

**INVERSE SCATTERING TRANSFORM
FOR THE DEGASPERIS–PROCESI EQUATION:
A RIEMANN–HILBERT APPROACH**

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ABSTRACT. We present the inverse scattering transform approach to the Cauchy problem on the line for the Degasperis–Procesi equation

$$u_{txx} - 2u_x + 2u_x u_{xx} + uu_{xxx} = 0$$

in the form of an associated Riemann–Hilbert problem. This approach allows us to give a representation of the solution to the Cauchy problem, which can be efficiently used in studying its long-time behavior.

1. INTRODUCTION

In this paper we present the inverse scattering approach, based on an appropriate Riemann–Hilbert problem formulation, for the initial value problem for the Degasperis–Procesi (DP) equation [17, 16]

$$u_t - u_{txx} + 3\omega u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad -\infty < x < \infty, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad (1.2)$$

where ω is a positive parameter. The DP equation arises as a model equation describing the shallow-water approximation in inviscid hydrodynamics in the so-called “moderate amplitude regime”: introducing two small parameters, the wave-amplitude parameter ε (characterizing the smallness of the wave amplitude) and the long-wave parameter δ (characterizing the smallness of the typical wavelength with respect to the water depth), in this regime we assume that $\delta \ll 1$ and $\varepsilon \sim \delta$. This regime can be characterized as to be more nonlinear than dispersive, which, in particular, allows “wave breaking”. This is in contrast with the so-called “shallow water regime” ($\delta \ll 1$ and $\varepsilon \sim \delta^2$), where nonlinearity and dispersion are so balanced that the solution of the initial value problem for the associated nonlinear equation (the Korteweg–de Vries equation) exists globally for all times, for all “nice” (sufficiently decaying and smooth) initial data.

Among the models of moderate amplitude regime, only two are integrable (admitting a bi-Hamiltonian structure and a Lax pair representation): they are the Camassa–Holm (CH) equation and the DP equation. Also, they are the only two integrable equations from the

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“ b -family” of equations

$$u_t - u_{txx} + b\omega u_x + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}.$$

The CH and DP equations correspond to $b = 2$ and $b = 3$, respectively.

The analysis of the CH equation by using the inverse scattering approach were started in [12, 15, 20]. A version of the inverse scattering method based on the Riemann–Hilbert (RH) factorization problem was proposed in [5, 7] (another Riemann–Hilbert formulation of the inverse scattering transform is presented in [13]). The RH approach has proved its efficiency in the study of the long-time behavior of solutions of both initial value problems [6, 4, 3] and initial boundary value problems [8].

In the present paper we develop the Riemann–Hilbert (RH) approach to the DP equation, following the main ideas developed in [7].

A major difference between the implementations of the Riemann–Hilbert method to the CH equation and the DP equation is that in the latter case, the spatial equation of the Lax pair is of the third order, which implies that in the matrix form one has to deal with 3×3 matrix-valued equations, while in the case of the CH equation, they have a 2×2 matrix structure, as in the cases of the most known integrable equations (KdV, modified KdV, nonlinear Schrödinger, sine–Gordon, etc.) Hence, the construction and analysis of the associated RH problem become considerably more complicated.

In a recent paper [14], the inverse scattering method for the DP equation based on a 3×3 matrix RH problem in the spectral k -plane is proposed, where the solution of the DP equation is extracted from the large- k behavior of the solution of the RH problem. Also, the dressing procedure is given for constructing N -soliton solutions, which is illustrated, particularly, by the explicit construction (in parametric form) of the 1-soliton solutions.

In our approach, we propose a different Riemann–Hilbert problem and give a different representation of the solution $u(x, t)$ of the initial value problem (1.1) in terms of a solution of this problem evaluated at a distinguished finite point of the plane of the spectral parameter. Remarkably, the formulae for $u(x, t)$ have the same structure as the parametric formulae for pure multisoliton solutions obtained in [21].

2. LAX PAIR AND EIGENFUNCTIONS

We assume that the initial function $u_0(x)$ in (1.2) is sufficiently smooth and decay fast as $|x| \rightarrow \infty$. Letting $u(x, t)$ be the solution of the DP equation, we introduce the “momentum” variable

$$m = m(x, t) \equiv u - u_{xx}.$$

It is known that, similarly to the case of the CH equation (see, e.g., [12]), the condition $m(x, 0) + \omega > 0$ for all x provides the existence of $m(x, t)$ for all t ; moreover, $m(x, t) + \omega > 0$ for all $x \in \mathbb{R}$ and all $t > 0$. This justifies the form

$$\left(\sqrt[3]{m + \omega}\right)_t = -\left(u\sqrt[3]{m + \omega}\right)_x \tag{2.1}$$

of the DP equation, which will be used in our constructions below.

The linear dispersion parameter ω can be scaled out to $\omega = 1$. Hence, in what follows, for simplicity, we assume $\omega = 1$.

2.1. Lax pairs.

Scalar Lax pair. The DP equation admits a Lax representation: this equation is actually the compatibility condition of two linear equations [16]

$$\psi_x - \psi_{xxx} = -z^3(m+1)\psi, \quad (2.2a)$$

$$\psi_t = \left(u_x - \frac{c}{z^3}\right)\psi - u\psi_x + \frac{1}{z^3}\psi_{xx}, \quad (2.2b)$$

where z is a spectral parameter, $\psi = \psi(x, t, z)$, and c is an arbitrary constant. In what follows we will see that it is convenient to choose $c = \frac{2}{3}$.

