where for the Hydrogen atom $V_V\ (\vec{r}) = -\frac{\alpha}{r}$ and is the time component of electromagnetic potential $A^\mu\ (\vec{r})$, $(\mu = 0, 1, 2, 3)$. If one introduces a Lorentz scalar potential $V_S\ (\vec{r})$, then

$$H = \vec{\alpha} \cdot \vec{p} + V_V\ (\vec{r}) + \beta (V_S\ (\vec{r}) + m) \tag{2}$$

The Dirac Hamiltonian (2) is invariant under a spin symmetry [7], $\left[H, \vec{S}\right] = 0$, provided that V_V $(\vec{r}) = V_S\ (\vec{r}) + U$, where U is a constant potential. Here the generators $\vec{S}$ form the spin $SU(2)$ algebra and are given by

$$\vec{S} = \begin{pmatrix} \vec{s} & 0 \\ 0 & u_p\ \vec{s}\ u_p \end{pmatrix} \tag{3}$$

where $\vec{s} = \vec{\sigma}/2$ are usual spin generators and $u_p = \frac{\vec{\sigma} \cdot \vec{p}}{p}$ is the helicity unitary operator. It is easy to check that

$$[S_i, S_j] = i\ \epsilon_{ijk}\ S_k \tag{4}$$

In the Pauli representation of Dirac matrices,

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = -i\alpha^1\alpha^2\alpha^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

By introducing $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, one can write the Dirac Hamiltonian (2) and the spin operator (3) as

$$H = \gamma^5 \beta \; \vec{\Sigma} \cdot \vec{p} + V_V \;(\vec{r}) + \beta (V_S \;(\vec{r}) + m) \tag{6}$$

$$\vec{S} = \frac{1}{2} \left[\beta \vec{\Sigma} + (1 - \beta) \, \vec{\Sigma} \cdot \vec{p} \; \vec{p} \frac{1}{p^2} \right] \tag{7}$$

Then since $[\beta, 1 - \beta] = 0$, $\left[\beta, \vec{\Sigma}\right] = 0$, $\left[\gamma^5, \vec{\Sigma}\right] = 0$, $\left[\gamma^5, \beta\right]_+ = 0$ and $\left[\vec{\Sigma} \cdot \vec{p}, \vec{\Sigma} \cdot \vec{p} \; p^i\right] = 0$

$$\begin{aligned} \left[H, \vec{S}\right] &= \frac{1}{4} \left[\gamma^5, \beta\right] \left[\vec{\Sigma} \cdot \vec{p} \;,\; \Sigma^i\right]_+ + \frac{1}{4} \left[\gamma^5, (1-\beta)\right] \left[\vec{\Sigma} \cdot \vec{p}, \frac{\vec{\Sigma} \cdot \vec{p} \; p^i}{p^2}\right]_+ \\ & \quad + \frac{(1-\beta)}{2} \left[V_V \;(\vec{r}), (1-\beta) \, \frac{\vec{\Sigma} \cdot \vec{p} \; \vec{p}}{p^2}\right] + \frac{\beta \,(1-\beta)}{2} \left[V_S \;(\vec{r}), \frac{\vec{\Sigma} \cdot \vec{p}}{p^2} p^i\right] \\ &= \frac{1}{2} \gamma^5 \beta \; 2 \; \delta^{ij} p^j - \frac{1}{2} \gamma^5 \beta \; 2 \; p^i \; + \frac{(1-\beta)}{2} \Sigma^j \; \left[\left(V_V \;(\vec{r}) - V_S \;(\vec{r})\right), \frac{p^j p^i}{p^2} \right] \end{aligned} \tag{8}$$

Thus $\left[H, \vec{S}\right] = 0$ if $\partial^i V_V \;(\vec{r}) = \partial^i V_S \;(\vec{r})$ or $V_V \;(\vec{r}) = V_S \;(\vec{r}) + U$. Further for spherically symmetric potentials, $V_V \;(\vec{r}) = V_V \;(r)$, $V_S \;(\vec{r}) = V_S \;(r)$, the Dirac Hamiltonian has an additional invariant algebra [7], $\left[H, \vec{L}\right] = 0$ where

$$\vec{L} = \begin{pmatrix} \vec{l} & 0 \\ 0 & u_p \, \vec{l} \, u_p \end{pmatrix} \tag{9}$$

and $\vec{l} = \vec{r} \times \vec{p}$ is the orbital angular momentum. One can write

$$\vec{L} = \vec{l} + \left[\frac{(1-\beta)}{2} \vec{\Sigma} - \vec{\Sigma} \cdot \vec{p} \; \vec{p} / p^2 \right] \tag{10}$$