1st matrix form. In order to control the behavior of solutions to (2.2) as functions of the spectral parameter z (which is crucial for the Riemann–Hilbert method), it is convenient to rewrite the Lax pair in matrix form. Introducing $\Phi = \Phi(x, t, z)$ by

$$\Phi = \begin{pmatrix} \psi \\ \psi_x \\ \psi_{xx} \end{pmatrix}$$

transforms the scalar Lax pair (2.2) into a Lax pair

$$\Phi_x = U\Phi, \quad (2.3a)$$

$$\Phi_t = V\Phi, \quad (2.3b)$$

where

$$U(x, t, z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^3 q^3 & 1 & 0 \end{pmatrix}, \quad (2.3c)$$

$$V(x, t, z) = \begin{pmatrix} u_x - \frac{c}{z^3} & -u & \frac{1}{z^3} \\ u + 1 & -\frac{c-1}{z^3} & -u \\ u_x - z^3 u q^3 & 1 & -u_x - \frac{c-1}{z^3} \end{pmatrix}, \quad (2.3d)$$

with

$$q = q(x, t) = (m+1)^{1/3}. \quad (2.3e)$$

From now on, (2.3) will be seen as a 3×3 matrix-valued linear system: a “matrix” solution of (2.3) is a collection $\Phi = (\Phi^{(1)} \ \Phi^{(2)} \ \Phi^{(3)})$ of three “vector” solutions $\Phi^{(j)}$. Now we notice that if $c = \frac{2}{3}$, then V is traceless. Thus in this case the determinant of a matrix solution to the equation $\Phi_t = V\Phi$ is independent of t . The analogous property of $\Phi_x = U\Phi$ is obvious.

The coefficient matrices U and V in (2.3) have singularities (in the extended complex z -plane) at $z = 0$ and at $z = \infty$. In order to control the large- z behavior of solutions of (2.3), we follow a strategy similar to that adopted for the CH equation [5, 7]. We transform (2.3) in such a way that:

- i) the leading terms for $z \rightarrow \infty$ in the Lax equations be diagonal, whereas the terms of order $O(1)$ be off-diagonal;
- ii) all lower order terms (including those of order $O(1)$) vanish as $|x| \rightarrow \infty$.

It is instructive to perform this transformation in two steps:

- (i) First, transform (2.3) into a system where the leading terms are represented as products of (x, t) -independent and (x, t) -dependent factors.
- (ii) Second, diagonalize the (x, t) -independent factors.

2nd matrix form. For the first step, introducing $\tilde{\Phi} = \tilde{\Phi}(x, t, z)$ by

$$\tilde{\Phi} = D^{-1}\Phi,$$

where

$$D(x, t) = \text{diag}\{q^{-1}(x, t), 1, q(x, t)\}$$

transforms (2.3) into a new Lax pair

$$\tilde{\Phi}_x = \tilde{U}\tilde{\Phi}, \tag{2.4a}$$

$$\tilde{\Phi}_t = \tilde{V}\tilde{\Phi}, \tag{2.4b}$$

where

$$\begin{aligned} \tilde{U}(x, t, z) &= q(x, t) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^3 & 1 & 0 \end{pmatrix} + \begin{pmatrix} \frac{qx}{q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{q} - q & -\frac{qx}{q} \end{pmatrix} \\ &\equiv q(x, t)U_\infty(z) + \tilde{U}^{(1)}(x, t) \end{aligned} \tag{2.4c}$$

and

$$\begin{aligned} \tilde{V}(x, t, z) &= -uq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^3 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3z^3} & 0 & \frac{1}{z^3} \\ 1 & \frac{1}{3z^3} & 0 \\ 0 & 1 & \frac{1}{3z^3} \end{pmatrix} \\ &\quad + \begin{pmatrix} -u\frac{qx}{q} & 0 & 0 \\ \frac{u+1}{q} - 1 & 0 & 0 \\ \frac{u_x}{q^2} & \frac{1}{q} - 1 + uq & u\frac{qx}{q} \end{pmatrix} + \frac{q^2 - 1}{z^3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\equiv -u(x, t)q(x, t)U_\infty(z) + V_\infty(z) + \tilde{V}^{(1)}(x, t) + \frac{1}{z^3}\tilde{V}^{(2)}(x, t). \end{aligned} \tag{2.4d}$$

Main matrix form. For the second step, it is important that the commutator of U_∞ and V_∞ vanishes identically, i.e., $[U_\infty, V_\infty] \equiv 0$, which allows simultaneous diagonalization of U_∞ and V_∞ . Indeed, we have

$$P^{-1}(z)U_\infty(z)P(z) = \Lambda(z), \tag{2.5a}$$

$$P^{-1}(z)V_\infty(z)P(z) = \frac{1}{z^3}I + \Lambda^{-1}(z) \equiv A(z), \tag{2.5b}$$

where I is the identity 3×3 matrix,

$$\Lambda(z) = \begin{pmatrix} \lambda_1(z) & 0 & 0 \\ 0 & \lambda_2(z) & 0 \\ 0 & 0 & \lambda_3(z) \end{pmatrix}, \tag{2.6a}$$

$$P(z) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1(z) & \lambda_2(z) & \lambda_3(z) \\ \lambda_1^2(z) & \lambda_2^2(z) & \lambda_3^2(z) \end{pmatrix}, \tag{2.6b}$$

and

$$P^{-1}(z) = \begin{pmatrix} (3\lambda_1(z)-1)^{-1} & 0 & 0 \\ 0 & (3\lambda_2(z)-1)^{-1} & 0 \\ 0 & 0 & (3\lambda_3(z)-1)^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1^2(z)-1 & \lambda_1(z) & 1 \\ \lambda_2^2(z)-1 & \lambda_2(z) & 1 \\ \lambda_3^2(z)-1 & \lambda_3(z) & 1 \end{pmatrix}. \quad (2.6c)$$

Here $\lambda_j(z)$, $j = 1, 2, 3$ are the solutions of the algebraic equation

$$\lambda^3 - \lambda = z^3, \quad (2.7)$$

so that $\lambda_j(z) \sim C_j z$ as $z \rightarrow \infty$, where the C_j 's are the cube roots of unity.

Thus, introducing $\hat{\Phi} = \hat{\Phi}(x, t, z)$ by

$$\hat{\Phi} = P^{-1} \tilde{\Phi}$$

leads to another Lax pair

$$\hat{\Phi}_x - q\Lambda(z)\hat{\Phi} = \hat{U}\hat{\Phi}, \quad (2.8a)$$

$$\hat{\Phi}_t + (uq\Lambda(z) - A(z))\hat{\Phi} = \hat{V}\hat{\Phi}, \quad (2.8b)$$

where

$$\hat{U}(x, t, z) = P^{-1}(z)\tilde{U}^{(1)}(x, t)P(z), \quad (2.8c)$$

$$\hat{V}(x, t, z) = P^{-1}(z)\left(\tilde{V}^{(1)}(x, t) + \frac{1}{z^3}\tilde{V}^{(2)}(x, t)\right)P(z). \quad (2.8d)$$

Commutator form. Notice that $\hat{U}(x, t, z) = O(1)$ and $\hat{V}(x, t, z) = O(1)$ as $z \rightarrow \infty$ since $U^{(1)}$ and $V^{(1)}$ are lower triangular matrices. Moreover, it can be checked directly that the diagonal entries of $\hat{U}(x, t, z)$ and $\hat{V}(x, t, z)$ are of order $O(1/z)$. This latter fact is important to establish the large- z behavior of $\hat{\Phi}$ [1].

On the other hand, $\hat{U}(x, t, z) = o(1)$ and $\hat{V}(x, t, z) = o(1)$ as $|x| \rightarrow \infty$, which suggests introducing a 3×3 diagonal function $Q(x, t, z)$ that solves the system

$$Q_x = q\Lambda(z), \quad (2.9a)$$

$$Q_t = -uq\Lambda(z) + A(z), \quad (2.9b)$$

as follows:

$$Q(x, t, z) = y(x, t)\Lambda(z) + tA(z) \quad (2.10)$$

with

$$y(x, t) = x - \int_x^\infty (q(\xi, t) - 1)d\xi. \quad (2.11)$$

The fact that the equations in (2.9) are consistent follows directly from the DP equation in the form (2.1): $q_t = -(uq)_x$. The normalization of $Q(x, t, z)$ is chosen in such a way that

$$Q(x, t, z) \sim x\Lambda(z) + tA(z) \quad \text{as } x \rightarrow +\infty. \quad (2.12)$$

The role of $Q(x, t, z)$ in the construction of the Riemann–Hilbert problem is to catch the large- z behavior of dedicated solutions of the Lax pair equations (2.8). Indeed, introducing the 3×3 matrix-valued function $M = M(x, t, z)$ by

$$M = \hat{\Phi}e^{-Q}$$

reduces (2.8) to the system

$$M_x - [Q_x, M] = \hat{U}M, \quad (2.13a)$$

$$M_t - [Q_t, M] = \hat{V}M. \quad (2.13b)$$

2.2. Eigenfunctions.

Fredholm integral equations. Particular solutions of (2.13) having well-controlled properties as functions of the spectral parameter z can be constructed as solutions of the Fredholm integral equation (cf. [1])

$$M(x, t, z) = I + \int_{(x^*, t^*)}^{(x, t)} e^{Q(x, t, z) - Q(\xi, \tau, z)} \left(\hat{U}M(\xi, \tau, z) d\xi + \hat{V}M(\xi, \tau, z) d\tau \right) e^{-Q(x, t, z) + Q(\xi, \tau, z)}, \quad (2.14)$$

where the initial points of integration (x^*, t^*) can be chosen differently for different matrix entries of the equation: Q being diagonal, (2.14) must be seen as the collection of scalar integral equations ($1 \leq j, l \leq 3$)

$$M_{jl}(x, t, z) = I_{jl} + \int_{(x_{jl}^*, t_{jl}^*)}^{(x, t)} e^{Q_{jj}(x, t, z) - Q_{jj}(\xi, \tau, z)} \left((\hat{U}M)_{jl}(\xi, \tau, z) d\xi + (\hat{V}M)_{jl}(\xi, \tau, z) d\tau \right) e^{-Q_{ll}(x, t, z) + Q_{ll}(\xi, \tau, z)}.$$

Choosing the (x_{jl}^*, t_{jl}^*) appropriately allows to obtain eigenfunctions that can be used in the construction of the Riemann–Hilbert problems associated with the initial value problems [7] as well as the initial boundary value problems [8].

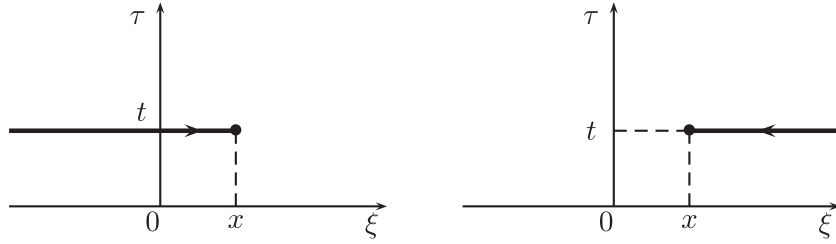


FIGURE 1. Paths of integration. Left: $(x^*, t^*) = (-\infty, t)$. Right: $(x^*, t^*) = (+\infty, t)$.