Then $\left[V_V \;(r), \vec{l}\right] = 0$, $\left[V_S \;(r), \vec{l}\right] = 0$ and it follows as above that

$$\left[H, \vec{L}\right] = 0, \qquad [L_i, L_j] = i \; \epsilon_{ijk} \; L_k \tag{11}$$

Thus $\vec{S}$ and $\vec{L}$ are separately constants of motion while $\vec{s}$ and $\vec{l}$ are not, but $\vec{s} + \vec{l}$ is.The above symmetries find applications in the hadron and nuclear spectroscopies [8] and $Q(\bar{Q})$ $q(\bar{q})$ meson spectroscopy [9] where spin-orbit splitting is seen to be suppressed [10]. Here Q is a heavy quark c or b and q is light quark u, d, or s,

Finally the Dirac Hamiltonian, which has relativistic dynamical spin and orbital angular momentum symmetries, is

$$H = \gamma^5 \vec{\Sigma} \cdot \vec{p} + (1 + \beta) V(r) + \beta m \tag{12}$$

where constant U can be absorbed in the mass term m. We note that

$$(H+m)=\gamma^5\vec{\Sigma}\cdot\vec{p}+(1+\beta)(V+m) \tag{13}$$

$$H^2-m^2=\vec{p}^2+2(1+\beta)V(m+V)+\gamma^5\left[(1-\beta)V\vec{\Sigma}\cdot\vec{p}+(1+\beta)V\vec{\Sigma}\cdot\vec{p}\right] \tag{14}$$

III. $SO(4)\times SU_\sigma(2)$ SYMMETRY FOR A COULOMB LIKE POTENTIAL

Since in the non-relativistic limit $\beta\to 1+O(p^2/m^2)$ and $\gamma^5\to O(p/m)$, the natural generalization of the Lenz's vector in the Schrodinger theory, for the relativistic case is

$$2m\vec{R}\to 2m\vec{\Gamma}$$

where

$$2m\vec{\Gamma}=(1+\beta)\,f(r)\vec{r}+\vec{\Lambda}+\left[(1+\beta)\,\gamma_5 g(r)\vec{r}\vec{\Sigma}\cdot\vec{p}+(1-\beta)\,\gamma_5\vec{\Sigma}\cdot\vec{p}g(r)\vec{r}\right] \tag{15}$$

where now

$$\vec{\Lambda}=\vec{p}\times\vec{L}-\vec{L}\times\vec{p} \tag{16}$$

$\vec{L}$ is the relativistic orbital angular momentum defined in Eq. (10). Since H involves $\gamma^5\vec{\Sigma}\cdot\vec{p}$, therefore $\vec{\Gamma}$ should involve such a term and the second term in square brackets appears to make the operator hermitian. The functions $f(r)$ and $g(r)$ are to be determined from

$$\left[H,\vec{\Gamma}\right]=0 \tag{17}$$

Now $\vec{L}$ commutes with H and also with V and βm and therefore it also commutes with $\gamma^5\vec{\Sigma}\cdot\vec{p}$, it follows that $\left[\gamma^5\vec{\Sigma}\cdot\vec{p},\vec{\Lambda}\right]=0$ since $\vec{p}$ commutes with $\vec{\Sigma}\cdot\vec{p}$. Thus using $(\vec{\Sigma}\cdot\vec{p})^2=p^2$,

$$\begin{aligned}2m\left[H,\vec{\Gamma}\right] &= \gamma^5\left[\vec{\Sigma}\cdot\vec{p},\ f(r)\ \vec{r}-2(V+M)g(r)\ \vec{r}\right]+\gamma^5\beta\left[\vec{\Sigma}\cdot\vec{p},\ f(r)\ \vec{r}-2(V+M)g(r)\ \vec{r}\right]_+\\ &\quad +(1+\beta)[V,\vec{\Lambda}]+(1+\beta)\left[p^2,g(r)\ \vec{r}\right]\end{aligned} \tag{18}$$

The condition (17) gives

$$f(r)=2(V+m)g(r) \tag{19}$$

and

$$(1+\beta)[V,\vec{\Lambda}]+(1+\beta)\left[p^2,g(r)\ \vec{r}\right]=0 \tag{20}$$

Now using Eq. (10) and the fact that $(1+\beta)(1-\beta)=0$,

$$(1+\beta)[V,\vec{\Lambda}] \;=\; (1+\beta)\left[V,\vec{p}\times\vec{l}-\vec{l}\times\vec{p}\right] \\ =\; (1+\beta)\frac{i}{r}\frac{\partial V}{\partial r}\left[\vec{r}\times\vec{l}-\vec{l}\times\vec{r}\right] \tag{21}$$

On the other hand

$$\left[p^2, g(r)\;\vec{r}\right] \;=\; -2i\left[g(r)+\frac{\partial g}{\partial r}\right]\vec{p}-\frac{i}{r}\frac{\partial g}{\partial r}\left[\vec{r}\times\vec{l}-\vec{l}\times\vec{r}\right]-\frac{1}{r}\left[\frac{\partial}{\partial r}(g(r)+r\frac{\partial g}{\partial r})\right]\vec{r} \tag{22}$$

It is important to point out that so far we have made no commitment to the form of potentials $V(r)$ and $g(r)$. The condition (20) is satisfied if

$$g(r)=V(r) \tag{23}$$

and

$$V(r)+r\frac{\partial V}{\partial r}=0$$

which gives $V(r)$ =constt. $\frac{1}{r}$ i.e, the Coulomb potential $-\frac{\alpha}{r}$. It is the constraint (20) which forces the relations (23) and as a result $V(r)$ has to be the Coulomb potential.

Hence

$$2m\vec{\Gamma}=2(1+\beta)V(V+m)\;\vec{r}+\vec{\Lambda}+\vec{F}$$

where

$$\vec{F}=\gamma_5\left[(1-\beta)V\;\vec{r}\;\vec{\Sigma}\cdot\vec{p}+h.c.\right] \tag{24}$$

In order to find $\frac{1}{4m^2}\left[\Gamma^i,\Gamma^j\right]$, we note that

$$\left[\Lambda^i,\Lambda^j\right]=-4\epsilon^{ijk}\;p^2L^k \tag{25}$$

$$\left[(1+\beta)^2V(V+m)x^i,\Lambda^j\right]-i\leftrightarrow j=-4(1+\beta)V(V+2M)i\epsilon^{ijk}L^k \tag{26}$$

where we have used, $r\frac{\partial V}{\partial r}=-V(r)$ for $V=-\frac{\alpha}{r}$ and that [c.f. Eq. (10)]

$$(1+\beta)l^k=(1+\beta)L^k$$

Further

$$\left[\Lambda^i,F^j\right]-i\leftrightarrow j=-4i\epsilon^{ijk}\;\gamma^5\left[(1-\beta)V\;\vec{\Sigma}\cdot\vec{p}+(1+\beta)\;\vec{\Sigma}\cdot\vec{p}V\right]L^k \tag{27}$$

$$\left[(1+\beta)V(V+m)x^i, F^j\right] - i \leftrightarrow j = 0 \tag{28}$$

$$\left[F^i, F^j\right] = -4(1+\beta)\epsilon^{ijk} L^k V^2 \tag{29}$$

Collecting the various terms and using Eq. (14) with $V = -\alpha/r$we see that

$$\left[\Gamma^i, \Gamma^j\right] = -\frac{H^2 - m^2}{m^2} i\epsilon^{ijk} L^k \tag{30}$$

Defining $K^i = \sqrt{\frac{m^2}{m^2 - H^2}}\Gamma^i$, we have finally.

$$\begin{aligned} \left[K^i, K^j\right] &= i\epsilon^{ijk} L^k \\ \left[K^i, L^j\right] &= i\epsilon^{ijk} K^k \\ \left[L^i, L^j\right] &= i\epsilon^{ijk} L^k \end{aligned} \tag{31}$$

which generates $SO(4)$ algebra. Further S^i commutes with L^j as well as with H and K^j. Thus defining

$$\begin{aligned} M^i &= \frac{L^i + K^i}{2} \\ N^i &= \frac{L^i - K^i}{2} \end{aligned}$$

we see that

$$\begin{aligned} \left[M^i, M^j\right] &= i\epsilon^{ijk} M^k \\ \left[N^i, N^j\right] &= i\epsilon^{ijk} K^k \\ \left[M^i, N^j\right] &= 0 \end{aligned} \tag{32}$$

Thus the invariance group for the Dirac Hamiltonian (12) for the Coulomb potential of the hydrogen atom is $SU_M \otimes SU_N \otimes SU_\sigma(2)$, where $SU_\sigma(2)$ is the group generated by S^i given in Eq. (4).