In particular, for the Cauchy problem considered in the present paper, it is reasonable to choose these points to be $(-\infty, 0)$ or $(+\infty, 0)$ thus reducing the integration in (2.14) to paths parallel to the x -axis (see Figure 1):

$$M(x, t, z) = I + \int_{(\pm)\infty}^x e^{Q(x, t, z) - Q(\xi, t, z)} [\hat{U}M(\xi, t, z)] e^{-Q(x, t, z) + Q(\xi, t, z)} d\xi, \quad (2.15)$$

or, in view of (2.10),

$$M(x, t, z) = I + \int_{(\pm)\infty}^x e^{-\int_x^\xi (q(\zeta, t) - 1) d\zeta \Lambda(z)} [\hat{U}M(\xi, t, z)] e^{\int_x^\xi (q(\zeta, t) - 1) d\zeta \Lambda(z)} d\xi. \quad (2.16)$$

Since $q - 1 > 0$, the domains (in the complex z -plane) where the exponential factors in (2.15) are bounded are determined by the sign of $\operatorname{Re} \lambda_j(z) - \operatorname{Re} \lambda_l(z)$.

A new spectral parameter. It is convenient to introduce a new spectral parameter k (see [14] and also [11]) such that

$$z(k) = \frac{1}{\sqrt{3}} k \left(1 + \frac{1}{k^6} \right)^{1/3}, \quad (2.17)$$

and fixed by the condition $z(k) \sim \frac{1}{\sqrt{3}} k$ as $k \rightarrow \infty$. We fix $\lambda_j = \lambda_j(k)$ as follows

$$\lambda_j(k) = \frac{1}{\sqrt{3}} \left(\omega^j k + \frac{1}{\omega^j k} \right) \quad \text{where } \omega = e^{\frac{2\pi i}{3}}. \quad (2.18)$$

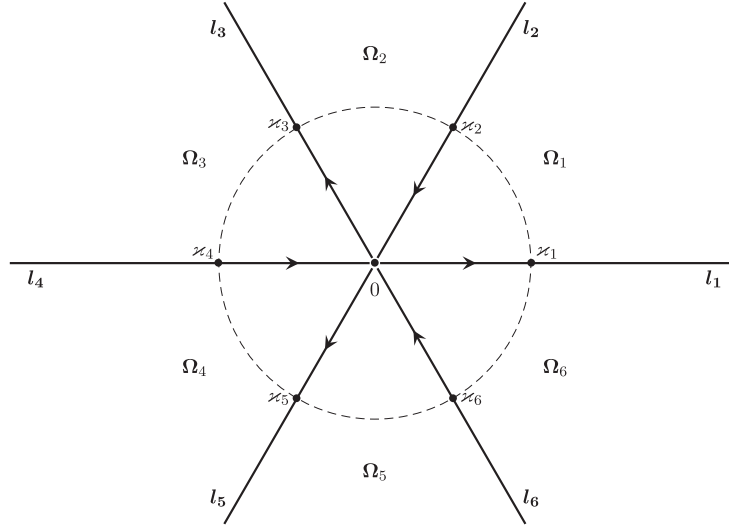


FIGURE 2. Rays l_ν , domains Ω_ν and points z_ν in the k -plane

The set $\Sigma = \{k \mid \operatorname{Re} \lambda_j(k) = \operatorname{Re} \lambda_l(k) \text{ for some } j \neq l\}$ consists of six rays

$$l_\nu = \mathbb{R}_+ e^{\frac{\pi i}{3}(\nu-1)}, \quad \nu = 1, \dots, 6$$

dividing the k -plane into six sectors

$$\Omega_\nu = \left\{ k \mid \frac{\pi}{3}(\nu-1) < \arg k < \frac{\pi}{3}\nu \right\}, \quad \nu = 1, \dots, 6.$$

In order to have a (matrix-valued) solution to (2.16) to be analytic in $\mathbb{C} \setminus \Sigma$, the initial points of integration ∞_{jl} are specified for each matrix entry (j, l) , $1 \leq j, l \leq 3$ as follows:

$$\infty_{jl} = \begin{cases} +\infty, & \text{if } \operatorname{Re} \lambda_j(z) \geq \operatorname{Re} \lambda_l(z), \\ -\infty, & \text{if } \operatorname{Re} \lambda_j(z) < \operatorname{Re} \lambda_l(z). \end{cases} \quad (2.19)$$

That means that we consider the system of Fredholm integral equations

$$M_{jl}(x, t, z) = I_{jl} + \int_{\infty_{j,l}}^x e^{-\lambda_j(z)} \int_x^\xi (q(\zeta, t) - 1) d\zeta [(\hat{U}M)_{jl}(\xi, t, z)] e^{\lambda_l(z)} \int_x^\xi (q(\zeta, t) - 1) d\zeta d\xi. \quad (2.20)$$

Proposition 2.1 (analyticity). *Let $M(x, t, k)$ be the solution of the system of equations (2.20), where the limits of integration are chosen according to (2.19). Then,*

- (i) M is piecewise meromorphic with respect to Σ , as function of the spectral parameter k .
- (ii) $M(x, t, k) \rightarrow I$ as $k \rightarrow \infty$.

Proof. The proof follows the same lines as in [1]. Notice that in order to have (ii) it is important that the diagonal part of \hat{U} vanish as $k \rightarrow \infty$. \square

Proposition 2.2 (symmetries). $M(x, t, k)$ satisfies the symmetry relations:

- (S1) $\Gamma_1 \overline{M(x, t, \bar{k})} \Gamma_1 = M(x, t, k)$ where $\Gamma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- (S2) $\Gamma_2 \overline{M(x, t, \bar{k}\omega^2)} \Gamma_2 = M(x, t, k)$ where $\Gamma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.
- (S3) $\Gamma_3 \overline{M(x, t, \bar{k}\omega)} \Gamma_3 = M(x, t, k)$ where $\Gamma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Hence, the values of M at k and at ωk are related by

$$M(x, t, k\omega) = C^{-1}M(x, t, k)C \quad \text{where } C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If $\lambda_i(k) = \lambda_j(k)$, $i \neq j$ for certain values of the spectral parameter k , then P at these values becomes degenerate (see (2.6c)), which in turn leads to singularities for \hat{U} and, consequently, for $\hat{\Phi}$ and M . These values are $\varkappa_\nu = e^{\frac{\pi i}{3}(\nu-1)}$, $\nu = 1, \dots, 6$. Taking into account the symmetries described in Proposition 2.2 leads to the following