The energy spectrum can now be easily determined

$$\Gamma^2 = \frac{H^2 - m^2}{m^2}\left(L^2 + 1\right) + \alpha^2 \frac{(H+m)^2}{m^2} \tag{33}$$

and $\vec{\Gamma} \cdot \vec{L} = \vec{L} \cdot \vec{\Gamma} = 0$, implying $\vec{K} \cdot \vec{L} = 0$ so that $M^2 = N^2$. In terms of Casimir operator, M^2, we can write Eq. (33) as

$$(m^2 - H^2)\vec{M}^2 = H^2 - m^2 + \alpha^2 (H+m)^2 \tag{34}$$

Now since $\vec{M}$ obey angular momentum commutation relations, $\vec{M}^2$ has eigenvalues $\mathfrak{m}(\mathfrak{m}+1)$, where $\mathfrak{m}$ can take on the values 0, 1/2, 1, $\cdots$. It is customary to use j for $\mathfrak{m}$, then Eq. (**??**), gives the energy eigenvalues

$$E = \pm m \frac{n^2 - \alpha^2}{n^2 + \alpha^2} \tag{35}$$

where $n = 2j + 1$, i.e. the energy spectrum is determined only by the principal quantum number n and states in the different j values are degenerate showing no spin-orbit splitting. The Dirac equation with vector and/or scalar Coulomb like potentials

$$V_V(r) = -\frac{\alpha_V}{r} \tag{36}$$

$$V_S(r) = -\frac{\alpha_S}{r} \tag{37}$$

is exactly solvable [11] and energy spectrum is given by

$$E = \frac{m}{\alpha_V^2 + (n - \delta_j)^2} \left\{ -\alpha_V \alpha_S \pm (n - \delta_j) \left[\alpha_V^2 - \alpha_S^2 + (n - \delta_j)^2 \right]^{1/2} \right\} \tag{38}$$

where

$$\delta_j = (j + \frac{1}{2}) - \left[(j + \frac{1}{2})^2 - (\alpha_V^2 - \alpha_S^2) \right]^{1/2} \tag{39}$$

For $\alpha_V = \alpha_S$, this reduces to Eq. (35). On the other hand for $\alpha_s = 0$,

$$E = \pm m \left\{ 1 + \frac{\alpha_V^2}{(n - \delta_j)^2} \right\}^{-1/2} \tag{40}$$

which is the well known energy spectrum for the Dirac equation for hydrogen atom, showing the fine-structure and spin-orbit splitting

IV. $SU(3)$ SYMMETRY FOR THE RELATIVISTIC HARMONIC OSCILLATOR

In non-relativistic quantum mechanics, the harmonic oscillator Hamiltonian

$$H = \frac{1}{2m} [p^2 + m^2 \omega^2 r^2], \tag{41}$$

which is symmetric in $p \longleftrightarrow x$, commutes with the quadrupole moment operator Q^{ij}, $[i, j = 1, 2, 3]$

$$Q^{ij} = \left[m^2 \omega^2 \left(x^i x^j - \frac{1}{3} \delta^{ij} x^2 \right) + \left(p^i p^j - \frac{1}{3} \delta^{ij} p^2 \right) \right] \tag{42}$$

Note that, being symmetric in i and j as well as traceless, Q^{ij} has five independent components while the orbital angular momentum $l^{ij} = x^i p^j - x^j p^i$, being antisymmetric in i and j, has three independent components. In order to go from tensor basis to the Gell-mann basis, we introduce

$$F_a = \left[m^2 \omega^2 \overline{x}^T \frac{\lambda_a}{2} \overline{x} + \frac{i}{2} m\omega \left(\overline{x}^T \frac{\lambda_a}{2} \overline{p} - \overline{p}^T \frac{\lambda_a}{2} \overline{x} \right) + \overline{p}^T \frac{\lambda_a}{2} \overline{p} \right] \tag{43}$$

where $a = 1, ...8$, and $\overline{x}$ and $\overline{p}$ are column matrices [belonging to representation 3 of $SU(3)$]

$$\overline{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \overline{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{44}$$