Proposition 2.3 (singularities). *The limiting values of $M(x, t, k)$ as k approaches one of the points $\varkappa_\nu = e^{\frac{\pi i}{3}(\nu-1)}$, $\nu = 1, \dots, 6$ have pole singularities with leading terms of a specific matrix structure.*

- (i) At $k = \varkappa_1 = 1$, as $k = 1 \pm i\varepsilon \rightarrow 1$, $\varepsilon > 0$,

$$M \sim \frac{1}{k-1} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_\pm & 0 & 0 \\ 0 & \alpha_\pm & 0 \\ 0 & 0 & \beta_\pm \end{pmatrix}, \quad (2.21)$$

where $\alpha_- = -\overline{\alpha_+}$ and $\beta_- = -\overline{\beta_+}$.

- (ii) At $k = \varkappa_2 = e^{\frac{\pi i}{3}}$, as $k = e^{\frac{\pi i}{3}} \pm i\varepsilon \rightarrow e^{\frac{\pi i}{3}}$, $\varepsilon > 0$,

$$M \sim \frac{1}{k - e^{\frac{\pi i}{3}}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{\beta}_\pm & 0 & 0 \\ 0 & \tilde{\alpha}_\pm & 0 \\ 0 & 0 & \tilde{\alpha}_\pm \end{pmatrix}, \quad (2.22)$$

where $\tilde{\alpha}_- = -\omega \overline{\tilde{\alpha}_+}$ and $\tilde{\beta}_- = -\omega \overline{\tilde{\beta}_+}$.

M has similar leading terms at the other polar singularities $\varkappa_3, \dots, \varkappa_6$ in accordance with the symmetry conditions stated in Proposition 2.2.

Indeed, consider, for example, the behavior of M at $k = 1$. We have $\lambda_1(1) = \lambda_2(1) = -1/\sqrt{3}$ and $\lambda_3(1) = 2/\sqrt{3}$. Moreover, as $k \rightarrow 1$, we have

$$\begin{aligned} 3\lambda_1^2 - 1 &= -2\sqrt{3}(k-1) + O((k-1)^2), \\ 3\lambda_2^2 - 1 &= 2\sqrt{3}(k-1) + O((k-1)^2), \\ 3\lambda_3^2 - 1 &= 3. \end{aligned}$$

Consequently, at $k = 1$,

$$P^{-1}(k) = \frac{1}{k-1} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} + O(1)$$

with some p_j .

Now recall that $\hat{\Phi} = P^{-1}\tilde{\Phi}$, where, on one hand, $\tilde{\Phi}$ satisfies a differential equation whose coefficients are regular at $k = 1$, and on the other hand, $\tilde{\Phi}$ satisfies the boundary condition (see (2.16) with signs chosen according to (2.19))

$$\tilde{\Phi} \sim P(k)e^{y(x,t)\Lambda(k)+tA(k)}, \quad x \rightarrow +\infty$$

for all k such that $\lambda_j(k) \neq \lambda_l(k)$ for all $j \neq l$. It follows that when k approaches 1 from the either side of l_1 , the first two columns of $\tilde{\Phi}$ coincide. Consequently, for the limiting values of $\hat{\Phi}$ (and thus M) we obtain (2.21) while the property $\alpha_- = -\overline{\alpha_+}$ and $\beta_- = -\overline{\beta_+}$ follows from the symmetry (S1) of Proposition 2.2. Similarly for the other points $\varkappa_\nu = e^{\frac{\pi i}{3}(\nu-1)}$, $\nu = 2, \dots, 6$.

Remark 2.4. While the set of singularities of $M(x, t, k)$ in the open domain

$$\Omega = \mathbb{C} \setminus \Sigma = \Omega_1 \cup \dots \cup \Omega_6$$

can be empty (for instance, this happens for all sufficiently small “potentials” $u(x, t)$), the singularities described in Proposition 2.3 are generic. This should be compared with [14], where the solutions of a system of integral equations similar to (2.16) are combined into a Riemann–Hilbert problem under the assumption that they have no singularities.

3. RIEMANN–HILBERT PROBLEM

3.1. Jump conditions.

1st RH problem. For k on the boundary between the adjacent domains Ω_ν , the limiting values of M , being the solutions of a system of differential equations (2.13) must be related by a matrix independent of x and t . Supplying the rays l_ν with the orientation, see Figure 2, we can write for the limiting values of M :

$$M_+(x, t, k) = M_-(x, t, k)e^{Q(x,t,k)}S(k)e^{-Q(x,t,k)}, \quad k \in l_1 \cup \dots \cup l_6. \quad (3.1)$$

Considering (3.1) at $t = 0$ we see that the conjugating matrix $S(k)$ is, in fact, determined by $u(x, 0)$, i.e., by the initial data for the Cauchy problem (1.1)–(1.2), via the solutions $M(x, 0, k)$ of the system (2.16) whose coefficients are determined by $u(x, 0)$. Thus the relation (3.1) can be considered as a “pre-Riemann–Hilbert problem” associated with (1.1): the data are $S(k)$,

and we seek for a piecewise meromorphic function M satisfying (3.1) for all x and t , in the hope that one can further extract the solution $u(x, t)$ to (1.1) from $M(x, t, k)$.

The conjugating matrix $S(k)$ has a particular matrix structure, cf. [14]. Indeed, the integral equations (2.16) allows studying the limiting values of M as $x \rightarrow \pm\infty$. Set $t = 0$ and consider, for example, the limiting values of $M_{\pm}(x, 0, k)$ for $k \in l_1 = \mathbb{R}_+$. The triangular structure of integration in (2.16) implies that

$$e^{-Q(x,0,k)}M_+(x,0,k)e^{Q(x,0,k)} \xrightarrow{x \rightarrow +\infty} S_1^+(k) = \begin{pmatrix} 1 & r_+(k) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.2)$$

and

$$e^{-Q(x,0,k)}M_-(x,0,k)e^{Q(x,0,k)} \xrightarrow{x \rightarrow +\infty} S_1^-(k) = \begin{pmatrix} 1 & 0 & 0 \\ r_-(k) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.3)$$

Then the symmetry (S1) from Proposition 2.2 implies that $r_-(k) = \overline{r_+(k)}$, and the jump matrix $S(k)$ from (3.1) for $k \in l_1$ takes the form

$$S(k) = (S_1^-(k))^{-1} S_1^+(k) = \begin{pmatrix} \frac{1}{r(k)} & 0 & 0 \\ r(k) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r(k) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.4)$$

where $r(k) := -r_+(k)$. Similarly, for $k \in l_4 = \mathbb{R}_-$, one has

$$S(k) = (S_4^-(k))^{-1} S_4^+(k) = \begin{pmatrix} \frac{1}{r(k)} & 0 & 0 \\ r(k) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r(k) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.5)$$

whereas the construction of $S(k)$ for $k \in l_{\nu}$ with $\nu \neq 1, 4$ follows from the symmetries of Proposition 2.2. Thus, similarly to the cases of, say, the KdV equation or the Camassa–Holm equation, the jump matrix on the whole contour is determined by a scalar function — the reflection coefficient $r(k)$ given for $k \in \mathbb{R}$. In turn, as it follows from (3.2) and (2.16) for $t = 0$, the reflection coefficient is determined by the initial condition $u_0(x)$.

2nd RH problem. The dependence of the matrix $e^Q S e^{-Q}$ relating M_+ and M_- in (3.1) on the parameters x and t suggests introducing the parameter $y = y(x, t)$ given by (2.11) and thus rewriting (3.1) as

$$\hat{M}_+(y, t, k) = \hat{M}_-(y, t, k) e^{y\Lambda(k) + tA(k)} S(k) e^{-y\Lambda(k) - tA(k)} \quad (3.6)$$

so that

$$M(x, t, k) = \hat{M}(y(x, t), t, k),$$

which provides explicit dependence of the relating matrix on the parameters (y, t) .

3rd RH problem. On the other hand, the particular matrix structure of the singularities at $k = \varkappa_{\nu} \equiv e^{\frac{\pi i}{3}(\nu-1)}$ suggests introducing a row vector-valued 1×3 Riemann–Hilbert problem

$$\mu_+(y, t, k) = \mu_-(y, t, k) e^{-y\Lambda(k) - tA(k)} S(k) e^{y\Lambda(k) + tA(k)} \quad (3.7a)$$

having the same jump conditions as in the matrix case but normalized at $k = \infty$ by the condition

$$\mu(y, t, k) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + o(1) \quad \text{as } k \rightarrow \infty. \quad (3.7b)$$

The transition from the 3×3 matrix RH problem to the 1×3 row vector RH problem can be viewed as the multiplication of the former by the constant row vector $(1 \ 1 \ 1)$ from the left, which suppresses the singularities at $k = \varkappa_\nu \equiv e^{\frac{\pi i}{3}(\nu-1)}$.

3.2. Residue conditions. In order to complete the formulation of the RH problem, one has to complete the jump condition (3.7a) and the normalization condition (3.7b) with the residue conditions at the possible poles of $\mu(y, t, k)$ or $\hat{M}(y, t, k)$ in $\mathbb{C} \setminus \Sigma$ where $\Sigma = \cup_{\nu=1}^6 l_\nu$.

Generically (see [1]), there are at most a finite number of poles lying in $\mathbb{C} \setminus \Sigma$, each of them being simple, with residue conditions of a special matrix form: distinct columns of M (distinct entries of μ) have distinct poles, and if k_n is a pole, then

$$\text{Res}_{k=k_n} \mu(y, t, k) = \mu(y, t, k_n) e^{y\Lambda(k_n) + tA(k_n)} v_n e^{-y\Lambda(k_n) - tA(k_n)}, \quad (3.7c)$$

where the 3×3 matrix v_n has only one non-zero entry at a position depending on the sector of $\mathbb{C} \setminus \Sigma$ containing k_n . For example, if $k_n \in \Omega_1$, a non-zero entry of v_n can be either $(v_n)_{12}$ or $(v_n)_{23}$. Then the positions (as well as the values) of the non-zero entries of poles in other sectors of $\mathbb{C} \setminus \Sigma$ are determined by the symmetries (S1)–(S3).

Similarly to the jump matrix $S(k)$, the residue conditions are determined by the initial values $u_0(x)$ (putting $t = 0$ in (3.7c)).

A closer look at the (y, t) -dependence of the exponential in the residue condition (3.7c) reveals the following. Suppose that the non-zero entry of v_n is $(v_n)_{jl}$, and let

$$v = \lambda_j - \lambda_l.$$

Then this exponential has the form (see (2.5))

$$e^{y(\lambda_j - \lambda_l) + t(A_j - A_l)} = e^{(\lambda_j - \lambda_l) \left(y - \frac{t}{\lambda_j \lambda_l} \right)} = e^{v \left(y - \frac{3t}{1-v^2} \right)}, \quad (3.8)$$

The last equality in (3.8) follows from the fact (see the definition (2.7) of λ_j) that $3\lambda_j \lambda_l = 1 - (\lambda_j - \lambda_l)^2$ for all $j \neq l$.

An analogy with the Camassa–Holm equation suggests making the conjecture that the actual positions of the poles associated with a global solution of the initial-value problem (1.1) with smooth, decaying initial data $u_0(x)$ are those for which the associated value of v is real and, moreover, $0 < v < 1$.

Due to the symmetry relations, it is enough to consider the location of the poles in one sector of $\mathbb{C} \setminus \Sigma$, say, in Ω_1 .