The superscript T denotes transpose so that $\overline{x}^T$ and $\overline{p}^T$ are row matrices. The tensor and Gell-Mann basis are related by

$$\begin{aligned} F_a &= \frac{1}{2}\sum_{i,j} (\lambda_a)_{ij} F^{ij} \\ F^{ij} &= \sum_a (\lambda_a)^{ij} F_a \end{aligned} \tag{45}$$

λ_a are 3×3 Gell-Mann matrices. The decomposition (43) corresponds to $3 \times 3 = 3 + \overline{6}$ where the representation 3 is antisymmetric and corresponds to orbital angular momentum l^{ij} while the representation $\overline{6}$ is symmetric and corresponds to the quadropole moment Q^{ij}. In particular $F_2 = m\omega l^3, F_5 = -m\omega l^2, F_7 = m\omega l^3$ while F_1, F_3, F_4, F_6 and F_8 corresponds to Q^{ij} e.g. $F_1 = m^2\omega^2 x^1 x^2 + p^1 p^2 = Q_x^{12} + Q_p^{12}$. Using the commutation relations, [I is 3×3 unit matrix]

$$[\overline{x}, \overline{p}^T] = iI \tag{46}$$

$$[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}] = if_{abc}\frac{\lambda_c}{2} \tag{47}$$

we obtain

$$[F_a, F_b] = if_{abc} m\omega F_c \tag{48}$$

where $a, b, c = 2, 5, 7$ and this corresponds to

$$[l^i, l^j] = i\epsilon^{ijk} l^k \tag{49}$$

On the hand for $a, b = 1, 3, 4, 6, 8$ in Eq. (47), c is restricted to $2, 5, 7$ and $f_{123} = 1, (f_{147}, f_{246}, f_{257}, f_{345}) = 1/2, (f_{156}, f_{367}) = -1/2, (f_{458}, f_{678}) = \sqrt{3}/2$, all others are zero.

Now for the relativistic harmonic oscillator where we have the Dirac Hamiltonian (12), with $V(r) = \frac{1}{2}m\omega^2 r^2$, qaudrupole moment operator takes the form

$$\Gamma^{ij} = (1+\beta)f(r)Q_x^{ij} + Q_p^{ij} + \gamma^5\left[(1-\beta)g(r)Q_x^{ij}\overrightarrow{\Sigma}.\overrightarrow{p} + (1+\beta)\overrightarrow{\Sigma}.\overrightarrow{p}\,g(r)Q_x^{ij}\right] \tag{50}$$

Since $[\gamma^5\overrightarrow{\Sigma}.\overrightarrow{p}, Q_p^{ij}] = 0$,

$$\begin{aligned} [H, \Gamma^{ij}] &= \gamma^5[\overrightarrow{\Sigma}.\overrightarrow{p},\ f(r)Q_x^{ij} - 2(V+m)g(r)Q_x^{ij}] + \gamma^5\beta[\overrightarrow{\Sigma}.\overrightarrow{p},\ f(r)Q_x^{ij} - 2(V+m)g(r)Q_x^{ij}]_+ \\ &\quad +(1+\beta)[V,\ Q_p^{ij}] + (1+\beta)[p^2,\ g(r)Q_x^{ij}] \end{aligned} \tag{51}$$

Thus $[H,\Gamma^{ij}]=0$ gives $[Q_x^{ij}=m^2\omega^2\left(x^ix^j-\frac{1}{3}\delta^{ij}x^2\right)$ and similar expression for Q_p^{ij}.]

$$\begin{aligned} f(r) &= 2(V+m)g(r) \\ g(r) &= \frac{1}{2m} \end{aligned} \tag{52}$$

with $V(r)=\frac{1}{2}M\omega^2r^2$. Thus

$$\Gamma^{ij} = (1+\beta)\frac{1}{m}(V+m)Q_x^{ij}+Q_p^{ij}+\frac{1}{2m}\gamma^5\left[(1-\beta)Q_x^{ij}\overrightarrow{\Sigma}\cdot\overrightarrow{p}+(1+\beta)\overrightarrow{\Sigma}\cdot\overrightarrow{p}Q_x^{ij}\right] \tag{53}$$