The exponential factor for $(v_n)_{12}$ contains

$$\begin{aligned} \lambda_1(k) - \lambda_2(k) &= \frac{1}{\sqrt{3}} \left(\omega k + \frac{1}{\omega k} - \omega^2 k - \frac{1}{\omega^2 k} \right) \\ &= \frac{\omega - \omega^2}{\sqrt{3}} \left(k - \frac{1}{k} \right) = i \left(k - \frac{1}{k} \right). \end{aligned} \quad (3.9)$$

It follows that the l.h.s. of (3.9) is real either for $\{k \mid \operatorname{Re} k = 0\}$ or for $\{k \mid |k| = 1\}$. The real axis does not intersect with Ω_1 whereas for $k = e^{i\varphi}$ with $0 < \varphi < \pi/6$, the associated v satisfies the inequalities $0 < |v| < 1$.

The exponential factor for $(v_n)_{23}$ contains

$$\lambda_2(k) - \lambda_3(k) = \frac{1}{\sqrt{3}} \left(\omega^2 k + \frac{1}{\omega^2 k} - k - \frac{1}{k} \right) = i \left(\omega k - \frac{1}{\omega k} \right). \quad (3.10)$$

Thus the admissible arc for the location of the poles in Ω_1 for this entry (23) is $\{k \mid k = e^{i\varphi}, \pi/6 < \varphi < \pi/3\}$. Notice that the symmetry relations provide that the associated residue conditions in Ω_3 are those for the (12) entry. Similarly for the other sectors Ω_ν .

The poles with the residue conditions described above are associated with the soliton long-time behavior of the solution of the Cauchy problem (1.1)-(1.2), the velocity of all solitons being greater than 3 (since $|v| < 1$ — the velocity c of a soliton being related to v as $c = 3/(1 - v^2)$, see (3.8)). On the other hand, the residue conditions with poles at $\{k \mid \operatorname{Re} k = 0\}$ give rise to the “loop solitons” [22]: in this case,

$$v = i \left(i\rho - \frac{1}{i\rho} \right) = - \left(\rho + \frac{1}{\rho} \right), \quad \rho \in \mathbb{R}$$

and thus $|v| > 2$; so they move, as opposite to the smooth solitons, in the negative direction. These solutions are not classical ones: in the (y, t) scale, they are given, similarly to the solitons, in an univalent way, but the transition to the original (x, t) scale makes them multivalued.

3.3. Solution of the Cauchy problem in terms of the solution of the RHP.

Assumption. In what follows we assume that the RH problem consisting in finding a piecewise, vector-valued function μ satisfying the jump condition (3.7a), the normalization condition (3.7b), and the appropriate residue conditions has a unique solution.

In order to obtain $u(x, t)$ from its solution $\mu(y, t, k)$, it turns out that it is convenient to evaluate $\mu(y, t, k)$ at distinguished finite points in the k -plane, more precisely, at the points

$$\kappa_\nu = e^{\frac{\pi i}{6}} + e^{\frac{\pi i}{3}(\nu-1)}, \quad \nu = 1, \dots, 6$$

characterized by the property $z(\kappa_\nu) = 0$.

Last Lax pair. Coming back to the system (2.3), let us introduce $\tilde{\Phi}^{(0)} = \tilde{\Phi}^{(0)}(x, t, z)$ by

$$\tilde{\Phi}^{(0)} = P^{-1}\Phi.$$

This reduces (2.3) to

$$\tilde{\Phi}_x^{(0)} - \Lambda(z)\tilde{\Phi}^{(0)} = \tilde{U}^{(0)}\tilde{\Phi}^{(0)}, \quad (3.11a)$$

$$\tilde{\Phi}_t^{(0)} - A(z)\tilde{\Phi}^{(0)} = \tilde{V}^{(0)}\tilde{\Phi}^{(0)}, \quad (3.11b)$$

where

$$\tilde{U}^{(0)}(x, t, z) = z^3 m(x, t) \begin{pmatrix} \frac{1}{3\lambda_1^2(z)-1} & 0 & 0 \\ 0 & \frac{1}{3\lambda_2^2(z)-1} & 0 \\ 0 & 0 & \frac{1}{3\lambda_3^2(z)-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (3.11c)$$

$$\tilde{V}^{(0)}(x, t, z) = P^{-1}(z) \begin{pmatrix} u_x & -u & 0 \\ u & 0 & -u \\ u_x - z^3 u(m+1) & 0 & -u_x \end{pmatrix} P(z). \quad (3.11d)$$

Now notice that $\tilde{U}^{(0)}(x, t, z)|_{z=0} \equiv 0$. Therefore, introducing $M^{(0)} = M^{(0)}(x, t, z)$ by

$$M^{(0)} = \tilde{\Phi}^{(0)} e^{-x\Lambda - tA},$$

and determining $M^{(0)}$ as the solution of a system of integral equations similar to the system (2.20) determining M , we have:

$$M^{(0)}(x, t, z)|_{z=0} \equiv I.$$

On the other hand, since $M^{(0)}$ and M are solutions of differential equations coming from the same system of differential equations (2.3), and since they have the same limit as $x \rightarrow +\infty$ for $k \notin \Sigma$:

$$M, M^{(0)} \xrightarrow{x \rightarrow +\infty} I,$$

it follows that they are related by

$$M(x, t, k) = P^{-1}(k) D^{-1}(x, t) P(k) M^{(0)}(x, t, k) e^{(x-y(x,t))\Lambda(k)}. \quad (3.12)$$

Particularly, at $k = \kappa_1 \equiv e^{\frac{\pi i}{6}}$ we have

$$\Lambda(e^{\frac{\pi i}{6}}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.13a)$$

$$P^{-1}(k) D^{-1}(x, t) P(k)|_{k=e^{\frac{\pi i}{6}}} = \frac{1}{2} \begin{pmatrix} 1 + \frac{1}{q} & 0 & -1 + \frac{1}{q} \\ 2(q - \frac{1}{q}) & 2q & 2(q - \frac{1}{q}) \\ -1 + \frac{1}{q} & 0 & 1 + \frac{1}{q} \end{pmatrix}. \quad (3.13b)$$

Now observe that

$$(1 \ 1 \ 1) P^{-1}(k) D^{-1}(x, t) P(k)|_{k=e^{\frac{\pi i}{6}}} = q(x, t) (1 \ 1 \ 1).$$

Combined with (3.12), this implies that the row vector solution $\mu(y, t, k)$ evaluated at $k = \kappa_1 \equiv e^{\frac{\pi i}{6}}$ takes the value

$$\begin{aligned} \mu(y, t, e^{\frac{\pi i}{6}}) &= q(x, t) (1 \ 1 \ 1) \begin{pmatrix} e^{-\int_x^\infty (q(\xi, t)-1)d\xi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\int_x^\infty (q(\xi, t)-1)d\xi} \end{pmatrix} \\ &= (q(x, t) e^{-\int_x^\infty (q(\xi, t)-1)d\xi} \quad q(x, t) \quad q(x, t) e^{\int_x^\infty (q(\xi, t)-1)d\xi}). \end{aligned} \quad (3.14)$$

Taking into account that in terms of functions of the variables (x, t) we have $u(y, t) = \frac{\partial x}{\partial t}(y, t)$ (this follows from (2.1) and from the definition (2.11) of the new variable y), the relation (3.14) provides a parametric representation of u .

Theorem 3.1. *Let $\mu \equiv \mu(y, t, k) = (\mu_1 \ \mu_2 \ \mu_3)$ be the solution of the Riemann–Hilbert problem (3.7a)–(3.7c), where $S(k)$ is the scattering matrix and $\{v_n\}$ are the residues associated with the initial data $u_0(x)$.*

Then the solution $u(x, t)$ of the Cauchy problem (1.1)–(1.2) for the Degasperis–Procesi equation can be expressed in terms of $\mu(y, t, k)$, evaluated at $k = e^{\frac{\pi i}{6}}$, in parametric form:

$$\begin{aligned} u(y, t) &= \frac{\partial}{\partial t} \log \frac{\mu_{j+1}}{\mu_j}(y, t, e^{\frac{\pi i}{6}}), \\ x(y, t) &= y + \log \frac{\mu_{j+1}}{\mu_j}(y, t, e^{\frac{\pi i}{6}}), \quad j = 1 \text{ or } 2. \end{aligned} \tag{3.15}$$

Notice that the structure of the parametric representation (3.15) is similar to that in the case of the Camassa–Holm equation. Moreover, this structure appears also in formulae for pure multisoliton solutions given in [21].

4. LONG-TIME ASYMPTOTICS

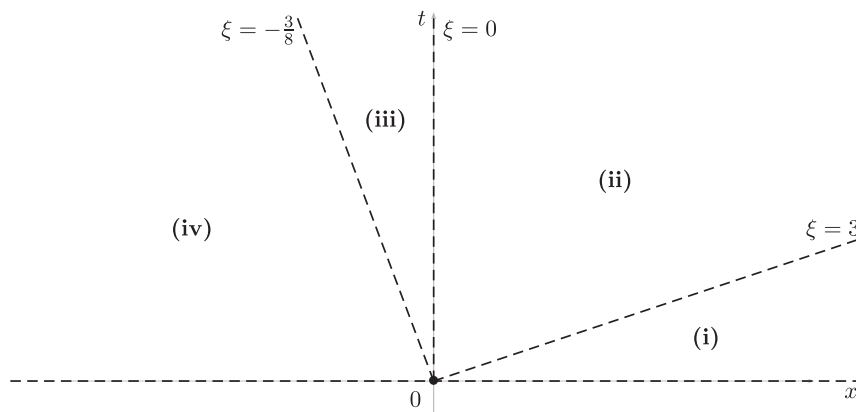
The analysis of the long-time behavior of the solution of the IVP is based on the analysis of the large- t behavior of the solution of the associated RH problem. The latter can be done in the framework of the nonlinear steepest descent method, whose key ingredient is the deformation of the original RH problem in accordance with the “signature table” for the phase functions involved in the jump matrix.

The analysis presented in the previous section shows that the structure of the jump matrix S , which is 3×3 , is essentially 2×2 : for each straight line of the contour Σ , a non-trivial block of $S(k)$ is 2×2 , see (3.4), (3.5) (under an appropriate change of basis), with only one exponential involved for each part, of the form (3.8). Now observe that this exponential is essentially the same as in the case of the Camassa–Holm equation (only the constant 2 is replaced by 3), provided ν plays the role of the spectral parameter. Consequently, the deformations of the each part of Σ are performed in the same way as the deformation of the real line in the case of the Camassa–Holm equation, leading to the similar asymptotic behavior of the solution, see [6, 4]:

- (i) in the sector $\frac{x}{t} > 3$, the long-time asymptotics is dominated by the solitons; in the case when there are no solitons, the asymptotics in this sector is fast decaying.
- (ii) for $0 < \frac{x}{t} < 3$, the asymptotics has the form of slowly decaying (as $t^{-1/2}$) modulated oscillations.
- (iii) for $-\frac{3}{8} < \frac{x}{t} < 0$, the asymptotics has the form of a sum of two slowly decaying modulated oscillations.
- (iv) $\frac{x}{t} < -\frac{3}{8}$, the asymptotics is fast decaying.

Besides, there are transition zones between the sectors, where the (slowly decaying) asymptotics is given in terms of the dedicated solutions of the Painlevé II equation.

The details of the asymptotics will be given elsewhere.

FIGURE 3. Sectors in the (x, t) -half-plane with different asymptotics

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