In the Gell-Mann basis this becomes

$$\Gamma_a = (1+\beta)\frac{1}{m}(V+m)F_a^x+F_a^p+\frac{1}{2m}\gamma^5\left[(1-\beta)F_a^x\overrightarrow{\Sigma}\cdot\overrightarrow{p}+(1+\beta)\overrightarrow{\Sigma}\cdot\overrightarrow{p}F_a^x\right] \tag{54}$$

Then using the commutation relations (44) and (45)

$$[\Gamma_a,\Gamma_b]=if_{abc}m^2\omega^2\left((1+\beta)(V+m)F_c+\frac{1}{2}\gamma^5\left[(1-\beta)F_c\overrightarrow{\Sigma}\cdot\overrightarrow{p}+(1+\beta)\overrightarrow{\Sigma}\cdot\overrightarrow{p}F_c\right]\right) \tag{55}$$

where $F_c=im\omega[\overline{x}^T\frac{\lambda_c}{2}\overline{p}-\overline{p}^T\frac{\lambda_c}{2}\overline{x}]$. Thus

$$[\Gamma_a,\Gamma_b]=if_{abc}m^2\omega^2\left[(H+m)F_c+\frac{1}{2}\gamma^5(1-\beta)[F_c,\ \overrightarrow{\Sigma}.\overrightarrow{p}]\right] \tag{56}$$

where $a,b=1,3,4,6,8$ while $c=2,5,7$. Thus F_c is essentially the orbital angular momentum $\overrightarrow{l}$. Now using Eq. (10)

$$\begin{aligned} (H+m)L^k &= \left[\gamma^5\overrightarrow{\Sigma}.\overrightarrow{p}+(1+\beta)(V+m)\right]\left[l^k+\frac{1}{2}(1-\beta)(\Sigma^k-\frac{\overrightarrow{\Sigma}.\overrightarrow{p}}{p^2}p^k\right] \\ &= (H+m)l^k+\frac{i}{2}\gamma^5(1-\beta)(\overrightarrow{\Sigma}\times\overrightarrow{p})^k \\ &= (H+m)l^k+\frac{1}{2}\gamma^5(1-\beta)\frac{1}{m\omega}\left[F_c,\overrightarrow{\Sigma}.\overrightarrow{p}\right] \end{aligned} \tag{57}$$

where we have used $\frac{1}{m\omega}\left[F_c,\overrightarrow{\Sigma}.\overrightarrow{p}\right]\sim\left[l^k,\overrightarrow{\Sigma}.\overrightarrow{p}\right]=\frac{i}{2}(\overrightarrow{\Sigma}\times\overrightarrow{p})^k$, Thus

$$\left[\widetilde{\Gamma}_a,\widetilde{\Gamma}_b\right]=if_{abc}\widetilde{F}_c \tag{58}$$

where now $\widetilde{F}_2=L^3,\widetilde{F}_5=-L^2,\widetilde{F}_7=L^1$ and $\widetilde{\Gamma}_a=\frac{\Gamma_a}{\sqrt{m\omega^2(H+m)}}$. Further

$$\left[\widetilde{F}_a,\widetilde{F}_b\right]=if_{abc}\widetilde{F}_c \tag{59}$$

$a,b,c=2,5,7$ and

$$\left[\widetilde{F}_a,\widetilde{\Gamma}_b\right]=if_{abc}\widetilde{\Gamma}_c \tag{60}$$

where $a = 2, 5, 7$ and $b, c = 1, 3, 4, 6, 8$.

The commutation relations (58, 59, 60) generate the $SU(3)$ algebra. Since $\overrightarrow{S}$ commutes with H as well as with all the above generators, the invariant algebra of the relativistic harmonic oscillator represented by the Dirac Hamiltonian (12) with $V(r) = \frac{1}{2}m\omega^2 r^2$ is $SU(3) \otimes SU_\sigma(2)$.

We now calculate the energy spectrum for which purpose we note from Eq. (52) that [$a = 1, 3, 4, 6, 8$]

$$\Gamma^2 = \Sigma\Gamma_a\Gamma_a = \frac{1}{3}(H^2 - m^2) - m\omega^2(H + m)(L^2 + 3) \tag{61}$$

where we have used the definitions of F_a^x and F_a^p, e.g. given in Eq. (41) $F_1^x = m\omega^2 x_1 x_2$ and $F_1^p = p_1 p_2$, Eqs. (13) and (14) with $V = \frac{1}{2}m\omega^2 r^2$, $(1 + \beta)(1 - \beta) = 0$, $\gamma_5(1 \pm \beta) = (1 \mp \beta)\gamma_5$ and Eq. (10). In terms of $\widetilde{\Gamma}_a$, Eq. (61) takes the form

$$3m\omega^2\left[\sum_a \widetilde{\Gamma}_a\widetilde{\Gamma}_a + L^2\right] = (H^2 - m^2)(H + m) \tag{62}$$

But $L^2 = \sum_b \widetilde{F}_b^2$, where $b = 2, 5, 7$. Thus Eq. (62) takes the form

$$3m\omega^2\widetilde{\Gamma}^2 = (H^2 - m^2)(H + m) - 9m\omega^2 \tag{63}$$

where $\widetilde{\Gamma}^2 = \sum_a \widetilde{\Gamma}_a\widetilde{\Gamma}_a + \sum_b \widetilde{\Gamma}_b\widetilde{\Gamma}_b$ is the invariant of the group SU(3) and as such is proportional to unit matrix.

Hence Eq. (63) gives the energy eigenvalues

$$(E - m)^2(E + m) - 9m\omega^2 = Cm\omega^2 \tag{64}$$

where C is to be fixed. This can be done in the following way. We take the non-relativistic limit of Eq. (14), $H \to H_{non-rel} + m$, which gives

$$H_{non-rel} = \frac{p^2}{2m} + 2V = \frac{p^2}{2m} + m\omega^2 r^2 \tag{65}$$

which as is well known gives the energy eigenvalues [note we have to replace ω by $\sqrt{2}\omega$ in the ordinary harmonic oscillator eigenvalues]

$$\mathcal{E}_N = \frac{1}{\sqrt{2}}\omega(2N + 3) \tag{66}$$

Now we take the non-relativistic limit of Eq. (64), $E \to \mathcal{E}_N + m$, which gives

$$m\omega^2(2N + 3)^2 - 9m\omega^2 = Cm\omega^2 \tag{67}$$

fixing $C = 4N(N + 3)$. Putting back in Eq. (62) the energy eigenvalue equation becomes

$$(\mathcal{E}_N - m)^2(\mathcal{E}_N + m) = 4\left(N + \frac{3}{2}\right)^2 m\omega^2 \tag{68}$$

where $N = 0, 1, \cdots$. This agrees with one obtained from the exact solutions of Dirac equation [12].

V. SUMMARY AND CONCLUSIONS

We have systematically reviewed the various dynamical symmetries of the Dirac Hamiltonian, clearly stating the conditions under which such symmetries hold. These symmetries include relativistic spin and orbital angular symmetries which hold when Dirac Hamiltonian with scalar $V_S(r)$ and vector $V_V(r)$ spherically symmetric potentials satisfy $\partial V_V/\partial r = \partial V_S/\partial r$ or $V_V = V_S + U$. Here U is a constant potential but can be absorbed in redefinition of the mass. Then if $V(r) = -\alpha/r$, as for the hydrogen atom the Dirac Hamiltonian has $SO(4) \otimes SU_\sigma(2)$ or equivalently $SU_M(2) \otimes SU_N(2) \otimes SU_\sigma(2)$ symmetry. Here $SU_\sigma(2)$ is the $SU(2)$ group generated by the relativistic spin S^i defined in Eq. (3) or Eq. (7). If $V(r) = \frac{1}{2}M\omega^2 r^2$, the harmonic oscillator potential, then the symmetry is $SU(3) \otimes SU_\sigma(2)$. We have used Dirac algebra of Dirac matrices and for the simple harmonic oscillation, Gell-Mann basis of $SU(3)$, which is more transparent and simple (at least for physicist with particle physics background) compared to the basis used in [4]. We have also calculated energy spectrum in each case from group theoretical consideration, which agrees with the exact solution of Dirac equation in each case.

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